Adiabatic Description of Long Range Frequency Sweeping

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A theoretical framework, built on the notion of nearly periodic fast particle orbits, is developed to describe long range frequency sweeping events in the 1D electrostatic bump-on-tail model with Krook collisions and dynamical friction (drag). Both the actual deviation from the linear mode frequency and the effect of particle trapping in the wave field due to an increasing wave amplitude affect the frequency sweeping rate. For upward sweeping holes, hooked and steady state frequency sweeping patterns are found as results of an interplay between Krook and drag collisions.

I. Introduction

Driven kinetic systems arise naturally far away from thermodynamic equilibrium, and magnetically confined plasmas typically exhibit a wide variety of instabilities that can be treated kinetically [1]. For example, fast ions can excite Alfvénic waves capable of enhancing energetic particle losses [2]. The role of dissipation in such systems is far from trivial: Near the instability threshold, coherent phase space structures (holes and clumps) with time dependent mode frequencies form spontaneously in the fast particle distribution function. Such frequency sweeping events are commonly observed [3, 4, 5, 6, 7], and have been modeled in nonlinear simulations of the 1D bump-on-tail (BOT) instability [8].

The BOT simulations confirmed the picture that fast particle Krook type collisions [9] and velocity space diffusion [10] suppress holes and clumps, whereas drag (dynamical friction) acts as a seed for hole/clump formation [11]. In fact, the collisional evolution of the near threshold BOT system is much more diverse than in the collisionless limit, with drag causing asymmetric frequency sweeping, exhibiting e.g. steady state and hooked hole behaviours [8]. These features resemble qualitatively observations from several experiments (see e.g. [12]), which emphasizes the importance of drag for the understanding of experimental results. The BOT simulations were however limited to short range frequency sweeping phenomena, in which the linear mode structure is preserved. In contrast, long range frequency sweeping involves significant changes in the mode amplitude and structure, which effectively alters the resulting sweeping rates. In the absence of fast particle collisions, this was recently described in [13] for the case when the wave field does not grow in time.

In this contribution, we develop a framework which generalizes the formalism in [13] to include the effects of fast particle sources and collisions, and to allow for a general evolution of the wave field. Specifically, we turn our attention to the special case of non-diffusive fast particle collisions. The work presented herein thus constitutes a necessary first step towards predictive modeling of long range frequency sweeping events in realistic plasma geometries.

II. Adiabatic Bump-On-Tail Model

The system under consideration is the one dimensional, spatially periodic BOT configuration. We investigate an electrostatic perturbation with period λ in a plasma containing three particle species: A population of static ions; a fluid background of cold electrons characterised by mass m_e, density n_e and perturbed fluid velocity v_e, which are subject to weak collisions with collision frequency $2\gamma_d$; a low density kinetic population of energetic electrons. The energetic electrons are assumed to suffer infrequent collisions with the background species, and the velocity gradient of their equilibrium distribution function F_0 provides a linear drive γ_L.

The starting set of equations is then given by the linearized fluid equation for the cold electrons[1]

$$\frac{\partial v_e}{\partial t} = -\frac{1}{m_e}\frac{\partial U}{\partial x} - 2\gamma_d v_e \ , \tag{1a}$$

the Poisson equation

$$\frac{\partial^2 U}{\partial x^2} = -\frac{e^2}{\epsilon_0}\left[\delta n_e + \delta n_f\right] \ , \tag{1b}$$

continuity equations describing conservation of particles within the cold and fast electron populations, respectively, i.e.

$$\frac{\partial n_j}{\partial t} + \frac{\partial}{\partial x}\left(n_j v_j\right) = 0 \ , \tag{1c}$$

and a kinetic equation, viz.

$$\frac{df}{dt} = \mathfrak{C}\left(f\right) = -\beta\left(f - F_0\right) + \frac{\alpha^2}{k}\frac{\partial}{\partial v}\left(f - F_0\right) + \frac{\nu^3}{k^2}\frac{\partial^2}{\partial v^2}\left(f - F_0\right) \ , \tag{1d}$$

describing the evolution of the fast electron distribution function f. Here, the electrostatic potential ϕ of the wave is represented by the potential energy $U \equiv -\left|e\right|\phi$, the perturbed fast electron density is given by

$$\delta n_f = \int \left(f - F_0\right) dv \ , \tag{2}$$

and the fast electron collision operator is written as a superposition of Krook, drag (dynamical friction) and velocity space diffusion, where the relative strengths are set by the collision frequencies β, α and ν, and $k \equiv 2\pi/\lambda$ is the wavenumber. Note that (1b) and (1c) can be combined to yield Ampére's law, which has traditionally been used to analyze the BOT model.

The system of equations (1), with (2), describes in full generality the BOT system, including the rapid evolution during the formation stage of phase space holes and clumps in the near threshold limit. In this contribution, we are however only concerned with long range sweeping and the associated slow evolution of already established phase space structures. Thus, the perturbation of interest to this investigation is a single travelling wave with a time dependent frequency, so that the U

$$U = U\left(x - s\left(t\right); t\right) \ , \tag{3}$$

where the wave phase velocity $\dot{s}\left(t\right)$ changes slowly in time compared to $s\left(t\right)$, i.e. $d\ln\dot{s}/dt \ll \omega = k\dot{s}$. Here, U is assumed to be periodic in its rapidly varying first argument, describing oscillations at the wave frequency ω, and slowly changing with respect to its

[1] This linear equation of motion is motivated as long as the mode amplitude is low enough.

second argument, describing the evolution of the wave amplitude and structure. Thus, $d\ln\omega_B/dt \ll \omega$, where ω_B, the bounce frequency of the resonant electrons trapped in the wave field, serves as a measure of the wave amplitude.

In fact, we invoke the so called adiabatic ordering

$$\left[\frac{d\ln\omega_B}{dt}, \frac{d\ln\dot{s}}{dt}\right] \ll \omega_B \ll \omega \ , \tag{4}$$

which has a number of consequences. To begin with, (4) allows the equations (1a) and (1c) (for the cold electrons) to be combined into

$$\delta n_e = \frac{n_{e0}}{m_e\dot{s}^2}U \tag{5}$$

The Poisson equation (1b) can then be written as

$$\frac{\partial^2 U}{\partial x^2} = -\frac{\omega_p^2}{\dot{s}^2}U - \frac{e^2}{\epsilon_0}\left[\int (f - F_0)\,dv - \left\langle \int (f - F_0)\,dv \right\rangle_\lambda\right] \ , \tag{6}$$

where $\langle \ldots \rangle_\lambda$ denotes averaging over the wavelength, and conservation of particles has been taken into account, so that $\langle U \rangle_\lambda = 0$. Thus, in the adiabatic limit (4), the perturbed potential energy U corresponds to a BGK wave [14] with a slowly changing frequency and amplitude, which always satisfies (6).

Secondly, the adiabatic ordering simplifies considerably the treatment of the fast electrons. The fast electron motion is governed by the wave frame Hamiltonian

$$\mathcal{H}_z = \frac{\left(p - m_e\dot{s}\,(t)\right)^2}{2m_e} + U\left(z \equiv x - s\,(t)\,;t\right) \ , \tag{7}$$

which, at any moment, defines a separatrix in (z, p) phase space centered around $p = m_e\dot{s}$, thus separating passing electrons (with wave frame energy $\mathcal{E}_z > U_{max}$) from trapped electrons (satisfying $\mathcal{E}_z < U_{max}$). Physically, the separatrix can be viewed as a rigid boundary: As the phase velocity of the wave changes due to the frequency sweeping, the trapped fast electrons are convected along in phase space while the passing electrons jump over the moving separatrix.[2] This natural separation of the fast electrons allows us to treat passing and trapped fast electrons individually.

For the passing electrons, we argue in the following manner: Due to the steep gradients in the fast electron distribution function inside the separatrix as compared with the equilibrium slope, hole and clump structures can be regarded as slowly evolving, modulated beams, moving in a smooth background. Hence, by far the largest part of the perturbed density derives from the trapped electrons. We thus approximate the passing electron distribution function with the equilibrium slope, meaning that $f - F_0$ vanishes everywhere outside the separatrix.

The trapped fast electrons, however, must still be treated fully kinetically. Formally, $\mathcal{H}_z$ is a function of time, so the fast electron energy in the wave frame is not a conserved quantity. Since the particle trajectories are nearly periodic, however, there are adiabatic

[2]Note that in the presence of weak fast electron velocity diffusion or when the wave amplitude is evolving slowly, the picture is somewhat altered. The separatrix is then more like a porous membrane which allows particles to gradually enter and leave the trapping region.

invariants defined by the phase space areas bounded by the lowest order particle orbits as $\mathcal{E}_z$, $\dot{s}$ and U are held constant. Adopting the trapped electron adiabatic invariant

$$J = \oint \sqrt{\frac{2}{m_e}\left(\mathcal{E}_z - U\left(z\right)\right)}\, dz \tag{8}$$

as an action variable, the Hamiltonian becomes independent of the corresponding angle θ, thus implying that to lowest order f is a slowly changing function of merely J. It is then advantageous to average the trapped electron kinetic equation over the fast bounce motion of the trapped electrons. Applying the bounce average

$$\langle \ldots \rangle_B \equiv \frac{1}{\tau_B}\int_0^{\tau_B} \ldots \; dt \; . \tag{9}$$

to (1d), one obtains the evolution equation

$$\frac{\partial\, \delta f}{\partial t} + \beta\, \delta f = -\frac{dF_0}{dv}\ddot{s} - \frac{\alpha^2}{k}\frac{dF_0}{dv} + m_e \frac{\nu^3}{k^2}\frac{\partial}{\partial J}\left[J \frac{dJ}{d\mathcal{E}_z}\frac{\partial\, \delta f}{\partial J}\right] \; , \tag{10}$$

valid for trapped fast electrons. Here, $\delta f \equiv \langle f \rangle_B - \langle F_0 \rangle_B$, which satisfies the boundary condition $\delta f = 0$ at the separatrix, and F_0 has been assumed globally linear in v.

Finally, in the adiabatic limit, the power

$$Q = 2\gamma_d m_e n_{e0} \int_0^{\lambda} v_e^2 \, dz = \frac{2\gamma_d n_{e0}}{m_e \dot{s}^2}\left\langle U^2 \right\rangle_\lambda \tag{11}$$

dissipated in the cold electron background due to the friction force will, to lowest order, balance the power released from the fast electrons during the slow mode evolution. If the separatrix is narrow, the power released by the fast particles as the separatrix travels in phase space (and due to drag collisions) is, to lowest order, given by

$$-\frac{\delta \mathcal{E}_k}{\delta t} \approx -m_e\, \dot{s}\left(\frac{d\dot{s}}{dt} + \frac{\alpha^2}{k}\right)\int_0^{\lambda}\int \delta f \, dv\, dz \; . \tag{12}$$

The power balance condition thus reads

$$m_e\, \dot{s}\left(\frac{d\dot{s}}{dt} + \frac{\alpha^2}{k}\right)\int_0^{\lambda}\int \delta f \, dv\, dz = \frac{2\gamma_d \lambda}{m_e \dot{s}^2}\left\langle U^2 \right\rangle_\lambda \; . \tag{13}$$

Having presented all the necessary theory, the wave evolution can now be found by solving self-consistently the kinetic equation (10), the bounce averaged Poisson equation (i.e. (6) with $(f - F_0)$ replaced by δf) and the power balance criterion (13) for δf, U and $\dot{s}$ in the following way:

1. For a given wave phase velocity at a certain instant of time, the kinetic equation (10) is used to calculate δf.

2. The Poisson equation (6) is solved, with inputs $\dot{s}$ and δf, to yield U.

3. The quantities U, δf and $\dot{s}$ are used in the power balance condition (13) to calculate the frequency sweeping rate, and the wave phase velocity is updated.

4. Return to 1.

Note that this bounce averaged, adiabatic method allows the system to be evolved efficiently by only resolving time-scales larger than the trapped particle bounce period, in contrast to earlier schemes [8, 15, 16].

III. Example: Non-Diffusive Case

Holes and clumps are formed within a narrow region of velocity space centered around the phase velocity $\dot{s}_0$ of the linear wave, and the frequency sweeping begins after the trapped fast electrons have become well phase mixed [17]. We can therefore model δf as initially flat inside the separatrix. Moreover, in the absence of diffusion, the kinetic equation (10) contains no derivatives with respect to J, so δf will remain flat unless the separatrix grows. The evolution of δf in time is then described schematically by a column, whose height varies as the phase velocity changes or due to Krook and drag collisions (which preserve the flatness of the distribution inside the separatrix) and whose width decreases as the separatrix shrinks. If the separatrix expands, on the other hand, new particles are captured whose δf by assumption equals 0. The trapped electron distribution function then develops a gradient in the proximity of the separatrix.

The simplest way of capturing the effect of the separatrix expansion is to describe the continuous evolution of δf by the inclusion of a discrete set of step-like, narrow columns in J-space. The simplicity of this model lies in the perturbed density integral, appearing in both (6) and (13), which can be carried out analytically with this special form of δf. Moreover, the trapped electron kinetic equation (10) simplifies to

$$\frac{dh_i}{dt} + \beta h_i = -\frac{dF_0}{dv}\ddot{s} - \frac{\alpha^2}{k}\frac{dF_0}{dv} \ , \tag{14}$$

which describes the evolution of each column height $h_i = h_i(t)$. Note that this scheme necessitates the creation of a new column every time the separatrix expands.

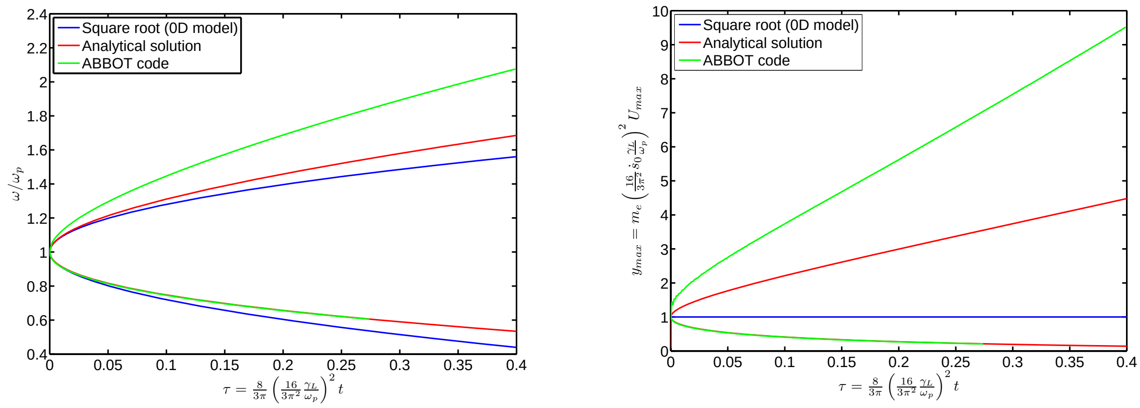

Figure 1: *Frequency and amplitude evolution for holes (sweeping up in frequency with growing amplitude) and clumps (sweeping down in frequency with decreasing amplitude) in the collisionless limit.*

In the following we present graphs produced by the non-diffusive ABBOT code (built on the scheme developed above) as well as an analytical solution which only incorporates the innermost column and its height h_0. These two solutions are compared with the analytical square root result obtained in the limit of small deviation in wave phase velocity from $\dot{s}_0$ [11]. Figure 1 shows the frequency sweeping and associated evolution of U_{max} for holes and clumps in the collisionless limit $\alpha = \beta = 0$. For the downsweeping clump, U_{max} is monotonically decreasing in time. Hence, the separatrix is shrinking during the entire simulation time, and the collisionless clump is well described using only one column. We can then utilize this case to benchmark the non-diffusive ABBOT code. It is seen that the ABBOT code reproduces remarkably well the analytical results. Note that, due to the frequency dependence of U in the analytical solution, the resulting sweeping rate eventually deviates from the square root predicted by the 0D model as the mode phase velocity shifts from $\dot{s}_0$ [13]. Turning to the hole evolution in the collisionless limit, we see that the analytical model and the ABBOT code each add separate new pieces to the square root sweeping of the 0D model. As for the clump, the frequency dependence of U taken into account in the analytical model results in a sweeping rate different from that of the square root. In the limit $t \to 0$, however, the square root is once again recovered. Accounting for fast electron capturing due to separatrix expansion, the ABBOT code predicts an even faster sweeping rate.

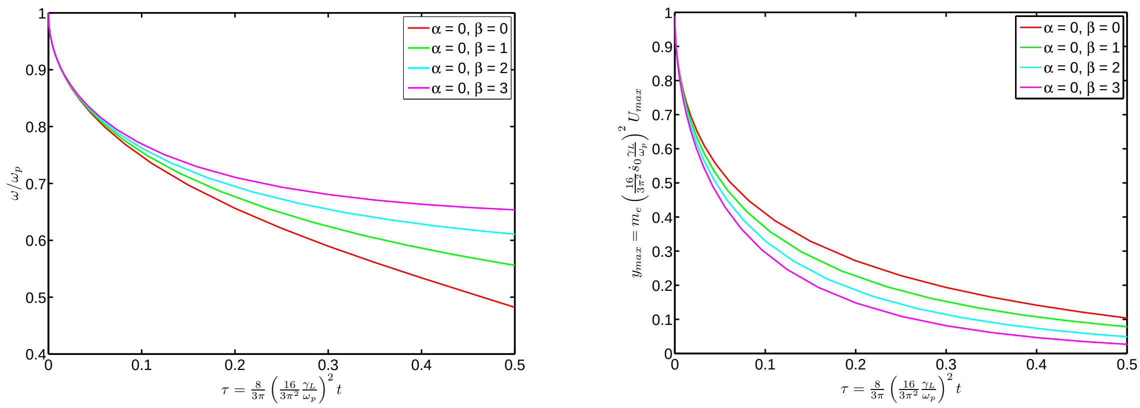

Figure 2: *Frequency and amplitude evolution for downsweeping clumps in the absence of drag, and with β varying from 0 to 3.*

Krook and drag collisions greatly enrich the observed behaviours of holes and clumps. For clumps, as in the collisionless limit, U_{max} is always decreasing in time, allowing their evolution to be described by the analytical model. To separate the effects of Krook and drag, we first show in Figure 2 how finite β affects the clump frequency sweeping and evolution of U_{max} when $\alpha = 0$. It is seen that finite β decreases the clump sweeping rate, and that this reduction is intimately linked with the increased rate at which the Krook collisions are abating U_{max}. Drag, on the other hand, affects the evolution through both (13) and (14). It thus imposes an additional sweeping rate proportional to $-\alpha$, as seen in Figure 3. In addition, the tendency of drag is to increase the rate of decay of the clump amplitude. Note, however, that in the limit $t \to 0$, all the curves in Figure 2 tend to the square root result, confirming the picture that collisions do not significantly alter the initial frequency sweeping rate. With a combination of Krook and drag collisions, the clump evolution consists of merely a superposition of the behaviors observed with Krook and drag alone, where Krook effectively decreases the clump frequency sweeping rate and drag tends to increase the latter. For holes, simulations with the non-diffusive ABBOT code reveal that Krook collisions alone act in a similar fashion to the clump case to reduce U_{max} and decrease the sweeping rate from the collisionless result presented in Figure 1.

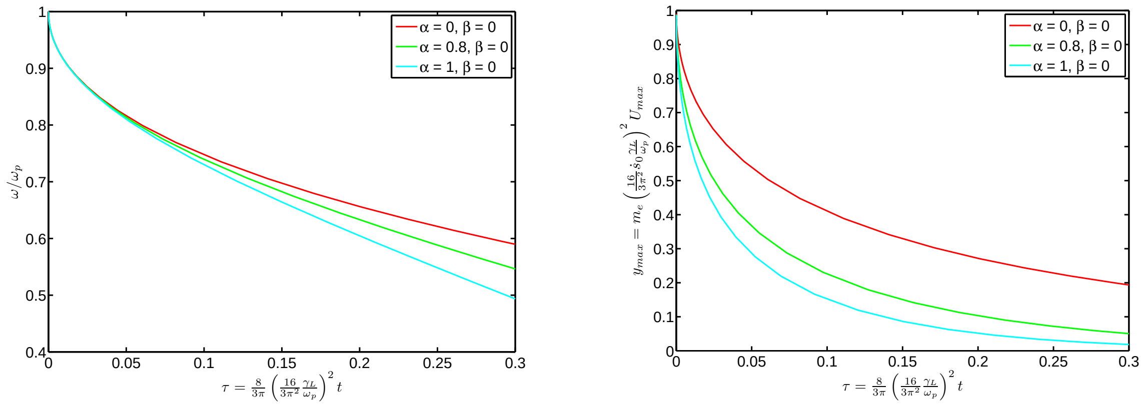

Figure 3: *Analytical frequency and amplitude evolution for clumps in the absence of Krook, i.e. when $\beta = 0$, and with α varying from 0 to 1.*

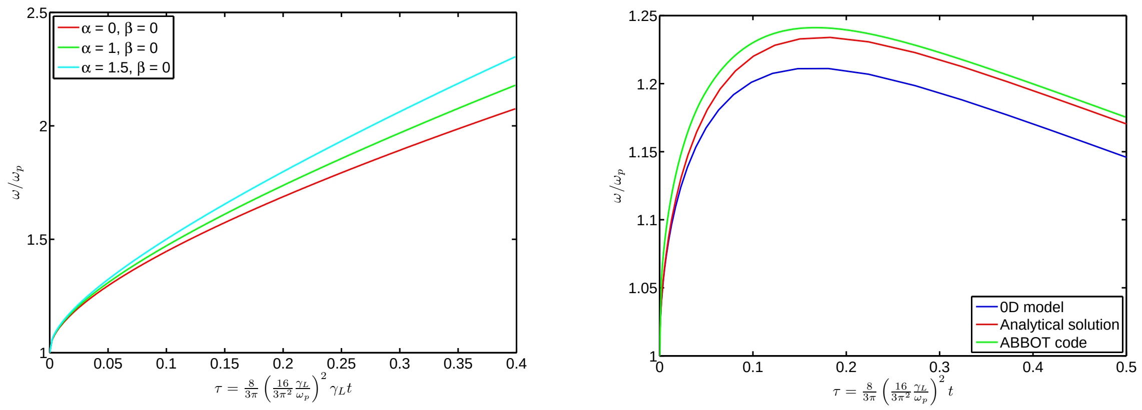

Figure 4: *Frequency evolution for a) a hole in the absence of Krook, and with α ranging from 0 to 1.5; b) a hole when $\alpha = 0.5$ and $\beta = 10$, displaying so called hooked behavior.*

Drag, however, affects hole structures completely differently as compared with clumps: The drag collision operator is asymmetric with respect to v, so instead of diminishing the mode amplitude, drag acts to deepen holes and thereby increase their associated mode amplitudes. It thus results in even faster growing amplitudes for upsweeping holes, with the total effect that the mode frequency sweeping rate increases (see Figure 4a), even though drag partially acts to decrease the sweeping rate through (13). Moreover, in the presence of both Krook and drag collisions, interesting frequency sweeping patterns are found for certain combinations of α and β. In Figure 4b, a so called hook is shown.

IV. Conclusions

The development of the adiabatic bump-on-tail model described in Section II constitutes an important step towards predictive modelling of nonlinear mode evolution in three dimensional experimental configurations. The essential new features in this work are the inclusion of collisions and allowing for a general evolution of the wave field, i.e that the mode amplitude is permitted to grow or decay as the mode evolves.

In the case when the wave amplitude does not grow, the evolution of the wave in the presence of Krook and and drag collisions can be treated using a particularly simple model, in which the holes and clumps are taken as flat top shaped functions inside the separatrix, thus allowing for an anlytical solution of the Poisson equation. The influence of particle trapping, brought about if the amplitude grows, is however surprisingly large. Only a modest amount of trapping is necessary to alter the wave field significantly, which is reflected in the enhanced hole sweeping rate and evolution of U_{max} in the collisionless case. The trapping effect must therefore be considered as equally important as the finite frequency corrections introduced in [13].

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