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Polarimetry with Phased Array Antennas: Theoretical Framework and Definitions

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***Abstract*—For phased array receivers, the accuracy with which the polarization state of a received signal can be measured depends on the antenna configuration, array calibration process, and beamforming algorithms. A signal and noise model for a dual-polarized array is developed and related to standard polarimetric antenna figures of merit, and the ideal polarimetrically calibrated, maximum-sensitivity beamforming solution for a dual-polarized phased array feed derived. A practical polarimetric beamformer solution that does not require exact knowledge of the array polarimetric response is shown to be equivalent to the optimal solution in the sense that when the practical beamformers are calibrated, the optimal solution is obtained. To provide a rough initial polarimetric calibration for the practical beamformer solution, an approximate single-source polarimetric calibration method is developed. The modeled instrumental polarization error for a dipole phased array feed with the practical beamformer solution and single-source polarimetric calibration was -10 dB or lower over the array field of view for elements with alignments perturbed by random rotations with 5 degree standard deviation.**

I. INTRODUCTION

The radio astronomy community is currently developing polarimetric aperture arrays and phased array feeds (PAFs) for large reflectors [1–4]. Accurate polarization state measurements for observed sources is critical to the science goals for current and planned phased array instruments. With a traditional waveguide feed, the polarization properties of the receiver are fixed at the time of manufacture and unwanted instrumental polarization can be calibrated by observing sources with known polarization parameters. For a phased array receiver, the polarimetric properties of each formed beam or image pixel can be adjusted on the fly by changing beamformer coefficients. If array output correlations are computed and stored, one set of observation data can be processed with multiple sets of beamformer coefficients tuned to optimize sensitivity, sidelobe level, or polarimetric accuracy. Exploiting this flexibility and achieving best possible system performance requires the development of a theory for polarimetric phased arrays, including figures of merit, optimal beamformer solutions, and practical calibration strategies.

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Key questions that must be answered by this theory include the following:

- How do astronomical performance criteria relate to the standard IEEE definitions for polarimetric antennas?
- What beamforming algorithm will simultaneously optimize for high SNR and polarimetric accuracy?
- Which requirements should be set on the antenna array and beamformer design to achieve optimal performance?
- How can a polarimetric array be accurately and efficiently calibrated?

This paper will consider the first two questions in detail and addresses the third empirically through a numerical study. An approximate single-source calibration scheme is presented to address the fourth issue. A full treatment of polarimetric calibration is beyond the scope of this paper and will be addressed in future work.

The first question arises because antenna engineers assess the polarimetric performance of antenna systems in terms of the axial ratio, cross-polarization discrimination (XPD), and cross-polarization isolation (XPI), while astronomers judge instrument performance and express system requirements in terms of Stokes parameters [5, 6]. Another challenge is that the standard IEEE definitions of the ARP, XPD and XPI have been established for single port systems, and these figures of merit must be extended to phased array systems that are capable of forming multiple dual-polarized beams simultaneously. Astronomical antenna applications also have unique constraints because radiation in terrestrial communication systems is typically highly polarized, whereas astronomical sources have a small but important polarized component of a few percent or less relative to the total signal flux density.

The starting point for answering these questions is the development of a signal and noise model for a polarimetric phased array receiver. This is accomplished in Sec. II. The treatment leads to the general problem of beamforming for rank-two signals and the concept of a polarimetric array beam pair. In Sec. III, the standard IEEE definitions for polarimetric figures of merit are related to the beam pair Jones matrix. In Sec. IV, the optimal beam pair solution is derived for a perfectly known instrument, and a practical beamforming method based on signal correlation matrix eigenvectors is considered for partially characterized polarimeters. An important result is that if sample estimation error (the error incurred by computing the array output correlation matrix from a finite number of voltage samples) is neglected, the practical eigenvector max-SNR method is equivalent to the optimal solution when polarimetrically calibrated. The performance of practical beamformers are compared to the ideal solution using a numerical model in Sec. V.

Voltage and field quantities are phasors with the $e^{j\omega t}$ convention. An overbar ($\overline{E}$) is used to denote three-dimensional field vectors, whereas vectors of voltages are typeset in boldface ($\mathbf{v}$). The superscript * designates the complex conjugate and ^{H} the conjugate transpose. $\langle \cdot \rangle$ denotes expectation over time.

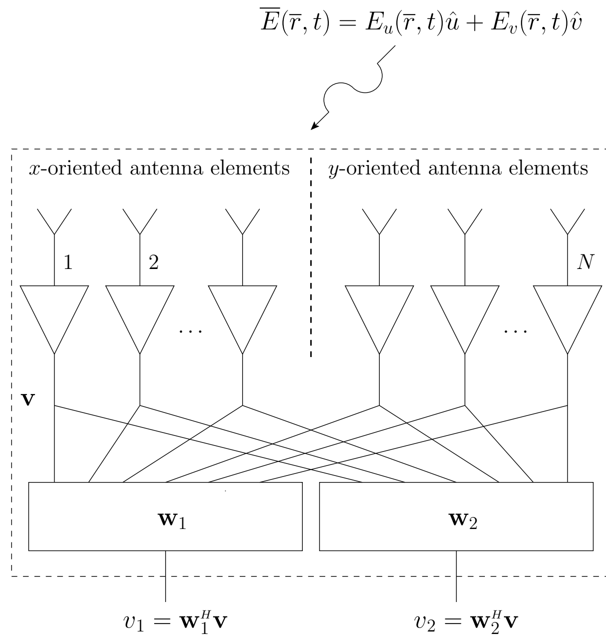

Fig. 1. A radio polarimeter comprised of a dual-polarized actively beamformed receiving antenna array.

II. Polarimetric Phased Array Model

The purpose of a radio polarimeter is to measure the polarization properties of an incident electromagnetic wave as a function of the angle of arrival. Figure 1 illustrates an N-element polarimetric beamforming array comprised of two groups of antenna elements with nominally orthogonal polarization. The antenna system is assumed to be illuminated by a point source radiating partially-polarized fields. The electric field intensity vector at the point $\overline{r}$ radiated by such a source can be approximated in the neighborhood of the receiver by the incident plane wave

$$\overline{E}\left(\overline{r},t\right) = \left[E_u\left(t\right)\hat{u} + E_v\left(t\right)\hat{v}\right]e^{j\overline{k}\cdot\overline{r}} \qquad (1)$$

where $\hat{u}$ and $\hat{v}$ are orthogonal unit vectors according to one of Ludwig's polarization definitions relative to the coordinate system of the array [7], and $\overline{k}$ is the wave vector corresponding to the angle of arrival. Since $\hat{u}$ and $\hat{v}$ are defined with respect to the coordinate system of the array, we are neglecting in this treatment a rotation of the polarization state from the astronomical coordinate system on the sky to the coordinate system of the array.

In the following theoretical development, we will consider the polarimetric calibration problem for one beam-steering direction. The calibration process must be applied to every formed phased array beam for each desired beam direction, as the numerical results make clear. Array calibration is accomplished with point sources, rather than extended sources, so all signal response quantities are assumed to arise from a point source. For an imaging array, each pixel represents a different beam steering direction and set of array beamforming coefficients. The figures of merit and beamforming procedures developed in this paper apply independently to each image pixel.

The source of interest is assumed to be a point source, and all results for figures of merit are calculated at the beam center. Since extended astronomical sources are common, beam polarization patterns are important, but this aspect of phased array polarimetry will be considered in future work.

The antenna output signals are amplified to form the N-element output voltage vector $\mathbf{v}$, which is subsequently combined into the output voltages v_1 and v_2 using the beamformer weight vectors $\mathbf{w}_1$ and $\mathbf{w}_2$ respectively, each of which is a column vector of size $N \times 1$. Together, $\mathbf{w}_1$ and $\mathbf{w}_2$ constitute a polarimetric beam pair for a given sky pointing direction[1].

The electric field components $E_u\left(\overline{r},t\right)$ and $E_v\left(\overline{r},t\right)$ are complex random processes in the phasor or complex baseband representation. The polarization state of the plane wave is determined by the covariance matrix of the two field components, which is the 2×2 Hermitian matrix

$$\mathbf{R}_{\overline{E}} = \begin{bmatrix} \langle |E_u|^2 \rangle & \langle E_u E_v^* \rangle \\ \langle E_u^* E_v \rangle & \langle |E_v|^2 \rangle \end{bmatrix}. \qquad (2)$$

[1] In the less general bi-scalar method, $\mathbf{w}_1$ and $\mathbf{w}_2$ are weight vectors of size $(N/2 \times 1)$ and are only associated to the x- and y-oriented antenna elements, respectively.

The covariance matrix has two real and one complex degrees of freedom, or four real degrees of freedom. The time-average power flux density of the incident wave is

$$S_{\rm sig} = \frac{\langle |E_u|^2\rangle + \langle |E_v|^2\rangle}{2\eta_0} = \frac{1}{2\eta_0}{\rm tr}[\mathbf{R}_{\overline{E}}] \quad (3)$$

where η_0 is the characteristic impedance of free space and tr is the matrix trace operation.

Some authors rearrange the covariance matrix to form the coherency vector [6]

$$\mathbf{c} = \mathrm{vec}\left(\mathbf{R}^*_{\overline{E}}\right) = \begin{bmatrix} \langle |E_u|^2\rangle \\ \langle E_u E_v^*\rangle \\ \langle E_u^* E_v\rangle \\ \langle |E_v|^2\rangle \end{bmatrix} \quad (4)$$

where $\mathrm{vec}(\mathbf{R}^*_{\overline{E}})$ is the vector obtained by stacking the columns of $\mathbf{R}^*_{\overline{E}}$. The relative magnitudes and phases of these quantities determine the degree of polarization and the polarization state of the polarized part of the wave.

A. Array Signal Response

With reference to Fig. 1, we will model the antenna array output signals $\{v_1, v_2\}$ in terms of the voltage responses of the array to the unit intensity, time-harmonic, linearly polarized waves

$$\overline{E}_u = \hat{u}e^{j\overline{k}\cdot\overline{r}} \implies \mathbf{v}_u \quad (5a)$$

$$\overline{E}_v = \hat{v}e^{j\overline{k}\cdot\overline{r}} \implies \mathbf{v}_v \quad (5b)$$

where $\mathbf{v}_u$ and $\mathbf{v}_v$ are vectors of the respective voltages induced by the two waves at the array receiver outputs before beamforming. Even though $\overline{E}_u^H\overline{E}_v = 0$, $\mathbf{v}_u$ and $\mathbf{v}_v$ may not be orthogonal due to possible non-ideal polarimetric response of the receiver.

The voltage responses $\mathbf{v}_u$ and $\mathbf{v}_v$ are not exactly known in practice, but are useful in developing a system model for a beamforming phased array antenna. We will show in Section IV-A that if $\mathbf{v}_u$ and $\mathbf{v}_v$ were known, the beam pair $\mathbf{w}_1$, $\mathbf{w}_2$ can be exactly calibrated polarimetrically. Experimental procedures for calibrating the array in the case that $\mathbf{v}_u$ and $\mathbf{v}_v$ are not known are discussed in Section IV-B.

Since $\hat{u}$ and $\hat{v}$ are orthogonal, it follows by linearity that for an arbitrarily polarized wave, the array signal voltage response vector can be written using $\mathbf{v}_u$ and $\mathbf{v}_v$ as

$$\mathbf{v}_{\rm s} = E_u\mathbf{v}_u + E_v\mathbf{v}_v \quad (6)$$

The $N\times N$ array output signal voltage covariance matrix is

$$\mathbf{R}_{\rm s} = \langle \mathbf{v}_{\rm s}\mathbf{v}_{\rm s}^H\rangle = \mathbf{v}_u\mathbf{v}_u^H\langle |E_u|^2\rangle + \mathbf{v}_u\mathbf{v}_v^H\langle E_uE_v^*\rangle + \mathbf{v}_v\mathbf{v}_u^H\langle E_u^*E_v\rangle + \mathbf{v}_v\mathbf{v}_v^H\langle |E_v|^2\rangle \quad (7)$$

which is of rank one for a fully polarized wave and of rank two for a partially polarized or unpolarized incident wave. Upon introducing the $N\times 2$ matrix

$$\mathbf{V} = [\mathbf{v}_u\ \mathbf{v}_v] \quad (8)$$

we can write (7) in the more compact form

$$\mathbf{R}_{\rm s} = \mathbf{V}\mathbf{R}_{\overline{E}}\mathbf{V}^H \quad (9)$$

Assuming that the phased array system noise can be characterized by the noise covariance matrix $\mathbf{R}_{\rm n}$, the complete array output voltage covariance matrix can be described as

$$\mathbf{R}_{\mathbf{v}} = \mathbf{R}_{\rm s} + \mathbf{R}_{\rm n} \quad (10)$$

This expression provides a signal and noise model for a polarimetric array. The precise form of the noise response $\mathbf{R}_{\rm n}$ for an array receiver is not important here, but is considered in detail in [8–10].

B. Beam Pair Signal and Noise Response

After beamforming, the two output voltages obtained with the beamformer pair are (*cf.* Fig. 1)

$$v_1 = \mathbf{w}_1^H\mathbf{v} \quad (11a)$$

$$v_2 = \mathbf{w}_2^H\mathbf{v}. \quad (11b)$$

The covariance matrix $\mathbf{R}_{\rm o}$ of the two beam outputs is

$$\mathbf{R}_{\rm o} = \left\langle \begin{bmatrix} v_1 \\ v_2 \end{bmatrix} \begin{bmatrix} v_1^* & v_2^* \end{bmatrix} \right\rangle = \begin{bmatrix} \langle |v_1|^2\rangle & \langle v_1v_2^*\rangle \\ \langle v_2v_1^*\rangle & \langle |v_2|^2\rangle \end{bmatrix}. \quad (12)$$

In terms of the array signal and noise covariance matrices, the beam pair output covariance matrix is

$$\mathbf{R}_{\rm o} = \begin{bmatrix} \mathbf{w}_1^H(\mathbf{R}_{\rm s}+\mathbf{R}_{\rm n})\mathbf{w}_1 & \mathbf{w}_1^H(\mathbf{R}_{\rm s}+\mathbf{R}_{\rm n})\mathbf{w}_2 \\ \mathbf{w}_2^H(\mathbf{R}_{\rm s}+\mathbf{R}_{\rm n})\mathbf{w}_1 & \mathbf{w}_2^H(\mathbf{R}_{\rm s}+\mathbf{R}_{\rm n})\mathbf{w}_2 \end{bmatrix}. \quad (13)$$

Introducing the $N\times 2$ vector

$$\mathbf{W} = [\mathbf{w}_1\ \mathbf{w}_2] \quad (14)$$

leads to the more compact expression

$$\mathbf{R}_{\rm o} = \mathbf{R}_{\rm o,s} + \mathbf{R}_{\rm o,n} \quad (15)$$

where

$$\mathbf{R}_{\rm o,s} = \mathbf{W}^H\mathbf{V}\mathbf{R}_{\overline{E}}\mathbf{V}^H\mathbf{W} \quad (16a)$$

$$\mathbf{R}_{\rm o,n} = \mathbf{W}^H\mathbf{R}_{\rm n}\mathbf{W}. \quad (16b)$$

This result provides a signal and noise model for the polarimetric beam pair $\mathbf{W}$.

III. Polarimetric Definitions

In this section, we consider the relationship between IEEE definitions for polarimetric figures of merit and the Jones and Mueller matrix formulations that are common in radio astronomy and remote sensing. We also consider the problem of quantifying the joint sensitivity of a polarimetric beam pair.

A. IEEE Definitions for Single Port Antennas

The IEEE definitions for polarization terms for single port antennas are as follows [5]:

Cross-polarization discrimination (XPD) is the ratio of the power level at the output of a receiving antenna, nominally co-polarized with the transmitting antenna, to the output of a receiving antenna of the same gain but nominally orthogonally polarized to the transmitting antenna.

Cross-polarization isolation (XPI) is the ratio of the wanted power to the unwanted power in the same receiver channel when the transmitting antenna is radiating nominally orthogonally polarized signals at the same frequency and power level.

Other figures of merit such as axial ratio can be applied to polarimetric phased arrays but will not be considered further in this paper.

B. Definitions for Phased Array Antennas

With reference to the polarimetric phased array model developed in Sec. II, the signal response of an array antenna to co- and cross-polarized incident fields E_u and E_v is described by the two beamformer output voltages

$$\begin{bmatrix} v_{1u} \\ v_{2u} \end{bmatrix} = \begin{bmatrix} \mathbf{w}_1^H \mathbf{v}_u \\ \mathbf{w}_2^H \mathbf{v}_u \end{bmatrix} \tag{17}$$

and

$$\begin{bmatrix} v_{1v} \\ v_{2v} \end{bmatrix} = \begin{bmatrix} \mathbf{w}_1^H \mathbf{v}_v \\ \mathbf{w}_2^H \mathbf{v}_v \end{bmatrix} \tag{18}$$

where $\mathbf{v}_u$ and $\mathbf{v}_v$ are the output voltage vectors of the receiving elements in response to pure $\overline{E}_u$ and $\overline{E}_v$ signals. The weight vectors $\mathbf{w}_1$ and $\mathbf{w}_2$ of the two beamformers are assumed to be defined such that they optimally receive or at least approximate the optimal reception of $\overline{E}_u$ and $\overline{E}_v$, respectively, according to some specified criterion to be discussed in Sec. IV. This implies that the polarizations of the two formed beams are nominally aligned with the $\hat{u}$ and $\hat{v}$ directions. While $\hat{u}$ and $\hat{v}$ are defined as real vectors in (1), the definitions in this section hold for any pair of waves $\overline{E}_u$ and $\overline{E}_v$ with orthogonal polarizations, whether linear, circular, or elliptical.

Ideally, one expects the two output voltage vectors to be directly proportional to $\overline{E}_u$ and $\overline{E}_v$, but this may not hold due to polarization leakage caused by imperfections in the element geometry or mutual coupling between the array elements, in particular when the antennas are placed in a finite array exhibiting strong truncation effects. In the latter case, the polarization characteristics of the embedded element patterns differ significantly from each other. The degree of polarization leakage is dependent not only on the array geometry and mechanical construction, but also on the values of the beamformer coefficients.

To quantify the beam-dependent polarization leakage, we extend the standard definitions in Sec. III-A as follows:

Cross-polarization discrimination is defined as the ratio of powers received at the beamformer outputs 1 and 2 due to the same incident field, i.e. $\overline{E}_u$ or $\overline{E}_v$:

$$\mathrm{XPD}_u = \frac{|v_{1u}|^2}{|v_{2u}|^2} = \frac{\mathbf{w}_1^H \mathbf{v}_u \mathbf{v}_u^H \mathbf{w}_1}{\mathbf{w}_2^H \mathbf{v}_u \mathbf{v}_u^H \mathbf{w}_2} = \frac{\mathbf{w}_1^H \mathbf{R}_{uu} \mathbf{w}_1}{\mathbf{w}_2^H \mathbf{R}_{uu} \mathbf{w}_2} \tag{19}$$

$$\mathrm{XPD}_v = \frac{|v_{2v}|^2}{|v_{1v}|^2} = \frac{\mathbf{w}_2^H \mathbf{v}_v \mathbf{v}_v^H \mathbf{w}_2}{\mathbf{w}_1^H \mathbf{v}_v \mathbf{v}_v^H \mathbf{w}_1} = \frac{\mathbf{w}_2^H \mathbf{R}_{vv} \mathbf{w}_2}{\mathbf{w}_1^H \mathbf{R}_{vv} \mathbf{w}_1} \tag{20}$$

where $\mathbf{R}_{uu} = \mathbf{v}_u \mathbf{v}_u^H$ and $\mathbf{R}_{vv} = \mathbf{v}_v \mathbf{v}_v^H$ are the signal covariance matrices in response to pure $\overline{E}_u$ and $\overline{E}_v$ signals of unit intensity.

Cross-polarization isolation is defined as the ratio of powers received at the same beamformer output, 1 or 2, due to orthogonally-polarized incident fields $\overline{E}_u$ and $\overline{E}_v$:

$$\mathrm{XPI}_1 = \frac{|v_{1u}|^2}{|v_{1v}|^2} = \frac{\mathbf{w}_1^H \mathbf{v}_u \mathbf{v}_u^H \mathbf{w}_1}{\mathbf{w}_1^H \mathbf{v}_v \mathbf{v}_v^H \mathbf{w}_1} = \frac{\mathbf{w}_1^H \mathbf{R}_{uu} \mathbf{w}_1}{\mathbf{w}_1^H \mathbf{R}_{vv} \mathbf{w}_1} \tag{21}$$

$$\mathrm{XPI}_2 = \frac{|v_{2v}|^2}{|v_{2u}|^2} = \frac{\mathbf{w}_2^H \mathbf{v}_v \mathbf{v}_v^H \mathbf{w}_2}{\mathbf{w}_2^H \mathbf{v}_u \mathbf{v}_u^H \mathbf{w}_2} = \frac{\mathbf{w}_2^H \mathbf{R}_{vv} \mathbf{w}_2}{\mathbf{w}_2^H \mathbf{R}_{uu} \mathbf{w}_2} \tag{22}$$

C. Jones Matrix Formulation

Unlike radio communications, for which antennas are fabricated to meet a fixed polarization purity requirement, applications such as radio astronomy and remote sensing that rely on accurate measurement of wave polarization states also require operational polarimetric calibration. When modeling system effects that contribute to the measured wave polarization state, it is convenient to use the Jones matrix formulation [11]. The Jones formulation allows matrix terms for various system components to be chained together into an overall Jones matrix that must be modeled or measured in order to infer the incident wave polarization state.

For a phased array, each polarimetric beam pair has an associated Jones matrix. The general relationship between the beam outputs and the incident electric field intensity vector is

$$\begin{bmatrix} v_1 \\ v_2 \end{bmatrix} = \begin{bmatrix} J_{11} & J_{12} \\ J_{21} & J_{22} \end{bmatrix} \begin{bmatrix} E_u \\ E_v \end{bmatrix} \tag{23}$$

where the two by two matrix on the right side of this equation is the beam pair Jones matrix, which we will denote in the following as $\mathbf{J}$. Using the notation developed above, for an array this relationship becomes

$$\begin{bmatrix} v_1 \\ v_2 \end{bmatrix} = \begin{bmatrix} \mathbf{w}_1^H(\mathbf{v}_u E_u + \mathbf{v}_v E_v) \\ \mathbf{w}_2^H(\mathbf{v}_u E_u + \mathbf{v}_v E_v) \end{bmatrix} = \begin{bmatrix} \mathbf{w}_1^H \mathbf{v}_u & \mathbf{w}_1^H \mathbf{v}_v \\ \mathbf{w}_2^H \mathbf{v}_u & \mathbf{w}_2^H \mathbf{v}_v \end{bmatrix} \begin{bmatrix} E_u \\ E_v \end{bmatrix} \tag{24}$$

The Jones matrix can be identified as

$$\mathbf{J} = \begin{bmatrix} \mathbf{w}_1^H \mathbf{v}_u & \mathbf{w}_1^H \mathbf{v}_v \\ \mathbf{w}_2^H \mathbf{v}_u & \mathbf{w}_2^H \mathbf{v}_v \end{bmatrix} = \mathbf{W}^H \mathbf{V} \tag{25}$$

where

$$\mathbf{W} = \begin{bmatrix} \mathbf{w}_1 & \mathbf{w}_2 \end{bmatrix}, \text{ and } \mathbf{V} = \begin{bmatrix} \mathbf{v}_u & \mathbf{v}_v \end{bmatrix}. \tag{26}$$

Using this result, the beam pair output signal voltage covariance matrix (16a) can be written as

$$\mathbf{R}_{\mathrm{o,s}} = \mathbf{J}\mathbf{R}_{\overline{E}}\mathbf{J}^H \tag{27}$$

The goal of polarimetric calibration is to transform a given beam pair $\mathbf{W}$ to a new beam pair $\mathbf{W}'$ for which the radiation patterns are steered to the same angle of arrival as the original beam pair but having Jones matrix $\mathbf{J}'$ as close as possible to the identity matrix. If $\mathbf{J}' \simeq \mathbf{I}$, the cross-polarization figures of merit are large in value. If the initial Jones matrix $\mathbf{J} = \mathbf{W}^H\mathbf{V}$ is known, it can be readily seen that the beam pair

$$\mathbf{W}' = \mathbf{W}\mathbf{J}^{H^{-1}} \tag{28}$$

is ideally polarimetrically calibrated and the realized Jones matrix is $\mathbf{J}' = \mathbf{W}'^H\mathbf{V} = \mathbf{I}$.

If the beam pair output signal coveriance matrix $\mathbf{R}_{\mathrm{o,s}}$ is measured for one signal with a known polarization state $\mathbf{R}_{\overline{E}}$, the relationship (27) does not uniquely fix the beam pair Jones matrix. Given known input and output polarization parameters, by substitution it can be shown that the general solution to (27) has the form

$$\mathbf{J} = \mathbf{R}_{\mathrm{o,s}}^{1/2}\mathbf{U}\mathbf{R}_{\overline{E}}^{-1/2} \tag{29}$$

where $\mathbf{U}$ is an arbitrary unitary matrix. This shows that with only a single calibrator source, there remains a 2×2 unitary degree of freedom in the Jones matrix. The physical interpretation of the unitary degree of freedom is discussed in [6, 12]. The ambiguity can be removed using knowledge of the nominal element polarizations (see Section IV-C), measurements of additional calibrator sources, or correlation with a second calibrated polarimetric antenna.

In terms of the elements of the Jones matrix, the cross-polarization figures of merit (19)–(22) are [13]

$$\mathrm{XPD}_u = \frac{|J_{11}|^2}{|J_{21}|^2} \qquad \mathrm{XPD}_v = \frac{|J_{22}|^2}{|J_{12}|^2} \tag{30a}$$

$$\mathrm{XPI}_1 = \frac{|J_{11}|^2}{|J_{12}|^2} \qquad \mathrm{XPI}_2 = \frac{|J_{22}|^2}{|J_{21}|^2} \tag{30b}$$

These expressions show that the standard antenna polarization figures of merit are measures of the magnitudes of the off-diagonal elements J_{12} and J_{21} relative to the diagonal elements J_{11} and J_{22}.

D. Mueller Matrix Formulation

While Jones matrices operate in the voltage phasor domain, 4×4 Mueller matrices represent transformations on wave polarization states in the correlation or Stokes parameter domain. Since the Mueller matrix formulation is commonly used in the astronomical literature, the treatment will be rehearsed here and placed into the mathematical framework developed in earlier sections.

For a signal characterized by the field coveriance matrix (2), the Stokes vector containing the four Stokes parameters is defined as [14, pp. 97–98]

$$\mathbf{S} = \begin{bmatrix} I \\ Q \\ U \\ V \end{bmatrix} = \begin{bmatrix} \langle|E_u|^2\rangle + \langle|E_v|^2\rangle \\ \langle|E_u|^2\rangle - \langle|E_v|^2\rangle \\ 2\,\mathrm{Re}\left[\langle E_u E_v^*\rangle\right] \\ 2\,\mathrm{Im}\left[\langle E_u E_v^*\rangle\right] \end{bmatrix} \tag{31}$$

Some authors define the Stokes parameters with a factor of $1/\eta_0$, where η_0 is the intrinsic impedance of space, so that the Stokes parameters have units of power density ($\mathrm{W/m^2}$).

It can be shown that [15, p. 29]

$$Q^2 + U^2 + V^2 \leq I^2. \tag{32}$$

The degree of polarization is

$$m = \frac{\sqrt{Q^2 + U^2 + V^2}}{I}. \tag{33}$$

For an unpolarized wave, $Q = U = V = 0$.

A Stokes polarimeter can be represented by the relationship

$$\mathbf{S}_{\mathrm{o}} = \mathbf{M}\mathbf{S}_{\mathrm{i}} \tag{34}$$

where $\mathbf{S}_{\mathrm{i}}$ is a vector of the Stokes parameters of the incident wave given by (31) and $\mathbf{M}$ is the Mueller matrix of the system. The vector of Stokes parameters referred to the beam output voltages is

$$\mathbf{S}_{\mathrm{o}} = \begin{bmatrix} \langle|v_1|^2\rangle + \langle|v_2|^2\rangle \\ \langle|v_1|^2\rangle - \langle|v_2|^2\rangle \\ 2\,\mathrm{Re}\,\langle v_1 v_2^*\rangle \\ 2\,\mathrm{Im}\,\langle v_1 v_2^*\rangle \end{bmatrix} \tag{35}$$

The beam pair is said to be polarimetrically calibrated if the Stokes parameters of the source, (31), and the measured Stokes parameters, (35), are identical, apart from a constant gain factor. In the following, we will include the gain calibration with polarimetric calibration, and consider the system to be polarimetrically calibrated when $\mathbf{M} = \mathbf{I}$.

The Mueller and Jones matrix formulations are connected by known formulas, which we will review here.

Stokes parameters are related to the coherency vector (4) by

$$\mathbf{S}_{\rm i} = \mathbf{T}\mathbf{c} \tag{36}$$

where

$$\mathbf{T} = \begin{bmatrix} 1 & 0 & 0 & 1 \\ 1 & 0 & 0 & -1 \\ 0 & 1 & 1 & 0 \\ 0 & -j & j & 0 \end{bmatrix} \tag{37}$$

The inverse of this transformation matrix is

$$\mathbf{T}^{-1} = \tfrac{1}{2} \begin{bmatrix} 1 & 1 & 0 & 0 \\ 0 & 0 & 1 & j \\ 0 & 0 & 1 & -j \\ 1 & -1 & 0 & 0 \end{bmatrix} \tag{38}$$

Using this relationship for the incident wave and the beam output voltages,

$$\mathbf{c}_{\rm o} = \mathbf{T}^{-1}\mathbf{M}\mathbf{T}\mathbf{c}_{\rm i} \tag{39}$$

Using the properties of the Kronecker product, (27) can be rearranged in the form

$$\mathbf{J}^* \otimes \mathbf{J}\, \mathrm{vec}\,(\mathbf{R}_{\overline{E}}) = \mathrm{vec}\,(\mathbf{R}_{\rm o,s}) \tag{40}$$

In view of (4), this becomes

$$\mathbf{J} \otimes \mathbf{J}^*\mathbf{c}_{\rm i} = \mathbf{c}_{\rm o} \tag{41}$$

Combining these results shows that the system Mueller matrix is related to the Jones matrix by (see also [16])

$$\mathbf{M} = \mathbf{T}(\mathbf{J} \otimes \mathbf{J}^*)\mathbf{T}^{-1} \tag{42}$$

In the present context, the Jones matrix $\mathbf{J}(\mathbf{w}_1, \mathbf{w}_2)$ is weight dependent, so that, by using (25) along with the properties of the Kronecker product,

$$\begin{aligned} \mathbf{M} &= \mathbf{T}(\mathbf{W}^H\mathbf{V}) \otimes (\mathbf{W}^T\mathbf{V}^*)\mathbf{T}^{-1} \\ &= \mathbf{T}(\mathbf{W}^H \otimes \mathbf{W}^T)(\mathbf{V} \otimes \mathbf{V}^*)\mathbf{T}^{-1}. \end{aligned} \tag{43}$$

If the polarimeter is ideally calibrated, then $\mathbf{J} = \mathbf{I}_{2\times 2}$, $\mathbf{M} = \mathbf{I}_{4\times 4}$, and $\mathbf{S}_{\rm o} = \mathbf{S}_{\rm i}$.

For a realistic system configured as a Stokes polarimeter, polarimetric calibration means measuring the beam pair Mueller matrix $\mathbf{M}$ and using that information to transform the weight pair $\mathbf{W}$, or in general adapt the beam pair covariance matrix $\mathbf{R}_{\rm o}$, such that the effective Mueller matrix becomes proportional to the identity matrix.

Polarimetric accuracy can be quantified by the error in the measured Stokes parameters for a given source or by the deviation of the Mueller matrix from the identity. The relative RMS Stokes error is

$$E_{\rm S} = \frac{\|\Delta\mathbf{S}\|}{I} = \frac{\sqrt{(\Delta I)^2 + (\Delta Q)^2 + (\Delta U)^2 + (\Delta V)^2}}{I} \tag{44}$$

This metric can be applied to computed Stokes parameters before the instrument is polarimetrically calibrated, to determine the raw or uncalibrated Stokes error, or it can be applied after calibration, to assess the performance of both the the calibration procedure and the native instrumental polarization properties.

As with the Jones formulation, source-independent measures of polarimetric accuracy are also desirable. By the definition of the induced operator norm, we have the bound

$$\frac{\|\Delta\mathbf{S}\|}{\|\mathbf{S}_{\rm i}\|} = \frac{\|(\mathbf{M}-\mathbf{I})\mathbf{S}_{\rm i}\|}{\|\mathbf{S}_{\rm i}\|} \le \|\mathbf{M}-\mathbf{I}\| \tag{45}$$

For weakly polarized sources, $\|\mathbf{S}_{\rm i}\| \simeq I$, and we have approximately

$$E_{\rm S} \le \|\mathbf{M}-\mathbf{I}\| \tag{46}$$

The quantity

$$E_{\rm M} = \|\mathbf{M}-\mathbf{I}\| \tag{47}$$

therefore is an approximate upper bound on the relative RMS Stokes error. $E_{\rm M}$ might be referred to as the Stokes instrumental polarization bound.

E. Beam Pair Sensitivity

The above treatment has considered traditional polarimetric formulations for the case of a phased array antenna. Another important aspect of beamforming for polarimetric arrays is that the beamformer coefficients control the beam output SNR and antenna sensitivity. For one beam, it is straightforward to determine the beamformer coefficients that maximize sensitivity given a knowledge of the array signal and noise responses. For a beam pair, the sensitivity of both beams should ideally be as high as possible. In particular, the beam sensitivities of an uncalibrated beam pair $\mathbf{W}$ and the polarimetrically calibrated beam pair $\mathbf{W}'$ are in general different, and there are many possible polarimetrically calibrated beam pairs that have lower sensitivity than the classical maximum-SNR beamformer solution. Therefore, it is of interest to characterize the sensitivity of both outputs of a beam pair with a metric that is independent of the beam pair polarimetric calibration.

For one beam $\mathbf{w}$, the sensitivity is

$$\frac{A_{\rm e}}{T_{\rm sys}} = \frac{k_{\rm b}B}{S_{\rm sig}} \frac{\mathbf{w}^H\mathbf{R}_{\rm s}\mathbf{w}}{\mathbf{w}^H\mathbf{R}_{\rm n}\mathbf{w}} \tag{48}$$

where B is the system noise equivalent bandwidth, $k_{\rm b}$ is Boltzmann's constant, and $S_{\rm sig}$ is the power flux density of the incident wave in W/m^2. As required by the definition of effective area, the incident wave is polarization matched with the polarization of the beam. The signal correlation matrix for a time-harmonic incident wave can be expressed as

$$\mathbf{R}_{\rm s} = \mathbf{V}\mathbf{E}\mathbf{E}^H\mathbf{V}^H \tag{49}$$

where $\mathbf{E} = [E_u \ E_v]^T$. The polarization of the incident wave is matched to the beam when $\mathbf{w}^H\mathbf{R}_\mathrm{s}\mathbf{w}$ is maximized, which occurs for the incident field state

$$\mathbf{E} = \sqrt{\frac{2\eta_0 S_\mathrm{sig}}{\mathbf{w}^H\mathbf{V}\mathbf{V}^H\mathbf{w}}}\mathbf{V}^H\mathbf{w} \tag{50}$$

where the scale factor follows from (3). Using these results in (48) leads to

$$\frac{A_\mathrm{e}}{T_\mathrm{sys}} = 2\eta_0 k_\mathrm{b} B \frac{\mathbf{w}^H\mathbf{V}\mathbf{V}^H\mathbf{w}}{\mathbf{w}^H\mathbf{R}_\mathrm{n}\mathbf{w}} \tag{51}$$

This same result can be obtained from (48) for an unpolarized wave ($\mathbf{R}_\mathrm{E} = \eta_0 S_\mathrm{sig}\mathbf{I}$) with S_sig in the leading scale factor replaced by half the power flux density of the unpolarized wave.

To define the beam pair sensitivity, we seek a figure of merit that is independent of the beam pair polarimetric calibration. This means finding bounds on the beam sensitivity (48) with $\mathbf{w}$ an arbitrary linear combination of $\mathbf{w}_1$ and $\mathbf{w}_2$. Using (51), the sensitivity is

$$\frac{A_\mathrm{e}}{T_\mathrm{sys}} = 2\eta_0 k_\mathrm{b} B \frac{\mathbf{a}^H\mathbf{W}^H\mathbf{V}\mathbf{V}^H\mathbf{W}\mathbf{a}}{\mathbf{a}^H\mathbf{W}^H\mathbf{R}_\mathrm{n}\mathbf{W}\mathbf{a}} \tag{52}$$

where $\mathbf{a} = [a_1 \ a_2]^T$ is an arbitrary vector. With $\mathbf{a}' = \mathbf{C}^{-1/2}\mathbf{V}^H\mathbf{W}\mathbf{a}$, where

$$\mathbf{C} = \mathbf{V}^H\mathbf{W}(\mathbf{W}^H\mathbf{R}_\mathrm{n}\mathbf{W})^{-1}\mathbf{W}^H\mathbf{V} \tag{53}$$

(52) becomes

$$\frac{A_\mathrm{e}}{T_\mathrm{sys}} = 2\eta_0 k_\mathrm{b} B \frac{\mathbf{a}'^H\mathbf{C}\mathbf{a}'}{\mathbf{a}'^H\mathbf{a}'} \tag{54}$$

This shows that the ratio of quadratic forms in (52) lies within the field of values of the matrix $\mathbf{C}$ [17]. Since $\mathbf{C}$ is Hermitian, the eigenvalues are real and the field of values is an interval on the real line. It follows that for any beamformer $\mathbf{w}$ in the subspace spanned by the beam pair the sensitivity is bounded by

$$2\eta_0 k_\mathrm{b} B \lambda_\mathrm{min} \le \frac{A_\mathrm{e}}{T_\mathrm{sys}} \le 2\eta_0 k_\mathrm{b} B \lambda_\mathrm{max} \tag{55}$$

where λ_min and λ_max are the eigenvalues of $\mathbf{C}$. By substituting $\mathbf{W}' = \mathbf{W}\mathbf{A}$, where $\mathbf{A}$ is an arbitrary invertible 2×2 matrix, it can be seen that the matrix $\mathbf{C}$ is independent of linear transformation of the beam subspace and hence of the beam pair polarimetric calibration.

Since the response $\mathbf{V}$ to orthogonal polarized waves is not directly available for a phased array in practice, it is of interest to express the sensitivity bound in terms of measurable array output quantities. In the Appendix it is shown that the sensitivity bound can be expressed in the form

$$\min_{\mathbf{S}_\mathrm{in}} \frac{k_\mathrm{b}B}{S_\mathrm{sig}}\mathrm{tr}[\mathbf{R}_\mathrm{o,n}^{-1}\mathbf{R}_\mathrm{o,s}] \le \frac{A_\mathrm{e}}{T_\mathrm{sys}} \le \max_{\mathbf{S}_\mathrm{in}} \frac{k_\mathrm{b}B}{S_\mathrm{sig}}\mathrm{tr}[\mathbf{R}_\mathrm{o,n}^{-1}\mathbf{R}_\mathrm{o,s}] \tag{56}$$

The calibration-independence of this form of the bound follows from the properties of the matrix trace.

We will refer to two beam pairs $\mathbf{W}$ and $\mathbf{W}'$ as sensitivity equivalent if they lead to the same upper and lower sensitivity bounds in (55) or (56). By the above derivation, if the beam pairs are related by a linear transformation, they are sensitivity equivalent, and a beam pair is sensitivity equivalent to its polarimetrically calibrated counterpart $\mathbf{W}\mathbf{J}^{H^{-1}}$. Finally, we observe that a polarimetrically calibrated beam pair does not necessarily achieve the upper bound in (55). The degree to which polarimetric calibration reduces beam sensitivity will be studied empirically in Section V.

IV. Polarimetric Beamforming

For a polarimetric phased array, calibration has two aspects:

1) determining the beamformer pair $\mathbf{W}$ so that sensitivity and other beam figures of merit are optimal, and;
2) determining the beam pair Jones matrix or the Mueller matrix so that the incident wave polarization state can be inferred from the beam pair outputs.

Combining 1) and 2) enables high sensitivity polarimetry. Beamformer coefficients applied to a phased array can be updated dynamically to accomplish various goals, such as sensitivity maximization as the noise environment changes, interference mitigation, or beam pattern sidelobe control. Consequently, the polarization properties of the array beam outputs are not necessarily fixed. This introduces the possibility of combining the beamformer optimization and polarimetric calibration steps, so that the beam pair satisfies a specified set optimality criteria and also is polarimetrically calibrated with both the Jones and Mueller matrices approximately equal to identity matrices.

We will consider the ideal case of a perfectly known system, in order to understand the ultimate performance limits of a given beam pair, as well as the practical case for which polarization properties of a beam pair must be calibrated empirically using observations of sources with known polarization parameters. To avoid sacrificing observation time for polarimetric calibration, one can observe with non-polarimetrically calibrated beam pairs having imperfect polarization discrimination. Therefore, it is important to find an approach for obtaining best possible polarimetric beam pairs using the single-source calibration data required for non-polarimetric array beamforming, and to characterize the polarization purity of beam pairs that are obtained without additional effort at polarimetric calibration.

A. Optimal Beamforming for a Perfectly Known System

Any beam pair that responds to two non-colinear incident polarizations can be polarimetrically calibrated, but the resulting beams may have poor sensitivity, which implies a low SNR and inaccurate measured Stokes parameters for a given integration time. We will show that there is a unique polarimetrically calibrated beam pair that minimizes estimation error in measured Stokes parameters for a point source at a given angle of arrival.

Measured Stokes parameters are a linear combination of the elements of a sample estimate of the 2×2 covariance matrix $\mathbf{R}_{\mathrm{o,s}}$ given by (16a). We will denote the sample estimated matrix for a given integration length K as $\hat{\mathbf{R}}_{\mathrm{o,s}}$. With the standard radiometric detection technique, $\hat{\mathbf{R}}_{\mathrm{o,s}}$ is the difference between an on-source measurement and off-source measurement of the beam pair output coveriance matrix:

$$\hat{\mathbf{R}}_{\mathrm{o,on}} = \mathbf{J}\hat{\mathbf{R}}_{\overline{E}}\mathbf{J}^H + \mathbf{W}^H\hat{\mathbf{R}}_{\mathrm{n,on}}\mathbf{W} \tag{57a}$$

$$\hat{\mathbf{R}}_{\mathrm{o,off}} = \mathbf{W}^H\hat{\mathbf{R}}_{\mathrm{n,off}}\mathbf{W} \tag{57b}$$

where $\hat{\mathbf{R}}$ signifies a sample-estimated coveriance matrix. The difference is

$$\hat{\mathbf{R}}_{\mathrm{o,diff}} = \hat{\mathbf{R}}_{\mathrm{o,on}} - \hat{\mathbf{R}}_{\mathrm{o,off}} = \mathbf{J}\hat{\mathbf{R}}_{\overline{E}}\mathbf{J}^H + \mathbf{W}^H(\hat{\mathbf{R}}_{\mathrm{n,on}} - \hat{\mathbf{R}}_{\mathrm{n,off}})\mathbf{W} \tag{58}$$

After polarimetrically calibrating the beam pair outputs, the output covariance matrix $\hat{\mathbf{R}}_{\mathrm{o,cal}}$ becomes

$$\hat{\mathbf{R}}_{\mathrm{o,cal}} = \mathbf{J}^{-1}\hat{\mathbf{R}}_{\mathrm{o,diff}}\mathbf{J}^{H^{-1}} = \hat{\mathbf{R}}_{\overline{E}} + \underbrace{\mathbf{J}^{-1}\mathbf{W}^H(\hat{\mathbf{R}}_{\mathrm{n,on}} - \hat{\mathbf{R}}_{\mathrm{n,off}})\mathbf{W}\mathbf{J}^{H^{-1}}}_{\hat{\mathbf{E}}_{\mathrm{n}}} \tag{59}$$

This is the same measurement equation that would be obtained with (58) subject to the calibrated beam pair

$$\mathbf{W}' = \mathbf{W}\mathbf{J}^{H^{-1}}, \text{ so that } \quad \mathbf{J}' = \mathbf{W}'^H\mathbf{V} = \mathbf{I} \tag{60}$$

Therefore, if Stokes parameters are computed by applying the beam pair inverse Jones matrix $\mathbf{J}^{-1}$ to the polarization parameters obtained from an uncalibrated beam pair $\mathbf{W}$, the measurement error due to system noise can be analyzed as if the beam pair were replaced by a polarimetrically calibrated beam pair.

The goal is to find the polarimetrically calibrated beam pair that minimizes the estimation error in (59), which we will write with (60) as

$$\hat{\mathbf{E}}_{\mathrm{n}} = \mathbf{W}'^H(\hat{\mathbf{R}}_{\mathrm{n,on}} - \hat{\mathbf{R}}_{\mathrm{n,off}})\mathbf{W}' \tag{61}$$

The matrix $\hat{\mathbf{R}}_{\mathrm{n}}$ is described stochastically by the Wishart distribution [18], which means that $\hat{\mathbf{E}}_{\mathrm{n}}$ is the difference of two identically distributed Wishart random matrices. The mean of $\hat{\mathbf{E}}_{\mathrm{n}}$ is the zero matrix. In the low SNR limit, the variance of the entries of $\hat{\mathbf{E}}_{\mathrm{n}}$ is

$$\mathrm{var}(\hat{E}_{\mathrm{n},kl}) = \frac{2}{P}R_{\mathrm{o,n},kk}R_{\mathrm{o,n},ll} \tag{62}$$

where P is the number of voltage samples that are averaged to produce $\hat{\mathbf{R}}_{\mathrm{o,on}}$ and $\hat{\mathbf{R}}_{\mathrm{o,off}}$. This result shows that in order to minimize estimation error in the Stokes parameters, the diagonal elements of the 2×2 beam pair output noise correlation matrix $\mathbf{R}_{\mathrm{o,n}}$ must be minimized.

This consideration reduces the polarimetric beamformer calibration problem to the joint constrained minimization problem

$$\operatorname*{argmin}_{\mathbf{w}_1} \mathbf{w}_1^H\mathbf{R}_{\mathrm{n}}\mathbf{w}_1, \ \operatorname*{argmin}_{\mathbf{w}_2} \mathbf{w}_2^H\mathbf{R}_{\mathrm{n}}\mathbf{w}_2 \tag{63}$$

$$\text{subject to } \mathbf{J} = \mathbf{W}^H\mathbf{V} = \mathbf{I} \tag{64}$$

Since $\mathbf{R}_{\mathrm{n}}$ is positive definite, we can add the two objective functions and recast the constrained minimization problem in the form

$$\operatorname*{argmin}_{\mathbf{W}} \mathrm{tr}(\mathbf{W}^H\mathbf{R}_{\mathrm{n}}\mathbf{W}), \ \text{subject to } \mathbf{W}^H\mathbf{V} = \mathbf{I} \tag{65}$$

The constrained optimization problem can be solved by setting to zero the matrix derivative

$$\nabla_{\mathbf{W}^H} \mathrm{tr}\left[\mathbf{W}^H\mathbf{R}_{\mathrm{n}}\mathbf{W} - (\mathbf{W}^H\mathbf{V} - \mathbf{I})\mathbf{\Lambda}\right] = \mathbf{0} \tag{66}$$

where $\mathbf{\Lambda}$ is a matrix of Lagrange multipliers, which will be chosen to satisfy the constraint in (64). Evaluating the derivative leads to

$$\mathbf{R}_{\mathrm{n}}\mathbf{W} - \mathbf{V}\mathbf{\Lambda} = \mathbf{0} \tag{67}$$

or

$$\mathbf{W} = \mathbf{R}_{\mathrm{n}}^{-1}\mathbf{V}\mathbf{\Lambda} \tag{68}$$

Using the constraint $\mathbf{W}^H\mathbf{V} = \mathbf{I}$, and solving for $\mathbf{\Lambda}$, we find that

$$\mathbf{\Lambda}^H\mathbf{V}^H\mathbf{R}_{\mathrm{n}}^{-1}\mathbf{V} = \mathbf{I} \tag{69}$$

from which it follows that

$$\mathbf{\Lambda}^H = \mathbf{\Lambda} = (\mathbf{V}^H\mathbf{R}_{\mathrm{n}}^{-1}\mathbf{V})^{-1}. \tag{70}$$

The solution to the optimization problem is found by substituting (70) in (68) to yield

$$\mathbf{W}_{\mathrm{opt}} = \mathbf{R}_{\mathrm{n}}^{-1}\mathbf{V}(\mathbf{V}^H\mathbf{R}_{\mathrm{n}}^{-1}\mathbf{V})^{-1}. \tag{71}$$

This pair of beamformer weight vectors minimizes the output system noise power while constrained to be polarimetrically calibrated with respect to a source with a given angle of arrival.

The optimal solution is related to a simple rank-two generalization of the classical maximum-SNR beamformer [19]:

$$\mathbf{W}_{\mathrm{maxSNR}} = \mathbf{R}_{\mathrm{n}}^{-1}\mathbf{V} \tag{72}$$

This beam pair realizes two maximum sensitivity beamformer weight vectors, with the Jones matrix $\mathbf{V}^H\mathbf{R}_\mathrm{n}^{-1}\mathbf{V} = \mathbf{V}^H\mathbf{W}_\mathrm{maxSNR} = \mathbf{J}_\mathrm{maxSNR}^H$. The optimal beamformer solution (71) can be written as

$$\mathbf{W}_\mathrm{opt} = \mathbf{W}_\mathrm{maxSNR}\mathbf{J}_\mathrm{maxSNR}^{H\,-1} \tag{73}$$

which shows that the beam pair $\mathbf{W}_\mathrm{maxSNR}$ is not in general polarimetrically calibrated but is sensitivity equivalent to the optimal beam pair.

B. Practical Polarimetric Array Calibration and the Eigenvector Method

In practice, the voltage response matrix $\mathbf{V}$ is difficult to measure for a PAF on a reflector antenna, since ideal, orthogonally polarized astronomical sources are unavailable. A practical polarimetric array calibration procedure using observations of unpolarized and partially polarized sources is needed.

For a dual-polarized array, neglecting estimation error, the signal covariance matrix $\mathbf{R}_\mathrm{s}$ is of rank two for an unpolarized source. The latter is readily concluded from (9) by taking $\mathbf{R}_{\overline{E}} = \mathbf{I}$, so that

$$\mathbf{R}_\mathrm{s} = \mathbf{V}\mathbf{R}_{\overline{E}}\mathbf{V}^H\big|_{\mathbf{R}_{\overline{E}}=\mathbf{I}} = \mathbf{v}_u\mathbf{v}_u^H + \mathbf{v}_v\mathbf{v}_v^H \tag{74}$$

which is a sum of two rank-one matrices. The eigenvalues and eigenvectors of $\mathbf{R}_\mathrm{s}$ are defined by

$$\mathbf{R}_\mathrm{s}\mathbf{v} = \lambda\mathbf{v} \tag{75}$$

Neglecting estimation error, two of the eigenvalues are nonzero. Veidt proposed to use the two principal eigenvectors $\mathbf{v}_{\mathrm{eig},1}$ and $\mathbf{v}_{\mathrm{eig},2}$ of $\mathbf{R}_\mathrm{s}$ as conjugate field match (CFM) beamformer weights [3]. We will refer to the beam pair

$$\mathbf{W}_\mathrm{CFM} = \mathbf{V}_\mathrm{eig} = [\mathbf{v}_{\mathrm{eig},1} \;\; \mathbf{v}_{\mathrm{eig},2}] \tag{76}$$

as the eigenvector CFM beamformer.

The eigenvector CFM method can be modified to form the maximum-SNR eigenvector beamformer weight vectors

$$\mathbf{w}_{\mathrm{eig},1} = \mathbf{R}_\mathrm{n}^{-1}\mathbf{v}_{\mathrm{eig},1} \tag{77a}$$
$$\mathbf{w}_{\mathrm{eig},2} = \mathbf{R}_\mathrm{n}^{-1}\mathbf{v}_{\mathrm{eig},2} \tag{77b}$$

This is the eigenvector max-SNR beamformer algorithm.

To compare this beamformer with the optimal solution presented in the previous section, we note that the voltage response vectors $\mathbf{v}_u$ and $\mathbf{v}_v$ must span the same subspace as the eigenvectors $\mathbf{v}_{\mathrm{eig},1}$ and $\mathbf{v}_{\mathrm{eig},2}$. This implies that

$$\mathbf{v}_u = A_{11}\mathbf{v}_{\mathrm{eig},1} + A_{21}\mathbf{v}_{\mathrm{eig},2} \tag{78a}$$
$$\mathbf{v}_v = A_{12}\mathbf{v}_{\mathrm{eig},1} + A_{22}\mathbf{v}_{\mathrm{eig},2} \tag{78b}$$

which is equivalent to

$$\mathbf{V} = [\mathbf{v}_{\mathrm{eig},1} \;\; \mathbf{v}_{\mathrm{eig},2}]\begin{bmatrix} A_{11} & A_{12} \\ A_{21} & A_{22}\end{bmatrix} = \mathbf{V}_\mathrm{eig}\mathbf{A} \tag{79}$$

Substitution in (71) gives

$$\begin{aligned}\mathbf{W}_\mathrm{opt} &= \mathbf{R}_\mathrm{n}^{-1}\mathbf{V}_\mathrm{eig}\mathbf{A}\left(\mathbf{V}^H\mathbf{R}_\mathrm{n}^{-1}\mathbf{V}_\mathrm{eig}\mathbf{A}\right)^{-1} \\ &= \mathbf{R}_\mathrm{n}^{-1}\mathbf{V}_\mathrm{eig}\left(\mathbf{V}^H\mathbf{R}_\mathrm{n}^{-1}\mathbf{V}_\mathrm{eig}\right)^{-1}\end{aligned} \tag{80}$$

which has a similar form as (71). Upon introducing $\mathbf{W}_\mathrm{eig} = \mathbf{R}_\mathrm{n}^{-1}\mathbf{V}_\mathrm{eig}$ and $\mathbf{J}_\mathrm{eig} = \mathbf{W}_\mathrm{eig}^H\mathbf{V}$, (80) can be written as

$$\mathbf{W}_\mathrm{opt} = \mathbf{W}_\mathrm{eig}\mathbf{J}_\mathrm{eig}^{H\,-1} \tag{81}$$

This result shows that the eigenvector max-SNR beam pair is sensitivity equivalent to the optimal beam pair.

A modification of the above approach can be obtained using the generalized eigenvalue form of the max-SNR beamformer:

$$\mathbf{R}_\mathrm{n}\mathbf{w} = \lambda\mathbf{R}_\mathrm{s}\mathbf{w} \tag{82}$$

Since $\mathbf{R}_\mathrm{s}$ is rank two, the generalized eigenvalue problem has two nonzero eigenvalues. The corresponding principle eigenvalues provide a maximum-SNR beam pair. It can be easily shown that this beam pair is also sensitivity equivalent to the optimal beam pair (71). The same beam pair can also be obtained by using the noise correlation matrix to pre-whiten the array output signals and using the eigenvector CFM method with the pre-whitened signal correlation matrix.

To summaruze, we have so far five polarimetric beamforming algorithms:

$$\mathbf{W}_\mathrm{opt} = \mathbf{R}_\mathrm{n}^{-1}\mathbf{V}(\mathbf{V}^H\mathbf{R}_\mathrm{n}^{-1}\mathbf{V})^{-1} \tag{83a}$$
$$\mathbf{W}_\mathrm{maxSNR} = \mathbf{R}_\mathrm{n}^{-1}\mathbf{V} \tag{83b}$$
$$\mathbf{W}_\mathrm{eig} = \mathbf{R}_\mathrm{n}^{-1}\mathbf{V}_\mathrm{eig} \tag{83c}$$
$$\mathbf{R}_\mathrm{n}\mathbf{w} = \lambda\mathbf{R}_\mathrm{s}\mathbf{w} \rightarrow \mathbf{W} \tag{83d}$$
$$\mathbf{W}_\mathrm{CFM} = \mathbf{V}_\mathrm{eig} \tag{83e}$$

The first and second beam pairs require knowledge of the array response $\mathbf{V}$ to orthogonally polarized incident waves, which is not directly available in practice. The third, fourth, and fifth are practical beamformers in the sense that they can be computed from measurable signal output correlation matrices for unpolarized or partially polarized sources. The first beam pair is exactly polarimetrically calibrated, but in general the others are not.

In the absence of estimation error, all of these beam pairs except for CFM are sensitivity equivalent. Sample estimation error in the array output correlation matrices may affect each beamformer algorithm differently, but the effect of estimation error on the beamformer weights can be driven to low levels by integrating the array output during the calibration phase for a long period of time, so this effect is not considered here.

C. Approximate Calibration with a Single Unpolarized Source

The practical beamforming algorithms, (83b)–(83e), are not polarimetrically calibrated. The eigenvectors computed from (75) are arbitrary up to a scale factor, and the resulting beam pair has a unitary degree of freedom according to (29). In order to give meaning to cross-polarization figures of merit, these degrees of freedom must be fixed by some concrete, repeatable algorithm.

Since each beam pair can be viewed as the output of a dual-polarized single pixel feed, any polarimetric calibration technique that is used for conventional feeds could be applied to calibrate the beam pair. Standard polarimetric calibration methods require observations of multiple sources with known Stokes parameters or tracking a polarized source over time as it rotates relative to the feed. Since these observations would have to be repeated with the telescope steered so that the source was at the center of each formed phased array beam, a straightforward implementation of standard polarimetric calibration could require many hours or even days. It is possible that efficient multiple-source calibration methods for phased arrays can be found, but a full treatment of efficient polarimetric calibration procedures for phased arrays is beyond the scope of this paper.

Since determining beamformer weights for a phased array requires observations of a bright (and typically unpolarized) source over a grid of telescope pointings at the center of each beam [20], the goal here is to find the best possible polarimetric calibration procedure given this already available observation data. We will show that a single unpolarized source observation per beam together with the known nominal element polarization can be used to calibrate each beam pair approximately.

Given an uncalibrated beam pair $\mathbf{W}$, the first step in the calibration procedure is to use the nominal polarization of array elements to find the approximate Jones matrix. The signal correlation matrix has the block form

$$\mathbf{R}_{\mathrm{s}} = \begin{bmatrix} \mathbf{R}_{xx} & \mathbf{R}_{xy} \\ \mathbf{R}_{yx} & \mathbf{R}_{yy} \end{bmatrix} \tag{84}$$

where x and y represent outputs from x-polarized elements and y-polarized elements, respectively. Neglecting estimation error, the matrices

$$\mathbf{R}_{\mathrm{s},x} = \begin{bmatrix} \mathbf{R}_{xx} & \mathbf{0} \\ \mathbf{0} & \mathbf{0} \end{bmatrix}, \quad \mathbf{R}_{\mathrm{s},y} = \begin{bmatrix} \mathbf{0} & \mathbf{0} \\ \mathbf{0} & \mathbf{R}_{yy} \end{bmatrix} \tag{85}$$

are rank one and have principal eigenvectors $\hat{\mathbf{v}}_x$ and $\hat{\mathbf{v}}_y$, respectively. These vectors are orthogonally rotated as needed to obtain $\hat{\mathbf{v}}_u$ and $\hat{\mathbf{v}}_v$, where the hat indicates that these are only approximate responses to orthogonally polarized waves, whereas $\mathbf{v}_u$ and $\mathbf{v}_v$ in (5) are exact. These vectors can be used directly as beamformer weights (the biscalar method), but we will employ them here to obtain the approximate Jones matrix

$$\hat{\mathbf{J}} = \mathbf{W}^H \hat{\mathbf{V}} \tag{86}$$

where $\hat{\mathbf{V}} = [\hat{\mathbf{v}}_u \;\; \hat{\mathbf{v}}_v]$. This approximate Jones matrix can be used to calibrate the beam pair to obtain

$$\mathbf{W}' = \mathbf{W}\hat{\mathbf{J}}^{H-1} \tag{87}$$

Since the eigenvectors $\hat{\mathbf{v}}_x$ and $\hat{\mathbf{v}}_y$ are only determined up to a scale factor, it remains to specify the complex scaling of the beamformer weight vectors $\mathbf{w}_1'$ and $\mathbf{w}_2'$ in the beam pair $\mathbf{W}'$. Assuming that the calibrator source is unpolarized, the magnitudes of the weight vectors can be fixed by equalizing the responses to the calibrator source. The overall phases of the weight vectors are set by dividing by the phase of the largest weight vector element, so that the largest weights are real. This leads to the approximately calibrated beam pair

$$\mathbf{w}_1'' = \mathbf{w}_1' \left(\frac{I/2}{\mathbf{w}_1'^H \mathbf{R}_{\mathrm{s}} \mathbf{w}_1'} \right)^{1/2} \frac{|w'_{1,\max}|}{w'_{1,\max}} \tag{88a}$$

$$\mathbf{w}_2'' = \mathbf{w}_2' \left(\frac{I/2}{\mathbf{w}_2'^H \mathbf{R}_{\mathrm{s}} \mathbf{w}_2'} \right)^{1/2} \frac{|w'_{2,\max}|}{w'_{2,\max}} \tag{88b}$$

In practice, the phase lengths of the receiver signal paths will be different, but this phase can be measured using injected calibration tones or other means and incorporated in the phase constraint. The magnitude scaling in (88) ensures that the diagonal elements of $\hat{\mathbf{J}}\hat{\mathbf{J}}^H$ are unity, which through (42) implies that the element M_{11} of the Mueller matrix is equal to unity.

This approximate single-source polarimetric calibration procedure is essentially equivalent to a polarimeter that uses the nominally orthogonally polarized outputs of a standard dual-polarized antenna (including a rotation from the astronomical coordinate system to the coordinate system of the antenna). As with a standard antenna, coupling between phased array elements and mechanical imperfections mean that the beam pair will not be exactly calibrated, and additional observations of polarized sources may be required to remove residual instrumental polarization effects. Numerical results will be given in the following section to assess the performance of this single-source polarimetric calibration method.

V. Numerical Results

The polarimetric figures of merit and beamforming algorithms will be illustrated for a 19×2 element hexagonal array of thin, lossless, x- and y-polarized crossed dipoles spaced $0.6\,\lambda$ apart and backed by a ground plane. The phased array feeds a 20-meter reflector with $f/D = 0.43$. The open-circuited element response is approximated using the analytical expression

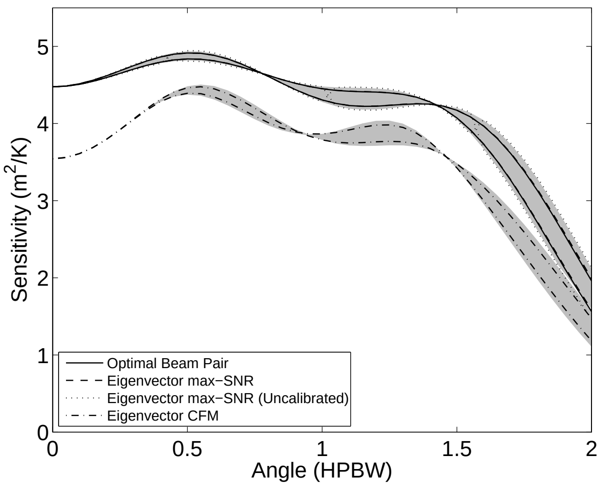

Fig. 2. Beam pair sensitivity for dipole array with perfectly aligned elements for beam steering angles from the feed boresight to 2 half power beamwidths (HPBW) from boresight. The cut is at an azimuth angle of 30° from the x-direction in the feed coordinate system. The sensitivity bound in Equation (55) is indicated with gray shading.

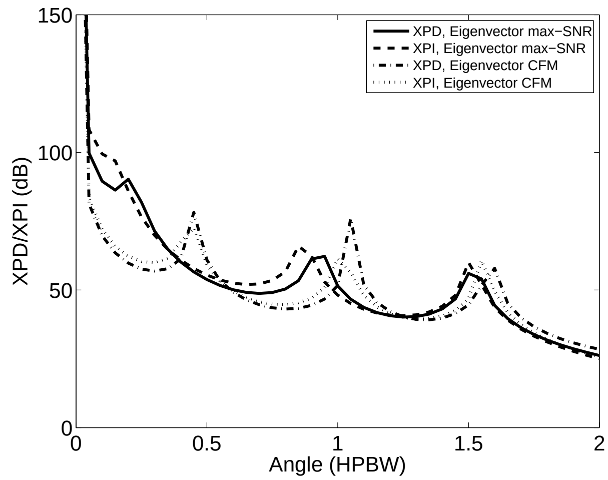

Fig. 3. Beam center cross-polarization figures of merit as a function of steering angle for dipole array with perfectly aligned elements.

for radiation by a dipole. Physical optics is used to compute secondary fields scattering from the reflector. The array mutual impedance matrix is approximated by conservation of energy from the element pattern overlap integrals. The PAF noise model includes sky noise, spillover, and receiver noise due to low noise amplifiers with parameters $\Gamma_{\rm opt} = 0$, $R_{\rm n} = 3.4\,\Omega$, and $T_{\rm min} = 33\,{\rm K}$. Ludwig's first convention [7] is used to define the u and v directions, which means that u and v are aligned with the array x- and y-coordinate frame.

The array is calibrated using the approach of [20] using observations of an unpolarized calibrator source for each desired beam steering direction. Beamformer weights are computed using the eigenvector max-SNR and CFM algorithms described above, and polarimetrically calibrated using the approximate single-source method of Section IV-C.

Results are given for two array configurations: (1) perfectly aligned elements and (2) perturbed element orientations with small random rotations away from the nominal x- and y-polarizations.

A. *Perfectly Aligned Elements*

The first study is a comparison of the sensitivity of the eigenvector max-SNR and CFM beam pairs to the optimal solution (Figure 2). The single-source calibration procedure of Section IV-C is used with both beamformers. The sensitivity of the max-SNR beam pair calibrated with the single-source procedure is nearly indistinguishable from that of the optimal solution. For the polarimetrically uncalibrated case, the max-SNR beam pair sensitivity is different, but still lies within the upper and lower bounds over arbitrary linear transformations of the beam pair. The beam pair sensitivity bounds are given by (55) and indicated in the figure with gray shading. For some beam steering angles, the sensitivity of one of the uncalibrated beams is larger than that of the optimal beam pair. This illustrates that the calibration constraint in (65) leads to a beam pair with sensitivity slightly smaller than is achieved by other uncalibrated beam pairs in the subspace spanned by the optimal beam pair. The single-source calibrated CFM beam pair sensitivity also lies within its corresponding bound (55), but the overall sensitivity is lower than that of max-SNR.

Cross-polarization figures of merit are shown in Figure 3. Results are given for each beam center as a function of beam steering angle. For the beam steered in the boresight direction, the polarimetric calibration is essentially perfect, but for beams steered off-boresight, the figures of merit decrease. Results are shown for the u polarization. Curves for the v polarized beams are nearly identical.

Error in measured Stokes parameters relative to the total source intensity for the single-source calibrated max-SNR and CFM beam pairs is shown in Figure 4. The source Stokes parameters are tabulated values from the NRAO C Band VLA/VLBA Polarization Calibration Database for 0371+331: $I = 5.0522\,{\rm Jy}$, $Q = 0.1869$, $U = 0.1109$, $V = 0$. The error closely follows the instrumental polarization bound (47) for both beam pairs.

It may be surprising that the perfectly aligned array, with no receiver gain imbalances, no mechanical defects, and exact analytical formulas for the element response, is only exactly calibrated for the boresight beam. For the boresight beam, the response vectors $\mathbf{v}_u$ and $\mathbf{v}_v$ in (7) are orthogonal, and the eigenvectors obtained from (75) are proportional to $\mathbf{v}_u$ and $\mathbf{v}_v$. Thus, at boresight

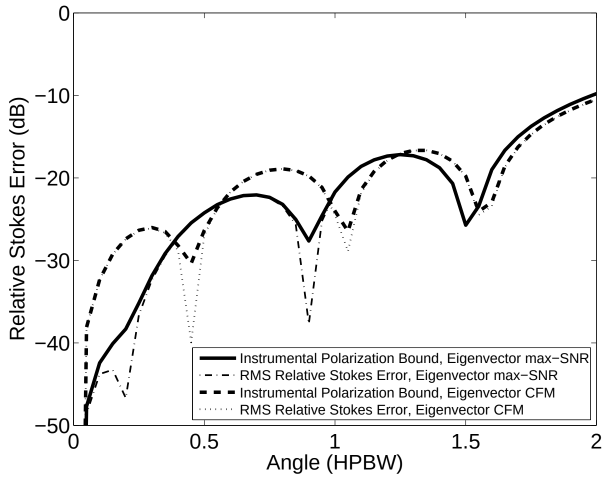

Fig. 4. Relative Stokes parameter error (44) and instrumental polarization bound (47) as a function of steering angle for dipole array with perfectly aligned elements.

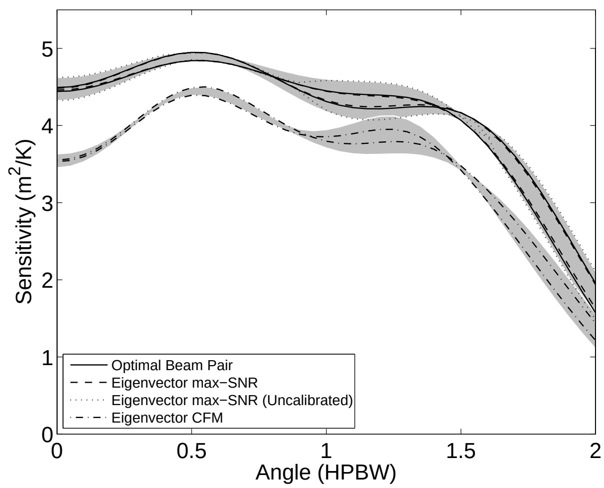

Fig. 5. Beam center sensitivity for dipole array with perturbed element orientations. The cut is at an azimuth angle of 30° from the x-direction in the feed coordinate system. The sensitivity bound (55) is indicated with gray shading.

the eigenvector max-SNR and CFM beam pairs can be exactly calibrated without making use of the biscalar transformation (87). For off-boresight steered beams, due to the depolarizing effect of the reflector, $\mathbf{v}_u$ and $\mathbf{v}_v$ are not orthogonal, and the eigenvector pair $\mathbf{v}_{\text{eig},1}$ and $\mathbf{v}_{\text{eig},2}$ for the perfectly aligned array require polarimetric calibration.

The Square Kilometer Array (SKA) design target for relative instrumental polarization is 25 dB for general polarimetric imaging and 40 dB for specialized applications (see [21] and other available SKA documents). For the perfectly aligned array, the instrumental polarization is better than 40 dB only for beam steering angles near boresight. For off-boresight beams, it is apparent that an additional polarimetric calibration beyond the approximate single-source method of Section IV-C is required.

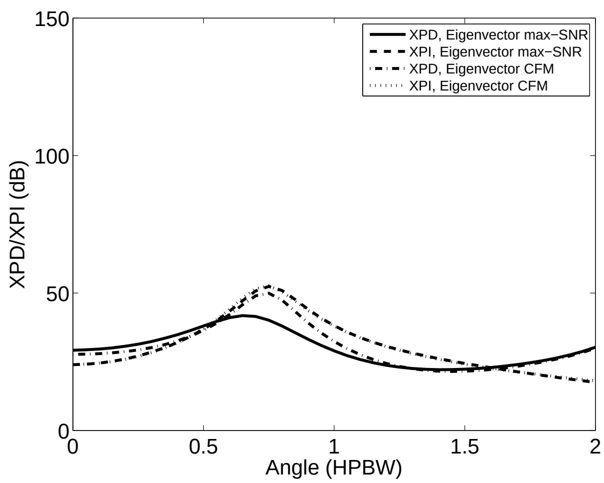

Fig. 6. Beam center cross-polarization figures of merit for dipole array with perturbed element orientations.

B. Imperfectly Aligned Elements

We now study the effect of mechanical imperfections on array feed polarimetric performance. A rotation in the x-y plane was applied to the orientation of each element in the array, with rotation angle chosen from a zero-mean normal distribution with 5° standard deviation. Figure 5 shows that the beam sensitivity is not reduced by the perturbation, but the upper and lower limits of the sensitivity bound (55) are no longer equal for the boresight beam. The cross-polarization figures of merit (Figure 6) and Stokes instrumental polarization (Figure 7) are significantly degraded. The instrumental polarization is poorer than 25 dB for many beam steering angles. For observations that require better polarimetric accuracy, the nominal calibration approach presented in this paper is not adequate, and a further calibration step for each beam pair similar to the methods used for traditional single-pixel feeds would be required.

VI. CONCLUSIONS

A signal and noise model for a dual-polarized beamforming array receiver has been used to develop polarization figures of merit and a theory of ideal polarimetric beamforming. An optimal polarimetrically calibrated maximum-SNR beamforming algorithm was given. For each angle of arrival, the algorithm provides a beam pair with maximum sensitivity subject to a polarimetric calibration constraint.

The optimal polarimetric beamforming algorithm requires exact knowledge of the responses of the array

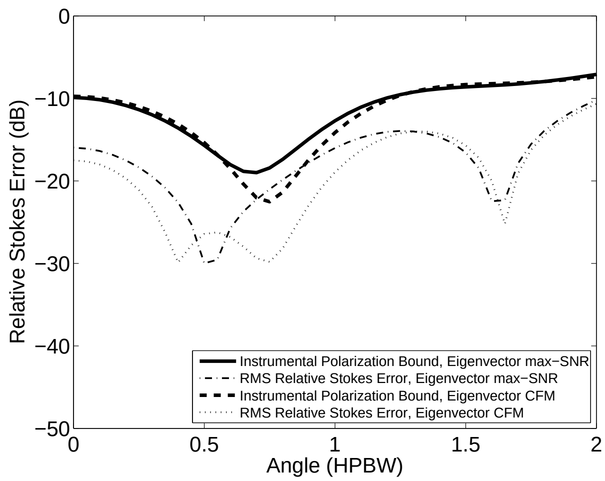

Fig. 7. Relative Stokes parameter error and instrumental polarization bound for dipole array with perturbed element orientations.

to orthogonally polarized waves from a given angle of arrival. In practice, the array responses to orthogonally polarized waves are not readily available. A practical eigenvector-based maximum-SNR beamformer solution has been shown to optimize sensitivity in the sense that when the beams are polarimetrically calibrated, in the absence of estimation error the beams become equivalent to the optimal solution. In general, the eigenvector maximum-SNR beam pair is not polarimetrically calibrated, and additional correction through estimation of the beam pair Jones matrix is required. An approximate single-source calibration procedure based on the approximate orthogonality of the array element polarizations was developed. The instrumental polarization of the approximately calibrated beams for a perturbed dipole phased array feed was better than 10 dB over the field of view. The approximate single-source method therefore provides a rough calibration method for routine observations that do not require high polarimetric accuracy.

In future work, more accurate polarimetric calibration methods for phased arrays should be studied. To achieve lower instrumental polarization than the single-source method used in this paper, standard polarimetric calibration techniques could be applied on a beam-by-beam basis to achieve lower instrumental polarization. To avoid time-consuming observations of multiple calibrator sources or long observations of a single polarized source for each formed beam, methods for reducing the number of required telescope pointings required to calibrate all formed beams are needed. Beam cross-polarization response patterns and the temporal stability of formed beam polarization responses are also of interest.

APPENDIX

Here we derive (56). By diagonalizing $\mathbf{C}$ in (53), it can be shown that $\mathrm{tr}[\mathbf{C}\mathbf{R}_\mathrm{E}]$ is bounded by

$$\lambda_{\min}\mathrm{tr}[\mathbf{R}_\mathrm{E}] \le \mathrm{tr}[\mathbf{C}\mathbf{R}_\mathrm{E}] \le \lambda_{\max}\mathrm{tr}[\mathbf{R}_\mathrm{E}] \quad (89)$$

In view of (3) and (55), the sensitivity of any beam in the beam pair subspace is therefore bounded by the largest and smallest values of

$$\alpha(\mathbf{R}_{\overline{E}}) = \frac{k_\mathrm{b}B}{S_\mathrm{sig}}\mathrm{tr}[\mathbf{V}^H\mathbf{W}(\mathbf{W}^H\mathbf{R}_\mathrm{n}\mathbf{W})^{-1}\mathbf{W}^H\mathbf{V}\mathbf{R}_{\overline{E}}] \quad (90)$$

Using the invariance of the trace with respect to the ordering of a product of two matrices, this can be expressed as

$$\alpha(\mathbf{R}_{\overline{E}}) = \frac{k_\mathrm{b}B}{S_\mathrm{sig}}\mathrm{tr}[(\mathbf{W}^H\mathbf{R}_\mathrm{n}\mathbf{W})^{-1}\mathbf{W}^H\mathbf{V}\mathbf{R}_{\overline{E}}\mathbf{V}^H\mathbf{W}] \quad (91)$$

Using (16), $\alpha(\mathbf{R}_{\overline{E}})$ becomes

$$\alpha(\mathbf{R}_{\overline{E}}) = \frac{k_\mathrm{b}B}{S_\mathrm{sig}}\mathrm{tr}[\mathbf{R}_\mathrm{o,n}^{-1}\mathbf{R}_\mathrm{o,s}] \quad (92)$$

This leads directly to the bound (56). Because of the properties of the trace, replacing $\mathbf{W}$ with the calibrated beam pair $\mathbf{W}' = \mathbf{W}\mathbf{J}^{\mathrm{H}^{-1}}$ does not change the value of the trace:

$$\begin{aligned}\mathrm{tr}[\mathbf{R'}_\mathrm{o,n}^{-1}\mathbf{R}'_\mathrm{o,s}] &= \mathrm{tr}[(\mathbf{J}^{-1}\mathbf{R}_\mathrm{o,n}\mathbf{J}^{H^{-1}})^{-1}\mathbf{J}^{-1}\mathbf{R}_\mathrm{o,s}\mathbf{J}^{H^{-1}}] \\ &= \mathrm{tr}[\mathbf{R}_\mathrm{o,n}^{-1}\mathbf{R}_\mathrm{o,s}] \quad (93)\end{aligned}$$

which shows that the trace of $\mathbf{R}_\mathrm{o,n}^{-1}\mathbf{R}_\mathrm{o,s}$ is invariant with respect to polarimetric calibration of the beam pair, and the bound in (55) is a polarimetric calibration-independent measure of the intrinsic beam pair sensitivity.

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