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Process Capability Analysis for Non-normal Processes with Lower Specification Limits

Master of Science

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Master's Thesis

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Abstract

Process capability analysis is an important element of any quality improvement initiative. However, estimating process capability is often problematic when it comes to non-normal distributions since the conventional methods sometimes give misleading results.

In this Master's thesis, a new method for estimating process capability of non-normal distribution with only lower specification limit is proposed. The proposed method only considers the left tail of the distribution rather than taking all of the data points. For estimating the process capability, it employs least squares technique and normal approximation to the selected observations from the left tail.

Furthermore, the proposed method is tested on log-normal, 3-parameter Weibull, and 3-parameter Gamma distributions. Simulations and real-world data analysis are used for verification and validation purpose, respectively. An easy and practical guideline is also developed. Following the proposed guideline, capability of a non-normal process can be estimated with a relatively high success rate.

Keywords: process capability analysis, non-normal distribution, lower specification limit.

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1. Introduction

In this section, a general description of process capability analysis and its importance in industry are provided. This is followed by the formulation of the problem of interest. The purpose and research questions draw the directions that the study follows and finally delimitations determine the scope of the study.

1.1. Background

The concept of quality evolved since the beginning of the twentieth century as the frontiers of quality have ever enlarged. What was once started as quality inspection – that is, checking finished products for defective units followed by scrapping or reworking – has evolved into quality management which encompasses actions for improving quality not only after production but actions before and during production as well (Hoerl and Snee, 2010a). As manufacturers started to realize the high costs of poor quality, which are related to both direct and indirect costs of quality defects, and feel the ever-growing pressure of customer demand and expectations; quality improvements become crucial for companies' survival and to keep their competitive edges (Feigenbaum, 1999; Bergman and Klefsjö, 2010).

There are indeed several different strategies – one can even name them philosophies – for deploying quality improvement efforts to various stages of manufacturing: TQM, Six Sigma, Design for Six Sigma, and so forth can be considered as examples (Snee, 2004; Nonthaleerak and Hendry, 2006). While differing in various attributes, these quality improvement strategies share some common features as well. Whether TQM or Six Sigma or any other strategy is employed, "Statistical Thinking" philosophy is amongst the common features for quality improvement (Hoerl and Snee, 2010b). Hoerl and Snee (2010a) discuss that the key elements of Statistical Thinking are process, variation, and data. These elements are intertwined with each other. For example, without proper contextual understanding of the data, faulty data analysis or analysis with no useful interpretations can be made. However, in order to develop such contextual insight first the process, which produces the data, has to be understood. Once the process and the context of data is appropriately comprehended, the direction chosen to reduce the variation is the utilization of statistical methods (Hoerl and Snee, 2010b). Through using statistical methods, companies aim at identifying and eliminating variation in their processes, measuring the performance of their processes and so forth (Bergman and Klefsjö, 2010). Statistical Process Control (SPC) and its two cardinal functions, control charts and process capability analysis (PCA), constitute the central pillar of statistical methods which are utilized in industry today (Wu*, et al.*, 2009; Hoerl and Snee, 2010a). A key element in PCA is to measure the capability of a process through process capability indices (PCIs), which make a statistical assessment possible (Kotz and Johnson, 1993).

PCA studies intrigue academics, practitioners, and also students. Practitioners sometimes cooperate with academics or students for investigating the capability of their processes. An example of such cooperation can be seen in Li and Chen (2009). Li and Chen (2009) have conducted a process capability analysis within the scope of their Master's thesis for Vestas, Denmark, the world leading supplier of wind power solutions. The study was based on the

assessment of the capability of the painting process of wind turbine blades. After a closer examination of the PCA methodology employed in the company, it was found out that the methodology of analyzing painting process capability was not accurate due to the non-normality of the process distribution. Further investigation showed that the process capability indices computed with non-normal data gave inconsistent results for different assumed non-normal distributions. This problem requires further research; therefore a theoretical Master's thesis was proposed in order to investigate an appropriate approach for dealing with non-normally distributed processes.

1.2. Problem Formulation

In order to be able to identify the problem, it is important to state the environment and the conditions where the problematic results first manifested themselves. In the following sub-sections, the blade painting process is described briefly and a short summary of the analysis of Li and Chen (2009) is provided. Then, the problem is defined based on the given information.

1.2.1. Blade Painting Process

Vestas manufactures the blades which are used as parts of the wind turbine. The manufacturing process can be summarized as (1) production of the parts of the blade, (2) assembly of the parts, (3) surface finishing treatment, and (4) painting of the blade. During surface finishing treatment, the blade goes through cleaning, grinding and repairing processes in order to make the blade ready for painting. The painting of the blade is essentially covering every facet of the blade with a layer of chemical material. This chemical layer provides protection from adverse environmental conditions, delays erosion and so forth. The performance of the painting process is crucial for the overall quality of the blades; therefore it is measured and analyzed.

Before starting the painting process, labels are stuck on every seven lightning receptors on two sides of the blade, resulting in total 14 labels. These labels are then used to measure the thickness of paint on the blade. Paint thickness is directly related to the protection of the blade from environmental conditions and it highly affects the durability and reliability of the blades. Painting operation is done by two painters manually using a spraying gun.

Li and Chen (2009) identified paint thickness as a quality characteristic of blade manufacturing process. It is used as the quality characteristic of interest when computing process capability indices. The paint thickness data were obtained by measuring the levels of paint on the labels which are placed on lightning receptors. For a single blade 14 measurements were taken from each receptor and these measurements were repeated three times. For each receptor, the average value of the three measurements were computed and checked whether the average satisfied the minimum level of paint thickness (100 μm). Otherwise, the blade would require re-painting.

1.2.2. Overview of the Findings from Li and Chen (2009)

Although paint thickness data does not have an upper specification limit, it has a lower specification limit, which is 100 μm. Since the painting is done manually, painters might

sometimes paint deliberately thicker in order to avoid re-work. Therefore some outliers might be found in the right tail of the process distribution.

Li and Chen (2009) have collected two batches of paint thickness data over two months time period. Thus, for each measurement point (i.e. lightning receptor), two batches of data have resulted in 28 measurement points. When the normality of the 28 measurement points were tested, it was found out that seven of them are not normally distributed. Therefore, it would be incorrect to analyze these non-normally distributed points together with the normally distributed ones because the results would be misleading. Li and Chen (2009) applied goodness-of-fit tests on non-normally distributed data sets and found out that some data sets fit log-normal, 3-parameter Weibull, and 3-parameter Gamma distributions. For illustration, the goodness-of-fit test results for one of the non-normally distributed receptor data is provided in Figure 1. *Minitab* software is utilized.

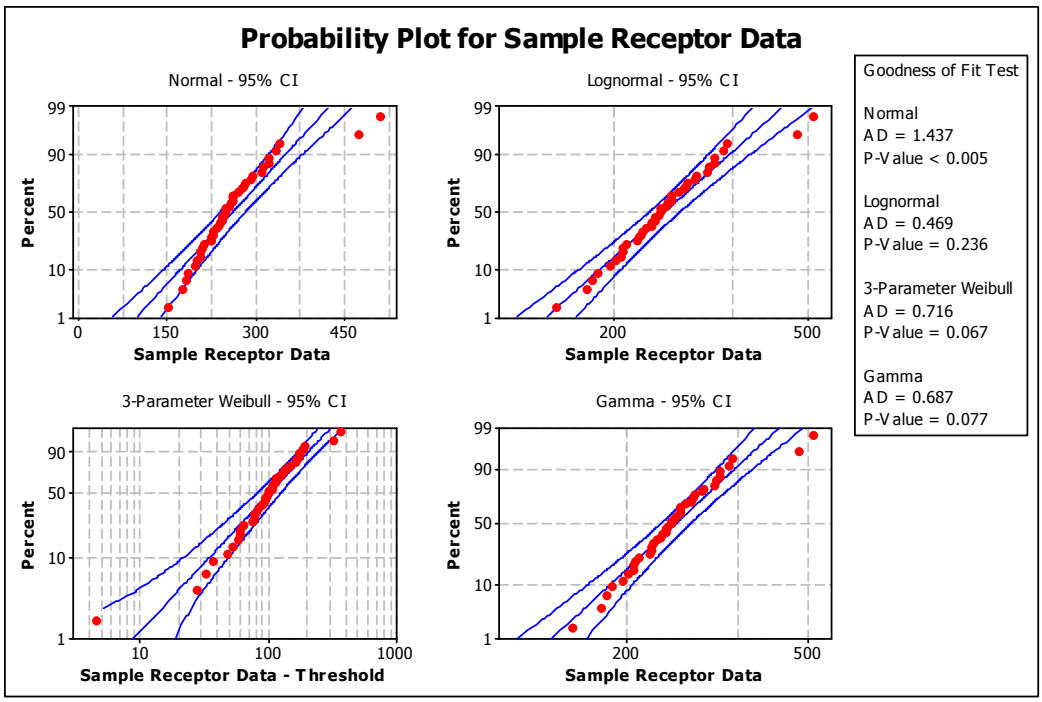

Figure 1. Goodness-of-Fit test of non-normal receptor data

1.2.3. Problem Definition

It has been stated in the previous section that analysis on the paint thickness data has revealed that the process did not always follow normal distribution pattern. For the normally distributed data, process capability indices, C_{pk} or P_{pk}, could be used without any problem. However, pooling the normal and non-normal data together and treating them as the same is a wrong approach. Although, non-normal distribution approximations may fit the paint thickness data well for some distributions such as log-normal, 3-parameter Weibull, and 3-parameter Gamma, when the process capability indices are calculated for the same process characteristic (i.e. paint thickness), varied C_{pk} values are obtained, depending on which non-normal distribution is assumed. Figures 2, 3, and 4 show the different C_{pk} results obtained

from the receptor sample data, which is used in the previous section, for log-normal, 3-p Weibull, and 3-p Gamma distributions, respectively.

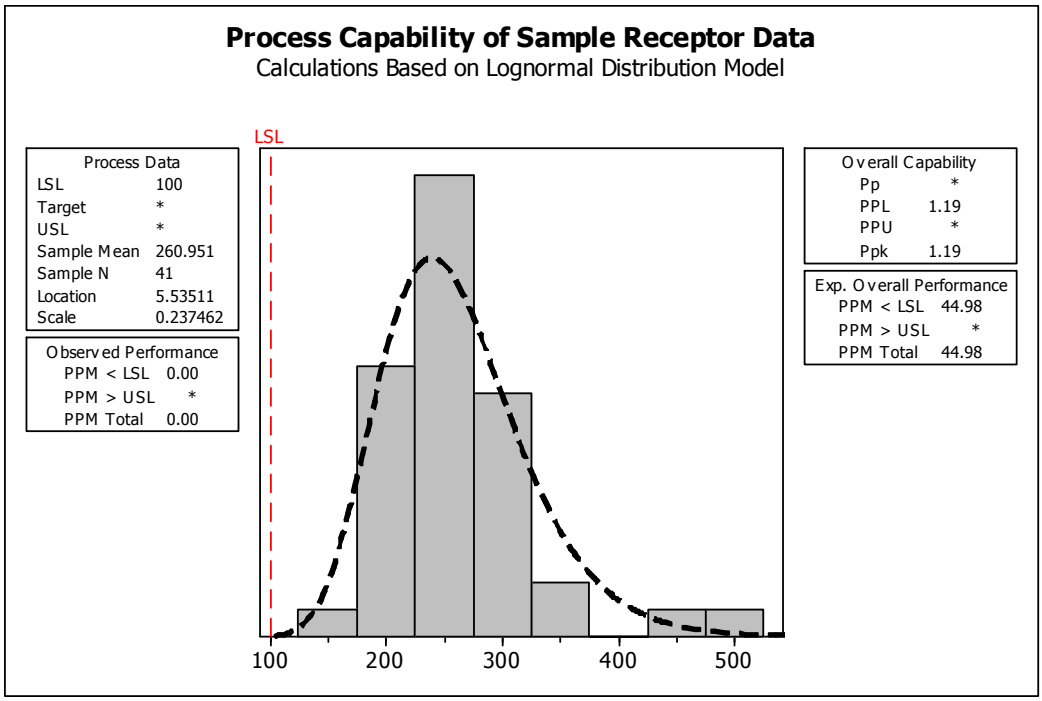

Figure 2. PCA of non-normal receptor data assuming log-normal distribution

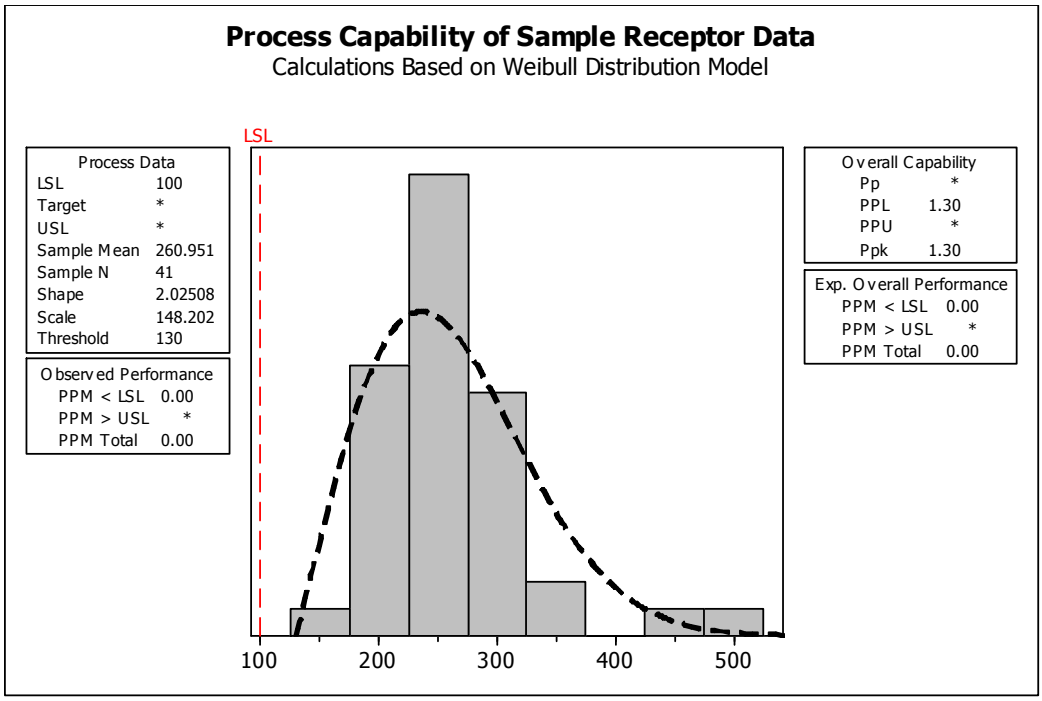

Figure 3. PCA of non-normal receptor data assuming 3-p Weibull distribution

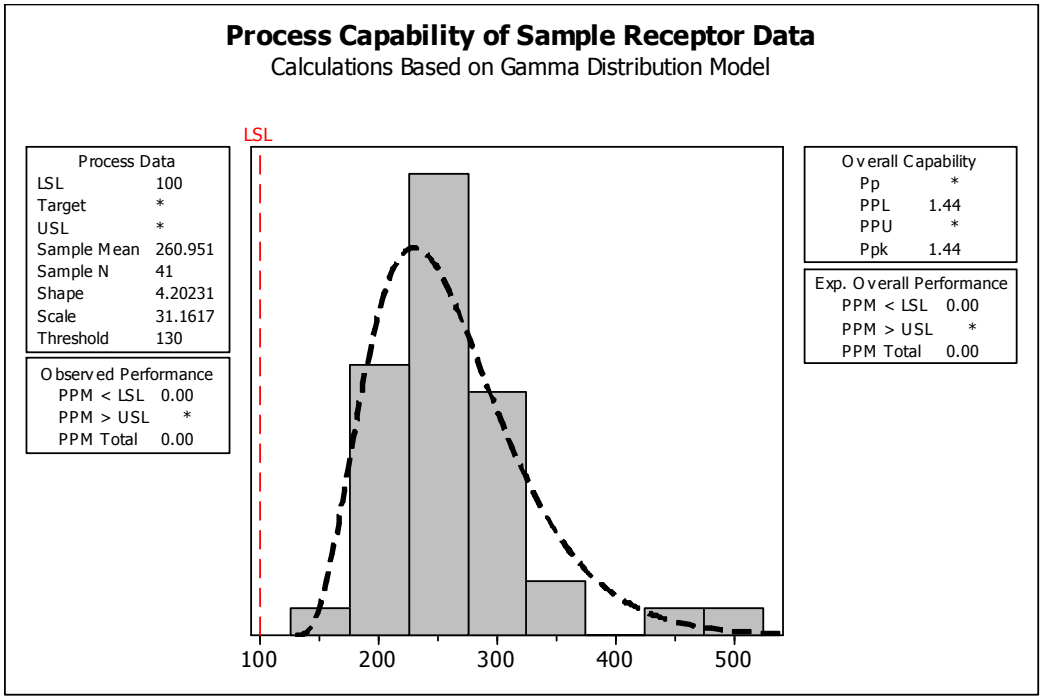

Figure 4. PCA of non-normal receptor data assuming 3-p Gamma distribution

It is shown in the figures that for the same process data, different P_{pk} (the software calculates only P_{pk} for non-normal distributions) results are obtained for different distributions. For example, assuming that the underlying distribution is log-normal, P_{pk} equal to 1.19 and expected ppm equal to 44.98 are obtained. However, assuming 3-p Weibull distribution the P_{pk} is 1.30 and expected proportion of non-conformity is zero. Furthermore, assuming 3-p Gamma distribution, expected proportion of non-conformity is 0 but P_{pk} is 1.44. Such results raise the question of which C_{pk} result should be assumed to reflect the capability and which ppm result should reflect the expected non-conformity of the process. In other words, without knowing which C_{pk} to be used, capability of the process cannot be correctly assessed.

The circumstances stated above draw the outline of the problem, which will be investigated in this master's thesis. In order to overcome the problem faced when dealing with non-normal data, a new approach will be developed for process capability analysis and will be tested through simulations. The process characteristic, which will be simulated in this thesis, shares common properties with the blade painting process in the company, particularly with paint thickness data. The process of interest investigated does not have two-sided specification limit but only a lower specification limit. Moreover, the process is non-normally distributed with a positive skew, which results in higher values in the right tail of the process. The first two moments of the process characteristic (i.e. mean and standard deviation) are chosen and simulated in such a manner that they would be consistent with the paint thickness data in the company. Finally, according to the results of the simulation procedure, the precision (i.e. measure of spread) and the accuracy (i.e. measure of bias) of the proposed method will be examined.

1.3. Purpose

The purpose of this master's thesis is to develop a non-normal process capability analysis method for processes with one-sided specification limit. The proposed method should be designed in such a manner that it would be applicable to a variety of non-normal distributions and should estimate the C_{pk} index within the acceptable levels of accuracy and precision.

The expected outcome of this study could be regarded in two aspects. The first aspect is to establish a better approach to estimate PCIs, which handles the misleading PCI results mentioned above. The second aspect is to be able to assess the performance of blade painting process in Vestas, which do not follow normal distribution, more accurately.

1.4. Research Questions

The focus of the study should be guided by research questions in order to fulfill the purpose at the end of the study. Considering the problem identified, two research questions are formulated as part of this thesis:

1. Is the process capability of a non-normally distributed process with lower specification limit could be estimated by only taking the left tail of the distribution into account?
2. What are the factors that affect the performance of the estimates?

The first question reflects the ability of assessing process capability with regard to the proposed estimation method. In order to answer this question, the performance of the estimation method has to be examined. The simulation procedure constructed in this study aims at providing the performance examination, which is required to evaluate the applicability of the proposed method. The second research question relates to the investigation of whether or not process characteristics (for example, population size and its skewness or kurtosis) and/or method properties (for example, number of observations taken from the) have an effect on the estimated results.

1.5. Delimitations

The delimitations in this thesis can be regarded in three aspects. First, since this is a theoretical thesis and no company is involved, the data required for the testing of the method is randomly generated by running simulations in the software "*R*".

Second, the capability index utilized in this study is the C_{pk} index based on percentile approach and the process investigated has only lower specification limit. Accordingly the C_{pk} computed through the simulations equals to the C_{pl}. For any other type of process capability indices and/or different process properties, such as two-sided specification limits, only upper specification limit, and so forth, the performance of the proposed method may be different.

Finally, the non-normal distributions investigated are limited to log-normal, 3-p Weibull, and 3-p Gamma distributions and the performance of the proposed method is evaluated only relative to these three distributions.

2. Research Methodology

Establishing a well-defined research methodology at the outset of a research project guides the researcher(s) about which research design to employ and which actions to take throughout the study. Bryman and Bell (2007) argues that any type of research methodology has different facets, such as the link between theory and research, epistemological and ontological considerations, which are inherent in its context. These facets deeply influence the way a research is conducted such as the choice of method, data collection techniques, analysis of data and so forth. The link between theory and research could be inductive, deductive or both which results in, in the most general sense, choosing a research strategy either qualitative, quantitative or mixed methods (Bryman and Bell, 2007).

2.1. Research Strategy: A Quantitative Approach

Quantitative research strategy can be described as a systematic collection of numerical data and analysis of the data through mathematically-driven, mostly statistical models in order to explain a certain phenomena (Muijs, 2004). As a result of the statistical approach, which is maintained in quantitative research, the data is used objectively to measure the reality of the phenomena (Williams, 2007).

As it is stated earlier in the purpose of this thesis, the aim is to develop a method for process capability analysis. At the most basic level, process capability indices are quantitative, dimensionless measures which indicate the performance of the process in relation with the process parameters and the process specifications (Kotz and Johnson, 1993). Therefore, in order to calculate any class of PCIs for a process with any type of underlying distribution, first one needs to collect numerical data about the process which would be followed by their statistical analysis. When these characteristics of the research are considered, it is seen that the most appropriate research strategy would be a quantitative research strategy.

According to Balnaves and Caputi (2001) in any type of research, one has to know what to research (i.e. problem exploration, description and explanation) and how to research (i.e. literature research, research design, research method, data analysis methods). In order to know these “what” and “how”, an organized quantitative research should entail the following steps:

- Step 1: Problem Definition
  The first thing to do is to formulate the research questions, which the research is supposed to answer at the end of the study. In the context of this study, two research questions are provided:
  1. Is the process capability of a non-normally distributed process with lower specification limit could be estimated by only taking the left tail of the distribution into account?
  2. What are the factors that affect the accuracy and precision of the estimates?
- Step 2: Literature Review
  Literature research is an invariable element of any kind of research project since through existing literature; researchers are able to assess the utility of their research questions and to elaborate on their research design (Bryman and Bell,

2007). A literature review is conducted in order to deepen the understanding of the subject. This review encompasses the literature in process capability with emphasis on non-normal data with given lower specification limit.

- Step 3: Research Design
- Step 4: Research Method
- Step 5: Data Analysis

Steps 3, 4, and 5 are critically important in a research methodology. Therefore, their concepts and their utilization in the context of this study are explained in detail in the upcoming sub-sections.

2.2. Research Design: An Experimental Approach

As the next step in defining the research methodology, a framework, which lays out the decisions about the collection and analysis of data, has to be built. This framework is called research design and it designs how the research method should be executed and how the analysis of data should be undertaken (Bryman and Bell, 2007).

Research designs for a quantitative approach could be classified into three categories: descriptive, experimental and causal comparative (Williams, 2007). Descriptive research analyzes the attributes of a phenomenon in their current environment. Causal comparative research examines the relationship between independent variable and dependent variable while no manipulation is introduced between them (Williams, 2007). On the other hand in experimental design, the independent variable is manipulated in order to see the effect on the dependent variable and draw cause-and-effect relationship between them (Bryman and Bell, 2007).

In the case of this master's thesis, the independent variable, which is the simulated data, is manipulated through the proposed method and the effect on the dependent variable, which is the resulting PCI value, is analyzed. Therefore, an experimental research design is found to be more appropriate.

2.3. Research Method: Data Collection through Simulations and Data Analysis

The last step in research methodology is to decide which research method to employ for collecting data (Bryman and Bell, 2007) and how to analyze the results. Since this is a theoretical thesis and no company is involved, the data required for the testing of the method is generated by running simulations in *R* software which is used for statistical computing and graphics. The data generated is then used to test the applicability of the proposed method.

The analysis of the results is based on the simulation results. *R*, *Minitab* and *JMP* are utilized in conjunction with each other for data analysis. Multiple criteria are used to evaluate the performance of the proposed method. These criteria are explained in detail in Section 5.

2.4. Evaluating the Research: Reliability, Validity and Verification

Even though, when a research methodology is developed for a particular study, one still needs to evaluate it. Two dominant evaluation criteria for quantitative research are reliability and

validity (Muijs, 2004). Reliability evaluates the consistency of the results of a study over time and whether or not the results are repeatable. Particularly in quantitative research, the reliability of a research instrument is assessed through the stability of a measurement over time and when replication is applied by someone else (Golafshani, 2003). Validity is concerned with the gap between what is intended to be measured and what is actually measured. If there is found to be no integrity between the conclusions of the measures, the research instrument is hardly valid (Bryman and Bell, 2007). This could be due to the inaccurate measurements of the research instruments and/or the instrument is not simply measuring what it should measure (Golafshani, 2003).

In order to provide reliability, the codes written for simulations are provided in Appendix A, B, and C for log-normal, 3-p Weibull, and 3-p Gamma distributions, respectively. Further, verification and validation studies are conducted in order to ensure the validity of this Master's thesis. Verification of the proposed method is obtained through simulations and for validation; the proposed method is applied on the real-world data, which are presented in Section 1.2.

3. Theoretical Study

In this section, relevant literature about non-normal process capability analysis is summarized.

3.1. Process Capability Analysis

Measuring a process' performance and acting upon the assessments based on the measurements are critical elements of any continuous quality improvement efforts (Spiring, 1995). Companies make assessments of process performance based on different indicators. Most common of these indicators can be described in terms of process yield, process expected loss and capability indices of a particular process characteristic (Chen, *et al.*, 2001). Among these indicators, Process Capability Indices (PCIs) have gained substantial attention both in academic community and several types of manufacturing industries since 1980s (Somerville and Montgomery, 1996; McCormack Jr, *et al.*, 2000; Kotz and Johnson, 2002; Wu, *et al.*, 2009). This increase in popularity is mainly due to the fact that companies require some numerical indicators of how well the process is performing regarding its specification limits (Anis, 2008). PCIs are unitless, statistical quantifications which compare the actual performance of a process characteristic in relation to its tolerance limits, which are determined by customers' requirements.

3.2. Process Capability Indices

The first process capability index proposed in the literature is the C_p index, which is defined as:

$$C_p = \frac{USL - LSL}{6\sigma} \tag{1}$$

where USL and LSL denote the upper and lower specification limits, respectively, and σ is the standard deviation of the process characteristic of interest. The C_p index measures the process spread in relation to its specification range. Since C_p does not take the process mean of the quality characteristic into account, it does not give any information about whether the process is centered (Bordignon and Scagliarini, 2002). In order to overcome this problem, a second generation PCI, the C_{pk} index, is introduced. The C_{pk} can be defined as:

$$C_{pk} = \min[\frac{USL - \mu}{3\sigma} ; \frac{\mu - LSL}{3\sigma}] = \min [C_{pu} ; C_{pl}] \tag{2}$$

where μ and σ are the mean and the standard deviation of the quality characteristic studied, respectively. The mean of the process characteristic has an influence on the C_{pk} index and therefore it is more sensitive to departures from centrality than the C_p index (Anis, 2008).

The C_p and C_{pk} indices are the most commonly used process capability indices in industry and they are called standard or basic PCIs (Deleryd and Vännman, 1999; Kotz and Johnson, 2002; Anis, 2008). The earlier practical applications of these basic PCIs require the fulfillment of two assumptions:

i. The process has to be in statistical control.
ii. The process characteristic of interest has to be normally distributed.

Utilization of the basic PCIs when these underlying assumptions are not satisfied may lead to incorrect assessments and misleading interpretations of process capability (Chen, 2000). There has been studies showing that varying results of process capability analysis could be obtained from different underlying distributions with the same mean and the same standard deviation (Kotz and Johnson, 1993). In other words, different non-normally distributed processes, which have the same mean and the same standard deviation, could give the same capability output but different proportion of non-conformity. This could be explained due to the fact that the basic PCIs only uses the mean and the standard deviation of a process but for non-normally distributed processes, the mean and the standard deviation are usually not enough to reflect the characteristics of a process.

3.3. Process Capability Indices for Non-normally Distributed Data

Non-normally distributed processes are not uncommon in practice. Combining this fact with the misleading results of applying basic PCIs to non-normal processes while treating them as normal distributions forced academicians and practitioners to investigate the characteristics of process capability indices with non-normal data (Kotz and Johnson, 2002; Spiring, *et al.*, 2003; Yum and Kim, 2010).

Literature review shows that the research for PCIs under non-normality has been grouped into two main streams: (1) examination of PCIs and their performances for various underlying distributions and (2) construction of new generation process capability indices and/or development of new approaches specially tailored for non-normally distributed outputs. Although much effort has been put into these studies, there is not yet any standard approach or standardized PCI accepted by academicians and practitioners when non-normal data is handled.

The former main stream of research focuses on exploring the properties of different PCIs under different conditions and provides comparisons between them and suggests some of them for specific circumstances. These specific circumstances can be exemplified by different underlying process distributions, one-sided or two-sided specifications limits, the corresponding proportion of non-conformity of PCIs, and so forth. The latter main stream of research attempts to provide new approaches or new PCIs which would be robustly applicable to non-normal data. However, there is no information about how widely these new indices are utilized by practitioners for some of the new indices require a rather involved statistical knowledge and might be rather confusing for practitioners.

The second stream of PCI research can be categorized into five groups (Shore, 1998; Kotz and Johnson, 2002):

1. Data transformation methods
2. Development of quality control procedures for certain non-normal distributions
3. Distribution fitting for empirical data
4. Development of distribution-free procedures
5. Construction of new PCIs.

In the following sub-sections previous research about these five categories will be explained briefly.

3.3.1. Data Transformation Methods

Data transformation approaches aim at transforming the non-normal process data into normal process data. Several methods have been proposed for approximating normally distributed data by using mathematical functions. Most known amongst these methods are Johnson transformation system, which is based on derivation of the moments of the distribution, and Box-Cox power transformation. The main rationale behind these methods is to first transform the non-normal data into normal data and then use standard process capability indices, which are based on the normality assumption, for the transformed data. However, transformation methods have drawbacks which inherent in their utilization. First, transformation methods are computing-extensive (Tang and Than, 1999). Second, practitioners are hesitant to use these methods because of the problems associated with translating the computed results with regard to the original scales (Kotz and Johnson, 2002; Ding, 2004).

3.3.2. Quality Control Charts for Non-normal Distributions

Some research has been devoted to investigate process capability with certain underlying non-normal distributions. In these studies, the question of whether or not the process is in statistical control is to be answered by the statistical control charts of that particular distribution and PCI estimations are derived from these control charts. There are control charts developed for log-normal and Weibull distributions (Shore, 1998). Lovelace and Swain (2009) proposed to use midrange and ratio charts in order to track central tendency and dispersion for log-normally distributed data and then estimate PCIs based on empirical percentiles. However, in reality the distribution itself cannot be identified precisely or it requires a great magnitude of data for a solid identification. Combining this with the unknown parameters, which have to be estimated in order to compute PCIs, using quality control charts and their respective PCIs for non-normal distributions are not highly favored by practitioners (Shore, 1998; Kotz and Johnson, 2002).

3.3.3. Distribution Fitting for Empirical Data

Distribution fitting methods use the empirical process data, of which the distribution is unknown. These methods later fit the empirical data set with a non-normal distribution based on the parameters of the empirical distribution (Shore, 1998). Clements' Method is one of the most popular distribution approaches (Kotz and Johnson, 2002). The method employs both a new process capability index based on percentiles and a distribution fitting approach. The basic C_p index is calculated by using the 99.865^{th} and the 0.135^{th} percentiles of the distribution of quality characteristic instead of 6σ, which is meaningful only when the process is normally distributed. Therefore, the percentile-based C_p is obtained by:

$$C_p = \frac{USL - LSL}{\xi_{0.99865} - \xi_{0.00135}} \tag{3}$$

where $\xi_{0.99865}$ and $\xi_{0.00135}$ denote the upper and lower 0.135^{th} percentiles of the process distribution, respectively.

Following the same logic, the C_{pk} index can be obtained using a percentile approach:

$$C_{pk} = min\left[\frac{USL - \xi_{0.5}}{\xi_{0.99865} - \xi_{0.5}} ; \frac{\xi_{0.5} - LSL}{\xi_{0.5} - \xi_{0.00135}}\right] = \min[C_{pu} ; C_{pl}] \quad (4)$$

where $\xi_{0.5}$ is the median of the process distribution, which is used instead of the process mean, because the process mean is not indicative of the centrality of a non-normal distribution especially when skewness of the distribution is taken into account (Anis, 2008). The denominators of C_{pu} and C_{pl} represent the distance between the upper or the lower 0.135^{th} percentile and the process median, which could be interpreted as 3σ under normality assumption (Tang and Than, 1999). Padgett and Sengupta (1996) named this C_{pk} index which is based on percentile approach as the C_{pk}-Extended (C_{pkE}).

In Clements' Method, the percentiles required in the equations (3) and (4) are obtained through estimating the first four moments (i.e. mean, standard deviation, skewness and kurtosis) of the unknown underlying distribution of the empirical data, matching these moments to a suitable Pearson system distribution and using the percentile values of the selected distribution (Clements, 1989).

There are several other distribution fitting approaches for different distribution families and various distribution types. For further information, Kotz and Johnson (2002) provide a wide range of references about the methods. However, there is a common drawback inherent in distribution fitting approaches. That is, in order to provide reliable skewness and kurtosis estimations, a rather large sample size of empirical data is required. Therefore, these methods may not be appropriate for cases dealing with relatively small sample sizes (Tang and Than, 1999; Chang*, et al.*, 2002).

3.3.4. Distribution-Free Approaches

These approaches either aim at establishing distribution-free specification intervals or at adjusting PCIs through heuristic methods (Kotz and Johnson, 2002). Chan et *al*. (1988) proposed obtaining distribution-free PCIs by using distribution-free specification interval estimations, which are assumed to be independent of the underlying distribution of the process. However, the construction of tolerance intervals is derived from normal distribution intervals. Therefore, this approach was criticized by Kotz and Johnson (1993) because of the dependency of a "distribution-free" approach on normal distribution.

A heuristic weighted variance method is proposed by Choi and Bai (1996). The essence of weighted variance method is to divide a non-normal skewed distribution into two different distributions such that the resulting distribution would be normally distributed with same mean but different standard deviations. This segmentation requires no assumption of the distribution; therefore it makes the approach distribution-free (Wu*, et al.*, 1999). The method modifies standard PCIs based on weighted variance method in such a manner that the skewness and kurtosis of a distribution are also taken into account (Choi and Bai, 1996).

3.3.5. New PCIs for Non-Normal Data

Significant amount of work has been put into developing PCIs which would be robust against the non-normality of the process.

Wright's Index, C_s, has been proposed as an index which is sensitive to skewness (Wright, 1995). The C_s index adds a skewness correction factor on the C_{pmk} index by taking the skewness of the process data into account.

A flexible PCI, C_{jkp}, is introduced by Johnson, et *al.*(1994). The index is based on the C_{pm} index and is assumed to be flexible because the asymmetry of a non-normal process is considered with regard to the difference of variability below and above the target value, which is reflected upon the index by treating the two specification limits differently (Deleryd, 1999; Wu and Swain, 2001).

More specific process conditions are also investigated by some researches. Vännman and Albing (2007) have constructed a new class of process capability index, $C_{MA}(\tau, v)$, which is specially designed for cases with only upper specification limit, a target value equal to zero and with a skewness influence over the zero-bound process distribution.

3.4. Confusion between Process Capability Indices and Process Performance Indices

In practical application and literature, confusion between process capability indices (PCIs) and process performance indices (PPIs), and about their notations (C_p, C_{pk} vs. P_p, P_{pk}) can sometimes rise. This problem is addressed in the discussion sections of Kotz and Johnson (2002). Although these two indices aim at indicating the performance of a process, they differentiate when the estimations of the indices are calculated. In other words, the method of estimating the standard deviation of a process is different for PCIs and PPIs. For standard PCIs, C_p and C_{pk}, the standard deviation of a process is estimated within sub-groups by using control charts. Since it uses the variation within sub-groups, this type of estimation measures the short-term variation of a process. This type of estimation is denoted as $\hat{\sigma}_{st}$ for preventing confusion and is obtained by

$$\hat{\sigma}_{st} = \bar{R}/d_2 \tag{5}.$$

On the other hand, the standard deviation estimations computed for PPIs (P_p, P_{pk}) are based on individual measurements and they indicate the long-term variation of a process. This type of estimation is denoted as $\hat{\sigma}_{lt}$ and is obtained by

$$\hat{\sigma}_{lt} = \sqrt{\frac{1}{n-1}\sum_{i=1}^{n}(x_i - \bar{x})^2} \tag{6}.$$

Even though many academicians prefer the estimation of long-term variation for process capability calculations, the notation C_p and C_{pk} is widely used in literature. However, in industry, especially in automotive industry, the P_p and P_{pk} notations are used for the second type of estimations as well (Kotz and Johnson, 2002). This dichotomy in notations creates confusion. A consensus on a standard notation is required in order to lessen and solve this confusion. For the sake of coherence with the academic society, "C_{pk}" notation will be used in this study.

3.5. Log-Normal Distribution

One of the non-normal distributions investigated in this study is the log-normal distribution. Suppose that there exists a normally distributed random variable, namely *W*. Then the distribution of *X*, a random variable, is assumed to follow a log-normal distribution if there exists a relationship between *X* and *W* such that $ln(X) = W$. In other words, *X* is log-normally distributed if the natural logarithm of *X* is normally distributed, that is, *W* is normally distributed (Montgomery and Runger, 2003). Log-normal distribution follows an asymmetrical and positively skewed behavior. It is a two-parameter distribution of which probability density function is obtained by

$$f(x;\ \theta, \omega) = \frac{1}{x\sigma\sqrt{2\pi}}\ exp\left[-\frac{(\ln x - \theta)^2}{2\omega^2}\right], \quad 0 < x < \infty \tag{7}$$

where θ and ω^2 are the mean and the variance of normally distributed *W,* respectively. It should be noted that the parameters of the log-normally distributed variable, *X* are θ and ω^2 which are the location and shape parameters, respectively. The mean (μ) and variance (σ^2) of *X* can be expressed as

$$\mu = e^{\theta + \omega^2/2} \quad \text{and} \quad \sigma^2 = e^{2\theta + \omega^2}\left(e^{\omega^2} - 1\right) \tag{8, 9).}$$

In order to avoid confusion, the parameters, the mean and variance of both normal and log-normal distributions are given in Table 1.

Table 1. Parameters, Mean and Variance of Normal and Log-normal Distributions

	Parameters	Mean	Variance
Normal Distribution	(θ, ω^2)	θ	ω^2
Log-Normal Distribution	(θ, ω^2)	$e^{\theta + \omega^2/2}$	$e^{2\theta + \omega^2}\left(e^{\omega^2} - 1\right)$

When the probability density functions with a constant θ = 1 and four different values of ω are analyzed in Figure 5, it is seen that as ω increases the skewness and the kurtosis of the distribution augment rapidly, whereas for small values of ω, the departures of normality in terms of skewness is smaller.

This shift in the distribution shape and the distribution skewness, which occurs as distribution parameters change, makes the log-normal distribution an eligible distribution to analyze non-

normal process data because it covers several tail behaviors and distribution types (Somerville and Montgomery, 1996; Chen, 2000).

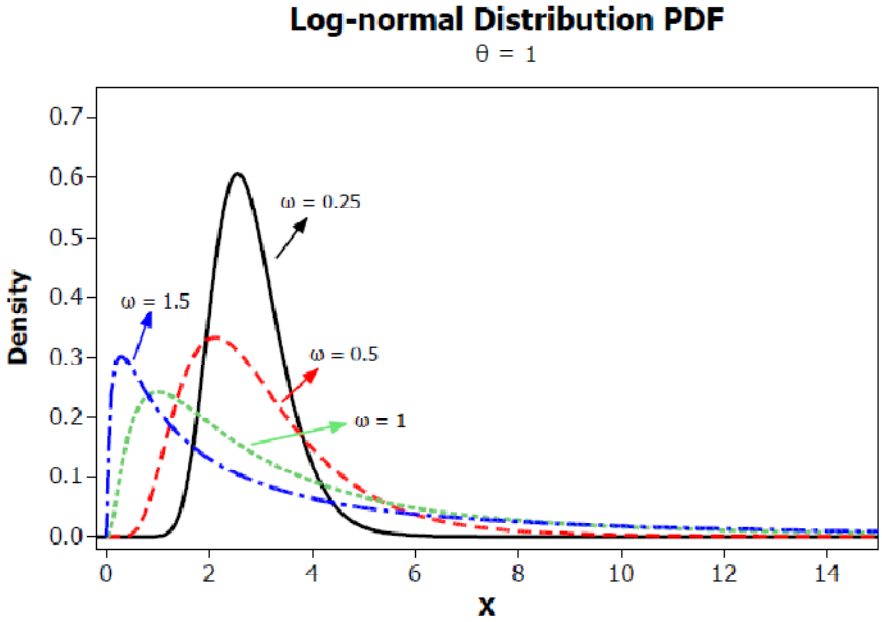

Figure 5. Log-normal distribution shapes with different scale values

Since the parameters of a log-normally distributed random variable X are θ and ω, in order to generate log-normally distributed random variates (x_i) with given mean (μ) and standard deviation (σ), θ and ω have to be expressed as functions of μ and σ. Following this logic; by taking the logarithms of μ and σ;

$$\ln(\mu) = \theta + {}^{1}/_{2}\,\omega^2 \tag{10}$$

$$\ln(\sigma^2) = (2\theta + \omega^2)\, ln\left(e^{\omega^2} - 1\right) \tag{11}$$

are obtained. From equation (10), θ is extracted and from equation (11), ω is extracted. Therefore, these two parameters are expressed as:

$$\theta = \ln(\mu) - {}^{1}/_{2}\,\omega^2 \tag{12}$$

$$\omega = \sqrt{\ln(\sigma^2 + \mu^2) - 2ln\,(\mu)} \tag{13}.$$

The simulations in this study are run with pre-determined process means and standard deviations of log-normally distributed X variables. Therefore the two transformations, which are shown above, are utilized for calculating the parameters of X in order to generate random variates of X.

3.6. Three-Parameter Weibull Distribution

Three-parameter Weibull distribution is commonly used in reliability applications in the industry. It represents several different shapes based on the chosen parameters which are the shape, the scale, and the location parameters. The location parameter (or the shift parameter)

positions the distribution on the x axis. The probability density function of the 3-parameter Weibull distribution is defined as:

$$f(x;\ \delta,\beta,\gamma) = \left[\frac{\beta(x-\gamma)^{\beta-1}}{\delta^{\beta}}\right] exp\left[-\frac{(x-\gamma)^{\beta}}{\delta^{\beta}}\right], \quad x \geq \gamma \tag{14}$$

where δ, β, and γ are shape, scale, and location parameters, respectively.

The mean and standard deviation of the 3-parameter Weibull distribution are defined as follow:

$$\mu = \gamma + \delta\,\Gamma\left(\frac{\beta+1}{\beta}\right) \text{ and } \sigma^2 = \delta^2\Gamma\left(\frac{\beta+2}{\beta}\right) - \delta^2\left[\Gamma\left(\frac{\beta+1}{\beta}\right)\right]^2 \tag{15, 16}.$$

where $\Gamma(x)$ is the gamma function.

The probability density function of 3-parameter Weibull distribution for different values of β when δ = 2 and γ = 1 is represented in Figure 6.

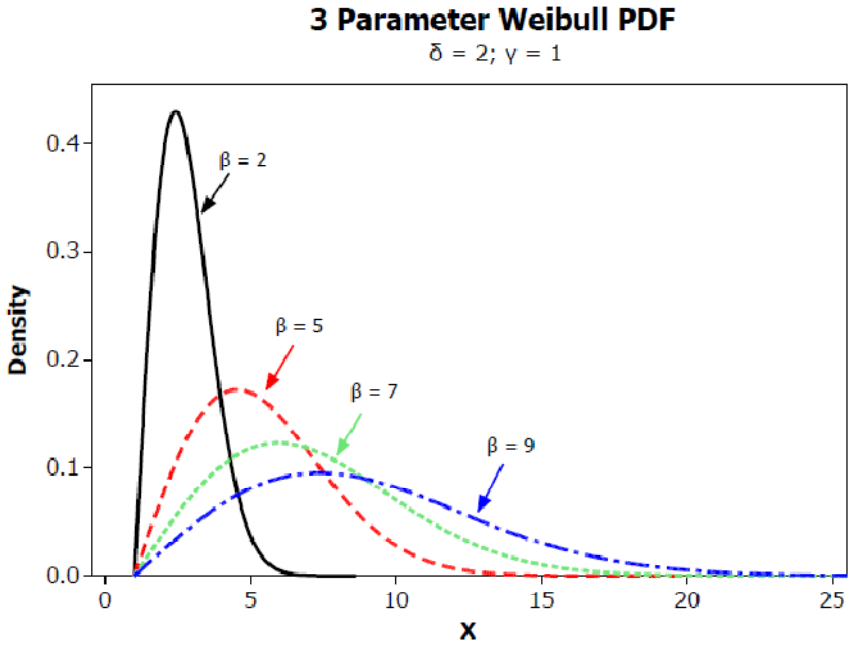

Figure 6. 3-parameter Weibull distribution shapes with different scale values

Note that γ affects the distribution's location of origin but not the shape nor the scale. Increasing or decreasing the value of γ slides the distribution. If γ > 0, the probability of obtaining a Weibull variate x smaller than γ is zero. Furthermore, as β increases the distribution gets more skewed to the right and gets less peaked.

Since, process mean and standard deviation are used as input values for simulations in this study, the parameters of 3-p Weibull distribution have to be extracted for a specific set of mean and standard deviation. In order to obtain the shape and scale parameters, "mixdist" package in *R* software is used. "`weibullpar`" function in this package compute the shape

and scale parameters for Weibull distribution given the mean, standard deviation, and location (Macdonald, 2010). The location parameter is chosen to be 130 due to the fact that it reflects the real-world data from Vestas. Further, since *R* does not provide an in-built function for 3-parameter Weibull random generator, "`rweibull3`" function in the package "FAdist" is utilized for random number generation (Aucoin, 2011).

3.7. Three-parameter Gamma Distribution

Three-parameter Gamma distribution (also known as Pearson Type III distribution) is widely used in modeling life data due to the variety of skewed shapes it can take based on the chosen distribution parameters. It is characterized by shape, scale, and location parameters. The probability density function of the 3-p Gamma distribution is defined as follows:

$$f(x;\, r, \lambda, \gamma) = \frac{[(x-\gamma)^{r-1}]}{\lambda^r \Gamma(r)}\, exp\left[-\frac{(x-\gamma)}{\lambda}\right], \quad x \geq \gamma \tag{17}$$

where r is the shape, λ is the scale, γ is the location parameters, and Γ(.) is gamma function. The mean and variance of 3-p Gamma distribution is obtained by

$$\mu = \gamma + r\lambda \quad \text{and} \quad \sigma^2 = r\lambda^2 \tag{18, 19).}$$

The probability density function of 3-p Gamma function for different shape parameters is shown in Figure 7. The location parameter does not have any effect on the shape of the distribution; it only changes the origin of the distribution. Note that as the shape parameter increases, the right skewness of the distribution decreases.

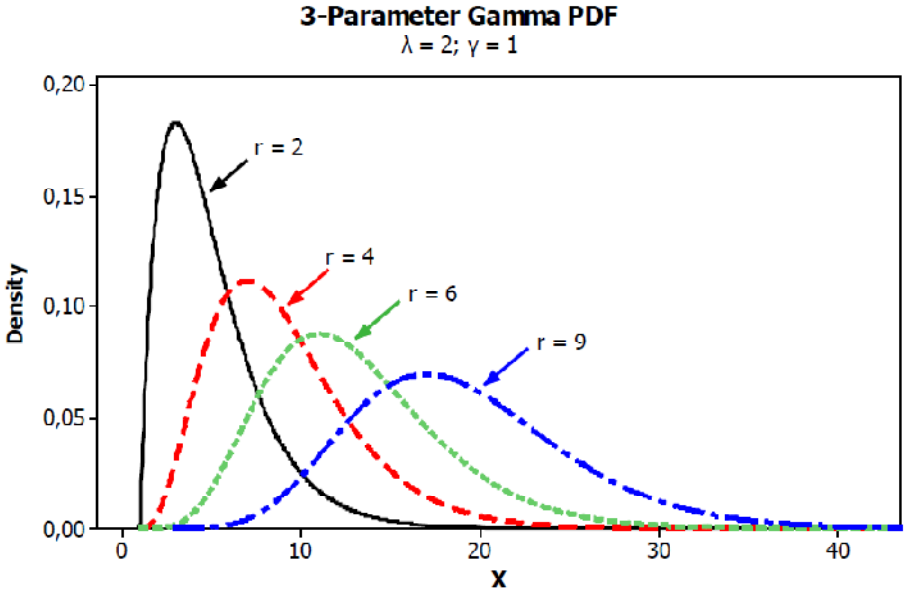

Figure 7. 3-parameter Weibull distribution shapes with different scale values

In order to be able to generate random 3-p Gamma variates with a given mean and standard deviation, the parameters of the distribution have to be transformed into expressions that can

be obtained through mean and standard deviation. From equation (18), the scale parameter (λ) is extracted as follows:

$$\lambda = \frac{\sigma^2}{\mu - \gamma} \tag{20}.$$

Using equation (20), the shape parameter r is obtained:

$$r = \frac{(\mu - \gamma)^2}{\sigma^2} \tag{21}.$$

These two transformations are used when computing the parameters of the 3-p Gamma distributions. Furthermore, the function "`rgamma3`" in the "FAdist" package is utilized for random number generation (Aucoin, 2011).

3.8. Conclusions from Theoretical Study

The literature research conducted in the context of this study shows that there has been substantial amount of research and developments in the area of process capability indices, especially when non-normally distributed data are of interest. New approaches for dealing with non-normality and new PCIs have been proposed by various researchers. The examinations and comparisons of these methods have also been provided. Amongst all these methods, however, not even one is accepted by both academicians and practitioners as a *de facto* standard to non-normally distributed process capability analysis. Furthermore, some of these methods are criticized by Kotz and Johnson (2002) due to their heavy statistical structures regardless of their practical relevance and utilization, which enlarges the gap between theory and practice.

Comparative studies of different approaches and different PCIs have been conducted to find the best performing process capability analysis for certain circumstances (Padgett and Sengupta, 1996; Tang and Than, 1999; Wu and Swain, 2001).

Padgett and Sengupta (1996) have examined the performance of the standard PCI, C_{pk}, and the PCI based on the percentile approach, which they named C_{pk}-Extended (C_{pkE}). Their analysis is based on the behaviors of estimators of these indices, which are obtained through simulations, when the process follows Weibull or log-normal distribution. According to the results, they conclude that the C_{pk} index performs poorly for the investigated conditions and that the capability index based on the percentile approach, C_{pkE}, should be used whenever the underlying distribution is Weibull or log-normal (Padgett and Sengupta, 1996).

Tang and Than (1999) have conducted an extensive study in which they compared the performances of distribution-free tolerance interval method, weighted variance method, Clements' method, Box-Cox transformation, Johnson transformation and Wright's PCI, C_s for log-normal and Weibull distributions. They argued that amongst the studied methods, the Box-Cox transformation method outperformed the other approaches for sample sizes greater than 100. However, when the considerations about this method (Section 3.3.1) are taken into

account, this method does not seem to be the best way to deal with non-normality (Tang and Than, 1999).

Wu and Swain (2001) have analyzed the weighted variance method, the Clements' method and the flexible index, C_{jkp}, through simulations for both symmetrical and skewed distributions. According to results, they argued that for symmetric distributions the performance of the Clements' method is superior to the others. However, it is noted that for skewed cases, none of the methods performed well (Wu and Swain, 2001).

One final remark should be made about the rarity of published works which investigate specific characteristics of distributions. For example, most of the work devoted to this research area is concerned with two sided specification limits. However, there is not much literature about one-sided specification limits. Vännman and Albing (2007) and Albing (2009) address the cases with one-sided specification limits in their papers and test their new PCI on processes which exhibit such behaviors.

4. Simulation Procedure

In this study, the performance of the proposed method is analyzed with regard to the C_{pk} index based on the percentile approach, which is defined in equation (4). Since the focus is on processes with only lower specification limits, the C_{pk} would be equal to C_{pl}. The rationale behind choosing this percentile-based index rather than the standard C_{pk} calculation is two-fold. First, it is due to the non-normality of the data for which the mean is not representative of the central tendency. Therefore it is more sensible to use the median ($\xi_{0.5}$) as an indicator of centrality rather than the mean. Second, the measure of variation utilized in the standard calculation of C_{pk}, which is 3σ, relates specifically to the normality assumption. It does not accurately represent the spread of non-normal data. Therefore, using percentiles for computing the spread would be more appropriate than 3σ.

Distributions studied in the simulations are log-normal, 3-parameter Weibull, and 3-parameter Gamma distributions. These distributions are mainly chosen due to findings from Li and Chen (2009). Furthermore, the characteristics of these three distributions represent a variety of distribution shapes and tail behaviors depending on the chosen parameters.

The simulation procedure is applied to a set of chosen mean and standard deviation values and three non-normal distributions which are stated above. The procedure is as follows:

1. Assume a non-normal distribution that fits the data:
   The simulations are tested on log-normal, 3-parameter Weibull, and 3-p Gamma distributions.

2. Set the distribution parameters and compute the targeted C_{pk} value:
   By using the process mean and standard deviation the corresponding distribution parameters are obtained. The targeted C_{pk} is computed based on equation (4) with respect to the assumed distribution. Targeted C_{pk}, which is referred to as "Ideal C_{pk}" henceforth, is then used to evaluate the performance of the proposed method through comparisons of the C_{pk} estimates obtained from simulations.

3. Generate random variates of size *N* for the assumed distribution:
   Random variates of size N = 50, 100, and 200 are generated for each underlying distribution according the distribution parameters determined in Step 2. For illustration, assume that a process with 100 data points is randomly generated for the underlying log-normal distribution with a mean of 260 and standard deviation of 80. The histogram of this process can be seen in Figure 8. Note that the distribution exhibits a positive skew, which reflects the characteristics of the real-world data.

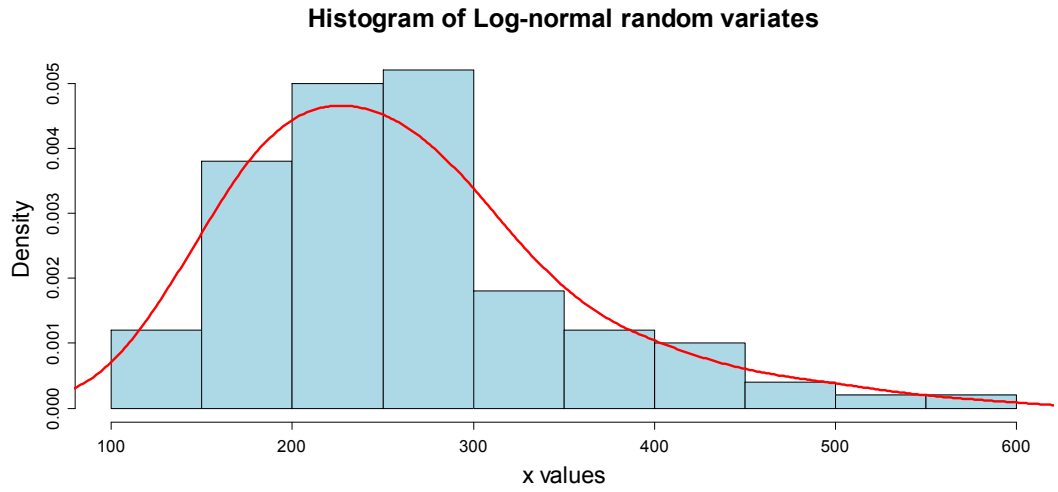

Figure 8. Histogram of Log-normal random variates with mean = 260 and standard deviation = 80.

4. Plot all the generated data into Normal Probability Plot:
   The generated random variates are then plotted into a normal probability plot.

5. Select the first *n* observations from the left tail of the distribution:
   The *n* ranges from 5 to 15. The simulation starts from the first five observations and continues by increasing the number of observations in subsequent iterations. The reasons for considering only a small part of the sample in the left tail are two-fold. First, the process has only lower specification limit; therefore the right tail's effect on process performance is not as important as the left tail's. Second, the right tail has the risk of involving a few assignable causes, such as the tendency of the painters to paint thicker deliberately in order to avoid being blamed.

6. Calculate the estimated mean and standard deviation of the selected *n* values using least square estimation:
   For the first *n* observations selected, a line that best fits them is drawn using linear regression approach. Using least squares estimation, the mean and standard deviation of the selected sample are estimated. In Figure 9, the normal approximation of the first 11 observations is shown.

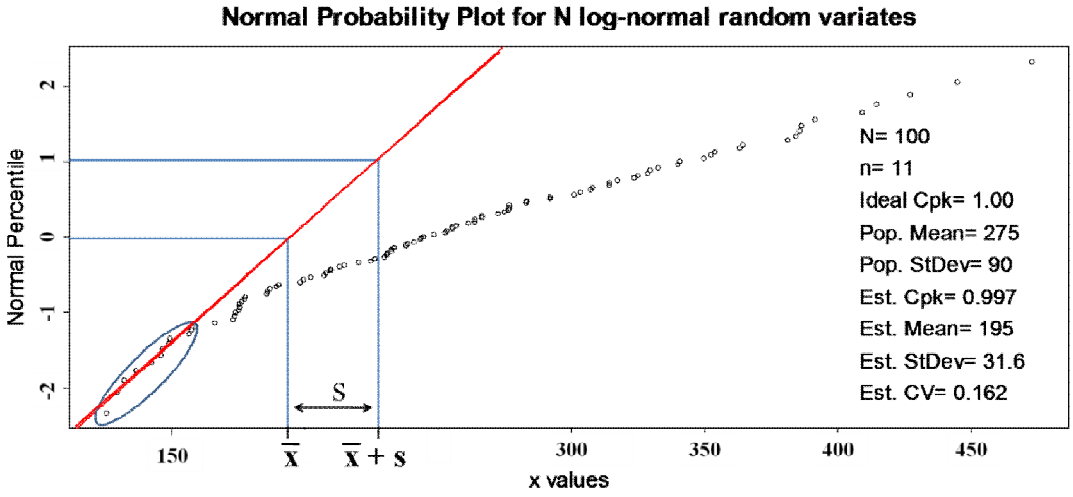

Figure 9. Illustration of the proposed model for N = 100 and observation size n = 11

For the sake of clarity, a simple example, which illustrates the computation of mean and standard deviation, is provided. Consider the NPP given in Figure 10, which has a regression line with the equation $Y = aX + b$.

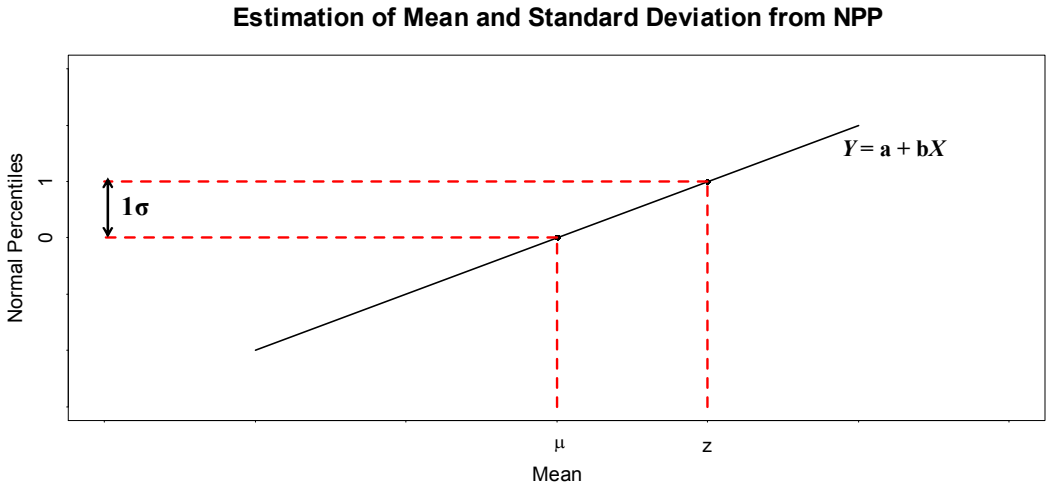

Figure 10. An example of estimating mean and standard deviation from NPP

When the Y axis is equal to 0, the corresponding value at the X axis is the mean of the distribution. Therefore,

$$y = a + bx \Rightarrow 0 = a + b\mu \Leftrightarrow \mu = {}^{-a}/_{b} \qquad (22)$$

is obtained. Since the difference between μ and z is created by the one unit shift of standard deviation, the following set of equations is obtained and standard deviation estimate is derived:

$$\begin{cases} 1\ \sigma = z - \mu \\ 1 = a + bz \end{cases} \Rightarrow \sigma = \left(\frac{1-a}{b}\right) - \left(\frac{-a}{b}\right) \quad (23).$$

Following the logic of this simple example, the mean and standard deviation of the selected values from the tail are estimated according to the normal distribution approximation based on the least square line.

7. Compute the coefficient of variation (CV):
   Using the estimated mean and standard deviation calculated in Step 6, the coefficient of variation (CV) is obtained. This results in total 11 estimates of mean, standard deviation, and CV (i.e. given a process mean and standard deviation, one set of values including estimated mean, standard deviation, and CV is computed for each observation size from 5 to 15).

8. Compute the standard C_{pk} estimate of the selected *n* values:
   Using the estimated mean and standard deviation obtained in Step 6, the C_{pk} estimate of the selected *n* observations is computed based on standard C_{pk} calculation (i.e. given a process mean and standard deviation, in total 11 C_{pk} estimates are computed).

9. Compare the Ideal C_{pk} defined in Step 1 to the C_{pk} estimates obtained in Step 8 for various *n*:
   In this step, the estimated C_{pk} values are analyzed with regard to their precision and accuracy when compared to the Ideal C_{pk} value.

5. Analysis and Results

In this section, the results of the simulations are analyzed. First, the proposed method is tested on normal distribution in order to see its applicability. In the following sub-sections, the method is applied to log-normal, 3-p Weibull, and 3-p Gamma distributions and their results are presented.

5.1. Experimentation on Log-normal Distribution

The applicability of using only a part of the left tail of a distribution for estimating the capability of a process is first tested on log-normal distribution. For this test, log-normally distributed 100 data points are generated and replicated 100 times. After plotting the data into a normal probability plot, first n observations are selected (n ranges from 5 to 15) and the mean and standard deviation of the n data points are estimated through the least squares method. By using the estimated mean and estimated standard deviation, the C_{pk} estimate of the process is computed. The results of the estimates are presented via violin plots. Violin plot representation is chosen over the traditional box-plot representation due to the fact that violin plots combine box plots with density trace (or smoothed histogram). Apart from accompanying the four main features of box plots (i.e. center, spread, asymmetry and outliers), the violin plots also make use of density traces, which enable a better graphical illustration of the distributional characteristics of the variables studied. This combination of box plot and density trace result in a fast and revealing comparison of data since violin plots give information about the peaks, valleys and bumps in the distribution (Hintze and Nelson, 1998). In order to build the violin plots for graphical illustration, "`violins`" function in "caroline" package is utilized in *R* (Schruth, 2011).

In Figure 11, the C_{pk} estimation results of a process sample with mean equals to 275 and standard deviation equals to 90 is shown. The 11 violin plots represent the number of observations taken from the left tail of the distribution starting from 5 to 15. The red dashed line indicates the Ideal C_{pk}, which is 1 in this case.

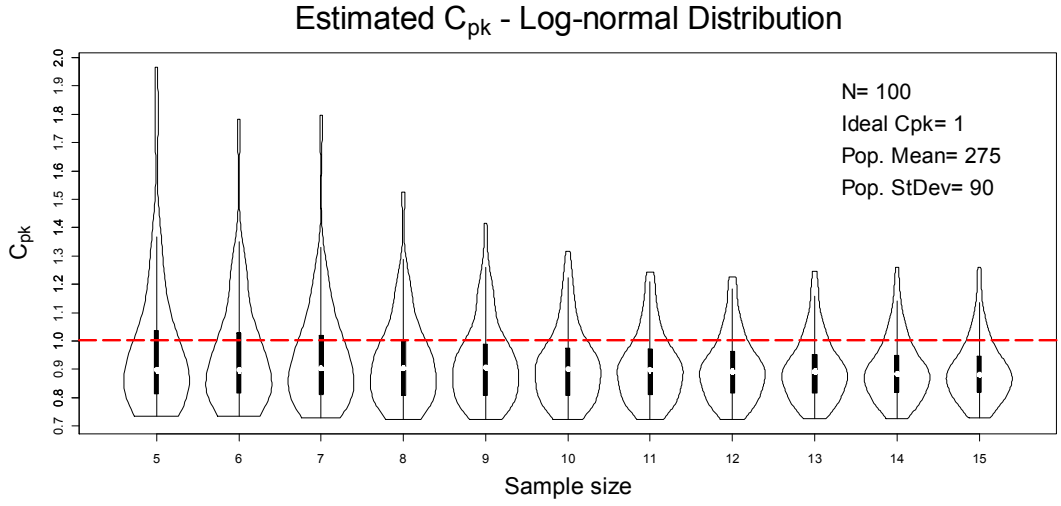

Figure 11. Violin Plots of C_{pk} estimates for log-normal distribution with process mean=250 and standard deviation=50

This experimentation shows that it is actually possible to estimate the capability of a log-normal process by taking only the left tail of the distribution into account. As it can be seen from Figure 11, the median of the estimates is around 0.9 for every sample size taken from the left tail. There are also some outliers, but their densities are low. However, the objective is to propose such a method that the C_{pk} estimates are found within close proximity of the red dashed line. Therefore, the conditions, which provide better estimates, should be investigated.

5.2. Proposed Guideline

In order to find out the conditions underlying good C_{pk} estimations, further investigation is conducted. First, the C_{pk} estimates which give a plus/minus 0.05 deviation from the Ideal C_{pk} are chosen for different sets of mean and standard deviation. The chosen estimates showed that there is a relationship between the CV estimates and C_{pk} estimates. It is found out that there is a range of CV values for different mean and standard deviation that gives plus/minus 0.05 accuracy in C_{pk} estimates. According to this relationship, a practitioner's guide is developed.

The proposed guideline is built for a range of Ideal C_{pk} values from 0.5 to 2.5. Randomly distributed samples assuming log-normal, 3-p Weibull, and 3-p Gamma distributions with different mean and standard deviation values are simulated 10000 times for each combination of mean and standard deviation for population sizes 50, 100, and 200 in order to obtain a three-dimensional contour plot where the estimated CVs are the response values and mean and standard deviation are the *x* and *y* axis. The mean and standard deviation values used for this study are provided in Table 2.

Table 2. Mean and standard deviation values used for CV contour plot

	Ideal C_{pk}	Min Mean	Max Mean	Min St. Dev.	Max St. Dev.
Log-Normal	0.5 – 2.5	180	300	30	90
3-p Weibull (location=130)	1.2 – 2.5	220	300	30	90
3-p Gamma (location=130)	1.3 – 2.5	220	300	30	85

Every combination of mean in increments of 10, and of standard deviation in increments of 5 is simulated. The mean and standard deviation values, as well as the location parameter of the Weibull and Gamma distributions, are chosen based on the nature of the problem defined in Section 1.2. In Section 5, only the graphs for sample size N=100 are provided. In Figures 12, 13, and 14, the CV contour plots for log-normal, 3-p Weibull, and 3-p Gamma distributions built for 100 data points are presented. The CV contour plots for N=50 and N=200 are given in Appendices D, E and F, respectively.

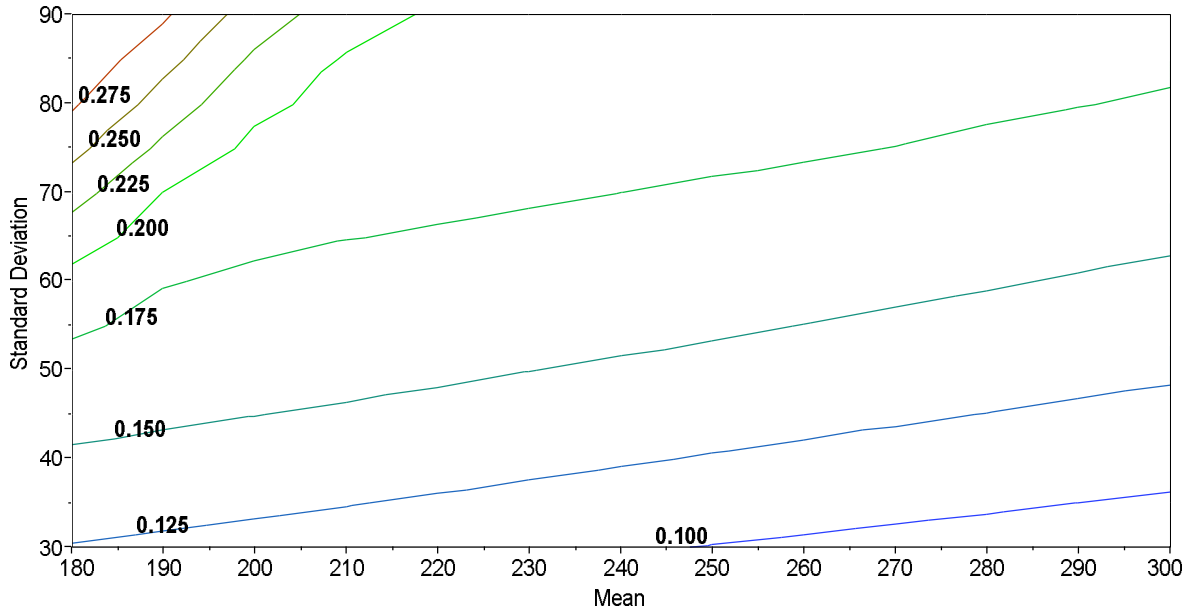

Figure 12. Contour plot for CV values vs. mean and standard deviation for Log-normal distribution and N = 100

Note that, for log-normal distribution and for sample size 100, there are nine CV ranges. On the other hand, for 3-p Weibull and 3-p Gamma distributions seven CV ranges are obtained.

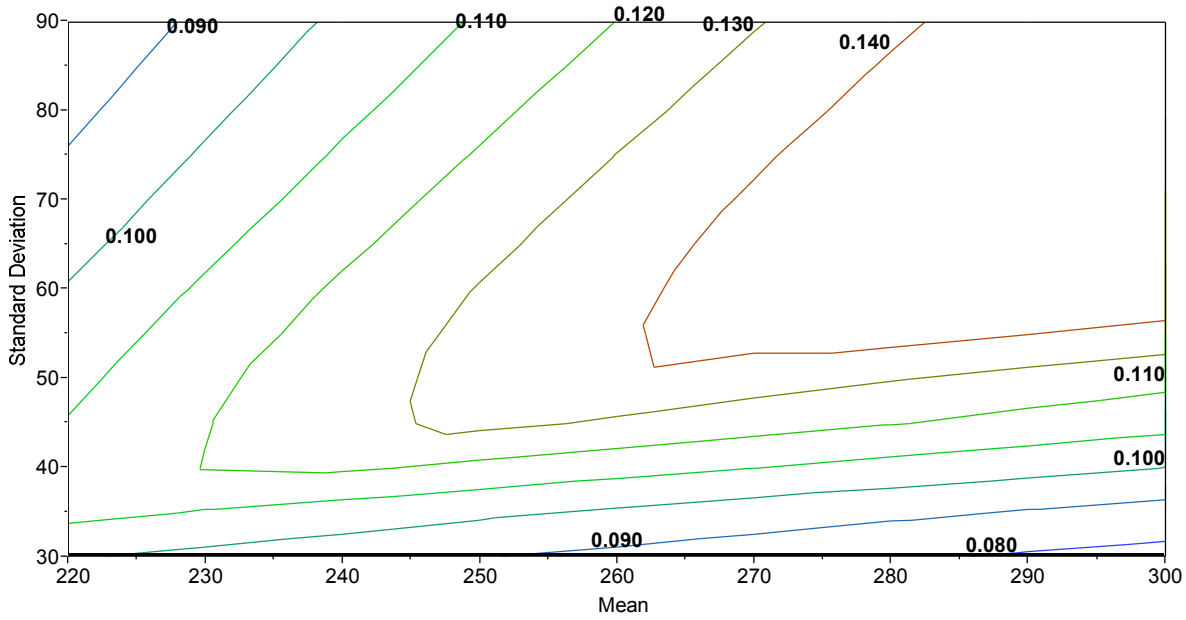

Figure 13. Contour plot for CV values vs. mean and standard deviation for 3-p Weibull distribution and N = 100

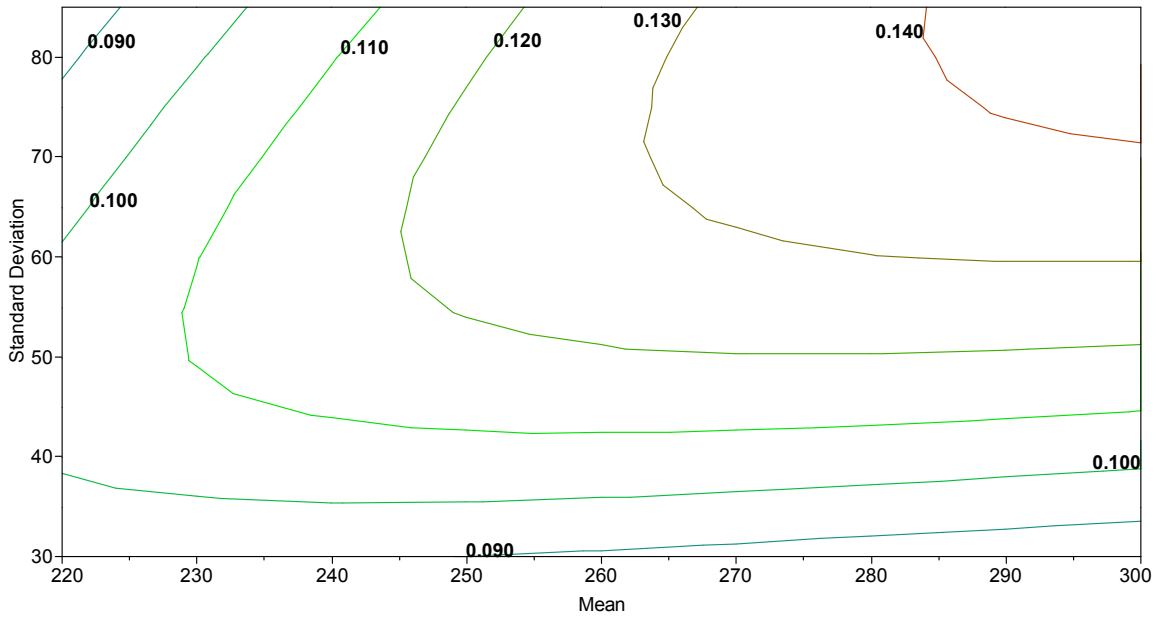

Figure 14. Contour plot for CV values vs. mean and standard deviation for 3-p Gamma distribution and N = 100

Following the CV contour plots, the practitioner's guideline is suggested step-by-step as follows:

1. Apply a goodness-of-fit test to the sample data in order to see if the data is non-normally distributed.
2. Use the process mean and standard deviation in order to find the suitable CV range in the contour plot. Note that different number of data points will require slightly different ranges (i.e. the ranges may differ for sample sizes 50, 100, and 200).
3. Plot all the points in a normal probability plot.
4. Start selecting the first n observations from the left tail of the distribution. It is recommended to start with the first 5 data points.
5. Draw a straight line that can cover as many selected points as possible. Estimate the mean and standard deviation of the line.
6. Compute the CV from the line and check if it falls within the range as prescribed in the contour plot. If it does, proceed to the next step. If not, add more data points and go back to the previous step.
7. Calculate the C_{pk} estimate for the selected observation set of size n based on the mean and standard deviation of the line.

5.3. Effect of Sample Size on the Proposed Method

As a preliminary analysis, the question of whether sample size taken from the left tail of the distribution has an effect on the proposed method is investigated. Therefore, for some data points (a data point means a specific set of mean and standard deviation), the individual normal approximations of every of observation size 5 to 15 are extracted from the simulations along with the information regarding the skewness and kurtosis of the empirical distributions.

Individual visualization analysis revealed that the sample size (from the first 5 observations to the first 15 observations) does indeed matter for the proposed method. It is found out that for the same empirical data set, some sample sizes may fail to meet the prescribed CV range, whereas the other sample sizes extracted from the same distribution may successfully fall within the range. In order to exemplify this situation, a failed and a successful case are provided in Figure 15 and 16 for 3-p Weibull distribution with mean and standard deviation equal to 290 and 53.

Figure 15 illustrates the linear regression applied to the first 10 observations taken from the left tail. It is labeled as a failed case since the estimated CV is computed as 0.114, whereas it should be between 0.13 – 0.14 to be qualified as a successful case (see Figure 13).

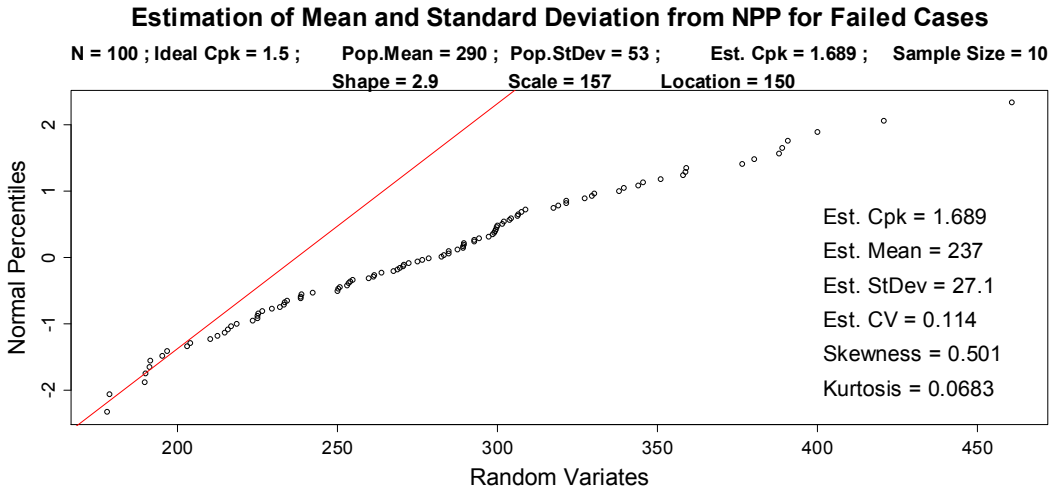

Figure 15. Visualization of a failed case of the first 10 sample, 3-p Weibull dist., mean = 290, stdev = 53.

On the other hand, Figure 16 represents a successful case as a result of the linear regression applied on the first 12 samples.

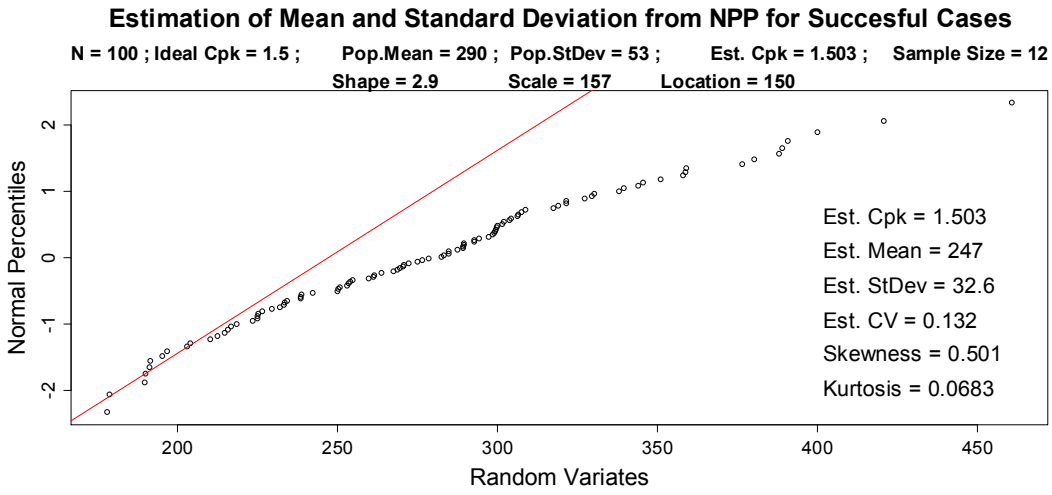

Figure 16. Visualization of a successful case of the first 12 sample, 3-p Weibull dist., mean = 290, stdev = 53.

Note that these two cases are derived from the same distribution with the same skewness and kurtosis values. This example shows that it is possible to find both successful and failed cases within the same distribution depending on the size of the samples taken from the tail. Similar examples are also found for log-normal and 3-p Gamma distributions (see Appendices G and H).

These findings imply that the practitioner would compute the CV estimates starting from the first 5 observations until it finds the first successful case where the estimated CV falls within the prescribed range and then compute the C_{pk} estimate for that sample size.

Since the sample size has an effect on the proposed method, it is interesting to see that given a set of mean and standard deviation, how often a set of samples makes CV estimations that would fall within the prescribed range. Following the 3-p Weibull example, the data point (290; 53) is replicated 10000 times and the frequencies of sample sizes, which estimate CV within the prescribed range, are shown in Figure 17.

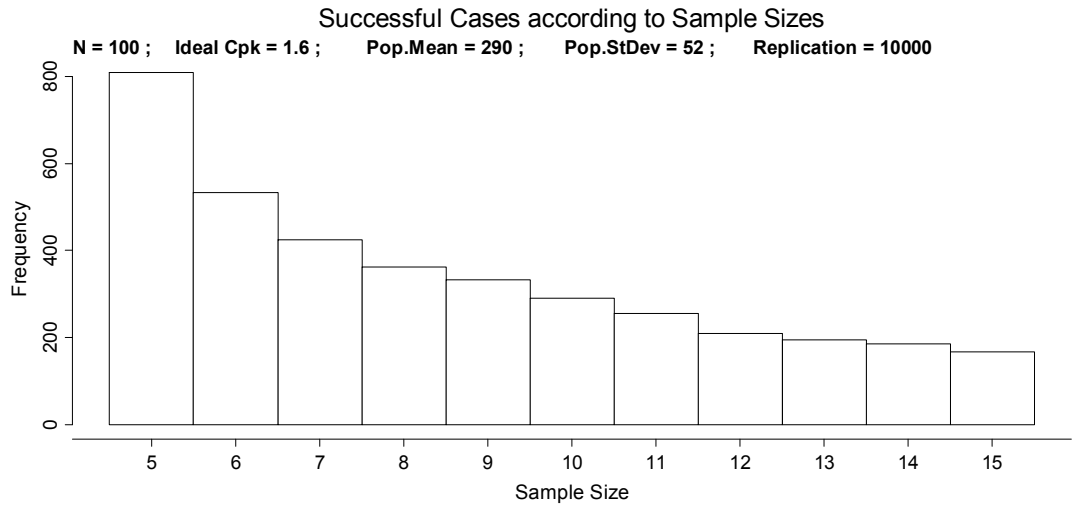

Figure 17. Histogram of successful results grouped according to sample size

It can be seen from Figure 17 that by using the first five observations, one can find CV estimates which fall within the prescribed CV range most of the time. However, taking the first successful C_{pk} estimation incorporates some risks as well. For instance, there is a possibility that better estimates can be obtained through other sample sizes.

Individual visualization also revealed that good C_{pk} estimations can sometimes be labeled as failed cases due to the fact that their corresponding CV estimates miss the prescribed CV range by very small differences.

5.4. Performance Analysis of the Proposed Method

In order to be able to assess the performance of the proposed C_{pk} estimation method, there is a need for certain evaluation criteria which reflect the performance of the proposed method in terms of precision, and accuracy. In this study, the chosen performance criteria are the mean squared error (MSE), variance and bias. These three indicators are widely used in relevant

literature for assessing the performance of estimators (Padgett and Sengupta, 1996; Deleryd, 1999; McCormack Jr, *et al.*, 2000). The MSE, variance and bias are obtained through the following formulas:

$$MSE = E\left[\left(\hat{\theta} - \theta\right)^2\right] \tag{24}$$

$$Var = E\left[\left(\hat{\theta} - \overline{\hat{\theta}}\right)^2\right] \tag{25}$$

$$Bias = \sqrt{MSE - Var} \tag{26}$$

where $\hat{\theta}$ is the estimator of θ and $\overline{\hat{\theta}}$ is the average of the estimates, $\hat{\theta}$.

The mean squared error of an estimator captures the difference between the estimator and the parameter value that is estimated. In this case, the MSE indicates the error that the C_{pk} estimates ($\widehat{C_{pk}}$) make by how much they differ from the Ideal C_{pk} value. Variance of an estimator indicates the dispersion of the estimates, thus reflecting the precision of the estimates. Bias, on the other hand, is the difference between MSE and variance and is an indicator of the accuracy of the estimates.

It is worth highlighting that the CV estimate does not always fall within the prescribed range. Therefore, another performance criterion, which would indicate how often this happens, is needed. This criterion, which is called "success rate", reflects the chance of finding the estimated CV value that falls within the CV range prescribed according to the process mean and standard deviation. The C_{pk} estimate is calculated if and only if the estimated CVs of the first *n* observations fall within the CV range (see Figure 12 - 14). Thus, success rate can be regarded as a measure of being able to estimate process capability via this proposed method.

For analysis, different data points (i.e. different sets of process sample mean and standard deviation) are chosen from each CV contour plot for population sizes N=50, 100 and 200 with log-normal, 3-p Weibull, and 3-p Gamma as underlying distributions. Furthermore, the data points are categorized according to their resulting Ideal C_{pk} values. Four Ideal C_{pk} ranges are used for this categorization: from 0.5 to 1, from 1 to 1.5, from 1.5 to 2 and from 2 to 2.5. For log-normal as the assumed distribution, 83 data points, for Weibull 60, and for Gamma 70 data points are tested. Each data point is then replicated 10000 times and the evaluation criteria discussed above are computed.

Once all the assessment results for every data point are grouped according to the CV range and the Ideal C_{pk} range, the minimum and maximum values of MSE, variance, bias, and success rate amongst different data points are recorded within a certain category. For MSE, variance and bias, the minimum and the maximum values can be regarded as the error margin that the practitioner would endure if this method is used for that specific category. For success rate, on the other hand, the minimum and maximum values indicate the highest and the lowest chance of being able to estimate C_{pk} with this method. In Table 3, 4, and 5 the results of the simulations for log-normal, 3-p Weibull and 3-p Gamma distributions are provided.

Table 3. Log-normal Distribution Simulation Results

		0.5<=Cpl ideal<=1								1<Cpl ideal<=1.5								1.5<Cpl ideal<=2.0								2.0<Cpl ideal<=2.5							
		MSE		Variance		Bias		Success Rate		MSE		Variance		Bias		Success Rate		MSE		Variance		Bias		Success Rate		MSE		Variance		Bias		Success Rate	
	RANGE	best	worst	best	worst	best	worst	best	worst	best	worst	best	worst	best	worst	best	worst	best	worst	best	worst	best	worst	best	worst	best	worst	best	worst	best	worst	best	worst
N = 50	0.275 - 0.300	0.040	0.045	0.007	0.008	0.181	0.190	49	48																						
	0.250 - 0.275	0.037	0.047	0.008	0.010	0.171	0.191	45	40																						
	0.225 - 0.250	0.031	0.043	0.007	0.010	0.155	0.181	41	32																						
	0.200 - 0.225	0.014	0.037	0.005	0.012	0.091	0.156	38	32																						
	0.175 - 0.200	0.005	0.014	0.004	0.006	0.017	0.089	38	29	0.004	0.008	0.003	0.003	0.030	0.068	43	38														
	0.150 - 0.175	0.006	0.010	0.006	0.005	0.000	0.073	44	30	0.005	0.013	0.004	0.004	0.017	0.095	48	39														
	0.125 - 0.150	0.006	0.008	0.006	0.008	0.000	0.014	39	31	0.005	0.013	0.005	0.004	0.010	0.093	52	39	0.009	0.024	0.008	0.008	0.039	0.127	52	48								
	0.100 - 0.125									0.011	0.018	0.011	0.012	0.025	0.080	45	38	0.012	0.031	0.012	0.018	0.010	0.115	55	42	0.054	0.073	0.019	0.019	0.188	0.232	58	57
	0.075 - 0.100																									0.060	0.091	0.044	0.057	0.059	0.217	57	44
		MSE		Variance		Bias		Success Rate		MSE		Variance		Bias		Success Rate		MSE		Variance		Bias		Success Rate		MSE		Variance		Bias		Success Rate	
	RANGE	best	worst	best	worst	best	worst	best	worst	best	worst	best	worst	best	worst	best	worst	best	worst	best	worst	best	worst	best	worst	best	worst	best	worst	best	worst	best	worst
N = 100	0.275 - 0.300	0.028	0.036	0.005	0.006	0.151	0.173	50	47																						
	0.250 - 0.275	0.028	0.040	0.005	0.006	0.153	0.184	51	50																						
	0.225 - 0.250	0.028	0.043	0.005	0.007	0.151	0.189	49	44																						
	0.200 - 0.225	0.024	0.040	0.005	0.008	0.137	0.179	46	34																						
	0.175 - 0.200	0.004	0.016	0.003	0.005	0.030	0.107	44	33	0.004	0.007	0.002	0.003	0.030	0.069	47	40														
	0.150 - 0.175	0.005	0.009	0.005	0.004	0.000	0.072	47	33	0.004	0.008	0.003	0.003	0.014	0.067	48	36														
	0.125 - 0.150	0.004	0.005	0.004	0.005	0.010	0.017	41	36	0.005	0.018	0.005	0.005	0.014	0.111	53	41	0.010	0.023	0.006	0.007	0.057	0.125	53	51								
	0.100 - 0.125									0.009	0.019	0.007	0.010	0.048	0.093	48	41	0.010	0.019	0.010	0.016	0.000	0.051	51	44	0.071	0.047	0.019	0.018	0.230	0.171	57	57
	0.075 - 0.100																									0.059	0.062	0.048	0.052	0.059	0.103	57	46
		MSE		Variance		Bias		Success Rate		MSE		Variance		Bias		Success Rate		MSE		Variance		Bias		Success Rate		MSE		Variance		Bias		Success Rate	
	RANGE	best	worst	best	worst	best	worst	best	worst	best	worst	best	worst	best	worst	best	worst	best	worst	best	worst	best	worst	best	worst	best	worst	best	worst	best	worst	best	worst
N = 200	0.275 - 0.300	0.013	0.030	0.004	0.006	0.093	0.154	41	27																						
	0.250 - 0.275	0.014	0.023	0.004	0.004	0.103	0.139	47	37																						
	0.225 - 0.250	0.015	0.025	0.003	0.004	0.110	0.146	51	47																						
	0.200 - 0.225	0.016	0.029	0.004	0.005	0.113	0.156	51	45																						
	0.175 - 0.200	0.005	0.015	0.003	0.005	0.042	0.101	49	40	0.004	0.006	0.002	0.002	0.039	0.065	50	47														
	0.150 - 0.175	0.004	0.008	0.004	0.004	0.010	0.065	48	36	0.003	0.011	0.003	0.003	0.010	0.089	52	37														
	0.125 - 0.150	0.004	0.004	0.004	0.004	0.000	0.017	47	38	0.004	0.014	0.004	0.005	0.000	0.096	53	41	0.007	0.022	0.007	0.007	0.000	0.120	52	49								
	0.100 - 0.125									0.007	0.013	0.005	0.009	0.040	0.066	48	43	0.012	0.038	0.012	0.013	0.000	0.158	56	48	0.035	0.067	0.016	0.018	0.137	0.222	56	55
	0.075 - 0.100																									0.057	0.082	0.054	0.049	0.054	0.181	58	49

Table 4. 3-parameter Weibull Distribution Simulation Results

		1<Cpl ideal<=1.5								1.5<Cpl ideal<=2.0								2.0<Cpl ideal<=2.5							
		MSE		Variance		Bias		Success Rate		MSE		Variance		Bias		Success Rate		MSE		Variance		Bias		Success Rate	
	RANGE	best	worst	best	worst	best	worst	best	worst	best	worst	best	worst	best	worst	best	worst	best	worst	best	worst	best	worst	best	worst
N50	0.14 - 0.15	0.0026	0.0075	0.0025	0.0047	0.0100	0.0533	26	21																
	0.13 - 0.14	0.0040	0.0087	0.0029	0.0026	0.0323	0.0782	27	20	0.0027	0.0032	0.0027	0.0030	0.0032	0.0155	27	25								
	0.12- 0.13	0.0040	0.0077	0.0038	0.0045	0.0158	0.0564	25	20	0.0031	0.0117	0.0030	0.0027	0.0089	0.0948	34	27								
	0.11 - 0.12	0.0058	0.0101	0.0049	0.0045	0.0290	0.0748	30	25	0.0039	0.0094	0.0038	0.0031	0.0063	0.0791	34	29								
	0.10 - 0.11	0.0049	0.0077	0.0042	0.0046	0.0268	0.0560	31	28	0.0041	0.0078	0.0041	0.0040	0.0000	0.0615	33	31	0.0048	0.0079	0.0041	0.0039	0.0259	0.0628	36	33
	0.09 - 0.10	0.0044	0.0051	0.0042	0.0049	0.0134	0.0138	31	29	0.0062	0.0066	0.0033	0.0053	0.0539	0.0354	41	30	0.0057	0.0230	0.0053	0.0051	0.0203	0.1338	40	34
	0.08 - 0.09	0.0051	0.1511	0.0041	0.0090	0.0323	0.3770	52	37									0.0091	0.0331	0.0073	0.0081	0.0417	0.1581	41	37
		MSE		Variance		Bias		Success Rate		MSE		Variance		Bias		Success Rate		MSE		Variance		Bias		Success Rate	
	RANGE	best	worst	best	worst	best	worst	best	worst	best	worst	best	worst	best	worst	best	worst	best	worst	best	worst	best	worst	best	worst
N100	0.14 - 0.15	0.0025	0.0047	0.0024	0.0018	0.0055	0.0545	40	32																
	0.13 - 0.14	0.0028	0.0044	0.0028	0.0022	0.0063	0.0469	45	35	0.0019	0.0045	0.0019	0.0020	0.0063	0.0505	38	35								
	0.12- 0.13	0.0027	0.0050	0.0027	0.0025	0.0089	0.0506	51	38	0.0023	0.0042	0.0022	0.0023	0.0071	0.0434	39	35								
	0.11 - 0.12	0.0027	0.0046	0.0026	0.0026	0.0105	0.0452	58	45	0.0029	0.0097	0.0026	0.0023	0.0179	0.0864	43	33								
	0.10 - 0.11	0.0024	0.0041	0.0021	0.0021	0.0173	0.0451	59	42	0.0022	0.0068	0.0022	0.0028	0.0045	0.0628	57	41	0.0035	0.0042	0.0033	0.0035	0.0152	0.0251	42	40
	0.09 - 0.10	0.0027	0.0036	0.0017	0.0025	0.0307	0.0329	63	55	0.0020	0.0051	0.0020	0.0043	0.0078	0.0283	62	40	0.0049	0.0251	0.0047	0.0041	0.0123	0.1448	50	39
	0.08 - 0.09	0.0020	0.0055	0.0020	0.0021	0.0000	0.0581	63	48									0.0072	0.0143	0.0070	0.0077	0.0141	0.0812	49	47
		MSE		Variance		Bias		Success Rate		MSE		Variance		Bias		Success Rate		MSE		Variance		Bias		Success Rate	
	RANGE	best	worst	best	worst	best	worst	best	worst	best	worst	best	worst	best	worst	best	worst	best	worst	best	worst	best	worst	best	worst
N200	0.14 - 0.15	0.0014	0.0026	0.0013	0.0014	0.0078	0.0346	50	43																
	0.13 - 0.14	0.0016	0.0018	0.0016	0.0015	0.0000	0.0170	56	49	0.0016	0.0028	0.0015	0.0016	0.0127	0.0346	46	44								
	0.12- 0.13	0.0012	0.0029	0.0010	0.0016	0.0164	0.0355	56	41	0.0018	0.0039	0.0016	0.0018	0.0032	0.0485	46	45								
	0.11 - 0.12	0.0013	0.0022	0.0010	0.0015	0.0032	0.0276	57	28	0.0022	0.0033	0.0016	0.0022	0.0130	0.0346	51	46								
	0.10 - 0.11	0.0012	0.0031	0.0011	0.0011	0.0105	0.0449	59	34	0.0030	0.0060	0.0023	0.0022	0.0259	0.0616	50	48	0.0030	0.0039	0.0029	0.0032	0.0127	0.0276	49	49
	0.09 - 0.10	0.0008	0.0050	0.0008	0.0011	0.0000	0.0621	31	27	0.0042	0.0047	0.0038	0.0032	0.0187	0.0386	51	49	0.0150	0.0150	0.0046	0.0046	0.1017	0.1017	52	50
	0.08 - 0.09																	0.0063	0.0104	0.0063	0.0075	0.0084	0.0539	53	52

Table 5. 3-parameter Gamma Distribution Simulation Results

		1<Cpl ideal<=1.5								1.5<Cpl ideal<=2.0								2.0<Cpl ideal<=2.5							
		MSE		Variance		Bias		Success Rate		MSE		Variance		Bias		Success Rate		MSE		Variance		Bias		Success Rate	
	RANGE	best	worst	best	worst	best	worst	best	worst	best	worst	best	worst	best	worst	best	worst	best	worst	best	worst	best	worst	best	worst
N50	0.14 - 0.15	0.0030	0.0060	0.0021	0.0029	0.0100	0.0566	32	25																
	0.13 - 0.14	0.0032	0.0067	0.0029	0.0035	0.0000	0.0566	31	26	0.0028	0.0047	0.0022	0.0032	0.0100	0.0469	33	25								
	0.12- 0.13	0.0031	0.0069	0.0030	0.0041	0.0100	0.0539	33	26	0.0026	0.0055	0.0026	0.0028	0.0000	0.0566	35	31								
	0.11 - 0.12	0.0039	0.0075	0.0036	0.0051	0.0000	0.0624	35	24	0.0031	0.0063	0.0028	0.0033	0.0000	0.0592	37	31								
	0.10 - 0.11	0.0047	0.0074	0.0038	0.0047	0.0000	0.0574	36	27	0.0038	0.0069	0.0036	0.0048	0.0100	0.0574	37	29	0.0040	0.0052	0.0039	0.0040	0.0000	0.0361	34	33
	0.09 - 0.10	0.0046	0.0061	0.0042	0.0053	0.0100	0.0436	39	31	0.0047	0.0111	0.0043	0.0045	0.0141	0.0825	41	35	0.0055	0.0166	0.0050	0.0059	0.0000	0.1034	40	35
	0.08 - 0.09	0.0041	0.0049	0.0035	0.0043	0.0245	0.0245	46	38									0.0102	0.0145	0.0082	0.0076	0.0447	0.0831	37	35
		MSE		Variance		Bias		Success Rate		MSE		Variance		Bias		Success Rate		MSE		Variance		Bias		Success Rate	
	RANGE	best	worst	best	worst	best	worst	best	worst	best	worst	best	worst	best	worst	best	worst	best	worst	best	worst	best	worst	best	worst
N100	0.14 - 0.15	0.0018	0.0040	0.0018	0.0019	0.0000	0.0458	42	36																
	0.13 - 0.14	0.0021	0.0043	0.0021	0.0025	0.0000	0.0424	46	37	0.0019	0.0038	0.0019	0.0026	0.0000	0.0346	39	37								
	0.12- 0.13	0.0025	0.0048	0.0024	0.0026	0.0100	0.0469	51	37	0.0021	0.0058	0.0020	0.0025	0.0100	0.0574	39	35								
	0.11 - 0.12	0.0028	0.0031	0.0025	0.0021	0.0173	0.0316	56	43	0.0025	0.0062	0.0025	0.0025	0.0000	0.0608	42	35								
	0.10 - 0.11	0.0023	0.0036	0.0023	0.0022	0.0000	0.0374	60	53	0.0029	0.0059	0.0027	0.0032	0.0100	0.0520	44	35	0.0046	0.0133	0.0035	0.0430	0.0332	0.0948	40	37
	0.09 - 0.10	0.0022	0.0034	0.0021	0.0023	0.0100	0.0332	63	56	0.0039	0.0108	0.0038	0.0036	0.0100	0.0812	43	40	0.0049	0.0171	0.0045	0.0053	0.0200	0.1086	40	39
	0.08 - 0.09	0.0038	0.0045	0.0021	0.0022	0.0412	0.0490	64	61									0.0098	0.0128	0.0069	0.0084	0.0374	0.0768	39	37
		MSE		Variance		Bias		Success Rate		MSE		Variance		Bias		Success Rate		MSE		Variance		Bias		Success Rate	
	RANGE	best	worst	best	worst	best	worst	best	worst	best	worst	best	worst	best	worst	best	worst	best	worst	best	worst	best	worst	best	worst
N200	0.14 - 0.15	0.0013	0.0026	0.0013	0.0012	0.0000	0.0374	46	42																
	0.13 - 0.14	0.0014	0.0030	0.0012	0.0015	0.0000	0.0412	49	43	0.0014	0.0032	0.0013	0.0014	0.0000	0.0436	44	41								
	0.12- 0.13	0.0014	0.0029	0.0011	0.0017	0.0000	0.0387	51	42	0.0017	0.0038	0.0017	0.0018	0.0000	0.0458	41	38								
	0.11 - 0.12	0.0012	0.0036	0.0010	0.0018	0.0100	0.0458	53	33	0.0018	0.0057	0.0017	0.0023	0.0100	0.0600	48	38								
	0.10 - 0.11	0.0011	0.0023	0.0010	0.0020	0.0100	0.0200	51	40	0.0025	0.0054	0.0020	0.0030	0.0141	0.0490	46	39	0.0043	0.0073	0.0031	0.0032	0.0332	0.0648	40	38
	0.09 - 0.10	0.0010	0.0063	0.0010	0.0012	0.0000	0.0714	30	20	0.0035	0.0097	0.0030	0.0034	0.0100	0.0819	45	42	0.0044	0.0092	0.0040	0.0050	0.0100	0.0648	40	37
	0.08 - 0.09																	0.0092	0.0122	0.0065	0.0079	0.0361	0.0755	40	38

Table 3, 4 and 5 are divided into matrix-like tables according to population sizes and Ideal C_{pk} ranges. The first column gives the CV range and the following columns give the simulation results. The maximum values are labeled as "best" and minimum values are labeled as "worse". The blank cells indicate that there are no data points which result in the specified Ideal C_{pk} within that CV range. For example, In Table 3, four data points are tested for the CV range 0.275 – 0.300. These data points are (181; 82), (188; 89), (181; 89) and (183; 86). The best values of MSE, variance, bias, and success rate are obtained with the data point (181; 82), whereas the worst values are obtained with (181; 89). The rationale of providing the minimum and maximum values is to show the worst and best performances of the method within that specific CV range.

When Table 3 is analyzed, it can be seen that for Ideal C_{pk} range from 0.5 to 1.0, MSE of the method improves starting from CV range 0.175 – 0.200. Accordingly, the bias results exhibit a similar behavior. Furthermore, it is observed that as Ideal C_{pk} gets higher, the performance of the method decreases slightly.

The log-normal simulation results report that the proposed method works better for CV ranges that are found in the middle of the CV contour plot (see Figure 12). In other words, for processes with low mean and high standard deviation (i.e. high CV and low Ideal C_{pk}) and for processes with high mean and low standard deviation (i.e. low CV and high Ideal C_{pk}), MSE is higher which indicates that the proposed method makes more errors when estimating the process capability. Furthermore, an overall impression of the proposed method for log-normal distribution is that the success rate can be as high as 1 in 2 chances, which is not bad.

For both 3-p Weibull and 3-p Gamma distributions, the results are grouped according to 7 CV ranges and 3 Ideal C_{pk} ranges (see Tables 4 and 5). Due to the chosen parameters, there is no set of mean and standard deviation that results in an Ideal C_{pk} value within the range 0.5 – 1.0. It can be seen from the Tables 4 and 5 that the method performs less well for Ideal C_{pk} range from 2.0 to 2.5 in terms of MSE, variance and bias. Moreover, as the CV values increase, small increases in MSE and bias are observed. This finding, which can be interpreted as the method returns better results for the values taken from the middle of the contour plot, is in accordance with the findings from log-normal simulation.

When the results for log-normal, 3-p Weibull and 3-p Gamma distributions are compared, it can be seen that the proposed method performs better for 3-p Weibull distribution and 3-p Gamma distribution in terms of MSE, variance and bias. In other words, the error margin is lower; precision and accuracy are higher when the method is applied to 3-p Weibull and 3-p Gamma distributions. On the other hand, the success rate of the method does not seem to differ substantially for different distributions, and for different population sizes.

After the analyzing the simulation results, two questions have arisen concerning the proposed method. The first question relates to the circumstances when the proposed method gives good results and when it gives bad results. It can be seen from the tables that different data points, which are taken from the same CV range, result in different values of MSE, variance and bias.

The second question relates to the success rate of the proposed method. In order to investigate these two questions, a further analysis is conducted on the proposed method.

In order to answer the first question, certain data points are selected from a specific CV range and analyzed individually. As for the second question, the coefficients of determination (R^2) of the regression line used in normal approximation are compared for both successful and failed cases. Moreover, skewness and kurtosis of the randomly generated distributions are analyzed in order to see whether there is a relationship between the success of the proposed method and the distribution's moments.

5.4.1. Studies Regarding the Good Results and the Bad Results

In order to investigate the reasons of why the proposed method performs better for some data points within the same CV range, individual data points are taken from different locations within the corresponding CV range. Different locations imply that some sets of mean and standard deviation are chosen from the borders of the corresponding CV range and some of them are chosen from the middle of the corresponding CV range.

Through simulation tests, it is found out that the proposed guideline leads to rather worse results for borderline cases in the contour plot compared to the middle values extracted. In order to exemplify this condition; some data points are taken for population size N=100. The data points are presented in Table 6. For each distribution, six data points are simulated 10000 times. Note that two sets of mean and standard deviation, which result in the same Ideal C_{pk} value, are tested within the same CV range. These points are chosen due to the fact that they would reflect as closely as possible the real-world data. Furthermore, if the data points are to be considered as pairs, it could be seen that a pair of data points are extracted within the same CV range and has the same Ideal C_{pk} value.

Table 6. Data points and their locations

Distribution	Mean	Standard Deviation	Ideal C_{pk}	CV Range	Data Point Location
Log-normal	260	80	1.0	0.175 – 0.200	Middle
	243	71	1.0	0.175 – 0.200	Lower contour border
	280	52	1.5	0.125 – 0.150	Middle
	241	40	1.5	0.125 – 0.150	Upper contour border
	293	40	2.0	0.100 – 0.125	Middle
	258	32	2.0	0.100 – 0.125	Lower contour border
3-p Weibull	290	53	1.5	0.13 – 0.14	Middle
	272	49	1.5	0.13 – 0.14	Lower contour border
	257	33	2.0	0.09 – 0.10	Middle
	263	35	2.0	0.09 – 0.10	Upper contour border
	290	33	2.5	0.08 – 0.09	Middle
	294	35	2.5	0.08 – 0.09	Upper contour border
3-p Gamma	260	58	1.5	0.12 – 0.13	Middle
	271	61	1.5	0.12 – 0.13	Right contour border
	285	40	2.0	0.10 – 0.11	Middle
	267	37	2.0	0.10 – 0.11	Lower contour border
	291	31	2.5	0.08 – 0.09	Middle
	297	32	2.5	0.08 – 0.09	Upper contour border

Figures 18, 19, and 20 illustrate the C_{pk} estimates via violin plots for log-normal, 3-p Weibull, and 3-p Gamma distributions, respectively. Each figure includes six violin plots, which correspond to the data points provided in Table 6. Three pairs of data illustrate different Ideal C_{pk} results, whereas the data points within the pair represent different locations in the CV contour plot.

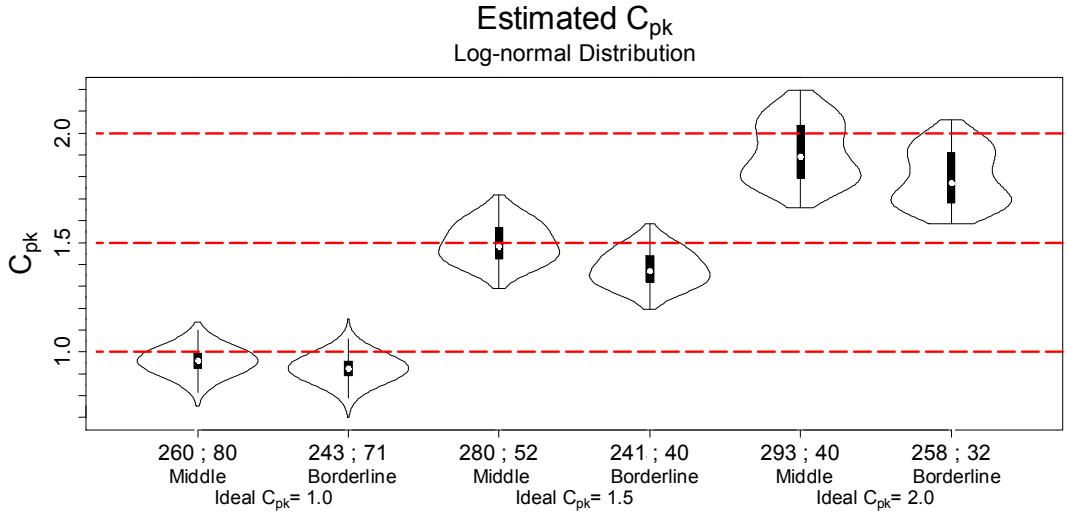

Figure 18. C_{pk} estimates for log-normal distribution, N=100

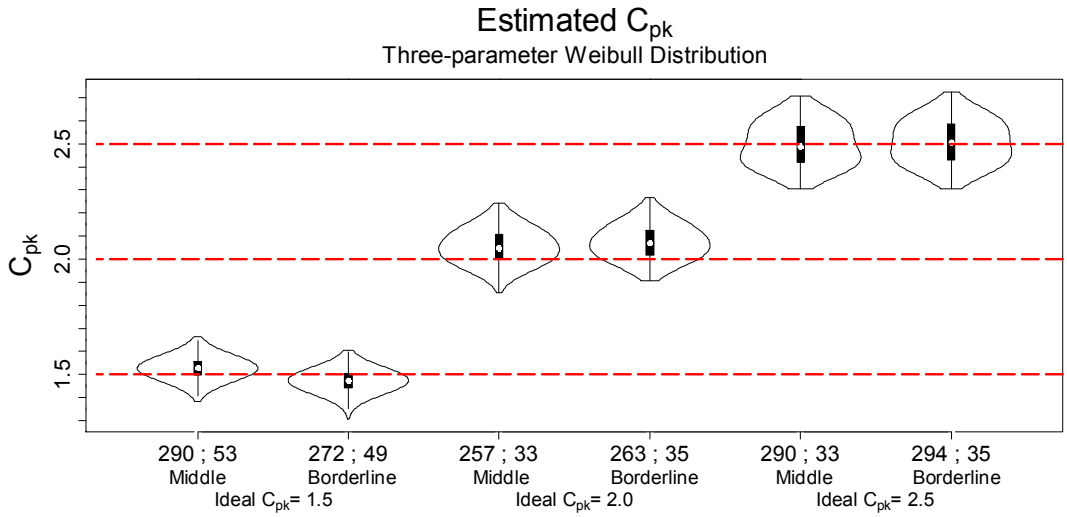

Figure 19. C_{pk} estimates for 3-p Weibull distribution, N=100

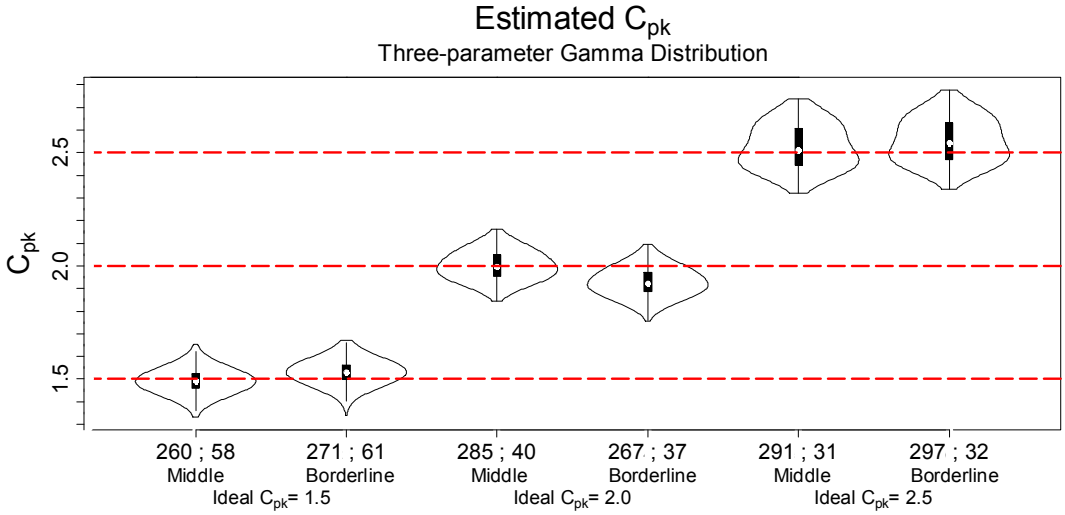

Figure 20. C_{pk} estimates for 3-p Gamma distribution, N=100

Figures 18-20 have two implications. First, even though two data points are taken from the same CV range and they return the performance of the proposed method differ. Second, as Ideal C_{pk} increases the precision and accuracy of the method decreases. Numerical indicators of Figures 18-20 are provided in Table 7. For every data point tested, the MSE, variance, bias, and success rate are provided. In addition to these, the first, second, and the third quartiles of the C_{pk} estimates along with the maximum and minimum values are given.

Table 7. Simulation Results for middle point and borderline cases

	(Mean; StDev)	Ideal C_{pk}	Min	Q1	Q2	Q3	Max	MSE	Var	Bias	Success Rate %
Log-normal	260; 80	1.0	0.753	0.923	0.959	0.994	1.135	0.0051	0.0029	0.0475	42
	243; 71	1.0	0.703	0.891	0.925	0.959	1.151	0.0070	0.0027	0.0657	44
	280; 52	1.5	1.289	1.427	1.485	1.566	1.717	0.0068	0.0068	0.0055	48
	241;40	1.5	1.194	1.319	1.370	1.440	1.584	0.0162	0.0053	0.1046	53
	293; 40	2.0	1.658	1.793	1.892	2.037	2.195	0.0195	0.0174	0.0454	53
	258; 32	2.0	1.586	1.683	1.771	1.912	2.062	0.0441	0.0149	0.1709	57
3-p Weibull	290; 53	1.5	1.382	1.498	1.528	1.557	1.665	0.0021	0.0020	0.0084	34
	272; 49	1.5	1.308	1.443	1.474	1.504	1.606	0.0044	0.0020	0.0494	37
	257; 33	2.0	1.853	2.001	2.050	2.108	2.243	0.0046	0.0045	0.0071	42
	263; 35	2.0	1.905	2.018	2.069	2.126	2.268	0.0111	0.0046	0.0809	37
	290; 33	2.5	2.307	2.418	2.489	2.576	2.706	0.0084	0.0076	0.0274	47
	294; 35	2.5	2.306	2.430	2.504	2.584	2.723	0.0108	0.0080	0.0531	44
3-p Gamma	260; 58	1.5	1.333	1.458	1.490	1.523	1.655	0.0022	0.0021	0.0100	39
	271; 61	1.5	1.338	1.499	1.531	1.563	1.673	0.0032	0.0023	0.0300	37
	285; 40	2.0	1.844	1.955	1.995	2.048	2.161	0.0039	0.0034	0.0224	39
	267; 37	2.0	1.755	1.886	1.923	1.972	2.094	0.0085	0.0032	0.0728	39
	291; 31	2.5	2.322	2.446	2.511	2.608	2.739	0.0086	0.0083	0.0173	39
	297; 32	2.5	2.340	2.471	2.543	2.632	2.776	0.0102	0.0079	0.0480	38

Table 7 proves the implications stated above. It is found out that the proposed method performs slightly worse for the data points extracted from the borders of the contours than the points taken from the middle of ranges. In order to exemplify this situation, consider the pair of data (290; 53) and (272; 49) for 3-p Weibull distribution. The former data point is taken from the middle of the CV contour, and the latter is from the borderline (see Table 6). It can be seen that although the variance of the estimates of these data points are almost the same, MSE and, accordingly, bias increases. These borderline cases are present in all of the three distributions.

Moreover, for higher Ideal C_{pk} values, the proposed method performs relatively worse. The MSE, variance, and bias of the estimates increase. This could be also observed from the spread of the estimates in terms of the maximum, minimum values, and the quartiles (see Table 7). Higher Ideal C_{pk} means higher process mean and lower process standard deviation. Therefore, it can be said that better estimates are obtained with moderate values of process mean and standard deviation. Finally, it can be seen from Table 6 that the method works slightly better for 3-p Weibull and 3-p Gamma distributions than for log-normal distribution.

5.4.2. Studies Regarding the Success Rate

It is interesting to investigate what make the success rate of the proposed method – that is, what the parameters, if they exist, affect the success or the failure of the method. Since the success or failure of the method depends on the CV estimate, it concerns how the normal approximation is done and the characteristics of the empirical distribution. Two areas, which are hypothesized to have an effect on the success of the method, are identified. These areas are the R^2 (coefficient of determination) results from the linear regression line and the skewness and kurtosis of the empirical distribution. In order to analyze these areas, three sets of mean and standard deviation are chosen for each of the three distributions, and simulations are conducted in order to see the relationship, if it exists, between the R^2, skewness, and kurtosis values of successful and failed cases. The data points and their corresponding Ideal C_{pk} values and CV ranges are given in Table 8. Each point is replicated 10000 times.

Table 8. Data points for R^2, skewness, and kurtosis analysis

	Mean	Standard Deviation	Ideal C_{pk}	CV Range
Log-Normal	260	80	1.0	0.175 – 0.200
N = 100	280	52	1.5	0.125 – 0.150
	293	40	2.0	0.100 – 0.125
3-p Weibull	290	53	1.5	0.13 – 0.14
N = 100	257	33	2.0	0.09 – 0.10
	290	33	2.5	0.08 – 0.09
3-p Gamma	260	58	1.5	0.12 – 0.13
N = 100	285	40	2.0	0.10 – 0.11
	291	31	2.5	0.08 – 0.09

5.4.2.1. Simulation Results of Coefficient of Determination (R^2)

Since R^2 is a measure of how well the regression line approximates the data points, an initial argument was that there could be a relationship between successful and failed cases with

respect to R^2 results. Again, a successful case implies the situation when the estimated CV falls within the prescribed CV range, whereas a failed case represents the situation when the estimated CV fails to meet the assigned range. The results are then illustrated via violin plots.

Only the graphical results of the data points for 3-p Weibull distribution is presented here (see Figure 21) due to the fact that results for log-normal and 3-p Gamma are almost the same. Therefore, same interpretations apply to these distributions. The R^2 plots for log-normal and 3-p Gamma distribution can be found in Appendix I.

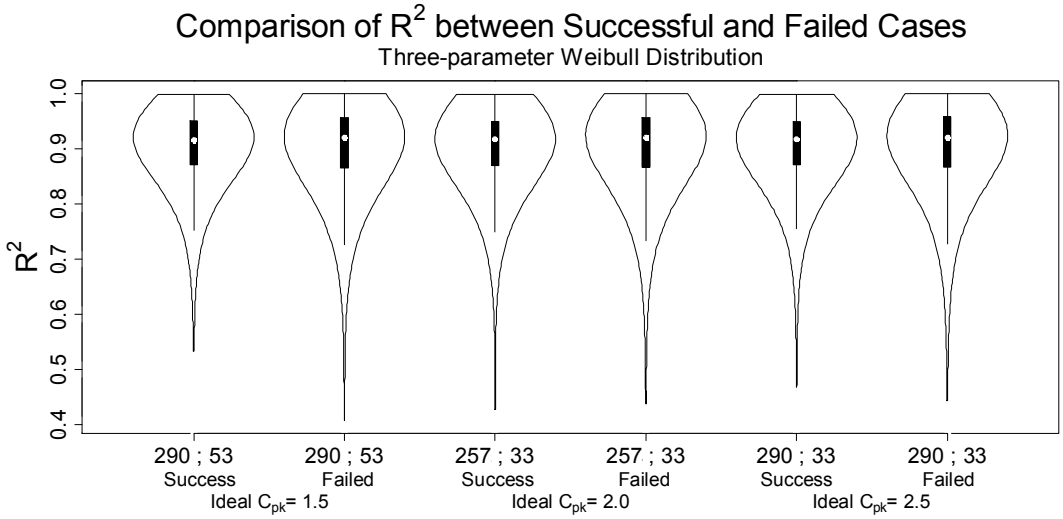

Figure 21. R^2 plot for 3-p Weibull distribution

Note that the thick vertical black lines inside the plots, which indicate the 25th and 75th quartiles, and the medians of the data, are almost the same for both successful and failed cases. This indifference of R^2 results between the successful and failed cases implies that the R^2 value of regression lines does not have an effect on the performance of the proposed method. In other words, there is no clear difference which indicates the success or the failure of estimating C_{pk} in terms of R^2.

5.4.2.2. Simulation Results for Skewness and Kurtosis

The second argument regarding the success of the method is related with the third and the fourth moments of the distribution. There have been studies in the literature analyzing the effect of skewness and kurtosis on process capability estimates (Choi and Bai, 1996; Deleryd, 1999; Chang*, et al.*, 2002). Therefore, as it is done with R^2, simulations are conducted in order to see the relationship, if it exists, between the skewness and kurtosis values of successful and failed cases.

Figures 22 and 23 illustrate the skewness and kurtosis plots for 3-p Weibull distribution. Since the results for log-normal and 3-p Gamma distributions do not reveal any other substantial information, they will not be presented here. Therefore, interpretations for Weibull are also applied to the other distributions. The skewness and kurtosis of skewness and kurtosis for log-normal and 3-p Gamma distributions can be found in Appendices J and K, respectively.

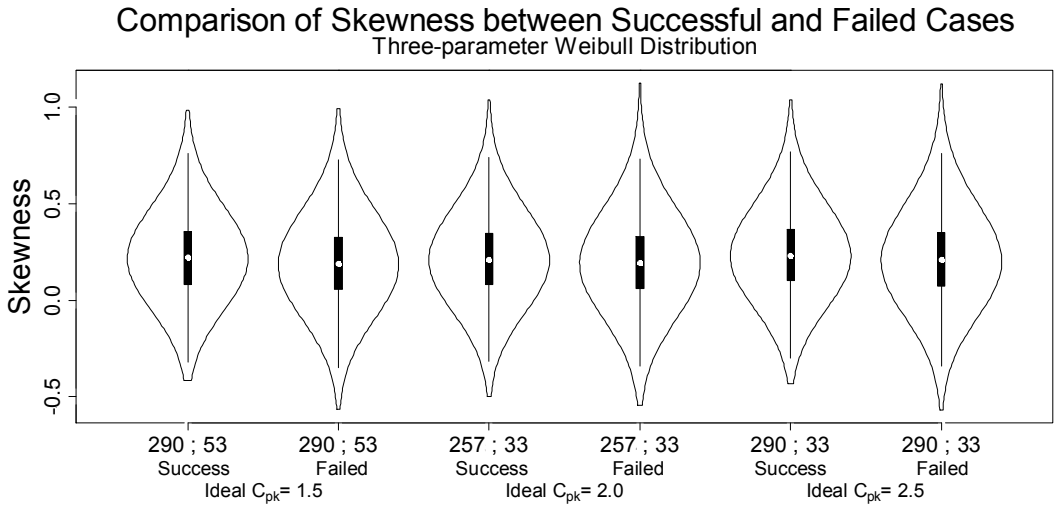

Figure 22. Skewness plots for 3-p Weibull distribution

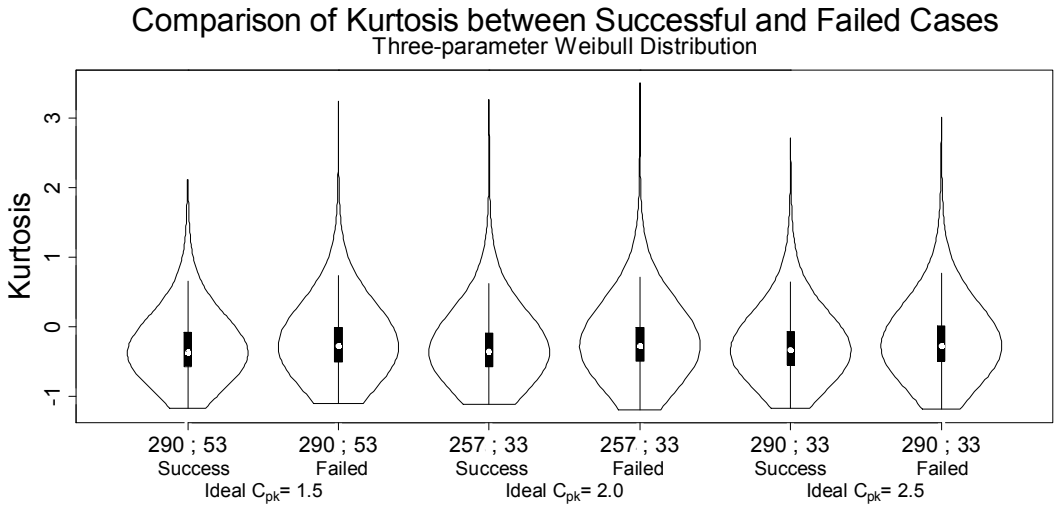

Figure 23. Kurtosis plots for 3-p Weibull distribution

Skewness and kurtosis plots exhibit similar interpretations with that of R^2 results. The first, second and third quartiles are almost the same, as well as the range of results for both successful and failed cases. However, some outliers are observed for kurtosis values but they are present both for successful and failed cases. The skewness and kurtosis of the distribution do not reveal any information about the reasons of failure of the method.

5.5. Validation: An Application on Real-World Data

In order to validate the proposed method, two batches of real-world data, of which the first batch is presented in Section 1.2, are used. Each batch of data consists of 41 data points; therefore the contour plots built for N=50 is used for analysis (see Appendices D, E, and F). The mean and standard deviation of Batch 1 is equal to 262 and 52, whereas for Batch 2, the mean and standard deviation are 260 and 69, respectively. It is worth noting that the the mean and standard deviation of the first batch is found in the middle of the contour plots. However,

the data point for the second batch is found close to the borderlines in the contour plots. The results are provided in Table 9.

Table 9. Results of the real-world data

		IdC_{pk} (Minitab)	IdC_{pk} (R)	Est. C_{pk}	Sample Size	CV Range
Batch 1	**Log-normal**	1.40	1.35	1.407	7	Prescribed
	3-p Weibull	1.37	1.36	1.407	7	Prescribed
	3-p Gamma	1.63	1.58	1.599	10	Prescribed
Batch 2	**Log-normal**	1.19	1.11	1.026	8	Widened
	3-p Weibull	1.30	1.29	1.026	8	Widened
	3-p Gamma	1.44	1.40	1.026	8	Widened

For the first data batch all results are obtained within the prescribed ranges. However, for the second data batch the proposed method did not give any results when the prescribed ranges are used. Therefore, as it is proposed in Section 4.1, the CV ranges are widened.

If these results are to be interpreted for the first batch, it could be said that the practitioner would have to take 7 samples from the tail of the distribution for log-normal and 3-p Weibull distribution and would estimate the process capability with a difference of 0.05 and 0.04, respectively if the Ideal C_{pk} from *R* is considered. For 3-p Gamma distribution, 10 observations from the tail are needed to estimate process capability with a difference of 0.01. However, the proposed method gives varied C_{pk} estimates itself depending on the assumed distribution.

On the other hand, the same C_{pk} estimate is obtained for the second data batch when CV ranges are widened. Although the estimations are worse in terms of comparison with the Ideal C_{pk} values, this example shows that with this method it is also possible to estimate the same process capability regardless of the assumed distribution. Furthermore, in this case another question arises about how accurate is the Ideal C_{pk} computation.

Since this method is designed to be applicable to any non-normal distributions, which distribution to assume for estimation is left at the discretion of the practitioner.

With respect to the research questions, the answers can now be provided. The first question related to the ability of estimating process capability of non-normal distributions with only taking the left tail of the distribution into account. As an answer to the first question, it can be said that the capability of a process can be estimated with only a few points from the tail since it is proven by verification through simulations and validation on real data.

The second research question was about the parameters that affect the performance of the proposed method. It is also found out that sample size (i.e. observations taken from the tail) affects the success or failure of the method in terms of estimating the CV within the prescribed CV range. For the same empirical data set, some sample sizes may be successful and some may not. R^2 results of the linear regression, and skewness and kurtosis of the

empirical distributions are also analyzed. However, these are not found to have an impact on the performance of the method. Furthermore, the process mean and standard deviation seem to affect the performance of the method.

6. Discussion

The proposed method and guideline has both strengths and weaknesses that require further investigation.

Although, the problem identified at the beginning of this study is based on the probability of obtaining varied C_{pk} results for the same process depending on the assumed distribution, the proposed method itself can give varied estimates. However, it has two practical advantages.

The first advantage of the method is that it estimates the process capability of a non-normal distribution by considering only few data points from the left tail and by avoiding probable assignable causes in the right tail. Second, it is an easy method to apply for practitioners. However, there seems to be some critical parameters that affect the accuracy and precision of the estimations. The mean and the standard deviation of the distribution, and consequently the CV, can be considered as critical parameters. For higher mean values and lower standard deviation values result in relatively worse C_{pk} estimates.

Another problematic circumstance is also identified for the proposed method. If the CV of the population is close to contour borderlines in the contour plot, then the C_{pk} estimates are rather poor compared to the C_{pk} estimates extracted from the CV values in the middle of the contours.

The conditions that parameterize the success rate of the proposed method are also investigated. It is shown that the sample size taken from the left tail of the distribution plays a critical role while estimating the process capability. Not every sample size may lead to a successful process capability estimate. If any sample size does not estimate the CV within the prescribed, the practitioner is recommended to take more data points from the left tail until it results in a successful estimate. Furthermore, it is found out that good C_{pk} estimations can sometimes be labeled as failed cases. In these cases, the CV ranges can be widened in order to obtain successful cases.

7. Conclusion

Estimating process capability when non-normal data is encountered can be rather problematic due to the violation of the basic assumption of normality. In this study, a new method was proposed for estimating process capability of non-normal data with only lower-specification limit. This method only takes the left tail of the distribution into account. The reason for this is justified by the fact that the process does not have upper specification limit; therefore the influence of the observations in the right tail is insignificant and could sometimes be misleading.

By simulating various values of process mean and standard deviation, a practitioner's guideline is suggested. This guideline can serve as an easy and practical tool for estimating process capability of non-normal data.

The proposed method is tested on log-normal, 3-parameter Weibull, and 3-parameter Gamma distributions. Moreover, an application on the real-world data taken from the company is provided. The results show that it is possible to estimate process capability with only taking the left tail into consideration with a relatively high success rate.

The proposed method does not always give the best accuracy and precision. For some borderline cases and high mean – low standard deviation situations, the performance of the method decreases. Furthermore, borderlines in the contour plots appear to be problematic. These conditions require further research in order to increase the performance of the method.

Another future research area could be devoted to building up confidence intervals for estimation. Moreover, the applicability of the proposed method to different process characteristics, such as negatively skewed distributions, processes with only upper specification limits, deserve further investigations.

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Appendix A: Simulation Code for Log-normal Distribution

```
lognormal_sim <- function (mean, stdev, LSL, N, replicate, minrange, maxrange) {
## Transformation of log-normal parameters
SdLN<-sqrt(log(stdev^2+mean^2)-2*log(mean))
MeanLN<-log(mean)-0.5*(SdLN^2)
## Compute theoretical Cpk
median <- qlnorm (0.5, MeanLN, SdLN)
lower_percentile <- qlnorm (0.00135, MeanLN, SdLN)
numer <- (median - LSL)
denom <- (median - lower_percentile)
IdealCpk <- numer / denom
## Prepare Y-axis for NPP
Srs<-c(1:N)/(N+1)
Y<-qnorm(Srs)
## Create a matrix for log-normal random variates
LnormDataMat <- matrix (0,nrow=replicate, ncol=N, byrow = TRUE)
## Generate Random Variates
LnormData <- replicate (replicate,rlnorm(N, MeanLN, SdLN))
LnormDataMat <- t(LnormData)
## Create matrices and lists for results
Results <- matrix (0, nrow = 11, ncol = replicate, byrow = TRUE)
Mean_Results <- matrix (0, nrow = 11, ncol =replicate, byrow = TRUE)
StDev_Results <- matrix (0, nrow = 11, ncol =replicate, byrow = TRUE)
est_CV_results <- matrix (0, nrow=11, ncol=replicate, byrow=TRUE)
coef1_Mat <- matrix (0, nrow=11, ncol=replicate, byrow=TRUE)
coef2_Mat <- matrix (0, nrow=11, ncol=replicate, byrow=TRUE)
Cpk_list <- list(0)
get.Cpk <- vector(mode="numeric")
MSE <- vector (mode="numeric")
Var <- vector (mode="numeric")
Bias <- vector (mode="numeric")
Success_Rate <- vector (mode="numeric")
## Sort each replicate of the log-normal variate matrix
LnormDataMat <- apply(LnormDataMat, 1, function(x) x[order(x)])
## Least Squared Estimation on every replication for sample sizes 5 to 15
LnormDataFrame <- data.frame (LnormDataMat)
       for (j in 5:15){
       coef1      <-      as.vector(sapply      (LnormDataFrame[1:j,],      function      (x)
(lsfit(x,Y[1:j])$coefficients[1])))
       coef2      <-      as.vector(sapply      (LnormDataFrame[1:j,],      function      (x)
(lsfit(x,Y[1:j])$coefficients[2])))
       coef1_Mat [j-4,] <- coef1
       coef2_Mat [j-4,] <- coef2 }
## Normal estimation of mean and standard deviation for sample sizes 5 to 15
Mean_Results <- (-coef1_Mat)/coef2_Mat
StDev_Results <- ((1 - coef1_Mat)/coef2_Mat) - Mean_Results
## Compute estimated Coefficient of variance:CV
est_CV_results <- StDev_Results/Mean_Results
## Compute the Cpk estimate
Results <- ((Mean_Results - LSL)/(3*StDev_Results))
## Subset the Cpk estimates which satisfy the prescribed CV range
## Compute the evaluation criteria
       for (i in 1:11){
              get.Cpk      <-      subset(Results[i,],      est_CV_results[i,]>=minrange      &
              est_CV_results[i,]<=maxrange)
              Cpk_list [[i]] <- get.Cpk
              MSE [i] <- round(sum((get.Cpk - IdealCpk)^2) / length(get.Cpk),digits=4)
              Var [i] <- round(sum((get.Cpk - mean(get.Cpk))^2) / length(get.Cpk),digits=4)
              Bias [i] <- round(sqrt(MSE[i] - Var[i]),digits=4)
              Success_Rate[i] <- length (get.Cpk)}
## Build a data table for simulation Results
IdCpk <- round(IdealCpk, digits=4)
Evaluation <- data.frame (MSE, Var, Bias,Success_Rate,IdCpk)
names (Evaluation) <- c("MSE","Variance","Bias","Success Rate","Ideal Cpk")
return (Evaluation)}
```

Appendix B: Simulation Code for 3-p Weibull Distribution

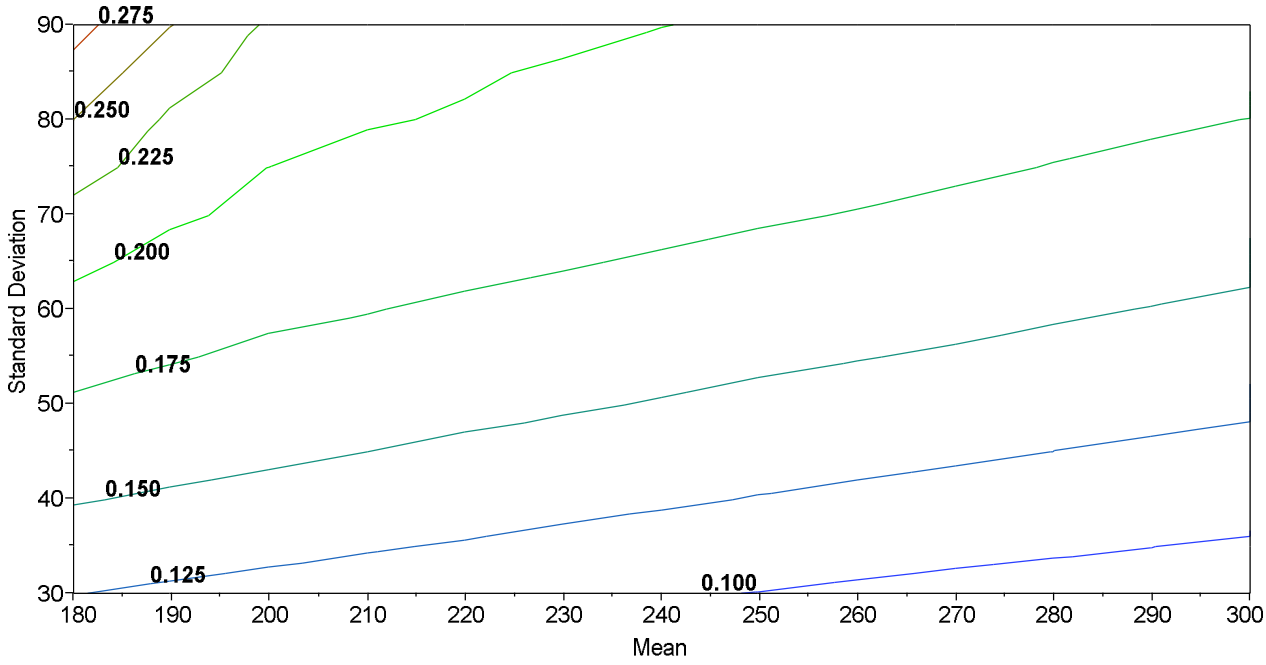
```
weibull_sim <- function (mean, stdev, LSL, N, replicate,minrange,maxrange) {
library(mixdist)
library(FAdist)
location <- 130
            repeat {
            wpar <- weibullpar(mean,stdev,location)
        location <- location + 5
        if(wpar$shape>=1 & wpar$shape<2.9) {break} }
## Computing theoretical Cpl
median <- qweibull3 (0.5, wpar$shape, wpar$scale, wpar$loc)
lower_percentile <- qweibull3 (0.00135, wpar$shape, wpar$scale, wpar$loc)
numer <- (median - LSL)
denom <- (median- lower_percentile)
IdealCpl <- numer / denom
## Preparing Y-axis for NPP
Srs<-c(1:N)/(N+1)
Y<-qnorm(Srs)
## Generate Random Variates
WeibullDataMat <- matrix (0,nrow=replicate, ncol=N, byrow = TRUE)
WeibullData <- replicate (replicate,rweibull3(N, wpar$shape, wpar$scale, wpar$loc))
WeibullData <- t(WeibullData)
## Create matrices for results
Results <- matrix (0, nrow = 11, ncol = replicate, byrow = TRUE)
Mean_Results <- matrix (0, nrow = 11, ncol =replicate, byrow = TRUE)
StDev_Results <- matrix (0, nrow = 11, ncol =replicate, byrow = TRUE)
est_CV_results <- matrix (0, nrow=11, ncol=replicate, byrow=TRUE)
coef1_Mat <- matrix (0, nrow=11, ncol=replicate, byrow=TRUE)
coef2_Mat <- matrix (0, nrow=11, ncol=replicate, byrow=TRUE)
Cpk_list <- list(0)
get.Cpk <- vector(mode="numeric")
Success_Rate <- vector (mode="numeric")
MSE <- vector (mode="numeric")
Var <- vector (mode="numeric")
Bias <- vector (mode="numeric")
## Sort each replicate (row) of the Weibull variate matrix
WeibullDataMat <- apply(WeibullData, 1, function(x) x[order(x)])
## Least Squared Estimation on every replication for sample sizes 5 to 15
WeibullDataFrame <- data.frame (WeibullDataMat)
       for (j in 5:15){
       coef1 <- as.vector(sapply (WeibullDataFrame[1:j,], function (x)
(lsfit(x,Y[1:j])$coefficients[1])))
       coef2 <- as.vector(sapply (WeibullDataFrame[1:j,], function (x)
(lsfit(x,Y[1:j])$coefficients[2])))
       coef1_Mat [j-4,] <- coef1
       coef2_Mat [j-4,] <- coef2 }
## Normal estimation of mean and standard deviation for sample sizes 5 to 15
Mean_Results <- (-coef1_Mat)/coef2_Mat
StDev_Results <- ((1 - coef1_Mat)/coef2_Mat) - Mean_Results
## Compute estimated Coefficient of variance:CV
est_CV_results <- StDev_Results/Mean_Results
## Compute the Cpk estimate
Results <- ((Mean_Results - LSL)/(3*StDev_Results))
## Subset the Cpk estimates which satisfy the prescribed CV range
## Compute the evaluation criteria
       for (i in 1:11){
       get.Cpk <- subset(Results[i,], est_CV_results[i,]>=minrange &
est_CV_results[i,]<=maxrange)
              Cpk_list [[i]] <- get.Cpk
              MSE [i] <- round(sum((get.Cpk - IdealCpk)^2) / length(get.Cpk),digits=4)
              Var [i] <- round(sum((get.Cpk - mean(get.Cpk))^2) / length(get.Cpk),digits=4)
              Bias [i] <- round(sqrt(MSE[i] - Var[i]),digits=4)
              Success_Rate[i] <- length (get.Cpk)}
## Build a data table for simulation Results
IdCpk <- round(IdealCpk, digits=4)
Evaluation <- data.frame (MSE, Var, Bias,Success_Rate,IdCpk)
names (Evaluation) <- c("MSE","Variance","Bias","Success Rate","Ideal Cpk")
return (Evaluation) }
```

Appendix C: Simulation Code for 3-p Gamma Distribution

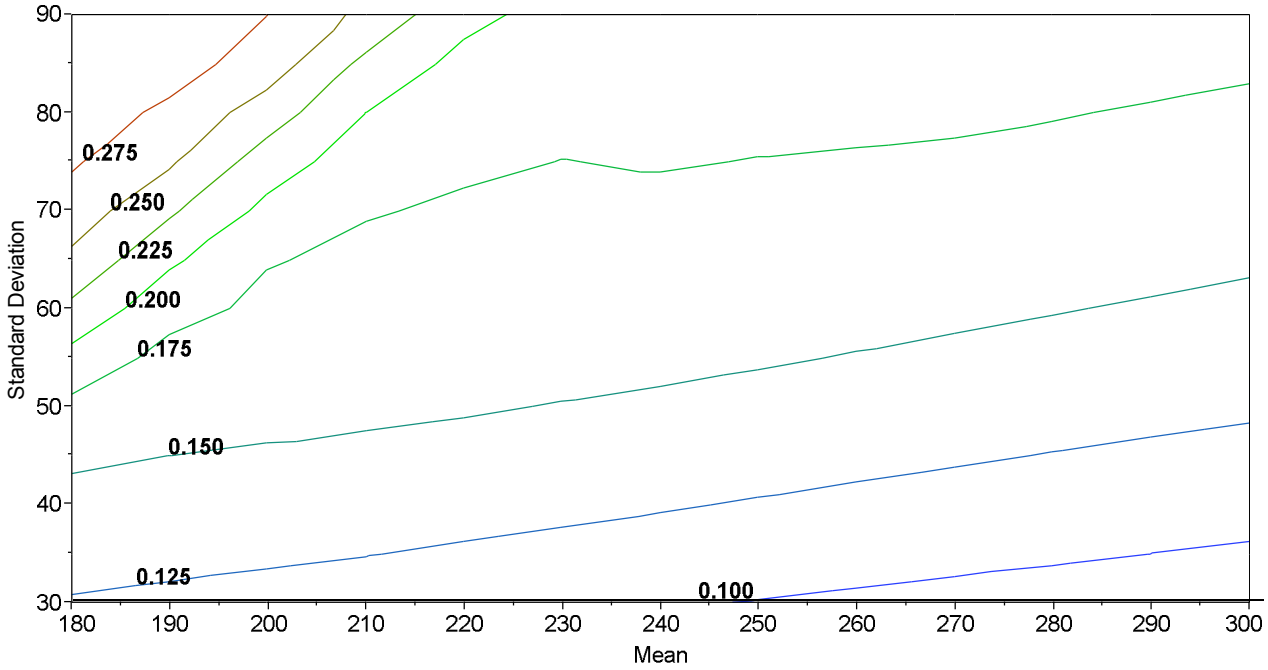
```
gamma_sim <- function (mean, stdev, LSL, N, replicate,minrange,maxrange) {
library(FAdist)
location <- 130
## Transformation of gamma parameters
r_shape <- ((mean-location)^2) / (stdev)^2
lambda_scale <- (stdev)^2/(mean - location)
## Computing theoretical Cpl
median <- qgamma3 (0.5, r_shape, lambda_scale,location)
lower_percentile <- qgamma3 (0.00135, r_shape, lambda_scale,location)
numer <- (median - LSL)
denom <- (median- lower_percentile)
IdealCpl <- numer / denom
## Preparing Y-axis for NPP
Srs<-c(1:N)/(N+1)
Y<-qnorm(Srs)
## Generate Random Variates
GammaDataMat <- matrix (0,nrow=replicate, ncol=N, byrow = TRUE)
GammaData <- replicate (replicate,rgamma3(N, r_shape, lambda_scale,location))
GammaDataMat <- t(GammaData)
## Create matrices for results
Results <- matrix (0, nrow = 11, ncol = replicate, byrow = TRUE)
Mean_Results <- matrix (0, nrow = 11, ncol =replicate, byrow = TRUE)
StDev_Results <- matrix (0, nrow = 11, ncol =replicate, byrow = TRUE)
est_CV_results <- matrix (0, nrow=11, ncol=replicate, byrow=TRUE)
coef1_Mat <- matrix (0, nrow=11, ncol=replicate, byrow=TRUE)
coef2_Mat <- matrix (0, nrow=11, ncol=replicate, byrow=TRUE)
Cpk_list <- list(0)
get.Cpk <- vector(mode="numeric")
Success_Rate <- vector (mode="numeric")
MSE <- vector (mode="numeric")
Var <- vector (mode="numeric")
Bias <- vector (mode="numeric")
## Sort each replicate (row) of the Weibull variate matrix
GammaDataMat <- apply(GammaDataMat, 1, function(x) x[order(x)])
## Least Squared Estimation on every replication for sample sizes 5 to 15
GammaDataFrame <- data.frame (GammaDataMat)
       for (j in 5:15){
       coef1 <- as.vector(sapply (GammaDataFrame[1:j,], function (x)
(lsfit(x,Y[1:j])$coefficients[1])))
       coef2 <- as.vector(sapply (GammaDataFrame[1:j,], function (x)
(lsfit(x,Y[1:j])$coefficients[2])))
       coef1_Mat [j-4,] <- coef1
       coef2_Mat [j-4,] <- coef2  }
## Normal estimation of mean and standard deviation for sample sizes 5 to 15
Mean_Results <- (-coef1_Mat)/coef2_Mat
StDev_Results <- ((1 - coef1_Mat)/coef2_Mat) - Mean_Results
## Compute estimated Coefficient of variance:CV
est_CV_results <- StDev_Results/Mean_Results
## Compute the Cpk estimate
Results <- ((Mean_Results - LSL)/(3*StDev_Results))
## Subset the Cpk estimates which satisfy the prescribed CV range
## Compute the evaluation criteria
       for (i in 1:11){
       get.Cpk <- subset(Results[i,], est_CV_results[i,]>=minrange &
est_CV_results[i,]<=maxrange)
              Cpk_list [[i]] <- get.Cpk
              MSE [i] <- round(sum((get.Cpk - IdealCpk)^2) / length(get.Cpk),digits=4)
              Var [i] <- round(sum((get.Cpk - mean(get.Cpk))^2) / length(get.Cpk),digits=4)
              Bias [i] <- round(sqrt(MSE[i] - Var[i]),digits=4)
              Success_Rate[i] <- length (get.Cpk)}
## Build a data table for simulation Results
IdCpk <- round(IdealCpk, digits=4)
Evaluation <- data.frame (MSE, Var, Bias,Success_Rate,IdCpk)
names (Evaluation) <- c("MSE","Variance","Bias","Success Rate","Ideal Cpk")
return (Evaluation)}
```

Appendix D: CV Contour Plots for Log-normal Distribution

N=50, CV vs. Mean and Standard Deviation

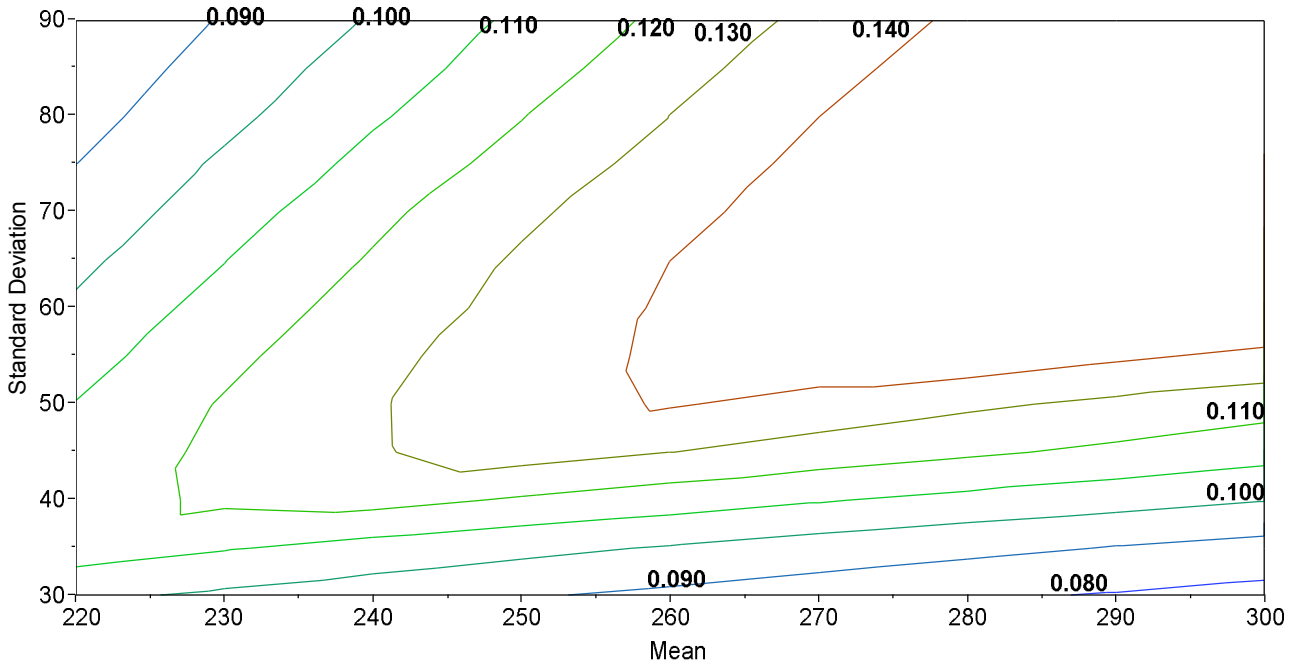


N=200, CV vs. Mean and Standard Deviation

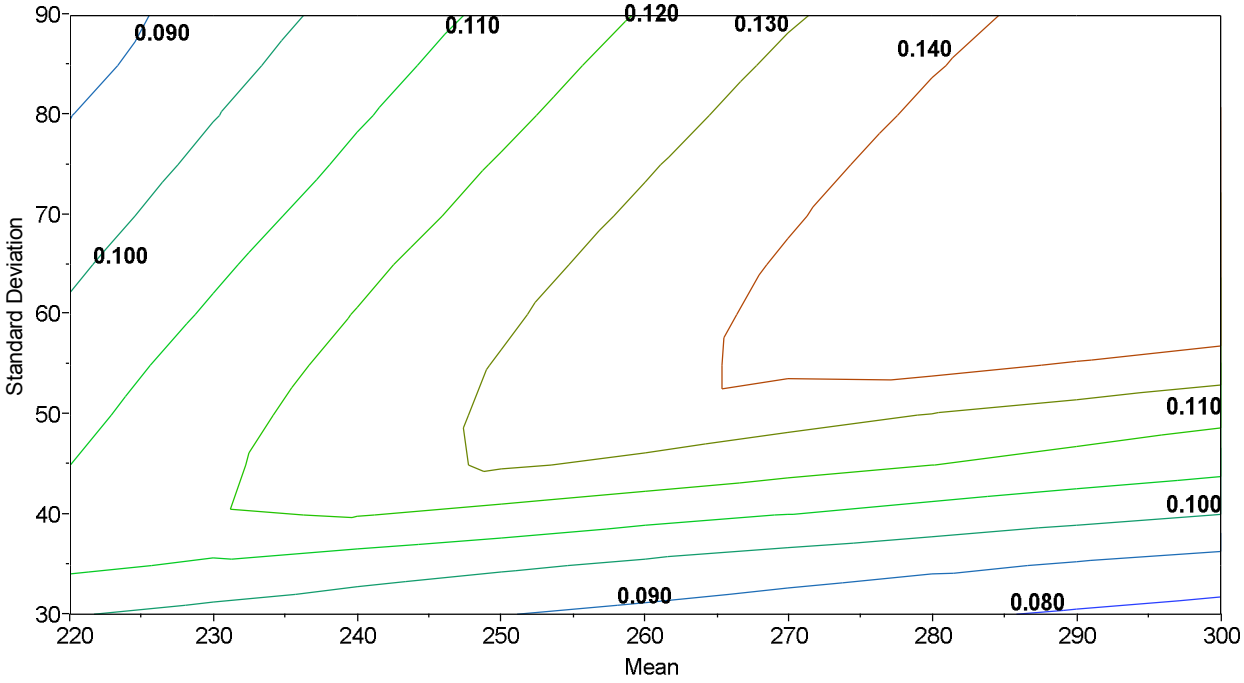


Appendix E: CV Contour Plots for 3-p Weibull Distribution

N = 50, CV vs. Mean and Standard Deviation

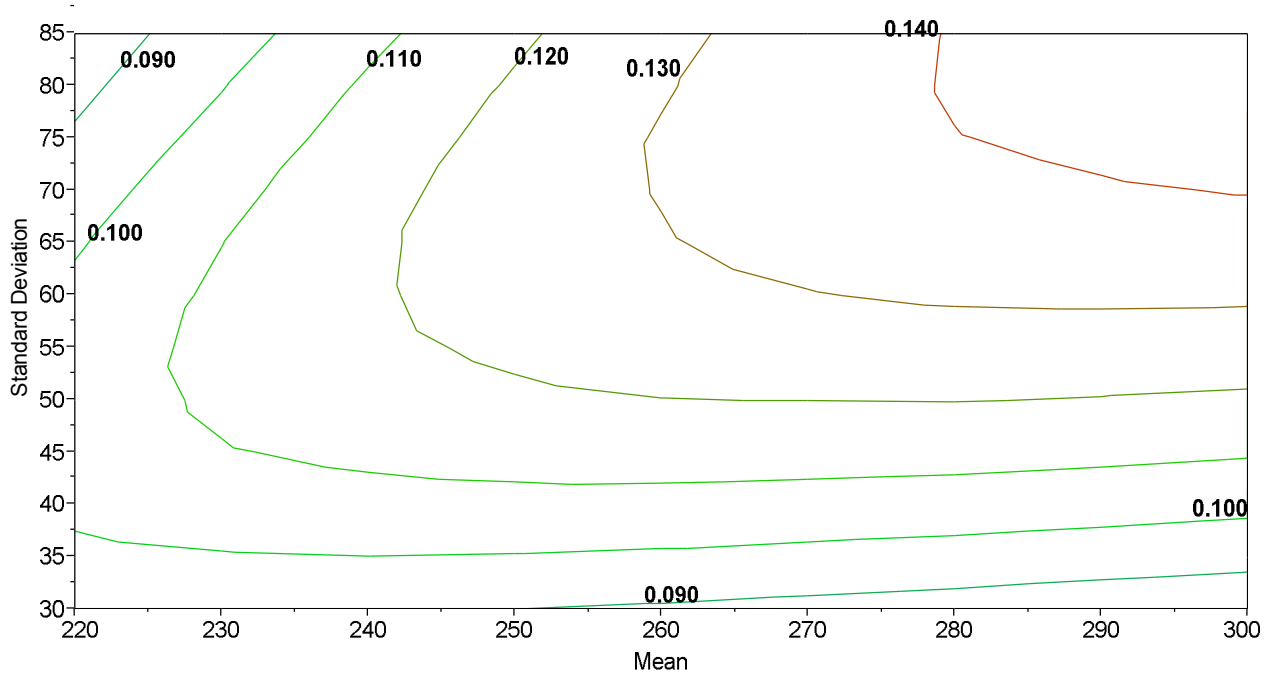


N = 200, CV vs. Mean and Standard Deviation

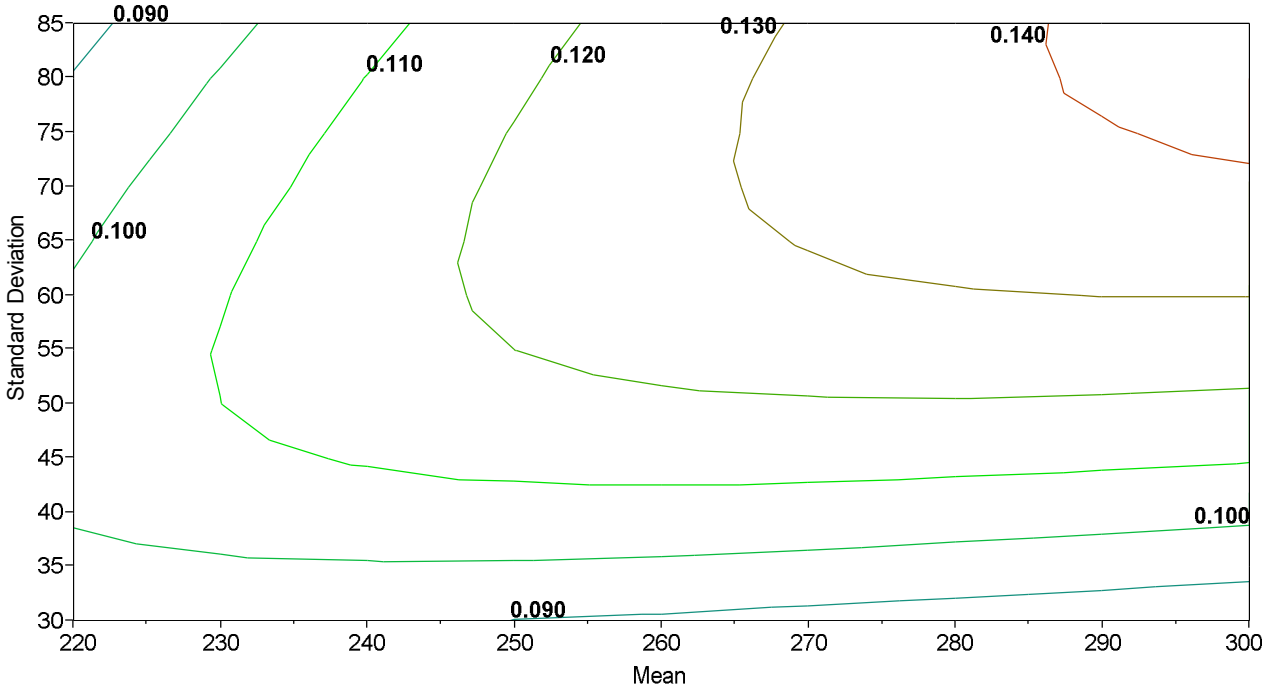


Appendix F: CV Contour Plots for 3-p Gamma Distribution

N = 50, CV vs. Mean and Standard Deviation

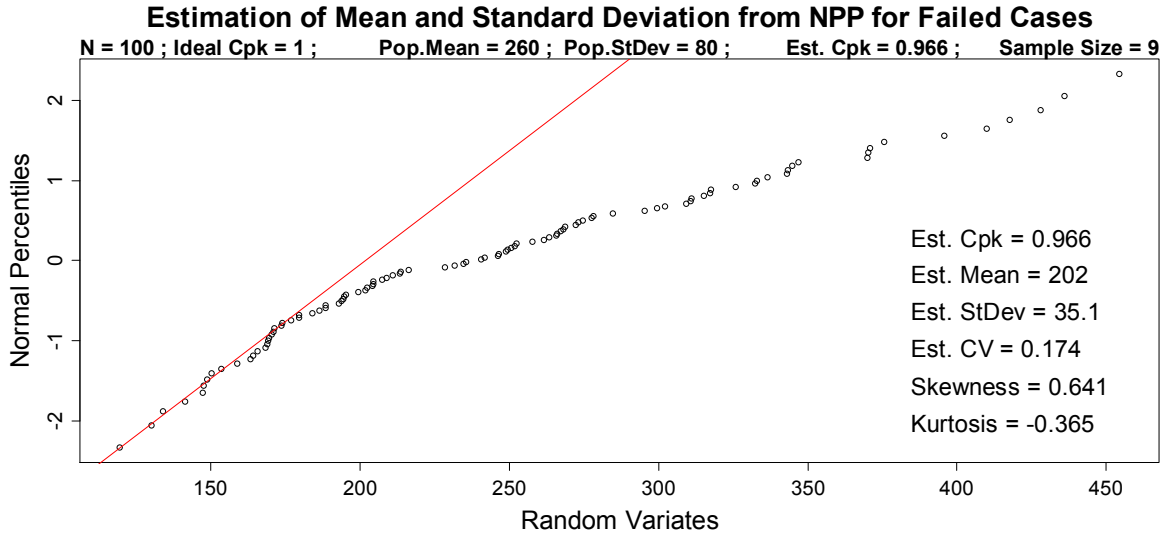


N = 200, CV vs. Mean and Standard Deviation

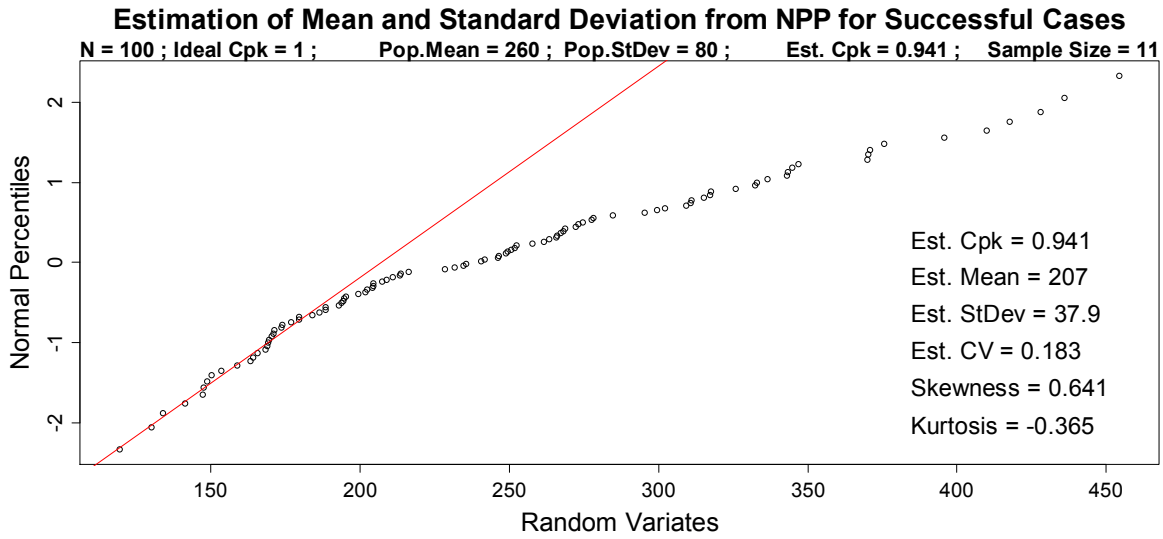


Appendix G: Linear Regression of Data Point (260; 80) for Log-normal Distribution, CV Range 0.175 – 0.200

Failed Case, Estimated C_{pk} = 0.966, Estimated CV = 0.174

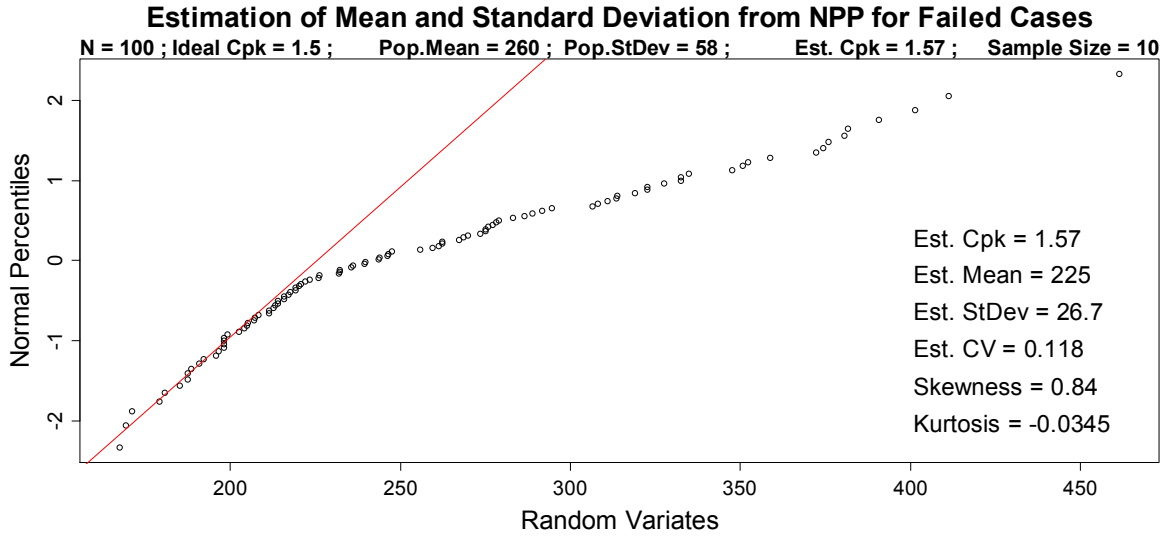


Successful Case, Estimated C_{pk} = 0.941, Estimated CV =0.183

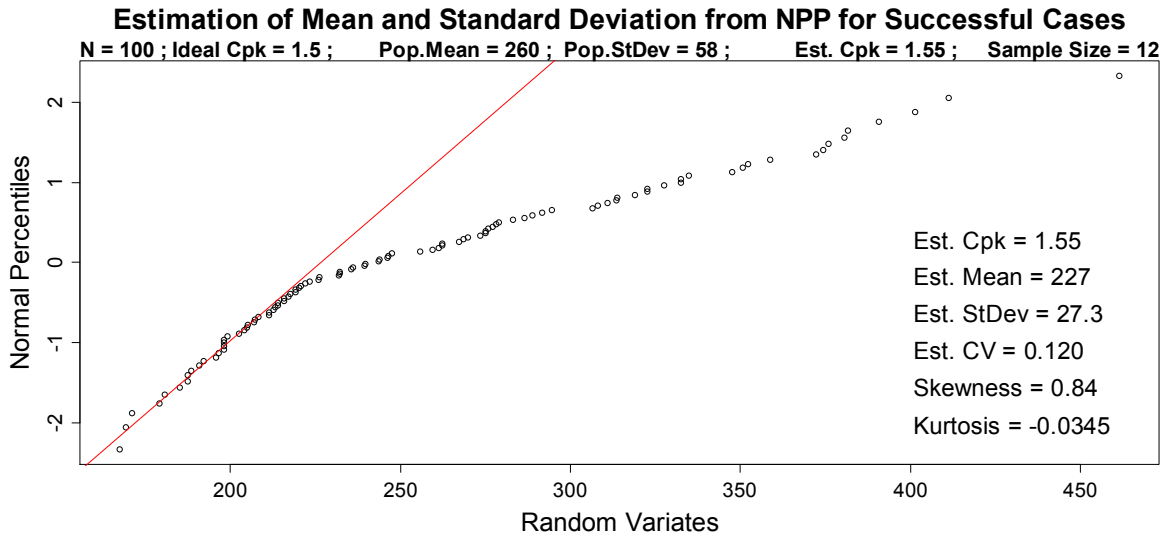


Appendix H: Linear Regression of Data Point (260; 58) for 3-p Gamma Distribution, CV Range 0.12 – 0.13

Failed Case, Estimated C_{pk} = 0.57, Estimated CV = 0.118

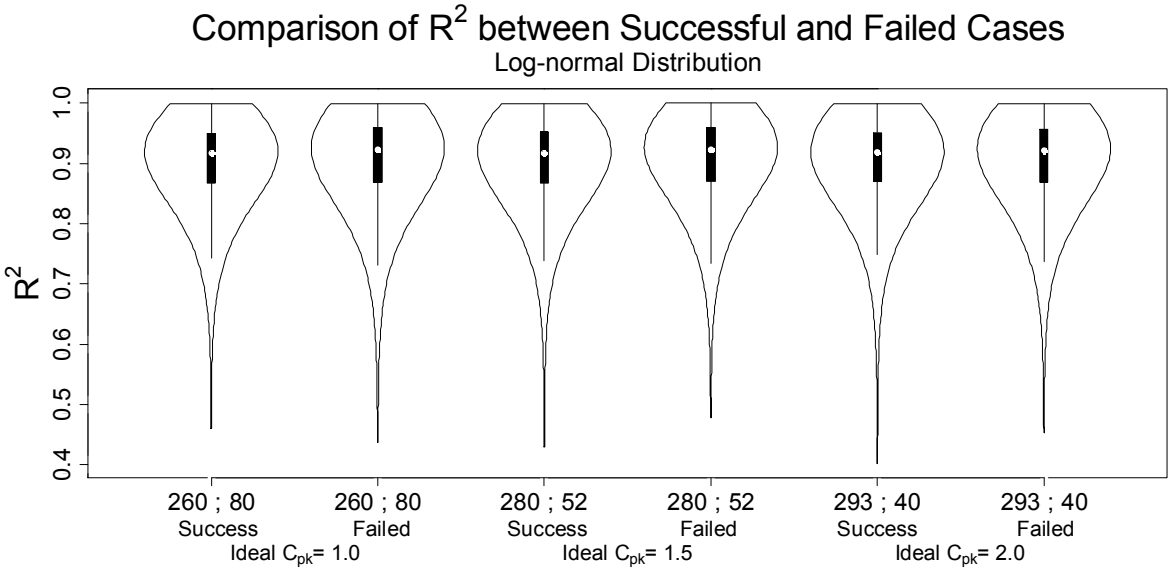


Successful Case, Estimated C_{pk} = 1.55, Estimated CV =0.120

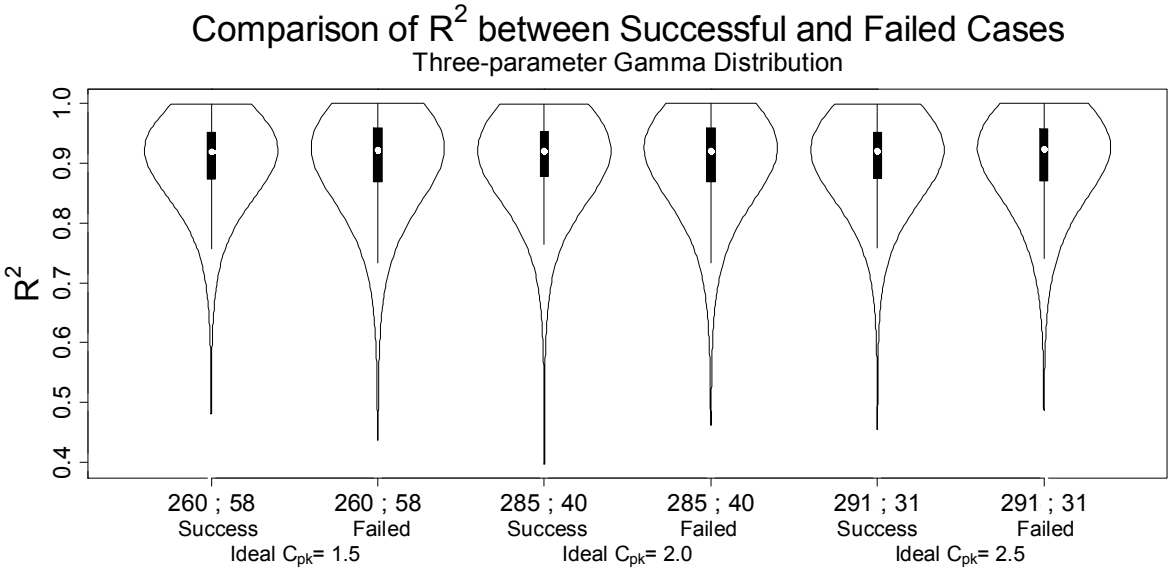


Appendix I: R^2 Simulation Results

Log-normal Distribution

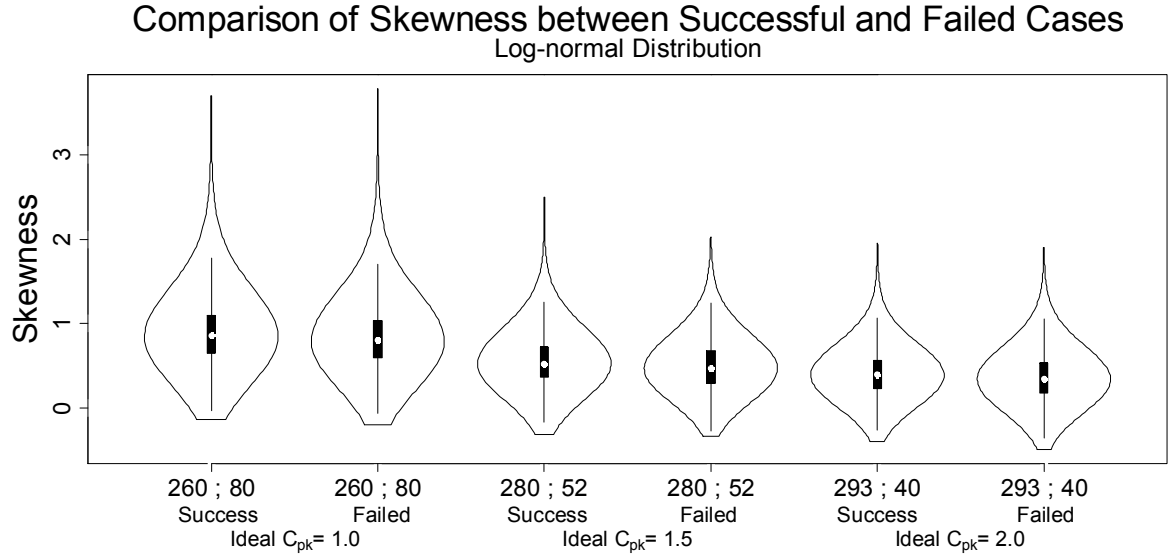


Three-parameter Gamma Distribution

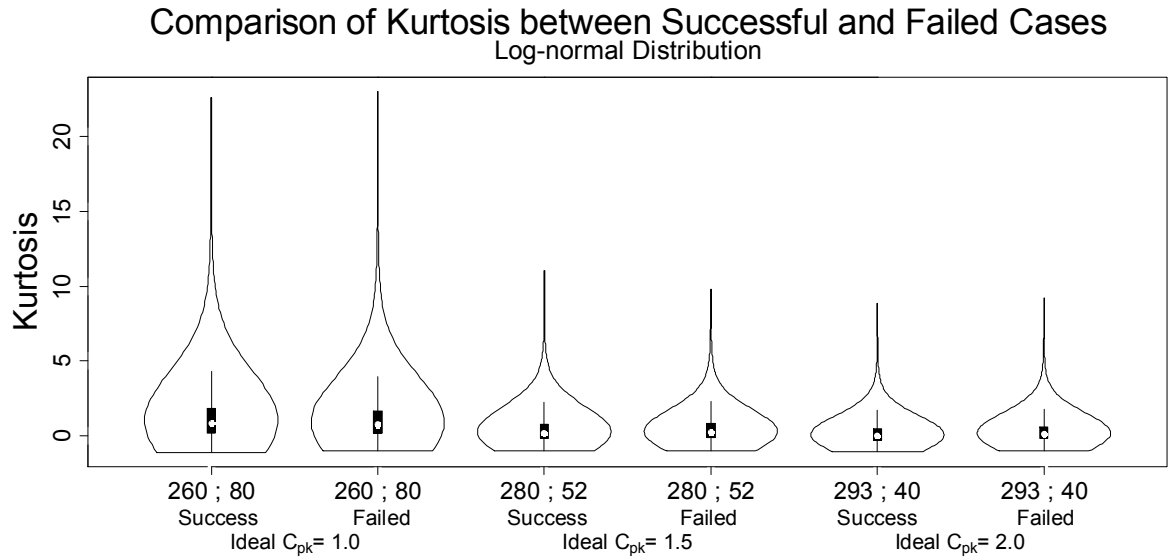


Appendix J: Skewness and Kurtosis Results for Log-normal Distribution

Skewness Simulation Results

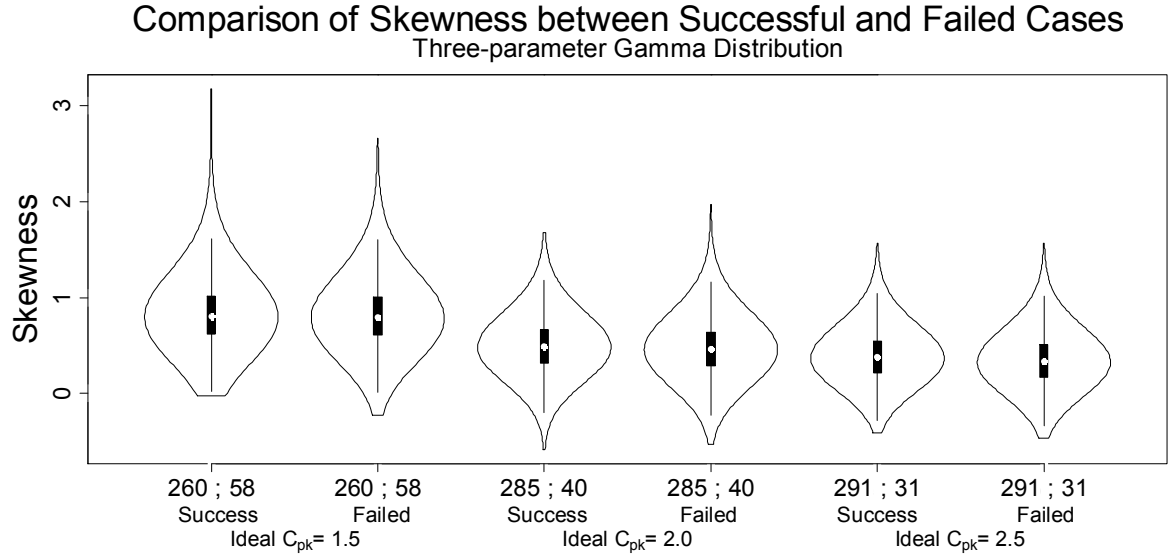


Kurtosis Simulation Results



Appendix K: Skewness and Kurtosis Results for 3-p Gamma Distribution

Skewness Simulation Results



Kurtosis Simulation Results