Nonlinear Propagation of Optical Pulses and Beams

by

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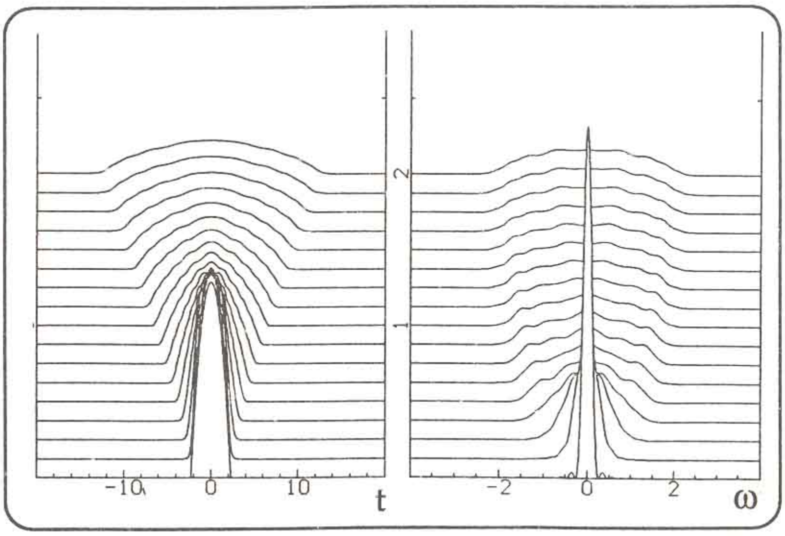

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Cover: *An intense parabolic pulse in a fiber can broaden without change of shape. This is discussed in paper F and chapter 6.2.*

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Abstract

The propagation of optical beams and pulses under the influence of nonlinear effects is characterized by a rich variety of phenomena and many potentially important applications. We analyse two main topics in this context: nonlinear beam propagation, and nonlinear pulse propagation in optical fibers.

The propagation of an optical beam is characterized by diffractive broadening. For Kerr-media, in which the refractive index increases with beam intensity, at a certain intensity, the beam may induce its own waveguide and propagate without broadening. For higher intensities, the nonlinearly induced refractive index causes *self-focusing* and in some cases even collapsing singularities. We demonstrate in this work that an analytical variational approach describes the dynamics of nonlinear beam propagation very well, in particular with respect to the phase modulation dynamics, which previous approaches are found to describe erroneously. The collapse can be removed by allowing the refractive index to saturate. Beam dynamics in saturable nonlinear media is therefore an important issue. Using the variational method, we manage to reproduce the essential features from numerical simulations, and to give a complete picture of optical beam dynamics in saturable nonlinear media. Another important effect in nolinear media is the *modulational instability*, which is well-known to break up broad beams into filaments. However, this instability can occur only in nonlinear focusing media. Considering a pulsed beam with a defocusing-in-time and focusing-in-space nonlinearity, we show that temporal breakup is possible due to the spatial instability, despite the fact that a purely temporal modulation is stable.

Pulse propagation in optical fibers is strongly affected by nonlinear effects for pulse durations around 10 ps and power levels around 0.1 W. In particular, the dispersive broadening of a pulse is enhanced in the normal dispersion regime of the fiber, and reduced in the anomalous dispersion regime. In the normal dispersion regime, the enhanced pulse broadening leads to *wave breaking*, which is steepening with subsequent oscillations in the pulse wings. We show in this work that some pulse shapes may be wave breaking free, i.e. propagate in a self-similar manner without change of shape. For anomalous dispersion, the nonlinearity can eliminate the dispersive broadening and create stable, non-broadening pulses, *solitons*. In this thesis we carry out the first systematic investigation of solitons perturbed by *fourth-order dispersion* (4OD), which is important for short pulses at particular carrier wavelengths. For positive 4OD, we find solitons to be unstable and to decay due to radiation. For negative 4OD, we demonstrate the existence of a new class of stable, soliton-like states. For the kind of short (subpicosecond) pulses which are important in this context, the *Raman effect*, which downshifts the soliton spectrum, cannot be neglected. In an investigation of the simultaneous action of positive 4OD and Raman downshift, we analyse a new pulse compression scheme, which in recent experiments was used to compress optical pulses from 95 to 55 fs.

Descriptors:

Self-focusing, Self-trapping, Optical collapse, Kerr-nonlinearity, Saturable nonlinearity, Paraxial-ray approximation, Modulational instability, Optical fibers, Self-phase modulation, Optical solitons, Optical wave breaking, Nonlinear Schrödinger equation, Group-velocity dispersion, Higher-order dispersion, Stimulated Raman scattering.

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03.40.Kf, 42.50.Ne, 42.50.Rh, 42.60.Jf, 42.65Jx, 42.65.Re, 42.81.Dp, 52.35.Mw, 52.35Sb, 78.20Ci, 84.40.Sr

List of published papers

This thesis comprises nine papers which have been separated into two groups: (i) nonlinear optical beam propagation, and (ii) nonlinear pulse propagation in optical fibers.

Nonlinear optical beam propagation

Paper A
M. Karlsson, D. Anderson, M. Desaix and M. Lisak
Dynamic effects of Kerr nonlinearity and spatial diffraction on self-phase modulation of optical pulses
Optics Letters **16**, 1373 (1991)

Paper B
M. Karlsson, D. Anderson, and M. Desaix
Dynamics of self-focusing and self-phase modulation in a parabolic index optical fiber
Optics Letters **17**, 22 (1992)

Paper C
M. Karlsson and D. Anderson
Super-Gaussian approximation of the fundamental radial mode in nonlinear parabolic-index optical fibers
Journal of the Optical Society of America B **9**, 1558 (1992)

Paper D
M. Karlsson
Optical beams in saturable self-focusing media
Physical Review A **46**, 2726 (1992)

Paper E
D. Anderson, M. Karlsson, M. Lisak, and A. Sergeev
Modulational instability dynamics in a spatial focusing and temporal defocusing medium
Physical Review E **47**, 3617 (1993)

Pulse propagation in optical fibers

Paper F
D. Anderson, M. Desaix, M. Karlsson, M. Lisak and M. L. Quiroga-Teixeiro
Wave-breaking-free pulses in nonlinear-optical fibers
Journal of the Optical Society of America B **10**, 1185 (1993)

Paper G
A. Höök and M. Karlsson
Ultrashort solitons at the minimum-dispersion wavelength: effects of fourth-order dispersion
Optics Letters **18**, 1388 (1993)

Paper H
M. Karlsson and A. Höök
Soliton-like pulses governed by fourth order dispersion in optical fibers
Optics Communications **104**, 303 (1993)

Paper I
A. Höök and M. Karlsson
Soliton Instabilities and Pulse Compression in Minimum Dispersion Fibers
IEEE Journal of Quantum Electronics **30**, 1831 (1994)

Parts of these papers have also been presented at the following international conferences:

"Nonlinear Guided-Wave Phenomena", Sept. 2-4, 1991, Cambridge, England; Paper A was presented.

"ICONO-91", Sept. 24-27, 1991, St. Petersburg, Russia; Excerpts from papers A and B were presented.

"1992 URSI Symposium on Electromagnetic Theory", Aug. 17-20, 1992, Sydney, Australia; A review of Papers A-D was presented.

"Nonlinear Guided-Wave Phenomena", Sept. 20-22, 1993, Cambridge, England; Paper H was presented.

Contents

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Chapter 1

Introduction

1.1 Nonlinear optics and self-action phenomena - physical and historical background

The invention of the laser in the late 1950's opened up the field of nonlinear optics. Before this technological breakthrough, coherent optical radiation sources were rather limited. However, with the laser came a source capable of producing intense monochromatic light. As a result, intensity dependent, i.e. nonlinear, optical phenomena could be investigated systematically. The first nonlinear optical experiment was carried out in 1961 by Franken et al. [1], and demonstrated second harmonic generation. Together with the theory by Bloembergen [2] and others, this marks the birth of nonlinear optical physics.

Similarly to other fields of nonlinear physics, nonlinear optics is characterized by a variety of complex phenomena; e.g. chaos, turbulence, shock-waves, solitons and instabilities. All such effects are generated via the interaction between optical waves and a nonlinear medium. An understanding of nonlinear optical processes involves two questions: i) how does the electromagnetic field affect the medium, and ii) how is the field affected by the medium response. Mathematically, stage i) is modelled through the constitutive relations between the electromagnetic fields, and stage ii) is described by Maxwell's equations. In order to significantly alter the properties of the medium, an intense optical wave, a "pump", is required. A weaker wave, a "probe", is then used to measure the medium response. If the two interacting waves have different frequencies, power can be transferred between them, and this forms the basis for the stimulated scattering- and parametric processes. Harmonic generation, like that Franken observed in the first nonlinear optical experiment, belongs to this class of phenomena, which has widespread applicability in e.g. optical amplifiers. The degenerate case, when the pumping wave affects itself through the nonlinear response of the medium is commonly denoted as *self-action phenomena*. Those phenomena are the main issue of this thesis.

The optical self-action effects are most easily described through a weakly intensity dependent refractive index. It was observed already by Kerr 1875 [3] that a stationary electric field could affect the refractive index of a material. This is known as the *electrooptic-* or *Kerr-effect*. Even prior to the development of the laser, it was pointed out by Buckingham [4] that the Kerr-effect could be generalized to AC-fields, and in particular, an intense beam could induce its own birefringence via this effect. A more rigorous theory was given by Maker and Terhune 1964 [5]. Several phenomena can arise due to an intensity dependence of the refractive index. The earliest examples were induced birefringence, leading to rotation of the polarization ellipse of the wave, and the *self-focusing* phenomenon. The latter is a crucial concept for this thesis, and is briefly explained below. A refractive index that increases with the optical intensity will cause a lower light-velocity in the central, high-intensity parts of a beam than in the beam wings. This will make the beam focus, and since it is the beam itself that causes the focusing effect, the phenomenon is known as *self-focusing*. Similarly, a medium in which the refractive index decreases with intensity is found to boost the diffractive spreading of an intense beam via *self-defocusing*.

The idea that the nonlinear response of a medium could prevent the diffractive spreading of an optical beam was originally suggested in 1962 by Askaryan [6]. Independently of this, Chiao, Garmire and Townes published the "landmark paper" on optical self-focusing 1964 [7]. They demonstrated a useful theory for the static self-focusing of optical beams, the so called *self-trapping* phenomenon. In the self-trapped state, the nonlinear focusing exactly balances the diffractive spreading, thus forming a non-spreading beam, trapped by itself. In waveguide theory, one would say that the beam is a mode of the waveguide it induces. A dynamic picture of the self-focusing process for cylindrical beams was soon suggested by Kelly [8], and it showed the inherent instability of the cylindrically self-trapped state. A small perturbation of the beam power would lead to either diffractive spreading, or a runaway self-focusing process, the so called *optical collapse*.

The early experimental results both agreed and disagreed with these findings. Hercher [9] observed optical collapse with subsequent material damage, prior to the theoretical explanations in 1964. In fact, the first application of the self-focusing theory was to avoid material damage caused by the collapse. Several experimental features were unexplaind, however. For instance, the experiments showed that the self-focusing of a broad beam, $\sim 100\mu m$ in diameter, led to beam break-up and formation of stable filaments of the order of a wavelength in width, which showed no collapsing instability. This stable feature could be explained by allowing the refractive index to saturate at high intensities [10]. Thus the filaments could be explained as self-trapped beams in a saturable nonlinear medium. The beam break-up phenomenon was explained by Bespalov and Talanov [11] 1966. Using linear stability analysis, they showed that the plane-wave solution in a nonlinear medium under certain circumstances is unstable to transverse modulations. In fluid mechanics, the same kind of instability was discovered simultaneously by Benjamin and Feir [12], were it is known as the *Benjamin-Feir instability*. In a nonlinear optical context it is commonly denoted as *modulation instability*, and its relation to the filamentation process was experimentally verified 1973 by Campillo et al. [13]. Another important effect that was found to accompany pulsed self-focusing was *self-phase modulation*, which manifests itself as strong spectral broadening of the self-trapped filaments [14].

It should be stressed that an immense amount of research was devoted to self-focusing during the latter half of the sixties, and the above is only a brief selection of some important result. One should also emphasize that the self-focusing scenario outlined above is not valid for all media. In fact a wide variety of effects (e.g. stimulated scattering processes, dielectric breakdown, etc.) can occur due to the intense fields that self-focusing give rise to. Strong sensitivity to inhomogeneities in the initial beams was also found in several experiments. Here we focus on the effects that arise due to the intensity-dependent refractive index only. It is worth emphasizing that a lot of the self-focusing features can be explained in terms of the simple model of a nonlinear refractive index.

1.2 The technological developments

Parallelling the previously mentioned developments was the breakthrough in the fabrication of low-loss optical fiber waveguides in the beginning of the 1970's [15]. As a result of this, optical fibers became a realistic alternative for signal transmission. An unexpected consequence was that low-loss fibers themselves was found to be a proper medium for the generation and observation of nonlinear phenomena, e.g. stimulated Brillouin- and Raman scattering [16], and self-phase modulation [17].

As transmission lines, optical fibers have many advantages over the conventional coaxial cable; higher bandwidth (by several orders of magnitude) and insensitivity to noisy environments are some examples. Fibers are therefore mostly considered for transmission line purposes, and have had a large impact on telecommunications. However, signal distorsions due to fiber *dispersion* were early recognized to limit the information carrying capacity [18]. The dispersion causes different frequencies to propagate with different velocities, thereby giving rise to pulse broadening. From a mathematical point of view, the dispersive broadening of a pulse is equivalent with the diffractive broadening of a beam. Thus, if the nonlinear refractive index can prevent diffractive brodening, (as in the self-focusing process), it can be used to counteract the dispersive pulse broadening in fibers as well. This simple but ingenious idea was proposed by Hasegawa and Tappert 1973 [19]. However, there are two important differences from the previously studied self-focusing. Firstly, the dispersive broadening in a fiber is a one-dimensional phenomenon, contrary to the cylindrical self-focusing process which occurs in two transverse dimensions. When this difference was investigated in detail, it was found to remove the collapsing instability. Moreover, the stationary, non-broadening pulse solutions was found to be very stable against almost any kind of perturbation. The particle-like behaviour of such pulses has given them the name *solitons*. The second difference between diffractive and dispersive broadening is that the fiber dispersion can have either sign; i.e. be either *normal*, or *anomalous*. It was found that bright solitons can be obtained only in the anomalous dispersion regime, which for fibers correspond to carrier wavelengths above 1.3 microns. Unfortunately, in the beginning of the seventies there were no lasers that could produce high enough intensity at those wavelengths, and the optical solitons seemed to be theoretically interesting, but of no practical value.

Nonlinear optical experiments is a branch of research that greatly relies on technological advances. New materials and lasers can create new and previously unexpected fields of research. On the other hand, theoretical ideas may be impossible to realise because of the lack of suitable equipment. This is the reason why it took seven years until the proper lasers were available to create optical soliton pulses in fibers. In a beautiful experiment 1980, Mollenauer, Stolen and Gordon [20] gave the optical soliton its experimental verification. The importance of this experiment is hard to overrate, since it proved that solitons were not just of academical interest as claimed by several sceptics. And indeed, the research in nonlinear optics and nonlinear waveguides increased tremendously during the eighties, especially with respect to optical solitons [21].

Another important application of the self-phase modulation effect in fibers is in optical pulse-compressors [22, 23]. By using a fiber in the normal dispersion regime, followed by a linear anomalously dispersive delay line, pulses have been compressed down to 6 fs [24]. Such short pulses contain only three optical cycles, and are spectrally "superbroadened" by the nonlinear self-phase modulation. There are promising applications is in e.g. spectroscopy and ultrafast measurements for these pulses.

The research on fiber-optical communication systems today is aimed towards extremely high bit-rates, of the order of 10-100 Gbits/s. At those time scales (picoseconds and below), electronic switching and modulation concepts are not fast enough, and all-optical switching methods, based on nonlinear self-action effects are promising alternatives [25]. Although the loss in today's fibers is very low ($\sim$ 0.3 dB/km), the optical signal still needs to be regenerated every 10-100 km. Optical amplifiers are therefore an essential device in long-distance fiber-optical communication systems. A particularly important recent development was the *Erbium-doped fiber amplifier* (EDFA) [26]. The EDFA:s have several advantages over conventional fiber amplifiers; e.g. polarization insensitivity, high gain, and low noise. They are therefore the likely choice in future optical communication systems.

1.3 Mathematical developments

The crucial difference between linear and nonlinear phenomena is that the *superposition principle* does not apply to nonlinear systems. Thus, for a nonlinear system, a sum of two solutions is not a solution itself. From a mathematical point of view this implies that the traditional means of solving partial differential equations, i.e. by expansion in Fourier sums, are not applicable to nonlinear equations. In fact there are no known, general analytical methods for finding solutions to nonlinear partial differential equations. Consequently, prior to the 1950's there was a rather limited knowledge of the rich dynamics embedded in seemingly simple nonlinear equations. However, the development of the computer changed this drastically. From the beginning of the sixties and onwards, computer simulations have become an important tool in physics.

Not only numerical simulations, but also the analytical theory of nonlinear equations made important progress in the sixties. An analytical breakthrough came in 1967 when Gardner, Greene, Kruskal and Miura [27] demonstrated an exact analytical method of solving the initial value problem for the nonlinear Korteweg-de Vries (KdV) equation. Prior to this, Zabusky and Kruskal [28] had discovered stationary, pulse-like solutions to the KdV equation. These pulse-like solutions had remarkable stability properties; two initially well-separated pulses could collide, interact nonlinearly, and emerge intact after the interaction. Thus, in this nonlinear system it seemed possible to superpose two separate solutions, despite the fact that the superposition principle is not valid! Zabusky and

Kruskal suggested the name *solitons* to these special solutions, due to their particle-like means of interaction. In the exact solution technique for the KdV equation that was later devised [27], the soliton solutions play a crucial role, somewhat analogous to the eigenfunctions of linear systems. The method is based on the fact that the eigenvalues of a certain scattering problem remain constant if the scattering potential function satifies the KdV equation. Furthermore, the solution to the KdV equation requires the reconstruction of the scattering potential from the scattering data. This important physical problem, *the inverse scattering problem*, had been solved a decade earlier by Gelfand, Levitan, Marchenko and others [29]. The ingenious way of solving nonlinear equations via linear scattering problems is somewhat analogous to the conventional Fourier transform, and the technique is commonly known as *the inverse scattering transform* (IST).

Further progress was made 1972 in an important paper by Zhakarov and Shabat [30], which demonstrated how the inverse scattering transform could be applied to another nonlinear partial differential equation, the *nonlinear Schrödinger (NLS) equation*. This equation is of particular importance in nonlinear optics, because it is the NLS equation and modifications of it that govern the self-action effects, including the formation of optical solitons in fibers. An important finding in the work by Zakharov and Shabat was that the light in a lossless optical fiber always can be decomposed into a stationary soliton part, and a radiation part that disperses away at long distances.

Despite its theoretical beauty, the IST suffers from two practical drawbacks. Firstly, it can be applied only to a limited number of equations. Modifications of these equations with e.g. additional terms of physical importance cannot be treated exactly. Secondly, the exact solutions that are obtainable are often very complicated and not very explicit. In fact, the only localized, exact solutions that are available are the soliton solutions. There is thus need for other approximate, analytical methods for solving the nonlinear partial differential equations that describe the self-action effects. During the sixties and seventies, several such schemes were proposed. The first approximate method, the "paraxial-ray approximation" were suggested by Wagner et al. 1968 [31], and are based on a Taylor expansion of the transverse profile. A moment theory for these equations were demonstrated by Zakharov 1972 [32] and by Lam et al. 1975 [33]. A variational approach was suggsted for the static self-trapping equation [34], and for the dynamic self-focusing problem by Tzoar et al. [35], and Anderson et al. [36].

1.4 The present thesis

Above, we have discussed nonlinear optical effects from a fundamental physical point of view, but we have also stressed that there are many potentially important applications for nonlinear optics; especially in high-bandwidth communication systems. There is obviously a great need for convenient mathematical models to aid experimentalists and engineers. In the present work, we put special emphasis on analytical results and methods, because of their general applicability. Numerical computations are also used, both to check analytical results, and to demonstrate phenomena beyond the reach of analytical investigations. The thesis is separated into two different parts; one dealing with the nonlinear propagation of beams, and one dealing with ultrashort pulse propagation in optical fibers.

In the case of nonlinear beams, we reexamine the self-focusing dynamics in bulk media (paper A) and graded-index waveguides (papers B-C). Self-focusing governed by a saturable nonlinearity is considered in paper D. Our analytical tool in these works is the variational method. In particular, with respect to the self-phase modulation dynamics, we find it to be more accurate than other analytical approaches. The results in these papers are explicit and useful for potential applications of nonlinear beams in e.g. switching and modulation schemes. In paper E, we predict and examine the dynamics of the modulational instability of a pulsed beam at normal dispersion. Despite the stability normally associated with normal dispersion, we show that the spatial instability may boost a spatio-temporal beam-pulse breakup.

The analyses of nonlinear pulse propagation cover two effects. One is the phenomenon of *wave breaking* of pulses in normally-dispersive fibers. This effect arises when the dispersive pulse broadening is nonlinearly enhanced. Due to this, the pulse will change shape during propagation, and acquire steep, ringing edges. The effect is commonly denoted optical wave breaking since it resembles the breaking of water waves. We demonstrate in paper F that *wave-breaking-free* pulses can exist, i.e. pulses that do not change shape during propagation. Due to their self-similar properties, such pulses may be of great importance in nonlinear pulse compression systems.

The effect of fourth-order dispersion (4OD) on optical soliton propagation has not been investigated previously. In papers G and H we discuss under which circumstances 4OD can be important, and how it affects optical solitons. By considering the simultaneous action of 4OD with the Raman downshift in an optical fiber, we also predict a novel pulse compression method (paper I).

This thesis is organized as follows: Chapter one provides an introduction to the field and present some historically important developments with respect to this work. Chapters two and three present derivations of the basic propagation equations for nonlinear beam- and pulse propagation, respectively. Nonlinear beam propagation is then discussed from a phenomenological point of view in chapter four, and papers A-E are put in their proper context. Chapter five is devoted to the important question of stability of nonlinear optical waves, and presents paper E. Chapter six discusses nonlinear pulse propagation in fibers, and provides the framework for papers F-I. Finally, the papers included in the thesis follow.

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Chapter 2

Nonlinear beam propagation - the physical framework

2.1 Maxwell's equations

The evolution in time and space of the electric field $\mathbf{E}(\mathbf{r},t)$ (Vm^{-1}) of an optical beam is governed by Maxwells equations, which in the absence of free currents and charges read

$$\begin{aligned}\nabla \times \mathbf{E}(\mathbf{r},t) &= -\frac{\partial \mathbf{B}(\mathbf{r},t)}{\partial t} \qquad \nabla \cdot \mathbf{D}(\mathbf{r},t) = 0\\ \nabla \times \mathbf{H}(\mathbf{r},t) &= \frac{\partial \mathbf{D}(\mathbf{r},t)}{\partial t} \qquad \nabla \cdot \mathbf{B}(\mathbf{r},t) = 0.\end{aligned} \tag{2.1}$$

Where **B**, **H** denote the usual magntic fields, and **D** the displacement field. Eliminating the **H** field using the constitutive relations $\mathbf{D} = \epsilon_0\mathbf{E} + \mathbf{P} = \epsilon\mathbf{E}$ and $\mathbf{B} = \mu_0\mathbf{H} + \mathbf{M}$, and assuming no magnetisation (i.e. $\mathbf{M}$=0), yields the wave equation

$$\nabla \times \nabla \times \mathbf{E}(\mathbf{r},t) + \frac{1}{c^2}\frac{\partial^2 \mathbf{E}(\mathbf{r},t)}{\partial t^2} + \mu_0\frac{\partial^2 \mathbf{P}(\mathbf{r},t)}{\partial t^2} = 0, \tag{2.2}$$

where $\mathbf{P}(\mathbf{r},t)$ (Cm^{-2}) is the induced polarisation. The first term in Eq. (2.2) can be approximated as

$$\nabla \times \nabla \times \mathbf{E}(\mathbf{r},t) = \nabla(\nabla \cdot \mathbf{E}) - \nabla^2\mathbf{E} \approx -\nabla^2\mathbf{E}, \tag{2.3}$$

i.e. it can be set equal to $-\nabla^2\mathbf{E}$ provided $|\nabla(\nabla \cdot \mathbf{E})| = |\nabla(\mathbf{E} \cdot \nabla\epsilon/\epsilon)| \ll |\nabla^2\mathbf{E}|$. The physical meaning of this requirement is that the beam is weakly guided [1, 2, 3], and this approximation is also known as the *weakly guiding approximation*. It has been made in the vast majority of the nonlinear optics literature, and we will adopt it here as well.

In order to find an equation governing the envelope of the electric field **E**, we need an additional relation between **E** and the induced polarisation **P**. The polarisation field can be seen as a macroscopic sum of the response of individual molecules and atoms to an applied electric field. In general, this response is a complicated nonlinear tensor relation, involving dependencies on both frequency and spatial coordinates. A quantum mechanical approach [4, 5, 6] is needed if all features are to be examined in detail. We avoid this by making two simplifying assumptions. Firstly, we divide the polarisation into a linear and a nonlinear (in $\mathbf{E}$) part, where only the linear part may have an explicit dependence on spatial coordinates. Thus we can write (using tensor notation)

$$\tilde{P}_i(\mathbf{r},\omega,\mathbf{E}) \equiv \tilde{P}_i^L(\mathbf{r},\omega,\mathbf{E}) + \tilde{P}_i^{NL}(\omega,\mathbf{E}). \tag{2.4}$$

The tilde notation indicates the Fourier transform, i.e.

$$\tilde{\mathbf{E}}(\mathbf{r},\omega) = \int_{-\infty}^{+\infty} \mathbf{E}(\mathbf{r},t)\exp[-i\omega t]dt. \tag{2.5}$$

The function relating the induced polarisation to the electric field is the *susceptibility*, χ. It is customary to expand the susceptibility in its different nonlinear terms, where the n:th suscebtibility $\tilde{\chi}^n_{1\ldots(n+1)}(\omega_1,...,\omega_n)$ is a tensor of rank n+1 and (in the frequency domain) a function of n frequency variables [6].

Our second simplification is that we restrict the analysis to *microscopically isotropic* media, i.e. the medium is assumed to have the same microscopic properties in all directions. This is true for e.g. gases, plasmas, liquids, and most amorphous solids. The latter includes optical fibers and a wide class of glasses. The assumption of isotropy will greatly simplify the susceptibilities, by causing several tensor elements to vanish. We will specify this in more detail in the subsequent discussion around the different susceptibilities.

2.2 The linear susceptibility

The linear part of the polarisation is most generally written in Fourier space as

$$\tilde{P}_i^L(\mathbf{r},\omega,\mathbf{E}) = \epsilon_0\tilde{\chi}^{(1)}_{ij}(\mathbf{r},\omega)\tilde{\mathbf{E}}_j \tag{2.6}$$

where $\tilde{\chi}^{(1)}_{ij}(\mathbf{r},\omega)$ is the first-order, or linear, susceptibility tensor. Assuming isotropic media implies that the off-diagonal elements of this tensor vanish, and that the diagonal elements are the same so that the induced polarisation becomes parallell to the applied electric field. This means that we can write the linear susceptibility as

$$\tilde{\chi}^{(1)}_{ij}(\mathbf{r},\omega) = \delta_{ij}\tilde{\chi}^{(1)}(\mathbf{r},\omega) \tag{2.7}$$

where δ_{ij} denotes the unity matrix and $\tilde{\chi}^{(1)}(\mathbf{r},\omega)$ is a scalar function. Note that the assumption of microscopic isotropicity does not contradict with the fact that we allow $\tilde{\chi}^{(1)}(\mathbf{r},\omega)$ to be spatially dependent. The reason is that the spatial dependence in χ is on a *macroscopic* scale, independently of the *microscopic* isotropy. By Fourier transforming Eq. (2.2), and applying the weakly guiding approximation we obtain

$$\begin{aligned}\nabla^2\tilde{\mathbf{E}}(\mathbf{r},\omega) + \frac{\omega^2}{c^2}(1+\tilde{\chi}^{(1)}(\mathbf{r},\omega))\tilde{\mathbf{E}}(\mathbf{r},\omega) \equiv \\ \nabla^2\tilde{\mathbf{E}}(\mathbf{r},\omega) + k^2(\mathbf{r},\omega)\tilde{\mathbf{E}}(\mathbf{r},\omega) = -\mu_0\omega^2\tilde{\mathbf{P}}^{NL}.\end{aligned} \tag{2.8}$$

The frequency and spatially dependent wavenumber $k(\mathbf{r},\omega)$ has been defined in the last equality which, in the absence of the nonlinear polarisation, is the Helmholtz vector wave equation. The wavenumber k is related to the refractive index n via $k(\mathbf{r},\omega) = \omega n(\mathbf{r},\omega)/c$. We now write $k(\mathbf{r},\omega)$ as a sum of one spatial part and one frequancy dependent part, i.e. $k(\mathbf{r},\omega) = k(\omega) + k(\omega_0)f(\mathbf{r})$. We emphasize that $k(\omega)$ in the time domain corresponds to an operator consisting of time derivatives. By Taylor expanding around the carrier frequency ω_0 we find

$$k(\mathbf{r},\omega) = k_0(1+f(\mathbf{r})) + \sum_{m=1}^{\infty} \frac{(\omega-\omega_0)^m}{m!} k^{(m)}_{\omega=\omega_0} \equiv \\ k_0(1+f(\mathbf{r})) + \Delta k(\omega-\omega_0) \tag{2.9}$$

or in the operator form

$$k(\mathbf{r},\omega_0 - i\frac{\partial}{\partial t}) = k_0(1+f(\mathbf{r})) + \sum_{m=1}^{\infty} \frac{(-i\frac{\partial}{\partial t})^m}{m!} k^{(m)}_{\omega=\omega_0} \tag{2.10}$$

where we have introduced the notation k_0 for $k(\omega_0)$, and the function $f(\mathbf{r}) \ll 1$ to describe the refractive index profile of a weakly guiding optical waveguide. We are now in position to peel off the rapidly oscillating factor of the electromagnetic wave by substituting

$$\mathbf{E}(\mathbf{r},t) = \frac{1}{2}\mathbf{F}(\mathbf{r},t)\exp[i(\omega_0 t - k_0 z)] + c.c. \tag{2.11}$$

so that the LHS of Eq. (2.8) becomes

$$\left(\frac{\partial^2}{\partial z^2} + \Delta k^2(\omega-\omega_0) - 2ik_0\frac{\partial}{\partial z} + 2k_0\Delta k(\omega-\omega_0) + 2k_0^2 f(\mathbf{r}) + \nabla_\perp^2\right)\frac{1}{2}\tilde{\mathbf{F}}(\mathbf{r},\omega-\omega_0) + c.c. \tag{2.12}$$

for the envelope $\mathbf{F}$ of the electric field. The envelope $\mathbf{F}$ will be regarded as slowly varying with z and t in comparison with the wavelength and wavenumber of the electromagnetic wave. As a result of this, several of the higher derivatives in Eq. (2.12) may be omitted. However, if the spatial and temporal derivatives are not treated with some degree of symmetry, unphysical results may arise [2]. A discussion around a proper way to omit the higher order derivatives will be carried out below.

2.3 The slowly varying envelope approximation

The slowly-varying envelope approximation is a crucial step in obtaining the governing equation for nonlinear optical beams and pulses. In principle, it simplifies the equation for dispersive wave-packet propagation from being second-order in z to first-order in z. As every approximation, however, it removes physical information from the system, and it is therefore important to know exactly what information is lost. Moreover, we will be interested in dispersive pulse propagation under the influence of higher (than second) order dispersion, and it is crucial to know which higher order derivatives shall be removed, and which shall be retained. The following describes how to apply the slowly varying envelope approximation to arbitrary dispersive order in a physically sound way, without introducing unphysical effects into the system. We follow essentially the approach used in e.g. Refs. [2, 7, 8].

Assume, for simplicity, that a component of the electric field of an electromagnetic wave is governed by the equation

$$\left(\frac{\partial^2}{\partial z^2} + \beta^2(\omega)\right) \tilde{E}(\omega, z) = 0 \tag{2.13}$$

where $\beta(\omega)$ is an arbitrary dispersion relation, corresponding to an operator in the time domain. To study the evolution of a wave-packet at the carrier frequency ω_0 we substitute $E(t,z) = \frac{1}{2}F(t,z)\exp[i(\omega_0 t - \beta_0 z)] + c.c.$ where F is the envelope function of the wave-packet, into Eq. (2.13). We obtain

$$\left(\frac{\partial^2}{\partial z^2} + \Delta\beta^2(\omega-\omega_0) - i2\beta_0\frac{\partial}{\partial z} + 2\beta_0\Delta\beta(\omega-\omega_0)\right) \tilde{F}(\omega-\omega_0, z) = 0 \tag{2.14}$$

where $\Delta\beta(\omega - \omega_0) = \beta(\omega) - \beta_0$. Now we make the restriction that F must describe a wave that propagates in one direction only. Since the two first terms of Eq. (2.14) is an operator sustaining propagation in both directions, those terms are omitted. This is a physically consistent way of omitting higher derivatives of F. The resulting equation for F reads, in the time domain

$$\begin{aligned}\left(-i\frac{\partial}{\partial z} + \beta(\omega_0 - i\frac{\partial}{\partial t}) - \beta_0\right) F(t,z) = \\ \left(-i\frac{\partial}{\partial z} + \sum_{m=1}^{\infty}\frac{(-i\frac{\partial}{\partial t})^m}{m!}\beta^{(m)}_{\omega=\omega_0}\right) F(t,z) = 0.\end{aligned} \tag{2.15}$$

Note also that the coefficient in front of the m:th time derivative in Eq. (2.15) is simply $\beta^{(m)}_{\omega=\omega_0}/m!$. In some texts one can find cross-terms like $\beta'^2_{\omega=\omega_0}$ in front of the second-order dispersion, or $\beta''_{\omega=\omega_0}\beta'_{\omega=\omega_0}$ in front of third-order dispersion. However, the the above analysis shows that these terms should not be present if the slowly varying envelope approximation is properly used. Thus, the approximation we do, and this is the essence of the slowly varying envelope approximation, is that we neglect the backscattered part of the envelope function F. This was originally pointed out by Shen [9].

Some important features regarding linear pulse propagation governed by Eq. (2.15) are worth emphasizing. Retaining terms to second order in the dispersion operator yields

$$i\frac{\partial F}{\partial z} + i\beta_0'\frac{\partial F}{\partial t} + \frac{\beta_0''}{2}\frac{\partial^2 F}{\partial t^2} = 0 \tag{2.16}$$

which describes a wave packet which moving with the group velocity $v_g = 1/\beta_0'$ in the z-direction. The factor proportional to β_0'' describes second-order, or group-velocity, dispersion (GVD). This causes the pulse to broaden during propagation, since the different frequency components have different group velocities [10, 11]. The GVD is defined as *normal* if $\beta_0'' > 0$, and *anomalous* if $\beta_0'' < 0$. This means that the group velocity increases (decreases) with frequency in the anomalous (normal) dispersion regime.

After these considerations, we can apply the slowly varying envelope approximation to Eq. (2.12), thus removing its first two terms:

$$\left(-i\frac{\partial}{\partial z} + [k(\omega_0 - i\frac{\partial}{\partial t}) - k_0] + k_0 f(\mathbf{r}) + \frac{1}{2k_0}\nabla_\perp^2\right)\mathbf{F}(\mathbf{r},t) + c.c. \tag{2.17}$$

The quantity in the square bracket is the dispersion operator, cf. Eq. (2.15).

2.4 The nonlinear susceptibility

We will now consider the nonlinear part of the polarisation in Eq. (2.8). We thus need to consider the properties of the n:th order susceptibility tensor $\chi^{(n)}$. This tensor has 3^{n+1} elements, and the number of independent and nonzero elements of the tensor can be reduced if the medium has certain symmetries. In particular, it can be shown using the transformation properties of the susceptibility tensors [6], that in an isotropic medium the tensor elements must fullfill $\chi^{(n)} = (-1)^{(n+1)}\chi^{(n)}$, and consequently all even-order tensors must vanish. In such media, the lowest order nonlinear susceptibility is the third order, $\chi^{(3)}_{ijkl}$, possessing 81 elements. Furthermore, the isotropy of the medium makes $\chi^{(3)}$ invariant under any rotational transformation. Using this condition it is straightforward to show that $\chi^{(3)}$ has the 21 nonzero elements

$$\begin{aligned} &\chi_{iikk} = \chi_{jjll} \qquad \chi_{ikki} = \chi_{jllj} \\ &\chi_{ikik} = \chi_{jljl} \qquad \chi_{iiii} = \chi_{jkjk} + \chi_{jjkk} + \chi_{jkkj} \end{aligned} \tag{2.18}$$

where i,j,k,l are all possible permutations of x,y and z. Obviously, of these elements only three are independent, e.g. χ_{xyxy}, χ_{xxyy} and χ_{xyyx}.

A difficulty with the nonlinear susceptibilities is their frequency dependence. For example, $\chi^{(3)}$ depends on the frequencies of all participating electric fields in the tensor product. In the time domain this is expressed in the most general way by the triple convolution integral:

$$\begin{aligned} P_i^{NL}(\mathbf{r},t) = \epsilon_0 \int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} &\chi^{(3)}_{ijkl}(t-t_1, t-t_2, t-t_3) \\ &\mathbf{E}_j(\mathbf{r},t_1)\mathbf{E}_k(\mathbf{r},t_2)\mathbf{E}_l(\mathbf{r},t_3)dt_1dt_2dt_3. \end{aligned} \tag{2.19}$$

This relation is of course very complicated to treat as it stands, but several simplifying approximations can be made. In the first approximation, we consider the *electronic Kerr-effect*, which is responsible for the main contribution to $\chi^{(3)}$ in dense glasses. The response time for this effect is extremely fast, being of the order of femtoseconds. If we are not going to study the propagation of subpicosecond pulses, it suffices to approximate the response with delta functions, i.e. to assume an instantaneous response. In the next chapter we will generalise the formalism to include a frequency dependent response; the so called *Raman scattering* process, but as a first approximation we consider only a frequency independent nonlinear susceptibility. Furthermore, since we will investigate how a wave oscillating at a single frequency interacts with itself, the triple integral in Eq. (2.19) degenerates into a single one. This integral can be carried out with the delta function response, and we are left with the tensor product

$$P_i^{NL}(\mathbf{r},t) = \epsilon_0 \chi^{(3)}_{ijkl} \mathbf{E}_j(\mathbf{r},t)\mathbf{E}_k(\mathbf{r},t)\mathbf{E}_l(\mathbf{r},t). \tag{2.20}$$

If we introduce an electric field oscillating at the frequency ω_0, the nonlinear polarisation specified by Eq. (2.20) will give rise to oscillations at the frequencys ω_0 and $3\omega_0$, a phenomenon known in nonlinear optics as *degenerate four-wave mixing*. Non-degenerate four-wave mixing arises when the three fields of Eq. (2.20) oscillate at different freqencies, but since we are interested in self-action effects, this is beyond the scope of the present study. The interested reader is referred to [11] for a recent review.

In order to get further we have to specify the vector properties of the electric field. If the electric field has components in the transverse (x and y) directions only, we have

$$\mathbf{E}(\mathbf{r},t) = \frac{1}{2}(F_x(\mathbf{r},t)\hat{x} + F_y(\mathbf{r},t)\hat{y})\exp[i(\omega_0 t - k_0 z)] + c.c. \tag{2.21}$$

By performing the tensor products implied in Eq. (2.20), and using the properties of the isotropic susceptibility $\chi^{(3)}$, we obtain the following expressions for the induced nonlinear polarisation

$$\begin{aligned} P_x^{NL} = \frac{\epsilon_0}{8}\chi^{(3)}_{xxxx}(3|F_x|^2F_x + 2|F_y|^2F_x + F_y^2F_x^*)\exp[i(\omega_0 t - k_0 z)] + \\ \frac{\epsilon_0}{8}\chi^{(3)}_{xxxx}(F_x^2 + F_y^2)F_x \exp[i3(\omega_0 t - k_0 z)] + c.c. \end{aligned} \tag{2.22}$$

The y-component is easily derivable from the above expressions by interchanging the x- and y- indicies. An interesting feature, evident from Eq. (2.22) is that the third harmonic contribution vanishes in the case of circularly polarised beams [12], i.e. for $F_x = iF_y$. We do, however, restrict the present analysis to linearly polarised beams in the x-direction, i.e. we use the polarisation in Eq. (2.22) with $F_y = 0$. A further approximation is that we neglect the oscillations at $3\omega_0$, which are so far away in the spectrum that they do not influence the slowly varying envelope F. Thus, using $\mathbf{P}^{NL} = \hat{x}P_x + c.c$, where P_x is given by Eq. (2.22) in Eq. (2.8), and applying the slowly varying envelope approximation, we obtain

$$\begin{aligned} \left(-i\frac{\partial}{\partial z} + [k(\omega_0 - i\frac{\partial}{\partial t}) - k_0] + k_0 f(\mathbf{r}) + \frac{1}{2k_0}\nabla_\perp^2)\right) F(\mathbf{r},t) = \\ \frac{3\chi^{(3)}_{xxxx}}{8c^2}\exp[-i\omega_0 t]\frac{\partial^2}{\partial t^2}(|F|^2F\exp[i\omega_0 t]) \cong -\frac{3\chi^{(3)}_{xxxx}k_0}{8n_0^2}|F|^2F \end{aligned} \tag{2.23}$$

where we in the last equality have neglegted derivatives of the nonlinear term, and denoted the refractve index at the carrier frequency with n_0. Eq. (2.23) is therefore not appliccable to the study of very short pulse propagation. However, for pulse durations larger than 10 ps in electronic Kerr-media, it represents a good description. In particular, Eq. (2.23) is the standard description of the self-focusing phenomenon [2, 13, 14]. For shorter pulse durations, or for slowly responding media, modifications with coupled equations and/or additional terms are used, see e.g. [2]. Finally, we can expand the dispersion operator to second order and write Eq. (2.23) as

$$i\frac{\partial F}{\partial z} = -\frac{k_0''}{2}\frac{\partial^2 F}{\partial t^2} + \frac{1}{2k_0}\nabla_\perp^2 F + k_0 f(\mathbf{r}) + \frac{3\chi^{(3)}_{xxxx}k_0}{8n_0^2}|F|^2 F \tag{2.24}$$

where we have transformed the variables to a retarded reference frame by the substitution $t \to t - zk_0'$, $z \to z$, thereby removing the $\partial F/\partial t$ term. This eqution will be our starting point in the investigation of nonlinear optical beam propagation.

We end this section with a somewhat simpler "phenomenological" derivation of equation (2.24). This is easily done by assuming the refractive index n to be intensity dependent [14], i.e.

$$n(\omega, \mathbf{r}, |F|^2) = n(\omega) + n_1(\mathbf{r}) + n_2|F|^2 \tag{2.25}$$

where n_2 is the nonlinear Kerr-coefficient. Note that $n_1 \ll n(\omega)$ and $n_2|F|^2 \ll n(\omega)$. Using this refractive index in the Helmholtz equation for the electric field, i.e.

$$\nabla^2 E + \frac{\omega^2 n^2(\omega, \mathbf{r}, |F|^2)}{c^2}E = 0, \tag{2.26}$$

where $E = \frac{1}{2}F\exp[i(\omega_0 t - k_0 z)] + c.c$, we obtain (after applying the same approximations as above)

$$i\frac{\partial F}{\partial z} = \frac{1}{2k_0}\nabla_\perp^2 F - \frac{k_0''}{2}\frac{\partial^2 F}{\partial t^2} + \frac{k_0}{n_0}(n_1(\mathbf{r}) + n_2|F|^2)F. \tag{2.27}$$

By comparing with Eq. (2.24), we can express the function $n_1(\mathbf{r})$ and the nonlinear refractive index coefficient n_2 as

$$n_1(\mathbf{r}) = n_0 f(\mathbf{r}) \qquad n_2 = \frac{3}{8n_0}\chi^{(3)}_{xxxx} \tag{2.28}$$

We emphasize that Eq. (2.27) contains the same physics as Eq. (2.24), and the only approximation involved are the assumption that the changes in the refractive index due to the spatial modulation (e.g. a waveguide) n_1 and the nonlinearity $n_2|F|^2$ are small. This condition is fulfilled in most practical cases. Moreover, we can easily generalize the formalism to hold for an arbitrary nonlinearity. This means that $n_2|F|^2$ can be replaced with $g(|F|^2)$ where g is an arbitrary, sufficiently small, function. In the previous formalism

Table 2.1: The Kerr-coefficient for a few different materials.

Material	n_2 $[m^2/V^2]$	*Reference*
Solids:		
Silicats and other glasses	$4 - 10 \times 10^{-23}$	[15]
SiO_2	1.2×10^{-22}	[11]
GaAs	7×10^{-20}	[16]
Ruby	1.6×10^{-23}	[17]
Liquids:		
Water	10^{-22}	[16]
Chinese tea	3×10^{-15}	[18]
Gases:		
Air	10^{-24}	[19]
Sodium vapor	-6×10^{-17}	[20]

with susceptibilities, this is equivalent with including higher (odd) order nonlinear terms.

The function $f(\mathbf{r})$ is intended as a model of the waveguiding index profile, i.e. for a circular fiber $f = f(r)$ where r is the transverse radius. There are, however, no formal objections against including the longitudinal coordinate in f, i.e. $f(r, z)$, which could be useful in models of fiber bendings, manufacturing defects, or other axial perturbations. The only restriction is that the induced refractive index must be small to allow for the weakly guided approximation.

The nonlinear index coefficient n_2, (m^2V^{-2}), sometimes called the *Kerr-coefficient*, is a basic material parameter in the investigations of nonlinear optical beam and pulse propagation. In most optical materials n_2 is very low, as indicated in table 2.1 below, where measured values of n_2 for a few materials are listed. We have invoked Chinese tea as an example of an organic liquid. It is a well known feature of organical compounds/solutions to have large Kerr-coefficients. This may be qualitatively understood by considering the fact that the induced polarisation is determined by the induced molecular dipole moments. Organic molecules are often large, and will therefore have large dipole moments which result in large values of n_2. The size of such molecules will, however, also increase the nonlinear response time due to their mechanical momentum. This makes them, unfortunately, of little use in pulsed and fast optics, although interesting effects may arise due to the delayed response, see e.g. [2]. When it comes to self-focusing, which is a main subject of this thesis, CW-beams are often considered, and organic materials could then be of interest. A lot of research is done in the area of new nonlinear optical materials, and there is certainly much to come.

Considering plasmas, the value of n_2 depends dynamically on many parameters such as temperature, plasma density, frequency etc. The most common approach is then to use a separate equation to describe the nonlinear index, see e.g. [21].

Materials with positive values of n_2 are often called focusing, because they give rise to nonlinear self-focusing of intense beams, a subject that will be discussed in a later chapter.

Negative values of the Kerr-koefficient are not unusual, and such medias are referred to as defocusing. Gases can often be defocusing, and Rubidium vapor is an example of a defocusing medium, see Ref. [22]. The magnitude of the nonlinearity is usually of the same order in defocusing media as in non-organic focusing media, but for gases there is also a dependence on the particle density (i.e. the pressure).

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Chapter 3

The description of pulse propagation in nonlinear optical fibers

3.1 Basic considerations

In this chapter, we will derive the equation governing nonlinear pulse propagation in optical fibers. This could be done with the method of the previous chapter as a starting point, but we will describe a somewhat different derivation here. The reason for this is twofold: Firstly, we utilize the coupled mode concept [1] in a way that is probably more accessible to the "fiber-optics" community, and secondly, we point out that although the derivations seem to differ from a *mathematical* point of view, the *physical* approximations are the same. We will defer the discussion around the physics of the approximations to the final section of this chapter, however. An important difference from the previous derivation is that we will formulate the field entirely in the frequency domain. Our derivation follows previously published frequency-domain derivations [2, 3, 4]. For readers requiring an even more rigorous derivation, we refer to the comprehensive work by Kodama and Hasegawa [5], the comments on this work [6], and the derivation given by Newell and Maloney [7].

Consider the electromagnetic fields $\tilde{\mathbf{E}}, \tilde{\mathbf{H}}$ of the light in an optical fiber. These fields are given by the source-free Maxwell's equations, which in the frequency domain are:

$$\begin{aligned} \nabla \times \tilde{\mathbf{E}} &= -i\omega\mu_0\tilde{\mathbf{H}} \\ \nabla \times \tilde{\mathbf{H}} &= i\omega\epsilon(\mathbf{r},\omega)\tilde{\mathbf{E}} + i\omega\tilde{\mathbf{P}}^{NL} \end{aligned} \tag{3.1}$$

where $\epsilon(\mathbf{r},\omega)$ models the refractive index profile and the material dispersion of the fiber, and the tilde notation denotes Fourier transform. The fiber has an infinite set of linear modes, given by the linear Maxwell's equations, i.e. Eqs. (3.1) with $\tilde{\mathbf{P}}^{NL} = 0$, and we denote the fundamental mode with $(\mathbf{e}, \mathbf{h})$. As will be shown in the next chapters, the amount of power required for the nonlinearity to affect the transverse mode profile is several (five to six) orders of magnitudes larger than that required to observe nonlinear-pulsed effects in the picosecond regime. Hence $\tilde{\mathbf{P}}^{NL} = 0$ is a very good approximation indeed. From Maxwell's equations for the mode $(\mathbf{e}, \mathbf{h})$ and the total field $(\tilde{\mathbf{E}}, \tilde{\mathbf{H}})$, it is straightforward to verify the relation

$$\nabla \cdot (\tilde{\mathbf{E}} \times \mathbf{h} - \mathbf{e} \times \tilde{\mathbf{H}}) = i\omega \tilde{\mathbf{P}}^{NL} \cdot \mathbf{e} \tag{3.2}$$

which is valid for any mode of the fiber.

We will now consider pulse propagation in the axial ($\hat{z}$) direction of the fiber, and for the coming discussion, we must distinguish between the forward- and backward-going modes. Those can be written

$$\left\{ \begin{array}{c} \mathbf{e}^f \\ \mathbf{h}^f \end{array} \right\} = \left\{ \begin{array}{c} \mathbf{e}_\perp(\mathbf{r},\omega) + \hat{z}e_z(\mathbf{r},\omega) \\ \mathbf{h}_\perp(\mathbf{r},\omega) + \hat{z}h_z(\mathbf{r},\omega) \end{array} \right\} \exp(-i\beta(\omega)z) \tag{3.3}$$

for the forward-going fundamental mode, and

$$\left\{ \begin{array}{c} \mathbf{e}^b \\ \mathbf{h}^b \end{array} \right\} = \left\{ \begin{array}{c} \mathbf{e}_\perp(\mathbf{r},\omega) - \hat{z}e_z(\mathbf{r},\omega) \\ -\mathbf{h}_\perp(\mathbf{r},\omega) + \hat{z}h_z(\mathbf{r},\omega) \end{array} \right\} \exp(i\beta(\omega)z) \tag{3.4}$$

for the backward-going fundamental mode. In these equations we have introduced the propagation constant $\beta(\omega)$, and the notation $\mathbf{e}_\perp, \mathbf{h}_\perp$ and e_z, h_z for the components of the fundamental mode. For clarity we neglect the linear attenuation (which is always present), but this is easily included later as an imaginary part of the propagation constant. Before we proceed, however, we briefly review the weakly guided approximation with respect to the fiber modes.

3.2 Weakly guiding fiber modes

The modes of a dielectric optical waveguide are in general *hybrid modes*, possessing both transversal and longitudinal components of the electric and magnetic fields. The simple linearly polarised modes (the so called LP-modes) that are often used in the literature are no real modes of the fiber, but “pseudo-modes” [8] for which the cross-sectional intensity and the polarisation state changes during propagation. Despite this fact, the LP-modes are often an accurate enough description, and we will see that they emerge as the lowest-order approximation in weakly guiding fibers.

If the fiber is weakly guided, i.e. if the refractive index difference of the fiber is small; $\Delta = (n_{core} - n_{clad})/n_{core} \ll 1$, then the mode formalism can be simplified considerably. By expanding the modes in powers of the small parameter Δ it can be shown [1, 8] that

$$\mathbf{e}_\perp = \mathbf{e}_\perp^{(0)} + \Delta \mathbf{e}^{(1)} + O(\Delta^2) \tag{3.5}$$

$$e_z = \Delta^{1/2} e_z^{(1/2)} + O(\Delta^{3/2}) \tag{3.6}$$

with similar relations for the magnetic field. We see that the longitudinal component of the electric field is much smaller than the transversal. Since $\sqrt{\Delta} \cong 1-5\%$ in typical fibers, the

$\hat{z}$ component of the electric field can be neglected in the lowest-order approximation. The leading terms of the transverse field $\mathbf{e}_\perp^{(0)}, \mathbf{h}_\perp^{(0)}$ describe the TEM wave in a homogeneous medium, which means that they are given by the scalar wave equation and related via

$$\mathbf{h}_\perp^{(0)} = \sqrt{\frac{\epsilon(\omega)}{\mu_0}}\hat{z} \times \mathbf{e}_\perp^{(0)} \equiv \frac{n(\omega)}{Z_0}\hat{z} \times \mathbf{e}_\perp^{(0)} \tag{3.7}$$

where $Z_0 \approx 120\pi$ is the wave impedance of vacuum. Similarly, the propagation constant can be expanded in powers of Δ as [1, 8]

$$\beta(\omega) = \omega\sqrt{\mu_0\epsilon(\omega)} + O(\Delta^{3/2}) \equiv \frac{\omega}{c}n(\omega) + O(\Delta^{3/2}). \tag{3.8}$$

To invoke the weakly guiding approximation therefore means that we neglect the higher-order (in Δ) terms in the equations above. The fundamental mode is then given by the $\mathbf{e}_\perp^{(0)}$ field, which we simply can choose to be linearly polarised. This constitutes the $LP_{01} - mode$ of the fiber.

3.3 Frequency formulation of the field

We emphasize again that these linear modes are not affected by the weak nonlinearity we study here. Thus, we can assume that the total field of a wavepacket propagating in the forward direction has the transverse mode profile of the linear mode, but we must allow for a slowly varying envelope A:

$$\left\{ \begin{array}{c} \tilde{\mathbf{E}} \\ \tilde{\mathbf{H}} \end{array} \right\} = \left\{ \begin{array}{c} A(\omega, z)\mathbf{e}_\perp(x, y, \omega) + \hat{z}E_z(\mathbf{r}, \omega) \\ A(\omega, z)\mathbf{h}_\perp(x, y, \omega) + \hat{z}H_z(\mathbf{r}, \omega) \end{array} \right\} \exp(-i\beta(\omega)z) \tag{3.9}$$

We aim to derive an equation for the slowly varying (along z) function A. This can be found from Eq. (3.2), by introducing the total field from Eq. (3.9) and the *backward* propagating mode from Eq. (3.4), which yields

$$\nabla \cdot (-2A(z, \omega)(\mathbf{e}_\perp \times \mathbf{h}_\perp) + \hat{z} \times \mathbf{v}(\mathbf{r}, \omega)) = i\omega\tilde{\mathbf{P}}^{NL} \cdot \mathbf{e}^b. \tag{3.10}$$

If instead the *forward* going mode had been used in Eq. (3.2), we would have obtained an equation for the longitudinal field components. However, we are not interested in these here. We need not be more specific about the vector $\mathbf{v}$ in Eq. (3.10), since the $\nabla \cdot (\hat{z} \times \mathbf{v})$-term vanishes after integrating over the transverse cross-section. Thus, the equation for A becomes

$$\frac{\partial A(z, \omega)}{\partial z} = -i\frac{\omega}{2}\frac{< \tilde{\mathbf{P}}^{NL} \cdot \tilde{\mathbf{e}}^b >}{< (\tilde{\mathbf{e}}_\perp \times \tilde{\mathbf{h}}_\perp) \cdot \hat{z} >} \exp(i\beta(\omega)z) \tag{3.11}$$

where <> denotes integration over the transverse coordinates. At this stage we can invoke the weakly guiding approximation which was elaborated above, and reduce Eq. (3.11) to

$$\frac{\partial A(z, \omega)}{\partial z} = -i\frac{\omega Z_0}{2n(\omega)}\frac{< \tilde{\mathbf{P}}_\perp^{NL} \cdot \mathbf{e}_\perp^{(0)} >}{< |\mathbf{e}_\perp^{(0)}|^2 >} \exp(i\beta(\omega)z). \tag{3.12}$$

Our next step is to choose a proper description for the electric field. This is a wavepacket centered around the carrier frequency ω_0, which in the time domain can be expressed as

$$\mathbf{E}(\mathbf{r},t) = \exp[i(\omega_0 t - \beta(\omega_0)z)]\frac{1}{2\pi}\hat{p}\frac{1}{2}\int \tilde{u}(z,\Omega)\kappa T(x,y,\omega_0+\Omega)\exp[i\Omega t]d\Omega + c.c. \quad (3.13)$$

where $\omega = \omega_0 + \Omega$, T is a scalar function describing the transverse mode profile and $\tilde{u}$ the field envelope. The polarisation state of the field is denoted by the unity vector $\hat{p}$, and κ is a normalisation constant that we will specify below. Fourier transforming the electric field yields

$$\tilde{\mathbf{E}}(\mathbf{r},\omega) = \hat{p}\frac{\kappa}{2}\tilde{u}(z,\Omega)T(x,y,\omega)\exp[-i\beta(\omega_0)z] + c.c. \quad (3.14)$$

By comparing this with the expression for the field in Eq. (3.9), we identify the transverse mode as $\hat{p}T(x,y,\omega) = \mathbf{e}_\perp(x,y,\omega)$ and the envelope function as

$$\frac{\kappa}{2}\tilde{u}(\Omega,z) = A(\omega_0+\Omega,z)\exp[-i\Delta\beta(\Omega)z] \quad (3.15)$$

where $\beta(\omega) = \beta(\omega_0) + \Delta\beta(\Omega)$. Using these changes of variables, Eq. (3.12) becomes

$$\frac{\partial\tilde{u}(z,\Omega)}{\partial z} = -i\tilde{u}(z,\Omega)\Delta\beta(\Omega) - i\frac{\omega Z_0}{n(\omega)\kappa}\frac{< \tilde{\mathbf{P}}^{NL}\cdot\hat{p}T >}{< T^2 >}\exp(i\beta(\omega_0)z). \quad (3.16)$$

A complication with this frequency-domain expression for the field is the modal frequency dependence in T, since the interpretation of the function u as the envelope of the electric field becomes less obvious. However, this can be remedied by the transverse averaging. If we interpret $\tilde{u}(z,\Omega)$ as the square root of power contained in the fiber cross-section, the normalisation constant κ becomes

$$\kappa^2(\omega) = \frac{4Z_0}{n(\omega) < T^2 >}. \quad (3.17)$$

It is physically more sensible to work with a global square root power than the local electric field. In this context, the local electric field is not of much interest, since the weakly guided approximation has already removed its finer structure.

3.4 The nonlinear susceptibility

According to the previous chapter, the lowest-order nonlinear susceptibility of a fiber is $\chi^{(3)}$, and in its most general form it is a convolution with the time dependence of all three ingoing fields. However, if we assume that we are far from medium resonances, the convolution will act only on the field intensity [2]. Assuming a linearly polarised electric field E in the $\hat{x}$ direction, we can write the $\hat{x}$ component P of the nonlinear polarisation (c.f. previous chapter) as the convolution [2, 9, 10, 11]

$$P_x^{NL}(\mathbf{r},t) = \epsilon_0\chi^{(3)}_{xxxx}\frac{1}{2}E(\mathbf{r},t)\int g(t-t')E^2(\mathbf{r},t')dt' + c.c. \quad (3.18)$$

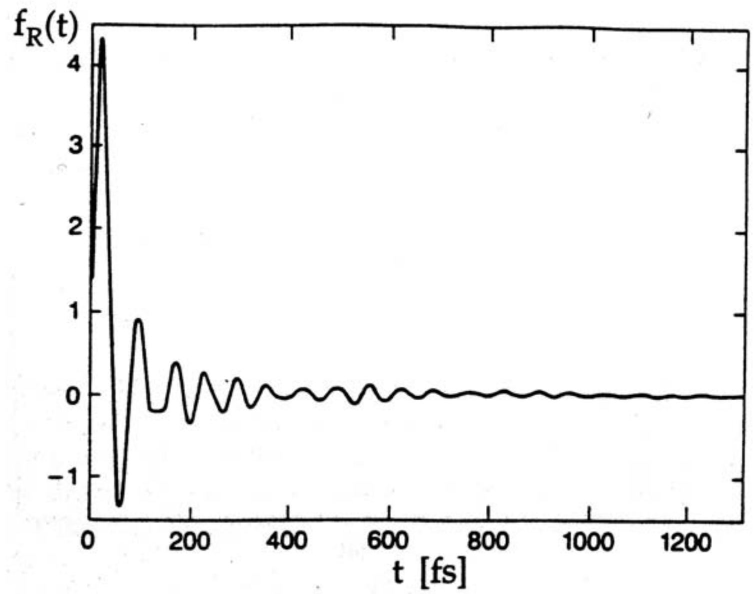

Figure 3.1: *The Raman response function of silica fibers.*

The function g(t) is the nonlinear response function, and it indicates how fast the medium responds to an applied external field. In optical fibers, there are two dominant contributions to the nonlinear polarisation, namely the electronic Kerr-effect, and the Raman effect. The electronic Kerr-effect is the polarisation of the electron cloud around the individual atoms, and it has an extremely fast respone time; of the order of femtoseconds. We can therefore approximate the Kerr-part of g(t) with a delta function.

The Raman contribution to the nonlinear susceptibility in fused silica originates from the interaction between the electric field and *optical phonons*, which are transversally oscillating molecular vibration modes in the medium [12]. The *Raman gain spectrum*, i.e. the Fourier transform of the Raman response, consists of several Lorentzian lines [13, 14], but we can approximate this with the the most dominant line around 13.2 THz. In the time domain, this corresponds to a response function of the form:

$$g_R(t) \sim \sin(t/t_1) \exp(t/t_2). \tag{3.19}$$

The constants have the measured values $t_1 = 12.2$ *(fs)* and $t_2 = 32$ *(fs)*, [2, 13, 14]. The measured Raman response of fused silica is shown in figure 3.1, and the approximation of Eq. in (3.19) is shown in figure 3.2 below. Although the agreement is good in general, there are several fine-structure details arising from other lines in the spectrum that are beyond this simple model.

We can thus express the total response function as a sum of the respective Kerr- and the Raman-responses,

$$g(t) = \alpha_1 \delta(t) + \alpha_2 g_R(t) \tag{3.20}$$

where the parameters α_1 and α_2 are dimensionless constants that will be determined later. Since our electric fields are given in the frequency domain, we write the Fourier transform

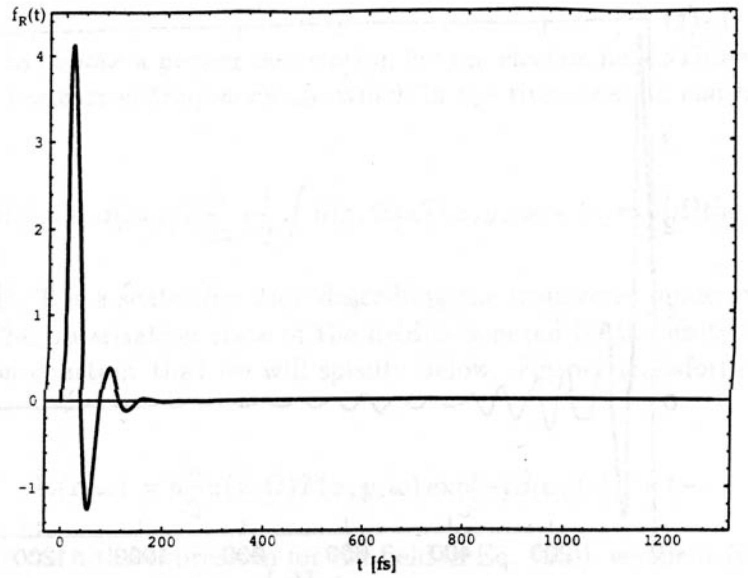

Figure 3.2: *The Lorentzian approximation to the Raman gain gives this response function.*

of the nonlinear polarisation as

$$\tilde{P}_x^{NL}(\mathbf{r},\omega) = \epsilon_0 \chi^{(3)}_{xxxx} \frac{1}{2}\frac{1}{(2\pi)^2} \int\int \tilde{g}(\omega_1-\omega_2) \times \\ \tilde{E}(\mathbf{r},\omega-\omega_1+\omega_2)\tilde{E}(\mathbf{r},\omega_1)\tilde{E}^*(\mathbf{r},\omega_2)d\omega_1 d\omega_2 + c.c. \qquad (3.21)$$

The next step is to insert the electrical field of Eq. (3.14) into this expression, neglect the oscillating term $\exp(\pm i3\omega_0 t)$ which correspond to waves that cannot propagate in the fiber [10], and perform the transverse integration:

$$< \tilde{P}_x^{NL}(\mathbf{r},\omega)T(x,y,\omega) >= \epsilon_0 \chi^{(3)}_{xxxx} \frac{1}{(2\pi)^2}\frac{3}{16} \exp[-i\beta(\omega_0)z] \times \\ \int\int \tilde{f}(\omega_1-\omega_2)\tilde{u}(z,\omega-\omega_1+\omega_2)\tilde{u}(z,\omega_1)\tilde{u}^*(z,\omega_2)S(\omega,\omega_1,\omega_2)d\omega_1 d\omega_2 + c.c. \qquad (3.22)$$

The function S comes from the transverse integration, and is defined by

$$S(\omega,\omega_1,\omega_2) =< T(x,y,\omega-\omega_1+\omega_2)T(x,y,\omega_1)T(x,y,\omega_2)T(x,y,\omega) > \times \\ \kappa(\omega-\omega_1+\omega_2)\kappa(\omega_1)\kappa(\omega_2). \qquad (3.23)$$

The fact that the function S appears in the integral is an additional difficulty that was pointed out originally in Ref. [4]. Its physical origin is due to the fact that the modal dispersion of the fiber is affected by the nonlinearity, whereas the material dispersion is not. Previous approaches put $S(\omega,\omega_1,\omega_2) \approx S(\omega,\omega,\omega)$ [2], or $S(\omega,\omega_1,\omega_2) \approx S(\omega_0,\omega_0,-\omega_0)$ [3], and moved S outside the integral. These approximations are valid if the frequency dependence of $T(x,y,\omega)$ is small over the pulse and the Raman bandwidth, which in most cases is a fairly good approximation. A proper treatment must, however, recognize that the integral gets its main contributions at $\omega_1 \approx -\omega_2 \approx \omega_0$, since the functions $u(\omega_1)$ and $u^*(\omega_2)$ are wavepackets centered at ω_0 and $-\omega_0$, respectively. Thus we can Taylor expand S around this point in (ω_1, ω_2) space. Note, however, that since the nonlinearity is small, the higher-order terms of this expansion constitute even smaller contributions to the governing equation, and can therefore be neglected. Thus, we make the approximation $S(\omega, \omega_1, \omega_2) \approx S(\omega, \omega_0, -\omega_0)$ and move S outside the integral. Finally, it should be noted that the Raman gain in Eq. (3.21) has been reduced with one third,

$$f(t) = \alpha\delta(t) + \frac{2}{3}\alpha_1 g_R(t) \equiv \alpha\delta(t) + (1-\alpha)f_R(t) \tag{3.24}$$

because one of the convolution integrals is approximately zero, see [2, 11]. This means that it is only two thirds of the Raman energy transfer (as defined in Eq. (3.18)) that is measured in a pump-probe configuration and that contributes to the nonlinear refractive index. We disregard this theoretical detail by normalising the function f so that $\int_0^\infty f(t)dt = 1$. Thus, f(t) becomes

$$f(t) = \alpha\delta(t) + (1-\alpha)\frac{t_1^2 + t_2^2}{t_1 t_2^2}\exp(-t/t_2)\sin(t/t_1) \tag{3.25}$$

and the constant α now denotes the relative contributions of the Kerr- and Raman effects to the total susceptibility. The value of α have been estimated to 0.82, see Refs. [14, 16].

We can now use the expression (3.21) for the nonlinear polarisation in Eq. (3.16) and obtain

$$\frac{\partial\tilde{u}(z,\Omega)}{\partial z} = -i\tilde{u}(z,\Omega)\Delta\beta(\Omega) - i\frac{3}{4n(\omega_0)}\chi^{(3)}_{xxxx}\epsilon_0 Z_0^2 \omega R(\omega) U^{NL} \tag{3.26}$$

where U^{NL} and $R(\omega)$ are given by

$$U^{NL} = \int\int \tilde{f}(\omega_1 - \omega_2)\tilde{u}(z, \omega - \omega_1 + \omega_2)\tilde{u}(z,\omega_1)\tilde{u}^*(z,\omega_2)d\omega_1 d\omega_2 \tag{3.27}$$

$$R(\omega) = \frac{\kappa(\omega - 2\omega_0)}{\kappa(\omega)n(\omega)}\frac{< T(\omega - 2\omega_0)T(\omega)T^2(\omega_0) >}{< T^2(\omega) >< T^2(\omega_0) >}. \tag{3.28}$$

Eq. (3.26) is the governing equation for the power envelope in the fiber. After transforming to the time domain it can be written as

$$\begin{aligned} i\frac{\partial u(z,t)}{\partial z} &= [\beta(\omega_0 - i\frac{\partial}{\partial t}) - \beta(\omega_0)]u(z,t) \\ &+\sigma(1 - \tau_{shock} i\frac{\partial}{\partial t})u(z,t)\int f(t-t')|u(z,t')|^2 dt' \end{aligned} \tag{3.29}$$

where

$$\sigma = \frac{3}{4n(\omega_0)}\chi^{(3)}_{xxxx}\frac{\omega_0}{cA_{eff}}\frac{Z_0}{n(\omega_0)} \equiv \frac{N_2}{A_{eff}}\frac{\omega_0}{c} \tag{3.30}$$

$$\tau_{shock} = \frac{1}{\omega_0} + \frac{R'(\omega_0)}{R(\omega_0)} \tag{3.31}$$

and the effective mode area is defined as

$$A_{eff} = \frac{< T^2(x,y,\omega_0) >^2}{< T^4(x,y,\omega_0) >}. \tag{3.32}$$

In deriving Eq. (3.29) we have expanded the factor in front of U^{NL} to first order around ω_0. The factor in the square bracket of Eq. (3.29) is the dispersion operator, which is defined by its Taylor expansion. In Eq. (3.30) we have defined the nonlinear refractive index with respect to power, $N_2 = 2n_2 Z_0/n(\omega_0)$, which is common in the literature. In silica fibers its value is $N_2 = 3.2 \cdot 10^{-16}(W^{-1}cm^2)$. The reason for introducing the factor τ_{shock} is that that term can give rise to an optical shock front, the so-called self-steepening effect [15]. The main contribution to τ_{shock} comes from the ω_0^{-1}-term, since $R'(\omega_0)/R(\omega_0)$ is negligible for wavelengths in the low-loss regime of the fiber[2]. Moreover, the shock-term is significant only for pulse durations of the order of tens of femtoseconds, so we will use $\tau_{shock} \cong \omega_0^{-1}$. This value of τ_{shock} is reasonable also from a physical point of view, because in that case Eq. (3.29) conserves the classical *photon number* i.e.

$$\frac{\partial}{\partial z}\int |\tilde{u}(\omega)|^2 \frac{d\omega}{\omega} = 0 \tag{3.33}$$

as proved in e.g. Ref. [2]. One cannot expect the optical energy to be conserved, since the Raman process transfers photons from high to low frequencies. We conclude the derivation by simplifying Eq. (3.29) to a form familiar to the "optical soliton community", namely a modified nonlinear Schrödinger equation. This is obtained by expanding the dispersion operator to third order, and by approximating the Raman gain with its low-frequency slope, which yields

$$\begin{aligned} i\frac{\partial u}{\partial z} &= -i\beta_0'\frac{\partial u}{\partial t} - \frac{\beta_0''}{2}\frac{\partial^2 u}{\partial t^2} + \sigma|u|^2 u \\ &+ \underbrace{i\frac{\beta_0'''}{6}\frac{\partial^3 u}{\partial t^3}}_{3OD} - \underbrace{\sigma\tau_R u\frac{\partial|u|^2}{\partial t}}_{Raman} - \underbrace{\frac{i\sigma}{\omega_0}\frac{\partial|u|^2 u}{\partial t}}_{shock}. \end{aligned} \tag{3.34}$$

This is the unperturbed nonlinear Schrödinger (NLS) equation (first line) with three extra terms (second line). These extra terms model the effects of third-order dispersion (3OD), Raman downshift, and self-steepening or optical shock formation. In a fiber, these terms have different strengths in different parameter regimes. The unperturbed NLS is a good model for pulse durations above 1 ps. However, the third-order dispersion term must be included for carrier wavelengths near the zero-dispersion wavelength, independently of the pulse duration. The Raman term is important for shorter pulses, of the order 1-0.05 ps. Using the gain of Eq. (3.24) the Raman coefficient becomes

$\tau_R = (1-\alpha)2t_1^2 t_2/(t_1^2 + t_2^2) \cong 1.5$ *(fs)*. However, the approximate model for the response does not model the initial slope of the Raman gain very well, since it aims at an overall agreement with the entire Raman gain spectrum. Estimations from the measured Raman gain curve show that the proper value is somewhat higher. However, there seem to be little consensus of the correct value of τ_R in the literature, and the values $\tau_R = 3$ *(fs)* [16], $\tau_R = 5$ *(fs)* [17] and $\tau_R = 6$ *(fs)* [18] have been used.

For even shorter pulses, below 50 fs, the shock term must be included, and the full Raman spectrum be taken into account. It has been argued [2] that the full equation (3.29) is a valid description down to pulse bandwidths of approximately one third of the carrier frequency ω_0, which corresponds to pulse durations above 6 fs.

3.5 Validity of the basic approximations

Equation (3.29), and simplifications such as Eq. (3.34) are widely used in the nonlinear-optics research. It is important to realize exactly what kind of approximations have lead to these equations and their regime of validity. In chapters 2 and 3, we introduced two fundamental approximations in the derivations of the equations governing pulses and beams; namely the *weakly guiding approximation* (WGA) and the *slowly varying envelope approximation* (SVEA). The underlying physics of these approximations will be discussed below.

The interpretation of the WGA is, in the case of beams, that the refractive index variation over the beam is weak. In this case, the refractive index has contributions from both the nonlinearity and from a possible linear material variation; i.e. a dielectric waveguide. We must therefore restrict the analysis to low beam amplitudes. Note, however, that this does not exclude beams guided by the material or the nonlinearity, it only requires the guiding to be weak. In the case of pulses in a fiber, the WGA was discussed in section 3.2. We concluded that if the fiber mode is weakly guided, then we can disregard from longitudinal field components and consider LP-modes, i.e. scalar electric fields. The same obviously holds for weakly guided *nonlinear* beams, since they are modes of their induced (weak) waveguides.

The SVEA have been employed in two different ways in this work. In chapter two, we derived a second-order wave equation from Maxwell's equations. This second-order wave equation sustains propagation of both forward- and backward-going waves. We simplified this second-order equation to a first-order one by discarding the backward-moving waves. However, a complication when omitting terms that sustain the backward wave is to drop the higher derivatives in t and z in a consistent way. In fact, the envelope does not necessarily have to be "slowly varying" as long as the backscattered light is negligible. In chapter three, we derived the governing equation *assuming* a forward-propagating mode, thus starting from a wave equation that was first-order in z. There was obviously no need to invoke the SVEA, since the backward-waves was discarded from the start. The physics of the seemingly different approaches of chapter two and three are therefore the same, and hence we obtain similar equations.

Furthermore, one may realize that the physical requirements for the WGA and the SVEA are the same! Obviously, the backscattered light is negligibly small as long as the induced refractive index in the material is small. This is exactly the same physical requirement as for the WGA. Our equations are therefore valid in all cases where the induced nonlinear index is small. This is also what is encountered in experiments, with one important exception, however. That is the collapse phenomenon of 2-D self-focusing. Here the induced refractive index becomes infinitely high, hence both the WGA and the SVEA breaks down. In fact, direct numerical simulations of Maxwells equations in this case have shown that the collapse gives rise to a large amount of backscattering [19, 20]. Thus, as a first approach to describe the dynamics of the 2-D collapse with the subsequent filamentation, one has to go beyond the SVEA, and allow for power loss through backscattering. These issues are beyond the present thesis, however.

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Chapter 4

The nonlinear propagation of optical beams

4.1 The governing equation

In chapter two, we derived the equation governing the slowly varying envelope F of the electric field of an optical beam in a nonlinear medium. In terms of the nonlinear refractive index it reads as Equation (2.27):

$$i\frac{\partial F}{\partial z} = \underbrace{\frac{1}{2k_0}\nabla_\perp^2 F}_{diffraction} - \underbrace{\frac{k_0''}{2}\frac{\partial^2 F}{\partial t^2}}_{dispersion} + \underbrace{\frac{k_0}{n_0}f(x,y)F}_{waveguiding} + \underbrace{\frac{k_0 n_2}{n_0}|F|^2 F}_{nonlinearity}. \tag{4.1}$$

where we have described the physical meaning of each term. The function f(x,y) is nonzero only in optical waveguides, were it models the radial index profile. Media with no index profile, i.e. in which f=0 shall be referred to as *homogeneous* or *bulk* media. In cylindrical, bulk media and in the absence of dispersion, this model was originally suggested by Kelly [36] 1965.

The properties of *linear*, dispersive beam propagation in homogeneous media are well known, since the governing equation is similar to the extensively studied diffusion equation [2]. It is easily shown that an initial pulse at z=0 will broaden in two ways: spatially in the x-y-plane due to diffraction, and temporally in the comoving t-frame due to dispersion. Thus, diffraction in space and dispersion in time are equivalent phenomena, with the difference that we may have either sign of the dispersive term. However, it has been shown in e.g. Ref. [3], that in the asymptotic limit as $z \to \infty$, an arbitrary input pulse will broaden independently of the sign of the dispersion. As an example we can derive an exact linear solution, the “spreading Gaussian”

$$\begin{aligned} F(x,y,t,z) = {} & \frac{x_0 y_0 t_0}{\sqrt{(x_0^2 - 2iz/k_0)(y_0^2 - 2iz/k_0)(t_0^2 + 2izk_0'')}} \times \\ & \exp\left[-\frac{x^2}{x_0^2 - 2iz/k_0} - \frac{y^2}{y_0^2 - 2iz/k_0} - \frac{t^2}{t_0^2 + 2izk_0''}\right] \end{aligned} \tag{4.2}$$

which describes a Gaussian pulse-beam that broadens due to diffraction and dispersion. The effect of the nonlinearity, which is important for larger amplitudes, will lead to an induced waveguide and thus a counteraction of the diffractive broadening. The dynamics of the dispersive broadening will, however, be strongly affected by the sign of the dispersion. The magnitude and sign of the dispersion is evaluated from the dispersion relation at the carrier wavelength, and in many media it can take on either sign. Anomalous dispersion, i.e. $k_0'' < 0$ corresponds to a positive sign in front of the time derivative in Eq. (4.1), and it is in that case a matter of scaling in order to have complete symmetry between the transverse spatial and the comoving temporal coordinates. It is then possible to have *self-trapping*, i.e. a perfect balance between the linear brodening and the nonlinear compression, in 1, 2 or 3 transverse dimensions [4]. This space-time symmetry requires that the beam width x_0 and the pulse duration t_0 are the same in normalised units, i.e.:

$$x_0 = \frac{t_0}{\sqrt{-k_0 k_0''}} \tag{4.3}$$

In typical glass waveguides $\sqrt{-k_0 k_0''} \cong 0.3(mm^{-1}ps)$ [4]. Note that we cannot consider arbitrary small k_0'', since the next term $\sim k_0'''$ in the dispersion relation has been neglected.

In the nonlinear regime, the exact analytical results that are available for Eq. (4.1) are rather limited. There is a trivial plane-wave solution that is independent of x,y and t:

$$F(z) = F_0 \exp[-i\frac{k_0 n_2}{n_0}|F_0|^2 z] \tag{4.4}$$

where F_0 is an arbitrary (complex) field strength that is constant over the cross-section and in time. If the phase-delays along z of this nonlinear wave and the linear Gaussian of Eq. (4.2) are compared in the case of anomalous dispersion, we find that they have different sign. The nonlinear solution has a negative phase-shift, and the linear pulse has a positive. This indicates that linear effects and nonlinear effects will counteract each other; and also that the phase velocities of linear and nonlinear waves differ with respect to sign. In a defocusing medium, where $n_2 < 0$, this does not hold, and the nonlinearity is found to boost the spreading. Moreover, the nonlinear plane-wave (4.4) can be shown to be unstable against transverse perturbations and will break into filaments when perturbed. This feature is known as *modulation instability* and will be discussed in chapter 5.

In the following, we will consider Eq. (4.1) in some different geometries and media. Since the self-focusing dynamics is strongly dependent on the transverse dimensionality, we treat each number of transverse dimensions separately.

4.2 Self-focusing in one transverse dimension - spatial solitons

We consider firstly the propagation of an intense optical beam in a slab waveguide. We assume a CW-beam or very weak dispersion so that the time derivative can be neglected. The waveguide is assumed to be guiding in the y-direction and infinite in the x-direction, i.e. it has an index profile $f(y)$. Thus, we have a planar waveguide in the y-direction, and we assume that the transverse field can be separated according to $F(x, y, z) = F(x, z)\exp[i\kappa^2 z]R(y)$, where R(y) is a linear mode of the waveguide corresponding to the eigenvalue κ, i.e.

$$\frac{1}{2k_0}\frac{\partial^2 R}{\partial y^2} + \frac{k_0}{n_0}f(y)R = -\kappa^2 R(y) \tag{4.5}$$

and F(x,z) is governed by

$$\frac{i}{k_0}\frac{\partial F}{\partial z} = \frac{1}{2k_0^2}\frac{\partial^2 F}{\partial x^2} + \frac{n_2\alpha}{n_0}|F|^2 F. \tag{4.6}$$

where, according to coupled mode theory, the constant $\alpha = \int R(y)^4 dy / \int R(y)^2 dy$ is of the order unity. For simplicity, if we assume a Gaussian mode profile which corresponds to $f(y) \sim -y^2$, i.e. a parabolic-index profile, we obtain $\alpha = 1/2$. Eq. (4.6) is the *Nonlinear Schrödinger (NLS) equation*, which can be solved exactly by means of the inverse scattering transform [5]. A deeper discussion around the properties of this equation is given in chapter 6. In the present discussion it is enough to note that a crucial role in the solution of the NLS equation is played by the *soliton* solution, i.e.

$$F(x, z) = \sqrt{\frac{2n_0}{k_0^2 x_0^2 n_2}}\,sech(\frac{x}{x_0})\exp[-i\frac{z}{2x_0^2 k_0}], \tag{4.7}$$

where the soliton width x_0 is arbitrary. Numerically, it was established early [6] that the soliton is very robust against perturbations. In fact, it was shown in Ref. [7] that the soliton (4.7) arises out of a wide range of initial conditions. Since this soliton is a spatial self-guided beam, it is often called a *spatial soliton*.

It is important to consider the validity of Eq. (4.6). The derivation of this equation is based on the fact that the mode profile in the y-direction is unaffected by the nonlinearity. This is valid only if there is enough asymmetry between the x- and y-axes of the beam. There are two practical ways to achieve this. The derivation given above assumes that the width in the y-direction is much smaller than the width in the x-direction, so that the beam tends to diffract mostly in the y-direction, where it is guided. The weaker diffraction in the x-direction then requires less power for soliton formation. This is the common way to generate solitons in slab waveguides [8, 9]. Note, however, that the NLS equation (4.6) becomes invalid for higher powers, when the soliton width is of the same order as the waveguide height [10]. The second way to create spatial solitons is to consider a bulk medium, and use a beam which is strongly elliptic, with the major axis in the y-direction. Then the diffraction in the y-direction is negligible in comparison to the diffraction in the x-direction. However, this situation is modulationally unstable [11] in the y-direction, and the beam will break into filaments. An ingenious way to get round this problem has been suggested by Barthelemy et al. [12]. If two slightly crossed beams are used, they create a transverse modulation, with a frequency *higher* than the upper cut-off for modulational instability. The purity of this experiment is, however, somewhat reduced by the beams' tendency to nonlinearly attract each other and deflect [13]. If the pulse dynamics is included in these spatial solitons, a pulse compression effect is observed, which have compressed 75 fs pulses to 19 fs [14].

A lot of research has been devoted to nonlinear beam propagation close to the intersection of two different media, i.e. near a discontinuity of the refractive index [15]. The main finding in these cases is two families of spatial solitons, corresponding to beams having their main parts on each side of the boundary. The family of beams in the fast medium, which has the lower refractive index, can be shown to be unstable, due to the tendency for optical rays to move towards regions with high refractive index. More general cases have also been analyzed; e.g. with three layers that differ in the linear or nonlinear parts of the refractive index. In particular, waveguides with the dominating nonlinear contribution in the cladding show bistability, i.e. there can be two or several modes at a given power level. We refer to the reviews by Mihalache et al. [16] and by Stegeman et al. [17] for a thorough treatment of nonlinear effects in planar waveguides.

The dynamics of nonlinear planar beams are often studied within the framework of the IST transform using a perturbation theory to allow for the medium inhomogeneities [15]. Variational methods are common in order to study the stationary modes and to determine their propagation constants, which in several cases is enough for the determination of stability [18]. Nevertheless, important insight into the beam dynamics can be obtained by generalising the variational method to treat the dynamic evolution of the beam parameters. This will be explained in the following section.

Finally we will consider some technical data for the stationary spatial soliton beams. The power P *(W)* contained in a spatial soliton is given by

$$P = \frac{n_0 w_0}{k_0^2 N_2 x_0} \qquad w_0 = (\int R^2 dy)^2 / \int R^4 dy \tag{4.8}$$

where w_0 is the guided width and $N_2 = n_2 Z_0 / n_0$ $(m^2 W^{-1})$ is the nonlinear refractive index with respect to power. Typical values of these parameters is, in a glass waveguide [8],

$$\begin{aligned} N_2 = 3.4 \times 10^{-8} (W^{-1} \mu m^2) \qquad w_0 = 3\mu m \\ \lambda = \frac{2\pi}{k_0} = 0.62\mu m \qquad n_0 = 1.53 \end{aligned} \tag{4.9}$$

which for a soliton width $x_0 = 15\mu m$ gives $P = 230kW$. This is five to six orders of magnitude greater than the power required for the creation of a temporal soliton (see chapter 6), and it reflects the fact that the strength of dispersion is much less than the strength of diffraction for normal waveguide parameters. Consequently, a higher power is required to overcome the diffraction. In the earliest reported experiments of spatial solitons in waveguides [9], the power level was reduced by two orders of magnitudes, because a liquid with a relatively high nonlinear refractive index was used as the nonlinear medium. In order to accomplish the required peak powers in a conventional glass waveguide, fs-pulses have to be used [8]. In that parameter regime, dispersion cannot be neglected as we have done here, and moreover, the nonlinear Raman effect must also be included. To my knowledge, there is no analysis of nonlinear "ultrashort beam propagation" in which all these effects are taken into account.

There are many potential applications for nonlinear planar guided waves. However, it is quite unpractical to have to generate so high powers and short pulses in order to observe the nonlinear effects. Therefore, materials with higher nonlinear coefficients are attracting strong interest; e.g. polymers and semiconductors may be promising alternatives to the conventional glass materials. This research is simplified by the fact that loss is not a critical design parameter, because the typical nonlinear length is of the order of centimeters for ps- and fs-pulses. A problem with these materials is, however, that the nonlinear response may not be fast enough to allow for ultrafast operation.

4.3 The variational approach to the solution of nonlinear evolution equations

Since the governing equation of the field envelope is nonlinear, the conventional separation techniques using Fourier expansions are not appropriate. If we want to examine the behaviour of a localised general input field $u(x, y, t, 0)$, other methods must be used. Direct numerical solutions are the most common way, but there is need also for approximate analytical methods. For instance, in self-action problems with more than one transverse dimension, the evolution equations are nonintegrable, and numerical schemes may be very time consuming. In that context approximate analytical methods are of great value.

Most analytical methods are based on an assumption of the transverse structure of the electric field, a *trial function*, which is then allowed to evolve along z. In the context of nonlinear self-action, a useful choice is

$$F(r, z) = A(z) f(\frac{r}{a(z)}) \exp[ib(z)r^2 + i\phi(z)] \tag{4.10}$$

where r denotes a transverse coordinate e.g. x, y, or $\sqrt{x^2 + y^2}$. The function f denotes some pulse-shape, and A(z) and a(z) are the amplitude and width, respectively, of the pulse. The phase is modelled with two functions; b(z) models a curved phase-front, and $\phi(z)$ is the longitudinal phase-delay. This description of the phase is cnsistent with the linear spreading solution (4.2) as well as with the purely nonlinear self-phase modulated solution (4.4). The purpose is now to find the real functions (A, a, b, ϕ). Several analytical methods have been suggested for this problem. The earliest one was the *aberrationless paraxial-ray approximation*. In this method the trial function is inserted in the evolution equation, where the nonlinear refractive index is Taylor expanded in the transverse direction, i.e. with respect to r. The two lowest-order terms in this expansions yield two complex ordinary differential equations from which the four sought functions can be obtained. This approach was originally suggested by Wagner et al. [19], although that work was restricted only to the functions A(z) and a(z). A generalisation to include the phase functions was later suggested by Akhmanov et al. [20]. A problem with this approach is that it is *local*, i.e. it exaggerates the importance of the field closest to the pulse maximum, with a subsequent loss of accuracy for the *global* pulse dynamics.

The *moment method*, or *virial theory* [21, 22, 23] remedies this drawback by considering the evolution of moments of the transverse coordinate, e.g. the second moment (RMS width) $< r^2 >= \int r^2|F|^2 ds / \int |F|^2 ds$ where $\int ds$ denotes transverse integration. However, it is only possible in some special cases to find moments as an exact explicit function of z; an important example is the RMS-width in 2-d self-focusing. This method thus made it possible to demonstrate the limits of the paraxial-ray method, and the critical power for self-focusing was found to be off by a factor of four. The moment method has not yet been generalised to include a proper phase description.

Another global approach is the *variational method*, which was originally used by Tzoar et al. [24], and later by Anderson et al. [25]. The variational method utilizes a Lagrangian L(F(r,z)), chosen to make the corresponding variational functional stationary, $\delta \int L ds dz = 0$, where F solves the evolution equation. Thus, the variational derivative $\delta L / \delta F = 0$ must be equivalent with the evolution equation, and this defines the Lagrangian. Since the r-dependence has been specified in the trial function F(r,z), and thereby in L(F), we can carry out the transverse integration, $< L >= \int L ds$ and obtain a reduced variational problem $\delta \int < L > dz = 0$ involving only z as independent variable. The four unknown functions are derived from the Euler-Lagragne equations

$$\delta < L > / \delta \bullet = 0; \qquad\qquad (\bullet = A, a, b, \phi). \tag{4.11}$$

This method has the features of being global, yielding the same accuracy as the moment method, and allowing for the phase functions to be determined. The accuracy of this method can be tested in the case when the evolution equation is the exactly integrable NLS-equation. It is found that the variational approach preserves the first four of the invariants to the NLS equation [26].

One shortcoming of all these methods is that they cannot describe changes of pulse shape during propagation, or pulse-splitting phenomena. This may seem as a serious drawback, since in some cases parts of the initial pulse are dispersed during the inital stages of propagation and the remaining energy is trapped as a stationary localised nonlinear wave, a *solitary wave* [7, 27]. In fact, the problem of determining the solitary wave content of an arbitrary initial condition is unsolved, apart from the NLS-case which is integrable. The accuracy of the described variational method can be improved by introducing more parameters to be optimised, but this will obviously require more calculations. Anyway, for the study of pulse evolutions in nonintegrable systems, the variational method as presented above is the most accurate analytical tool known today. We will consequently employ it in the forthcoming analyses of nonlinear beam propagation.

4.4 Self-focusing in two transverse dimensions

4.4.1 Self-trapping, collapse and exact solutions

Self-focusing in two transverse dimensions is physically relevant for e.g. cylindrical CW beams, pulsed propagation in a slab waveguide or other cases with only two transverse dimensions. The self-focusing phenomenon was investigated rather extensively during the late sixties and early seventies, and we restrict ourselves to the most relevant works for this thesis. A deeper discussion is given in the reviews by Shen [29] and Marburger [30]. The governing equation in the 2-d case is

$$i\frac{\partial F}{\partial z} = \frac{1}{2k_0}\nabla_\perp^2 F + \frac{k_0 n_2}{n_0}|F|^2 F. \tag{4.12}$$

where we assume a bulk medium. The most commonly analyzed case is the propagation of a cylindrically symmetric beam, and it is then convenient to use the cylindrical coordinates (r, θ) for which

$$\nabla_\perp^2 = \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} = \frac{1}{r}\frac{\partial}{\partial r}(r\frac{\partial}{\partial r}) + \frac{1}{r^2}\frac{\partial^2}{\partial \theta^2}. \tag{4.13}$$

In bulk media, the *stationary* solution to Eq. (4.12) is found by the substitution $F(r, z) = R(r)\exp[-i\delta z]$, which yields the eigenvalue problem

$$\delta R(r) = \frac{1}{2k_0}\nabla_\perp^2 R(r) + \frac{k_0 n_2}{n_0}|R(r)|^2 R(r). \tag{4.14}$$

Note that the eigenvalue $\delta > 0$ because of the boundary condition $R(r) \to 0$ as $r \to \infty$. This eigenvalue problem was originally formulated and numerically solved by Chiao, Garmire and Townes [28]. Haus [31] pointed out that the problem has an infinite number of discrete modes, corresponding to several rings surrounding the central maximum. An interesting fact, formulated by Akhmanov et al. [32] is that the power contained in the N:th mode grows approximately like $2N^2 - 1$, for no apparent reason. Anderson et al. [33] showed by use of the variational method that the shape of the fundamental mode is close to sech, i.e. the fundamental soliton-shape. And indeed, although optical solitons do not exist for the 2-d NLS, self-focusing and solitons are very closely related phenomena.

If R is assumed to be real, it has been shown [22] that Eq. (4.14) has no elliptically shaped solutions, and it has been conjectured that only axisymmetric solutions exist. Recently it was proven that any fundamental nonlinear mode in 2 transverse dimensions must be circularly symmetric [34]. This does not rule out the possibility of having azimuthally dependent solutions, however. And in fact, allowing R to be complex, a set of non-axisymmetric exact analytical solutions to Eq. (4.14) have been found in terms of elliptical functions [35]. However, the physical relevance of these solutions, which depend on the cylindrical angle θ, are yet to be found. The angular dependence is $\sim \exp[i\theta/3]$ and thus the solution has discontinous phase jumps. Moreover, the power of these beams is not finite, i.e. $\int r|R|^2 dr$ diverges.

The dynamics of Eq. (4.12) was considered early by Kelly [36], and Talanov [37] who found that beams with a power above a certain critical power P_c would collapse within a certain distance, known as the *self-focusing distance*. For lower powers, diffraction is stronger than self-focusing, and lead to monotonic broadening of the pulse. Beams having exactly the power P_c are *self-trapped* and propagates without focusing or broadening. Note, however that the self-trapped state is unstable; a small deviation in power from P_c leads to monotonic collapse or diffraction. The critical power P_c varies for different initial beam shapes, but the lowest value [22] corresponds to the self-trapped shape calculated by Chiao et al. [28]. In our units, we can find this beam power to be

$$P = \frac{2.92 n_0}{k_0^2 N_2} \cong 1MW \tag{4.15}$$

where we have used the material parameters of Eq. (4.9).

The 2-d NLS equation is a special case in the sense that the radial and longitudinal dependences can be separated, and an exact, self-similar solution can be obtained for the whole beam dynamics. This was originally noted by Suydam [22] and Glass [38], and it follows from an additional symmetry property of the NLS-equation in 2-d that was found by Talanov [39]. The similarity solution is assumed to be of the power-conserving form

$$F(r,z) = f(\frac{r}{a(z)})a(z)^{-1}\exp[iS(\frac{r}{a(z)},z)] \equiv \frac{f(\rho)}{a(z)}\exp[iS(\rho,z)]. \tag{4.16}$$

If this is inserted in Eq. (4.12), we find the following equations for S and f:

$$S(\rho,z) = -\frac{k_0}{2}a(z)a'(z)\rho^2 + \phi(z) \tag{4.17}$$

$$\frac{1}{\rho}\frac{\partial}{\partial\rho}(\rho\frac{\partial f(\rho)}{\partial\rho}) - (\beta+\alpha\rho^2)f(\rho) + \frac{2k_0^2 n_2}{n_0}f(\rho)^3 = 0 \tag{4.18}$$

where

$$\alpha = k_0^2 a(z)^3 a''(z) \tag{4.19}$$

$$\beta = 2k_0 a(z)^2 \phi'(z). \tag{4.20}$$

The cofficients α and β must be independent of z if the separation ansatz is to be consistent. Assuming the initial condition to be that of a plane wave, i.e. $\phi(0) = a'(0) = 0$, the functions a(z) and $\phi(z)$ become

$$a(z) = a_0\sqrt{1+\alpha\frac{z^2}{a_0^4 k_0^2}} \tag{4.21}$$

$$\phi(z) = \frac{\beta a_0}{2\sqrt{\alpha}}\arctan(\frac{z\sqrt{\alpha}}{a_0^2 k_0}). \tag{4.22}$$

The localised solutions of the eigenvalue problem (4.18) relate the constants α, β to the central amplitude f(0). Moreover, physical (localised) solutions must have $\alpha \geq 0$ [22], which for $a'(0) = 0$ implies diffracting solutions only. However, this does not rule out collapsing solutions, because if $a(z) = a'(0)z + a(0)$ and $a'(o) < 0$, then $\alpha = 0$ and we can describe collapsing solutions in a self-similar manner. We will elaborate more on this in the context of blow-up instabilities in the next chapter. The above also tells us that the collapse of an initially plane wave with $a'(0) = 0$ cannot be described in a self-similar fashion. In other words, the beam will change shape during the collapse. It was also found in the early numerical computations, see e.g. [36], that the collapsing beam tends to peak more in the center than in the wings. This means that there can be at least two, possibly several, spatial scales that determine the collapse dynamics. Despite an immense theoretical effort [40], there is not yet a full understanding of the dynamics in the collapse governed by the NLS equation in two transverse dimensions. We will discuss this further in chapter 5.

Since the above is a scalar theory, it was pointed out early that vectorial effects may be important [41]. Self-trapped TE and TM-modes have also been numerically obtained [42], and they are characterised by their ring-shape, i.e. that the field is zero at r=0. We emphasize, however, that this is not in contradiction with the present scalar theory. Since we can view a self-trapped beam as a mode of its induced waveguide, we have in the weakly-guiding limit a consistent scalar theory [43], equivalent to the theory of weakly guided modes in fibers. During the collapse process, however, the induced waveguide becomes strong, and the scalar theory breaks down.

4.4.2 Self-focusing in bulk media (Paper A)

In **Paper A**, the variational method described earlier is used to find the dynamical evolution of an intense circular beam in a Kerr-medium. The method is found to provide very good quantitative agreement with numerical results. All the well-known results from self-focusing theory are reproduced, e.g. that collapse occurs above a critical power P_c and monotonic diffraction takes place below this power. However, previously unknown information about the longitudinal phase-shift was also obtained, and these analytical predictions have recently been verified numerically [44, 45]. The result is in contrast to another analytical approach, the paraxial-ray approximation, which is shown in paper A to predict erroneous sign of the longitudinal phase-delay. We find that the correct sign of the phase in presence of both diffraction and nonlinearity is the same as for the pure self-phase modulated plane wave, i.e. that given by Eq. (4.4). Consequently, the inclusion of diffraction leaves the phase dynamics qualitatively unaffected, as is physically expected.

Paper A does also consider the propagation of a pulse in a non-dispersive self-focusing medium, and a pulse compression effect is predicted which we call “compression by erosion”. The effect arises from the fact that the wings of the pulse have lower power than the center and diffract away more rapidly. Thus energy is removed, eroded, from the wings, leaving a sharp central peak. A similar effect was numerically predicted for the 1 dimensional self-focusing case; compare Figs. 3-4 of paper A with Figs. 4a-b of Ref. [46].

4.4.3 Self-focusing in waveguides, (Papers B and C)

When considering the effect of a nonlinearity in a linear waveguide, it is important to realize that the beam will be guided by *two* contributions to the refractive index profile; the linear guide and the nonlinearly induced guide. This means that the propagation constant, the modal area and the shape of a linear mode of the guide will be altered in the nonlinear regime. In optical fibers, the propagation constant is of great importance, since it determines the dispersion characteristics of the fiber. The problem of how the nonlinearity affects the dispersion of the fibers has been adressed by Okamoto et al. [47] and Sammut et al. [48]. It is found that significant modifications of the dispersion characteristics requires very high powers (~ 100 kW), so that the refractive index profile is severely altered. For low powers this is elegantly expressed in terms of the fiber parameter V. This parameter is well known as the normalised frequency of the fiber, and defined as $k_0 a(n_{co}^2 - n_{cl}^2)$, where a is the core radius. In the nonlinear regime, V simply generalises to [49, 50, 51]

$$V_{NL} = \frac{V_{lin}}{\sqrt{1 - P/P_c}}. \tag{4.23}$$

were P_c is the previously defined critical power for self-focusing. This illustrates the important fact that a weakly nonlinear guided wave has an equivalent linear wavegude mode. Anyway, for powers below 1 kW, $P/P_c < 0.001$ and the nonlinear contribution to the fiber dispersion is negligible.

The dynamic evolution of a CW-beam in a nonlinear parabolic-index waveguide is governed by the equation

$$i\frac{\partial F}{\partial z} = \frac{1}{2k_0}\frac{1}{r}\frac{\partial}{\partial r}(r\frac{\partial F}{\partial r}) - \frac{k_0 g r^2}{n_0}F + \frac{k_0 n_2}{n_0}|F|^2 F. \tag{4.24}$$

which is derived from Eq. (4.1) by using $f = -gr^2$, where g is the graded-index parameter. This model was originally considered by Bendow et al. [52]. In that paper equation (4.24) was solved approximately by using the paraxial-ray method. It was found that for low powers, a Gaussian beam stays in the waveguide, and oscillates during propagation. Above the self-focusing power, however, the beam collapses, similarly to the behaviour in bulk media. The collapse distance, however, is somewhat shorter in the waveguide. The longitudinal phase-shift was not considered in this work.

In **Paper B**, we solve the same problem with the variational method. We reproduce the above results more accurately, and we derive expressions for the phase behaviour of the beam and the critcal power for self-focusing. The phase is again found to be qualitatively different from the paraxial-ray predictions. We find, similarly to the bulk medium, that the phase-shift is positive for low powers and negative for high powers. It vanishes at a beam power of 2/3 the critical power, similarly to the result of the bulk media found in paper A.

The properties of beams in nonlinear fibers are further examined in **Paper C**. The paper utilizes the same variational method as earlier to solve Eq. (4.24), but with an important improvement. A super-Gaussian trial function is used to model the radial mode profile, i.e.

$$F(r, z) = A \exp\left(-\frac{1}{2}(\frac{r}{a})^{2m} + ibr^2 + i\phi\right) \tag{4.25}$$

The inclusion of the parameter m enables us to answer the question "How is the shape of the fundamental mode altered by the nonlinearity?". It is shown for stationary profiles that the super-Gaussian coefficient m only depends on the incoming power relative to the critical power for self-focusing. For dynamical propagation, that is when A, a, b, and ϕ are functions of z, we show that the equations for these functions receive only minor modifications in the coefficients. The qualitative behaviour is thus the same as in paper B. In the case of stationary propagation, A, a, b, and ϕ do not depend on z, and can be expressed as functions of m. Two limits are found for m; the linear limit with zero nonlinearity corresponds to the well-known Gaussian solution (m=1), and the strongly nonlinear limit with no waveguiding (g=0) which is approximated with m=ln(2). Any relative strengths between the waveguiding and nonlinear forces will correspond to a value of m between ln(2) and 1. By this method we have managed to get a description of how the shape of the stationary mode depends on the relative strength of the waveguiding and the nonlinearity, respectively.

It is noteworthy that the equation governing the stationary shape of a beam in a parabolic-index fiber, Eq. (4.24), is the same as the equation for the shape of the similarity solution for nonlinear self-focusing in bulk media derived above, Eq. (4.18). Thus, the different shapes of the stationary modes in a parabolic-index fiber are the same as the different shapes of the diffracting self-similar solutions of Eq. (4.18). This is a quite unexpected feature of paper C which was not recognized in time for its publication. For a more thorough discussion of this equivalence we refer to [53]. Recently the inverse relation was pointed out. That is, the propagation of a beam in a parabolic-index fiber was transformed to the propagation of a beam in a bulk Kerr-medium [54].

4.4.4 Self-focusing with a saturable nonlinearity (Paper D)

In the earlier analyses, we have shown that an optical beam propagating with sufficient power in a Kerr-medium will collapse catastrophically. The singular collapse point is unphysical, involving infinite electric field amplitude and zero width. Furthermore, the governing equation does not provide any information about the beam properties beyond the collapse distance. In order to reestablish the physical behaviour, i.e. to avoid infinite electric fields, we assume the induced refractive index to saturate. The refractive index is modelled as

$$n^2 = n_0^2 + \frac{n_0 n_2 |F|^2}{1 + \frac{n_0 n_2 |F|^2}{n_{sat}^2 - n_0^2}} \tag{4.26}$$

which means that for high amplitudes the index saturates at n_{sat}. We do not linger on the physical mechanism behind the saturation, but mereley note that it exists in several media [55, 56, 57]. From a physical point of view, we can qualitatively discuss beam dynamics in such a medium. Since the index saturates at n_{sat}, the beam cannot focus to zero width. Instead, there is a minimum width below which the diffraction "overtakes" the waveguide defined by n_{sat} and n_0, see [60]. The governing equation for the beam becomes

$$i\frac{\partial F}{\partial z} = \frac{1}{2k_0}\frac{1}{r}\frac{\partial}{\partial r}(r\frac{\partial F}{\partial r}) + \frac{k_0 n_2 |F|^2/n_0}{1 + n_0 n_2 |F|^2/(n_{sat}^2 - n_0^2)}F. \tag{4.27}$$

Other functional forms of the saturation mechanism should not alter the qualitative behaviour. This kind of evolution equation, was originally suggested by Wagner et al. [19]. The analytical analysis in that paper used the paraxial-ray method, but did not consider the phase dynamics of the solutions. Numerically, this equation was solved by Marburger et al. 1968 [58], and it was shown that the collapse process was indeed terminated due to the saturation. The moment method has also been applied to saturble media, and the existence of a minimum beam radius was shown [59]. The stationary mode profiles to Eq. (4.27) was solved numerically by Chen [60], and it was found that different degrees of saturation gave different self-trapped shapes. Those could be well fitted with in turn hyperbolic secant, Gaussian, and cosine-profiles for increasing degree of saturation [60].

In **Paper D** we adress these issues. The variational method is again our analytical tool. We start with an analysis of the self-trapped mode profiles via a super-Gaussian ansatz. This enables us to analytically explain the numerical results of Ref. [60]. We also explain the physical reason for the different mode profiles in the different regimes of saturation. The simplest way to explain this feature is to consider the different induced waveguides in which the beam is self-guided. Absence of saturation is equivalent to the common Kerr-self-trapping. The shape of this self-trapped mode is nearly hyperbolic secant, as shown by Anderson et al. [33]. A low degree of saturation will yield an approximate parabolic index waveguide which has a Gaussian eigenmode. A high degree of saturation corresponds to a step-index guide. The reason for the accurate cosine-fit in this case is that the eigenfunction for the step-index-guide is two joint Bessel-functions, which behaves like a cosine function near zero.

The above indicates that a Gaussian profile is a good approximation for the mode profile in a saturable medium. Therefore we use the Gaussian as trial function when examining the beam dynamics. In the limit of low powers it is found that the beam diffracts monotonically just like in a Kerr-medium (cf. paper A). For powers above the critical power for self-focusing, however, the behaviour is more like that of a beam in a Kerr-fiber, with oscillations of the width if the stationary conditions are not exactly fulfilled (cf. paper B). This is also found when Eq. (4.27) is solved numerically [61]. Considering the behaviour of the on-axis phase-shift we find differences as compared to the nonlinear fiber, however. The phase-shift of a beam in the Kerr-fiber makes a leap at the nodes, where the amplitude of the beam has a maximum (paper B, Fig. 2). In this nonlinearly induced waveguide however, the phase-shift decays slower at the nodes (paper D, Fig. 12a), i.e. the contrary behaviour. The reason for the difference is that a saturable guided beam behaves qualitatively like a linear beam in a fiber, and the beam dynamics in the strongly saturated limit can be shown to be the very near that of a beam in a step-index fiber[43].

The latter is indeed a well-known and widely studied problem.

The investigation of the phase of these beams may prove important in future devices based on the interaction between nonlinearly guided beams. Nonlinear beams interact via the induced refractive index, i.e. via the overlap of their tails. Like interacting solitons, they will repel (attract) each other if they are out of (in) phase [62, 63]. In this context it is crucial to know the phase of each beam, since it is the relative phase between the beams that determine the sign of the interaction force.

4.5 Self-focusing in three transverse dimensions

The propagation of a pulsed beam is governed by Eq. (4.1), where all second order derivatives are included. For cylindrical beams, we consider two different cases depending on the sign of k_0'', i.e. if we have normal or anomalous dispersion. In the case of anomalous dispersion, we can have complete symmetry between the temporal and spatial coordinates, and the governing equation becomes, in spherical symmetry:

$$\frac{i}{k_o}\frac{\partial F}{\partial z} = \frac{1}{2\rho^2}\frac{\partial}{\partial \rho}(\rho^2 \frac{\partial F}{\partial \rho}) + \frac{n_2}{n_0}|F|^2 F = 0 \tag{4.28}$$

where $\rho^2 = x^2 k_0^2 + y^2 k_0^2 - k_0 t^2 / k_0''$ is the normalised spherical radius. It was noted very early that this equation had stationary solutions [64, 65] of the form $F = R(\rho)\exp[-i\kappa z]$, where $R(\rho)$ is localised. Similarly to the case of the cylindrical self-trapped beam, these spherical solutions are unstable to perturbations of the power and will either broaden or collapse. A saturable nonlinearity was also early suggested to remedy the collapse [64, 65]. When investigating the stability of the stationary solutions in the saturable case, it was found that not all were stable, however. The stability criterion (see chapter 5) can be formulated as $\partial P/\partial \kappa > 0$ for stable propagation, where P is the integrated energy $P = \int R(\rho)\rho^2 d\rho$ [64].

In a nonlinear optics context, the equation (4.1) was suggested by Silberberg [4] in a bulk media, and he termed the spherical solutions *light-bullets*. The solutions were later examined by Desaix et al. [66] using the variational approach, and expressions for the collapse distance were given. The shape of the spherical light-bullet is in fact closer to Lorentzian than sech, and this is associated with the fact that the Lorentzian is an exact solution for the nonlinear self-trapping equation in four transverse dimensions [53, 67]. A way to avoid the collapse and observe light-bullets in experiments can be to use a pump-probe configuration, as suggested by Blagoeva et al. [68]. In this scheme, an intense pump beam provides the necessary refractive index modulation for the probe to be self-trapped in time and space.

In the case of normal dispersion, the relative sign between the dispersive and diffractive terms differs, and there is no space-time symmetry. Instead, the governing equation is of the form

$$\frac{i}{k_0}\frac{\partial F}{\partial z} = \frac{1}{2k_0^2}\frac{1}{r}\frac{\partial}{\partial r}(r\frac{\partial F}{\partial r}) - \frac{k_0''}{2k_0}\frac{\partial^2 F}{\partial t^2} + \frac{k_0 n_2}{n_0}|F|^2 F. \tag{4.29}$$

where k_0'' is positive. This equation has been investigated numerically by several authors [44, 69, 70]. One might suppose that the collapse in the radial direction should be counteracted by the broadening in the temporal direction. This is also observed in some parameter regimes [44]. It is possible to show that the RMS width of a field governed by Eq. (4.29) broadens monotonically in time during propagation [69]. However, this does not rule out the collapsing behaviour, and it is found that the collapse dynamics may be quite complicated. An initial pulse breaks up into subpulses, each of which collapses and compresses. A detailed understanding of this "fractal collapse" process has yet to be given.

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Chapter 5

Stability of nonlinear optical waves

A very important property to consider for nonlinear guided waves is the question of stability. A nonlinear beam may possess many attractive properties, but if it is unstable these may be of no practical use. There is a wide variety of instabilities that may occur in nonlinear optics, and in this chapter we restrict ourselves to the most important ones in the context of nonlinear guided beams and pulses. The interested reader is referred to e.g. Refs. [1, 2, 3] for a more general treatment. In the present context, we define instability as: *A stationary beam or pulse is unstable when a weak perturbation can grow to destroy its stationary intensity distribution.* This is not a rigorous mathematical definition, but nevertheless it will suit our purposes. The practical implications of this is that if we can perturb a beam in such a way that the perturbation grows to significantly modify the beam, it may be impossible to generate the beam in practice. We use "may", because the mathematical definition of stability can in some cases be too rigorous for practical applications; i.e. there are examples of beams which mathematically can be proven unstable, but nevertheless are found experimentally. There can be at least two reasons for such a result:
(i) The instability is not realized in the experiment, e.g. the kind of perturbation that triggers the instability is not excited in the experimental situation.
(ii) The instability is triggered, but its growth rate is too slow to be important in the length- and time scales of the experiment.

Another complication arises when numerical methods are used. There might be cases when a beam is theoretically stable, but when the numerical discretisation induces some kind of artificial instability. We will provide examples of this below. In general, however, "experimental stability" is more difficult to accomplish than both "theoretical" or "numerical" stability. The perturbations and combinations of perturbations which arise in experiments are often much more complex than the idealized situations which are considered in theory. However, it should be emphasized that optical fibers, being essentially one-dimensional, allow comparatively controlled experiments in nonlinear dynamics to be carried out, as compared to other areas of physics, e.g. fluid- or plasma physics. Furthermore, an instability may not be of a purely negative character. We will see below that there are applications where instabilities can be used as an advantage. In the following discussion, we divide the instabilities into different groups for simplicity. Firstly we discuss instabilities of guided beams in homogeneous- and inhomogeneous media, respectively. Then we go on to study modulational- and resonant instabilities. It should be emphasized that there is no well-defined borderline between the instabilities studied here. On a deeper physical level they can often be shown to be related. In fact, *Fermats principle*, which states that rays of light move towards regions of higher refractive index can be used to explain the physics behind many of the instabilities studied here. Moreover, we emphasize that the instabilities discussed in this chapter are not purely theoretical concepts, but have been verified experimentally.

5.1 Stability of self-guided beams in bulk media

A stable stationary state can be defined as the state a beam reaches asymptotically after long distance of propagation, or mathematically, in the limit $z \to \infty$. Slight perturbations of the initial conditions will not alter the fact that the beam approaches the stable state when propagating. Therefore, perturbations around such a state will not grow, and the beam is stable. The most familiar example of this kind of stability in nonlinear optics is the fundamental soliton, governed by e.g. equation (4.6)

$$\frac{i}{k_0}\frac{\partial F}{\partial z} = \frac{1}{2k_0^2}\frac{\partial^2 F}{\partial x^2} + \frac{n_2 \alpha}{n_0}|F|^2 F. \tag{5.1}$$

The soliton (and higher order solitons) emerges as asymptotic solution of Eq. (5.1) [4]. Small perturbations in the initial conditions will be radiated away during the initial stages of the propagation, and asymptotically for large z, a soliton will emerge. From the IST theory for solving this equation, it is possible to deduce directly from the initial condition what kind of solitons that will emerge, see chapter 6.

On the other hand, the 2-d self-trapping in a Kerr-medium, governed by the eigenvalue problem (4.14)

$$\delta R = \frac{1}{2k_0}\nabla_\perp^2 R + \frac{k_0 n_2}{n_0}|R|^2 R \tag{5.2}$$

(cf. paper A and ch. 4.4) is an example of a stationary state which is not asymptotically stable [5]. A beam cannot evolve into this self-trapped state, but the self-trapping has to be excited initially. Furthermore, a slight perturbation in the amplitude of the self-trapped beam will make it either broaden or collapse monotonically. This means that the 2-d self-trapping in Kerr-media is unstable against perturbations in beam power.

The collapse process inherent in the higher-dimensional NLS equations is interesting both from a mathematical and a physical point of view, and it has recieved great attention in the literature. In 2-d, the earliest studies appeared in the context of nonlinear optics [5], whereas collapse in 3-d was investigated for the first time in the context of Langmuir waves in plasmas [6]. It was recently recognized by Silberberg that the same equation applies to pulsed light beams in dispersive media, [7]. For a thorough treatment on the collapse process, we refer to the reviews by Berkshire and Gibbon [8] and by Rasmussen and Rypdal [9]. Below, we will review some of the most important properties of the collapse.

The governing equation for multidimensional nonlinear optical beams is given by e.g. Eq. (2.27)

$$i\frac{\partial F}{\partial z} = \frac{1}{2k_0}\nabla_\perp^2 F + \frac{k_0 n_2}{n_0}|F|^2 F \tag{5.3}$$

where the dimensionality of $\nabla_\perp^2$ can be D=1,2 or 3. It was shown in e.g. Ref. [9] that collapse can only occur for $D \geq 2$. In the special case $D = 2$ the moment method (virial theory) [6] yields the following equation for the RMS width $< r^2 >= \int r^2 |\psi|^2 r dr$

$$\frac{d^2 < r^2 >}{dz^2} = 2\int \frac{1}{k_0^2}|\nabla_\perp F|^2 - \frac{n_2}{n_0}|F|^4 r dr = 2H = constant \tag{5.4}$$

where we have defined the Hamiltonian H, which is independent of z [9, 10, 11]. Since collapse corresponds to $< r^2 >= 0$, it is in this particular case possible to deduce the collapse from the initial condition. Obviously, a sufficient condition for collapse is $H < 0$. In the case D=3, the right-hand side of Eq. (5.4) is not independent of z, and a similar criterion is not possible. Moreover, the $D = 2$-case has an additional symmetry, Talanov's "lens transformation" [12] which enables a self-similar description of the collapse. This solution was derived in chapter 4.4, eqs. (4.16-4.20), and from it we can construct an exact blow-up solution to eq. (5.3) as

$$F(r,z) = a(z)^{-1} R(\frac{r}{a(z)}) \exp[i(-\frac{a'(0)k_0 r^2}{2a(z)} - \frac{\delta}{2k_o a'(0)a(z)}] \tag{5.5}$$

$$a(z) = a(0) + a'(0)z \tag{5.6}$$

where R is the solution to the eigenvalue problem solved by Chiao et al. [13], i.e. Eq. (5.2). δ is the eigenvalue and $a(0) > 0, a'(0) < 0$ are real constants. It is worth emphasizing, however, that this self-similar solution only describes one particular blow-up scenario. We can physically interpret this scenario as the singularity caused by focusing the 2-d self-trapped beam to a point by using a thin lens. All other blow-up cases will lead to non-self-similar evolutions, characterized by a change of shape and strong peaking of the central parts of the beam. In particular, the blow-up in 3-d is always of this kind [14]. Using the variational method [15], approximate collapse criteria can be found for the 3-d case where the virial theory fails.

The collapsing singularity can be removed by allowing the refractive index to saturate. The self-trapped beams in such media will then be governed by an equation which we can write in normalised form as

$$\delta\psi = \frac{1}{2r^{(D-1)}}\frac{\partial}{\partial r}(r^{(D-1)}\frac{\partial \psi}{\partial r}) + \frac{|\psi|^2\psi}{1+|\psi|^2} \tag{5.7}$$

where the eigenvalue $\delta > 0$ is determined by the boundary condition $\psi \to 0$ as $r \to \infty$. This eigenvalue problem was investigated numerically by Vakhitov and Kolokolov 1973,

[16] and the dispersion diagram showing the beam energy $P(\delta) \equiv \int |\psi|^2 r^{(D-1)} dr$ vs. the eigenvalue δ was constructed for the fundamental self-trapped states. Quite surprisingly, a qualitative difference between the cylindrical (D=2) and spherical (D=3) beams was found. The spherical beam was found to have a local minimum at δ_0, i.e. $P'(\delta_0) = 0$, whereas in the cylindrical case $P'(\delta) > 0$ for all δ (see e.g. Paper D, Fig.4). Moreover, the stability of these fundamental self-guided modes was considered, and it was found that the modes were stable provided $P'(\delta) > 0$. This means that cylindrical self-trapped beams in a saturable media are stable for all $\delta > 0$, but spherical self-trapped beams are stable only for $\delta > \delta_0$. Later, Kolokolov generalised this, and showed that $P'(\delta) < 0$ is a sufficient condition for instability of the fundamental self-guided mode, independently of the nonlinearity. Physically, this is associated with Fermats principle, which states that light rays move towards regions with higher refractive index. Similar results were obtained more recently by Mitchell and Snyder [18], in which the stability criterion of the fundamental mode of an arbitrary nonlinear waveguide were formulated. Note, however, that there are cases when the stability cannot be deduced directly from the $P(\delta)$-plot. As an example, consider the fundamental mode of the D-dimensional beam in a Kerr medium. It was shown e.g. in Ref. [15] that

$$P(\delta) \sim \delta^{1-D/2} \tag{5.8}$$

from which we can deduce instability for D=3, and possible stability for D=1. In the cylindrically symmetrical case, D=2, P does not depend on δ and we cannot deduce instability from the $P(\delta)$-plot. However, we know by other means that the D=2-case is unstable (see paper A), and that the D=1-case is stable, since it corresponds to the NLS-soliton solution. In conclusion, we may note that the condition $P'(\delta) > 0$ is a necessary, but not sufficient stability criterion. It is obviously more difficult to prove stability than instability, since one has to prove stability to all possible perturbations. A discussion around the concept of stability should therefore be related to a certain class of perturbations.

5.2 Stability in nonlinear waveguides

Another way of causing stability of the diffracting-collapsing situation of Kerr-self-focusing in 2-d is to use a linear waveguide (see Paper B). This prevents the diffraction but not the collapse, so that beams are stably self-trapped below the critical power for self-focusing. Above the critical power they collapse, as in the bulk medium. In paper B, we derive an approximate solution for the width as function of z, and it is obvious that changes in power below P_c only causes oscillations of the width, and not monotonic diffraction/collapse.

A lot of research have been devoted to the stability of waves in nonlinear planar wave guides, with nonlinearities in either the core or the cladding or both. The same stability condition as above have been found, i.e. stability for $P'(\delta) > 0$ [19] - [23]. Mitchell et al. [18] have shown that this criterion is universally appliccable to all fundamental modes of nonlinear waveguides. A controversy over the stability of a beam in a waveguide surrounded by nonlinear material was recently settled in favour of the theory, and it was demonstrated that a previously observed instability was in fact a numerical artifact [24]. Concerning the more difficult problem of the stability of higher-order modes, no simple universal stability criterion has been obtained. It is most likely that this issue will have to be analyzed on a case-by-case basis [25].

5.3 Modulational instability

The modulation instability (MI) is a well known phenomenon occurring in nonlinear dispersive media. The phenomenon has been studied in such diverse areas as in plasma physics [26], fluid mechanics [27] and, as here, in nonlinear optics, were it was originally predicted by Bespalov and Talanov 1966 [28]. The concept arises from the fact that a periodic perturbation, a modulation, on a stationary wave envelope in a nonlinear material can exhibit exponential growth. It was not pointed out until the early eighties that the phenomenon could arise in nonlinear optical fibers as well, governed by the NLS equation [29, 30, 31]. Below, we will discuss MI separately in the contexts of beam propagation and pulse propagation in fibers, respectively.

5.3.1 Filamentation of beams, MI in several dimensions

Starting out from Eq. (2.27) in normalised form, assuming anomalous dispersion and space-time symmetry, we have for an arbitrary nonlinearity f,

$$i\frac{\partial \psi}{\partial z} = \frac{1}{2}\nabla_{\perp}^{2}\psi + f(|\psi|^{2})\psi \tag{5.9}$$

where we can allow the transverse dimensionality to be D=1,2 or 3. It is easy to see that

$$\psi_s = A\exp[-if(|A|^2)z], \tag{5.10}$$

with A being a constant amplitude, is a stationary solution to Eq. (5.9). It is no restriction to take A to be real. Perturbing this stationary solution, viz.

$$\psi = (A + \epsilon(x,y,t,z))\exp[-if(A^2)z] \tag{5.11}$$

where $|\epsilon(x,y,t,z)| \ll A$, and linearising in ϵ yields

$$i\frac{\partial \epsilon}{\partial z} = \frac{1}{2}\nabla_{\perp}^{2}\epsilon + A^2 f'(A^2)(\epsilon + \epsilon^*) \tag{5.12}$$

where we have expanded the nonlinearity around the background amplitude. Now the perturbation is assumed to be of the form

$$\epsilon(x,y,t,z) = (a+ib)g(x,y,t)\exp(\gamma z) \tag{5.13}$$

where a,b and γ are real constants, and $g(x,y,t)$ is an eigenfunction of the transverse Laplacian, i.e. $\nabla_{\perp}^{2} g = -\kappa^2 g$. Inserting Eq. (5.13) into (5.12), and eliminating a and b, we find that the growth rate γ of the perturbation is given by

$$\gamma(\kappa) = |\kappa|\sqrt{A^2 f'(A^2) - \frac{\kappa^2}{4}} \quad (5.14)$$

We see that γ is real and positive within two unstable sidebands, having "cut-off" at $\kappa = \pm 2A\sqrt{f'(A^2)}$. This implies that the eigenfunctions of the tranverse Laplacian constitute a class of unstable perturbations which grow exponentially. These eigenfunctions are all of oscillatory nature, and the instability is therefore called *modulational instability*. Noteworthy in the cases of cylindrical/spherical symmetry, is the existence of higher order eigenfunctions of the transverse Laplacian, which also lead to instability. These are periodic in the azimuthal direction, i.e. such perturbations break up the angular symmetry. This kind of angular instability was observed experimentally very early [32, 33] and identified as a probable cause for the filamentation of beams. In a Kerr-medium (f(x)=x), the term $f'(A^2)A^2$ is directly proportional to the intensity A^2, and the upper limit of MI will increase with the background intensity. In a defocusing medium, which has f(x)=-x, Eq. (5.14) has no real roots, and it is therefore stable against modulations. If a saturable medium is considered, e.g. f(x)=x/(1+x), we find that $A^2 f'(A^2) \to 0$ when $A \to \infty$, and MI is supressed at high background intensities [34, 35].

Two important facts should be emphasized at this point. Although it is not physically possible to have an infinitely wide background, MI can still be observed on a finite background, provided the background is wide enough for the unstable wavelengths. These are of the same order as the width of the fundamental self-trapped beam of the background amplitude. In other words, the tendency to create filaments out of a broad initial beam is a manifestation of MI. The second important point is that not only constant backgrounds, but also solutions that are broad enough in one direction are modulationally unstable in that direction. For instance, the NLS in 2-d have soliton-like solutions of the kind $sech(ax + by)$, where a and b are arbitrary constants. In the (x,y)-plane, these solutions are localised in the (a,b)-direction, but constant in the perpendicular (b,-a)-direction. They are therefore modulationally unstable along this direction. This fact was originally pointed out by Zakharov and Rubenchik 1974 [36]. Modulational instability may seem disastrous, but in fact it can be used as an advantage. For instance, a self-trapped CW-beam in a saturable medium will break up into pulses [34, 37], which makes MI a convenient light-bullet generator, or, using the imaginative title of Ref. [34], an "optical machine gun".

5.3.2 MI in a temporally defocusing medium (Paper E)

As seen above, MI is strongly dependent on the relative sign between the dispersion / diffraction and the nonlinearity. In a bulk medium with normal dispersion, the diffraction and dispersion act with different signs with respect to the nonlinearity. An arbitrary input beam would focus in space and defocus in time, and it is not obvious how MI will develop in such media. Simulations reveal that spatial collapse can still occur, although the background broadens in time [38, 39]. The governing equation for a pulse in a homogeneous Kerr-medium with normal (defocusing) dispersion is in normalised form

$$i\frac{\partial\psi}{\partial z} = \frac{1}{2}\nabla_{\perp}^2\psi - \frac{1}{2}\frac{\partial^2\psi}{\partial t^2} + |\psi|^2\psi. \tag{5.15}$$

The standard MI approach, as outlined above, can be applied to this equation. This was done by Liou et al. [40], and the growth rate was found to be given by

$$\gamma = \frac{1}{2}\sqrt{(k_x^2 + k_y^2 - \Omega^2)(4A^2 - k_x^2 - k_y^2 + \Omega^2)} \tag{5.16}$$

where k_x, k_y are the wavenumbers of the transverse modulations, Ω is the modulation frequency in time, and A is the background amplitude. However, this result assumes an infinite background. In fact, it can be shown that a localised background solution to Eq. (5.15), which is present in experiments, must always expand in time. A more exact treatment of MI in this kind of medium should therefore take this fact into account.

In **Paper E** we derive an approximate background solution to Eq. (5.15) which is assumed to be constant in the x- and y-directions on which the perturbations can grow. This background, which expands in time, has a self-similar character, i.e.

$$\psi_b(z,t) = \sqrt{I(z)\left(1 - (\frac{t}{\tau(z)})^2\right)}\exp[i\frac{\alpha(z)t^2}{2} + \int_0^z I(z')dz'] \tag{5.17}$$

where I, τ, and α can be expressed as functions of z. Asymptotically it is found that the intensity I(z) decays as z^{-1}, and as a first approach we consider perturbations in the transverse spatial directions when the background amplitude decays as z^{-1}. We find that exponential growth is possible in the initial stages of the propagation, when the background is still strong. After some distance, however, the background amplitude has decreased so that the perturbation modes are no longer unstable, but become oscillatory. Given a specific distance, it is also posible to find the wavenumber which corresponds to the strongest perturbation. Finally an analysis of modulations both in time and in space is carried out, and it is found that such a modulation also can exhibit exponential growth. This means that one can have break-up in time due to the spatial instability, despite the stability to pure temporal modulations. In other words, the spatial modulation triggers the temporal modulational instability, a result which agrees with numerical simulations, [38, 39].

5.3.3 MI in nonlinear optical fibers

In chapter 3, we derived the equation governing pulses in fibers as Eq. (3.29)

$$i\frac{\partial u(z,t)}{\partial z} = [\beta(\omega_0 - i\frac{\partial}{\partial t}) - \beta(\omega_0)]u(z,t) + \sigma u(z,t)|u(z,t)|^2 \tag{5.18}$$

where we have neglected the shock-term and the Raman response of the fiber. We will demonstrate how the theory for MI can be applied to a fiber with arbitrary dispersive properties, and we therefore keep the full expansion of the dispersion operator, i.e.

$$[\beta(\omega_0 - i\frac{\partial}{\partial t}) - \beta(\omega_0)] = \sum_{m=1}^{\infty} \frac{(-i\frac{\partial}{\partial t})^m}{m!}\beta_0^{(m)}. \tag{5.19}$$

Assuming a perturbed cw-field $u(t, z) = (A + \epsilon(z, t)) \exp(-iA^2 z)$, we can carry out the same derivation as above. Writing $\epsilon(z, t) = a\cos(kz - \Omega t) + ib\sin(kz - \Omega t)$, it is straightforward to obtain the dispersion relation $k(\Omega)$ for the perturbation as

$$\left(k - \sum_{n\ odd} \frac{(\Omega)^n}{n!}\beta_0^{(n)}\right)^2 = \left(2\sigma A^2 + \sum_{n\ even} \frac{(\Omega)^n}{n!}\beta_0^{(n)}\right) \sum_{n\ even} \frac{(\Omega)^n}{n!}\beta_0^{(n)} \tag{5.20}$$

in which the summation should be done for $n > 0$. The perturbation exhibits exponential growth if the perturbation wavenumber is complex, i.e. for the LHS of Eq. (5.20) negative. Thus, the condition for the existence of unstable frequencies Ω can be written as

$$-2\sigma A^2 < \sum_{n\ even} \frac{(\Omega)^n}{n!}\beta_0^{(n)} < 0. \tag{5.21}$$

Several important features can be deduced from Eqs. (5.20-5.21).

(i) A necessary (but not sufficient) condition for instability is that at least one of the even coefficients $\beta_0^{(2m)}$ is negative.

(ii) A sufficient (but not necessary) condition for instability is that the lowest order coefficient β_0'' is negative. The physical implication of this is that the background wave propagates in the regime of anomalous GVD.

(iii) The odd dispersive orders do not contribute to the onset of MI. Instead, the odd orders, which yield the real part of k, contributes to the phase velocity of the perturbation.

The simplest example of modulational instability described by the above formulas is when higher than second-order (group-velocity) dispersion contributions are negligible. We find

$$Im(k) = \sqrt{-(2\sigma A^2 + \beta_0''\frac{\Omega^2}{2})\beta_0''\frac{\Omega^2}{2}} \tag{5.22}$$

from which it is obvious that β_0'' must be negative for instability. In other words, an intense continous wave in the wavelength range of anomalus dispersion will be broken up into a train of soliton-like pulses. The cut-off frequencies for the unstable sidebands are given by $\Omega_{co} = \pm 2A\sqrt{-\sigma/\beta_0''}$, and the maximum growth rate is obtained at

$$\Omega_{max} = \pm A\sqrt{-2\sigma/\beta_0''}. \tag{5.23}$$

This modulation frequency corresponds to the duration of the fundamental soliton with amplitude A, and MI can therefore be seen as the tendency to create solitons out of a long input pulse. These spectral properties of MI were noted in the early theoretical studies [29, 30, 31]. Later on, higher-order dispersive effects were incorporated. In agreement with the above formulas, Vyshloukh et al. [41] showed that third-order dispersion does not affect the growth rate of the instability, and recently the effects of fourth-order dispersion were found to give rise to novel features of the MI, e.g. additional sidebands

[42, 43, 44]. The higher order nonlinear terms of the NLS equation in fibers have also been incorporated into the theory of MI. The self-steepening term was included by Shukla et al. [45]. Potasek [46] added the Raman term, and Blow et al. [47] considered the full model, including the entire Raman gain spectrum. This is important, since the Raman gain severely modifies the MI gain spectrum.

The above theory is based on linear stability analysis. When the perturbation starts to grow, the assumption of a small perturbation becomes invalid, and this theory breaks down. The modulations cannot grow indefinetely, however, since depletion of the background will limit the growth. It should therefore be valuable to have a picture of the full dynamics of MI. Numerically, this problem was treated by Yuen and Ferguson 1978 [48]. The dynamics of MI in the one-dimensional (integrable) NLS equation was demonstrated to yield *recurrence*. This means that the inital growth of the modulation depletes the background heavily, and then the reverse process takes place so that after a certain propagation distance the system is back to its initial state. A similar phenomenon was earlier observed by Fermi, Pasta and Ulam in the context of nonlinear lattice vibrations, and it is commonly known as *Fermi-Pasta-Ulam recurrence*. Further progress was made by Infeld 1981 [49], who analyzed the dynamic growth of two discrete sidebands surrounding the carrier background. Analytical expressions for the oscillating (along z) amplitudes in terms of elliptical functions can be obtained in this way. A similar method was recently applied to optical fibers by Trillo et al. [50]. It has also been shown that two different kinds of periodic evolutionary patterns exist for MI, which are separated by a homoclinic orbit [51]. This orbit coincides with an exact aperiodic analytical solution found by Akhmediev et al. [52]. In fact, it has been shown that the inverse scattering transform, which is valid only for localised input pulses, can be generalised to periodic initial conditions [53]. Therefore, a wide class of exact analytical solutions for the MI problem of the NLS equation exists [52, 54]. Recurrence phenomena have also been observed in the 2-d NLS equations [55, 56], but then with an important difference: The system does not return *exactly* to the initial state, but to a point near this in phase space. This phenomenon is termed quasi-, or pseudo-recurrence. Pseudo-recurrence can occur only in non-integrable systems [53], and it is associated with the trajectories of strange-attractors in nonlinear dynamical systems. It may be added that MI can occur also in coupled systems of NLS equations, where it leads to break-up of continous waves of different frequencies [57] or polarisation states [58] in nonlinear fibers. Those instabilities can arise also in the regime of normal dispersion. Since coupled NLS equations are non-integrable in general, we may only expect pseudo-recurrence for those instabilities.

There are several possible applications of the modulational instability. It was recognized early that MI in fibers could be utilized as a means of generating high-repetition rate pulse-trains [30]. This was later experimentally verified in experiments by Tai et al. [59, 60] in which repetition-rates of 0.3 THz were reported. The modulational instability have also been utilized in switching [61], and lasing [62] configurations.

5.4 Resonant instabilities

In this section we restrict the treatment to perturbations of soliton pulses in optical fibers. However, the criterion for instability (phase matching) that we use is of much more general validity, and we will show how it can be used to explain the modulational instability as well. For pulse propagation in an optical fiber, we derived in chapter (3) the nonlinear Schrödinger (NLS) equation

$$i\frac{\partial u(z,t)}{\partial z} = \frac{1}{2}|\beta_0''|u(z,t) + \sigma u(z,t)|u(z,t)|^2 + \hat{P}u \tag{5.24}$$

were we have assumed anomalous dispersion ($\beta_0'' < 0$) and left room for an arbitrary perturbation operator $\hat{P}$, which can be linear or nonlinear. When $\hat{P}$ can be neglected, the NLS equation has the stable soliton solution

$$u_s(z,t) = \sqrt{\frac{|\beta_0''|}{\sigma t_0^2}} sech(\frac{t}{t_0}) \exp[-i\frac{|\beta_0''|}{2t_0^2}z] \equiv \sqrt{\frac{|\beta_0''|}{\sigma t_0^2}} sech(\frac{t}{t_0}) \exp[-ik_{sol}z] \tag{5.25}$$

where the pulse duration t_0 can be chosen arbitrarily, and the nonlinear propagation constant k_{sol} is positive due to self-phase modulation. The stability of the soliton solution was established very early [63], and can be viewed as being due to the fact that the soliton exist in a regime in k-space that is forbidden for linear waves. Weak linear wave solutions $\sim \exp[i\Omega t - k_{lin}z]$ of Eq.(5.24) obey the dispersion relation

$$k_{lin}(\Omega) = -|\beta_0''|\frac{\Omega^2}{2} \tag{5.26}$$

so that $k_{lin} < 0$. It is well-known in the theory of wave interactions, that if energy is to be transferred between two waves, they should be *phase-matched*, i.e. have the same propagation constant k. In a quantum mechanical description, this corresponds to *momentum conservation* in the four-photon mixing process. Obviously, if k_{lin} and k_{sol} have different signs, then solitons and linear waves cannot be phase-matched, and the soliton cannot transfer energy to the linear waves. The soliton will therefore remain as a nonlinearly trapped wave-packet. However, certain perturbations $\hat{P}$ can open up a way to phase-match the soliton with the linear waves, and the result will be that the soliton loses energy by radiation. This is a resonant process in the sense that there is one (or several) particularly unstable frequencies that will drain energy from the soliton, and we therefore term these processes *resonant instabilities*. We will give two particularly important examples of perturbations giving rise to resonant instabilities, namely higher-order dispersion and periodic amplification.

5.4.1 Instabilities due to higher-order dispersion

Higher-order corrections to the dispersion operator can be of two kinds, depending on the carrier wavelength and the dispersive properties of the fiber. If the carrier wavelength is close to the zero-dispersion wavelength λ_0, then third-order dispersion (3OD) dominates, and this is modelled in the NLS equation (5.24) with

$$\hat{P}^{3OD} = i\frac{\beta_0'''}{6}\frac{\partial^3}{\partial t^3}, \tag{5.27}$$

cf. Eq. (3.34). In most fibers $\beta_0''' > 0$. However it is possible to draw fibers in which the group velocity dispersion has a minimum, i.e. $\beta_0''' = 0$, at a certain frequency, and around that frequency the next higher-order dispersion contribution will be the fourth-order dispersion (4OD). In the NLS equation (5.24) this corresponds to

$$\hat{P}^{4OD} = \frac{\beta_0''''}{24}\frac{\partial^4}{\partial t^4}. \tag{5.28}$$

Obviously, the linear dispersion relation will be modified by the inclusion of higher order dispersion effects, so that

$$k_{lin}^{3OD}(\Omega) = -|\beta_0''|\frac{\Omega^2}{2} + \frac{\beta_0'''}{6}\Omega^3 \qquad k_{lin}^{4OD}(\Omega) = -|\beta_0''|\frac{\Omega^2}{2} + \frac{\beta_0''''}{24}\Omega^4. \tag{5.29}$$

The inclusion of (3OD) thus yields a linear propagation constant k_{lin} that can take on either sign. Following the above discussion, the frequency Ω_{uns} that has the same propagation constant as the soliton, $k_{sol} = k_{lin}(\Omega_{uns})$, will be *unstable* which means that linear waves are generated at this frequency. Since the group-velocity $v_g = d\omega/dk$ at the unstable frequency Ω_{uns} in general differs from the group velocity of the soliton, energy will be lost from the soliton by this radiation. To lowest order in the small dimensionless parameters $\beta_0'''/|\beta_0''|$ and $\beta_0''''/|\beta_0''|$, the unstable frequency is given by

$$\Omega_{uns} \approx \frac{3|\beta_0''|}{\beta_0'''} \qquad \Omega_{uns} \approx \pm\sqrt{\frac{12|\beta_0''|}{\beta_0''''}}. \tag{5.30}$$

From these expressions we see that an unstable frequency always exists in the case of 3OD. For 4OD, $\beta_0'''' > 0$ is required for instability, and in that case two frequencies, symmetrically placed around the carrier, are unstable. The physical meaning of the requirement $\beta_0'''' > 0$ is that two zero-dispersion frequencies exist around the carrier, so that the soliton is situated in a spectral "well" of anomalous dispersion. We term this "positive" 4OD. This, and "negative" 4OD are discussed in more detail in the next chapter and in papers G-I of this thesis.

We can also estimate the amplitude of this radiation in a very simple manner. Since the radiation is generated by the soliton at the frequency Ω_{uns}, it will obviously be proportional to the spectral amplitude of the soliton at this frequency. The Fourier transform of the soliton (5.25) is

$$\tilde{u}_s(z,\omega) = \pi\sqrt{\frac{|\beta_0''|}{\sigma}}sech(\frac{\pi t_0\omega}{2})\exp[-ik_{sol}z]. \tag{5.31}$$

Note that the spectral amplitude of the soliton is independent of the arbitrary duration t_0. The radiation amplitude is now proportional to $|\tilde{u}_s(z,\Omega_{uns})|$ which is

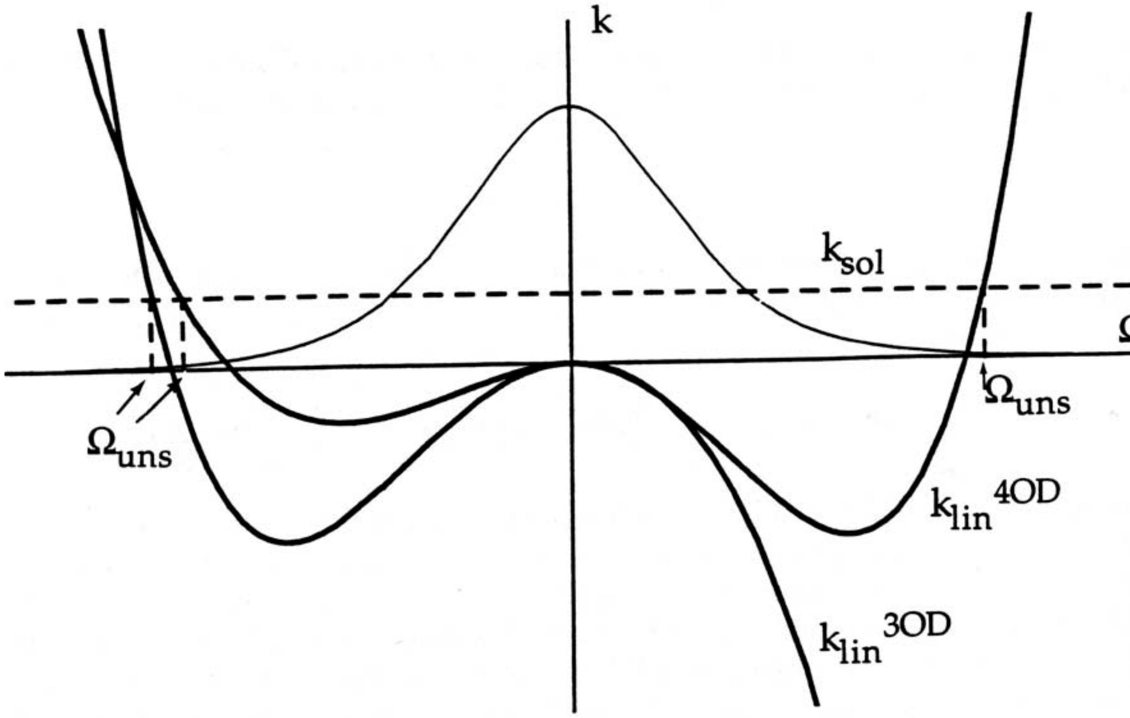

Figure 5.1: *The unstable frequency is defined by the requirement that the soliton wavenumber and the linear wavenumber are equal. This is often approximated with $k_{lin} = 0$. The soliton spectrum is included as the thin line for convenience.*

$$|\tilde{u}_s(z, \Omega_{uns})| \approx 2\pi\sqrt{\frac{|\beta_0''|}{\sigma}} \exp[-\frac{3\pi|\beta_0''|t_0}{2\beta_0'''}] \tag{5.32}$$

and we see that the radiation amplitude is exponentially small when $\epsilon = \beta_0'''/|\beta_0''|t_0$ is small, i.e. when the main part of the soliton spectrum lies in the anomalous-dispersion regime. The expression for 4OD is similar to this.

The fact that solitons perturbed by 3OD will radiate at a discrete frequency was demonstrated 1986 in numerical simulations by Wai et al. [64]. They identified the resonant frequency as given by Eq. (5.30), but did not identify the correct resonance condition, $k_{sol} = k_{lin}$ which in fact makes the unstable frequency weakly dependent on the soliton width t_0, see Fig. (5.1) and Eq. (5.25). This condition was identified by Kuehl et al. [65] and recently by Elgin [66]. The radiation due to 4OD was originally demonstrated in **paper G** of this thesis, and we will discuss this in more detail in the next chapter.

We found form Eq. (5.32) that the radiation amplitude is proportional to $\exp[-1/\epsilon]$, where $\epsilon \sim \beta_0'''/|\beta_0''|t_0$ is a small dimensionless parameter. This function cannot be approximated with a power series in ϵ, and this fact makes calculations of the radiation amplitude using conventional perturbation theories difficult. Quite cumbersome perturbation methods that develop the perturbation series "beyond all orders" have been applied to this problem [65, 67], but recently somewhat simpler approaches have been suggested [68, 69]. A further complication in the 3OD-case is the *spectral recoil* effect. This follows from the invariance of the spectral center-of-mass (momentum) of the perturbed NLS equation. Thus, if an unstable sideband grows in the normal-dispersion region, the soliton will be pushed, "recoiled", further into the anomalous region, i.e. away from the unstable frequency. This decreases the radiation amplitude and stabilizes the soliton. A soliton generated too close the zero-dispersion wavelength λ_0 will therefore stabilize itself through this radiation process. A physically interesting fact is that this linear radiation is equivalent with Cherenkov radiation, since the phase velocity of the soliton exceeds the phase velocity of the linear waves [69]. The important problem of what kind of soliton that will emerge from an arbitrary pulse launched close to λ_0 is yet unsolved, although important results were obtained in simulations by Wai et al. [64, 70]. Experimental verification of the above theoretical predictions for 3OD has been given by a number of authors, see e.g. Gouveia-Neto et al. [71].

5.4.2 Instabilities due to periodic amplification

When soliton pulses are used in practical applications for long-distance telecommunications, the effects of loss cannot be neglected. Therefore, the optical pulses must be amplified periodically along the distance of propagation. The first numerical investigations of periodically amplified solitons was done in the mid-eighties by Hasegawa [72] and Mollenauer et al. [73]. It was found that the stability of the soliton pulses was strongly dependent on the amplification period. In particular, if the amplification period is of the same order as the inverse soliton wavenumber, the soliton could lose power due to resonance in a manner similar to that described above. We will consider two types of amplification that are commonly applied: *distributed amplification*, and amplification by means of *lumped amplifiers*. The former applies to distributed Erbium-doped fibers, or Raman-pumped fibers. In such systems, the signal power varies approximately sinusoidally through each amplification period. As a simple model, we take

$$\hat{P}_{dist} = i\cos(k_{amp}z) = \frac{i}{2}(\exp(ik_{amp}z) + \exp(-ik_{amp}z)) \tag{5.33}$$

in Eq. (5.24). The amplification wavenumber is $k_{amp} = 2\pi/L$ where L is the amplifier spacing. In a system using lumped amplifiers, we can approximate the gain with a train of delta functions, spaced the distance L along the fiber, so that

$$\hat{P}_{lump} = -i\gamma + i\exp(\gamma L)\sum_{n=1}^{\infty}\delta(z - nL) = -i\gamma + i\exp(\gamma L)\sum_{n=-\infty}^{\infty}\exp(ink_{amp}z) \tag{5.34}$$

where γ is the linear loss of the fiber, and the factor $\exp(\gamma L)$ is necessary to give zero net gain over each period. The above models of gain and loss in a communication system obviously constitute periodic perturbations in z, with the characteristic wavenumbers $\pm k_{amp}$ for distributed gain, and nk_{amp} for lumped amplifiers. Those wavenumbers can phase-match the soliton to linear waves via the phase-matching condition $k_{sol} \pm k_{amp} = k_{lin}$ for distributed gain, and $k_{sol} + nk_{amp} = k_{lin}$ for the lumped amplifiers. Using k_{lin} from equation (5.26) we find the unstable frequencies as

$$\Omega_{uns}^{dist} = \pm t_0^{-1}\sqrt{8\frac{z_0}{L} - 1} \tag{5.35}$$

$$\Omega_{uns}^{lump} = \pm t_0^{-1}\sqrt{8n\frac{z_0}{L} - 1} \tag{5.36}$$

where we have introduced the soliton period $z_0 = \frac{\pi t_0^2}{2|\beta_0''|}$. These expressions for the unstable frequencies are defined only when the square roots are real. A measure of the radiation is obtained from the soliton's spectral value at these frequencies, i.e.

$$u_r^{dist} \sim sech(\frac{\pi}{2}\sqrt{8\frac{z_0}{L} - 1}) \tag{5.37}$$

$$u_r^{lump} \sim \sum_{8n\frac{z_0}{L}>1} sech(\frac{\pi}{2}\sqrt{8n\frac{z_0}{L} - 1}) \tag{5.38}$$

These expressions reproduce the qualitative features of the numerical calculations [74], i.e. that the radiation grows indefinitely with L/z_0 for lumped amplifiers, but decreases for large L/z_0 in the case of distributed gain. The reason for this is that lumped amplifiers excite more unstable sidebands than distributed amplifiers. There is a strong resonance at $L = 8z_0$, since most of the spectral energy of the soliton lies at $\Omega = 0$. We also observe that for small values of L/z_0, the unstable sidebands are far away from the soliton spectrum, and soliton propagation when the amplifier spacing is much less than the soliton period is therefore stable. In fact, it has been shown by Hasegawa and Kodama [75] (see also Ref. [76]) that if $L \ll z_0$ the loss can virtually be neglected and the unperturbed, renormalised NLS equation is a valid desription for pulse propagation in fibers. It seems quite clear that a proper system has to operate in the region $L/z_0 \ll 8$, although there has been some controversy over the optimun choice of z_0 in a properly designed system [77, 76]. An interesting possibility that, to my knowledge, has not been investigated is *aperiodically* spaced amplifiers. This would undoubtedly decrease the strength of the resonances, but it is unknown to what degree. The useful picture with the phase matching condition $k_{sol} + k_{amp} = k_{lin}$ was originally suggested by Gordon [78], and the unstable frequencies given by Eq. (5.36) was derived by Kelly [79] and later by Elgin [66]. In the case of distributed gain, Kaup [80] derived a perturbation- inverse scattering scheme which enabled an analytical expression of the numerical features of Ref. [73].

5.4.3 Interpretation of the modulational instability as a resonant process

Finally, we wish to point out the fact that the modulational instability can be explained in terms of the above resonance condition. MI is the interaction between an intense pump wave with amplitude A, which has the wavenumber $k_{pump} = \sigma A^2$ and a weak probe at a frequency Ω_{uns}. The dispersion relation for the probe is then

$$k_{probe} = -|\beta_0''|\frac{\Omega^2}{2} + 2\sigma A^2 \tag{5.39}$$

where the factor $2\sigma A^2$ comes from the cross-phase modulation between the pump and the probe. From the resonance condition $k_{pump} = k_{probe}(\Omega_{uns})$ we find

$$\Omega_{uns} = \pm A\sqrt{2\sigma/|\beta_0''|} \tag{5.40}$$

which is equivalent to the expression (5.23) for the maximum-growth-rate frequency. This means that MI can be viewed as a three-wave mixing (3WM) process that is phase-matched by the self-phase modulation. This is particularly useful when analyzing the dynamics and recurrence of MI, which can be done from the coupled 3WM equations [81]. From this theory several important features of MI can be related to well-known features of 3WM, e.g. parametric amplification or energy conversion between the harmonics and the pump [82]. Another interesting application, suggested by Garth and Pask [83] was to use the frequency dependence of the phase-matching condition to determine the dispersive properties of the fiber, e.g. the zero-dispersion wavelength. Recently this teory was extended to include the nonlinear coupling between the polarisation states of the fiber [84].

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Chapter 6

Nonlinear pulse propagation in optical fibers

6.1 Introduction - bright and dark soliton pulses

In chapter 3, we derived the equation governing the slowly varying envelope u(z,t) ($W^{1/2}$) of a pulse in an optical fiber as Eq. (3.29), which can be approximated as

$$
\begin{aligned}
i\frac{\partial u}{\partial z} &= -\frac{\beta_0''}{2}\frac{\partial^2 u}{\partial t^2} + \sigma|u|^2 u \\
&+i\frac{\beta_0'''}{6}\frac{\partial^3 u}{\partial t^3} + \frac{\beta_0''''}{24}\frac{\partial^4 u}{\partial t^4} - \sigma\tau_R u\frac{\partial(|u|^2)}{\partial t} + i\gamma u - \frac{i\sigma}{\omega_0}\frac{\partial(|u|^2 u)}{\partial t}
\end{aligned}
\tag{6.1}
$$

where t is the retarded time, z the axial distance, ω_0 the carrier frequency and σ, τ_R are positive constants that were defined in chapter 3. We have expanded the dispersion operator to fourth order, since one important part of this thesis is to consider the effect of higher (i.e. third and fourth) order dispersion. The coefficient γ has been incorporated to account for linear loss ($\gamma < 0$) or amplification ($\gamma > 0$). The loss in a fiber is typically $\sim 0.2 - 0.3(dB/km)$ for the wavelengths of interest here, which corresponds to $\gamma \cong 2 - 3 \times 10^{-5}(m^{-1})$. For fiber lengths less than 1 km this can be ignored. The terms on the first line of Eq. (6.1) constitutes the *Nonlinear Schrödinger (NLS) equation*, and was derived for fibers originally by Hasegawa and Tappert 1973 [1, 2]. The NLS equation is a universal nonlinear propagation equation in the sense that it arises in many different fields of physics, and it has the for nonlinear partial differential equations unusual and nice feature that it is *integrable*. This means that its initial-value problem can be solved exactly. The only restriction is that the initial condition must be *localised*, i.e. $\int_{-\infty}^{+\infty} |u(0,t)|^2 dt$ must exist. In this chapter, we will discuss pulse propagation governed by Eq. (6.1) which is a very good model for pulse propagation in silica fibers.

We will divide the discussion into two separate cases, depending on the sign of the group-velocity dispersion β_0''. These cases are commonly denoted normal (anomalous) dispersion and corresponds to $\beta_0'' > 0$ ($\beta_0'' < 0$). In standard fibers, there is a *zero-dispersion wavelength* $\lambda_0 \cong 1.3\mu m$, at which $\beta_0'' = 0$. Below (above) this carrier wavelength lies the normal (anomalous) dispersion regime.

It is convenient to work with the NLS equation in normalized form, and we introduce the "soliton normalizations" [3]

$$q = u\sqrt{\frac{\sigma}{L_d}} \qquad \tau = \frac{t}{t_0} \qquad \xi = \frac{z}{L_d} \equiv \frac{z|\beta_0''|}{t_0^2} \tag{6.2}$$

where t_0 is the input pulse width and L_d the dispersive length. In the case of anomalous dispersion, the NLS equation in normalized units becomes

$$i\frac{\partial q}{\partial \xi} = \frac{1}{2}\frac{\partial^2 q}{\partial \tau^2} + |q|^2 q. \tag{6.3}$$

We can expect this equation to have stationary pulse-shaped solutions, because the induced refractive index is higher over the pulse centre, thus yielding a temporal "waveguiding" effect, (cf. spatial solitons) that counteracts the dispersive broadeing. The mathematical theory for the solution of this equation, which demonstrates the *inverse scattering transform (IST)* for the NLS, was given 1972 in an important paper by Zakharov and Shabat [4]. They found that a crucial role is played by the *soliton* solution to eq. (6.3):

$$q(\xi, \tau) = A\, sech(A(\tau - V\xi)) \exp[-iV\tau + i\frac{(V^2 - A^2)\xi}{2}], \tag{6.4}$$

where A is an arbitrary soliton amplitude, and V is an arbitrary frequency shift. Note that the soliton moves with the "velocity" V in the retarded reference frame due to this shift. This reflects the fact that the NLS equation is invariant under the Galileian transformation. The IST reveals that all localized solutions to the NLS equation consist of solitons and dispersive radiation. Thus, asymptotically as $z \to \infty$, the solution consists of a discrete number of solitons only. Early numerical simulations [1] also demonstrated the stable-attractor properties of the soliton-solution. The theory also shows that if N solitons are present in the total field we have an "N-soliton" solution. This can be either N well-separated soliton pulses, or if the pulses are clumped together, an oscillating N-soliton structure, a *breather*. The oscillations of a higher-order soliton can be seen as the beating between separate solitons with different wavenumber. Solitons are therefore solutions that can be *superposed*. In particular, solitons can collide and emerge unaffected after the collision in spite of the fact that they are goverened by an equation for which the linear superposition principle is not valid! This is only possible in the limited number of integrable equations, e.g. the NLS-, KdV- or Sine-Gordon equations.

In the case of normal dispersion the normalized governing equation becomes

$$i\frac{\partial q(\xi, \tau)}{\partial \xi} = -\frac{1}{2}\frac{\partial^2 q}{\partial \tau^2} + |q|^2 q. \tag{6.5}$$

Here, the nonlinearity boosts the dispersive broadening of a pulse. However, if we consider *dark pulses*, i.e. an intensity dip on a constant background level, we realize that the nonlinearity can be viewed as a defocusing "guide" which counteracts the dispersive broadening. Such dark pulse solutions of Eq. (6.5) are known as *dark solitons*, in contrast to the bright soliton (6.4) and were obtained for the first time by Zakharov and Shabat [5], and independently by Hasegawa and Tappert [2]:

$$q(\xi, \tau) = (iV + A \tanh(A(\tau - V\xi))) \exp[-i(A^2 + V^2)\xi] \tag{6.6}$$

where A is the amplitude and V the velocity of the dark soliton. The dark solitons have similar stability properties as the bright ones, but they can be difficult to generate experimentally because of the necessary phase modulation. Bright solitons in fibers were observed for the first time by Mollenauer et al. 1980 [6], whereas dark solitons were experimentally verified later, by several groups 1987-88 [7]. Dark solitons can be of two kinds, "grey" or "black" depending on whether the dark minimum reaches zero or not. We emphasize that dark solitons could prove a choice as information carriers in fibers, since their mutual interaction is less than that of the bright solitons [8], and they can therefore be more densely packed. However, since dark solitons is not a main topic of this thesis, we will not discuss this any further. There are several fundamental differences between bright and dark solitons, and the interested reader is referred to the recent reviews by Weiner [9] and Kivshar [10].

The nonlinear Schrödinger equations described above have several families of exact solutions apart from the solitons, see e.g. [11, 12, 13]. However, the solitons are the only exact *localized* solutions, and in this work we restrict ourselves to bright-pulse propagation in optical fibers. Because of the qualitative difference between the normal- and anomalous-dispersion regimes, we study those cases separately below. The most important applications that we will consider are optical pulse compression in the normal dispersion regime, and optical soliton communication systems in the anomalous dispersion regime.

6.2 Pulse propagation in normal dispersion

6.2.1 Pulse dynamics - the wave breaking phenomenon

Pulse propagation in the anomalous dispersion regime is characterized by temporal and spectral broadening, due to the self-phase modulation effect [14]. For fibers, the first simulation and experiment of bright-pulse propagation in the normal dispersion regime was presented by Nakatsuka et al. [15] 1981. This experiment, together with that of Mollenauer et al. [6] for anomalous dispersion, demonstrated that the NLS equation is a very good model for pulse propagation in low-loss fibers. Further work by e.g. Nelson et al. [16] also proved excellent agreement between experiment and numerical results. The theory for nonlinear pulse propagation in normally dispersive fibers is strongly connected with the development of fiber-optical pulse compressors. It was demonstrated by Nakatsuka, Grischkowsky and Balant 1981 [15, 17] that the spectrally broadened pulses from a normal-dispersion fiber could be used in compression schemes. This will be discussed further below.

The simulations [15]-[18] of the pulse propagation showed that the evolution of an initial pulse of the form

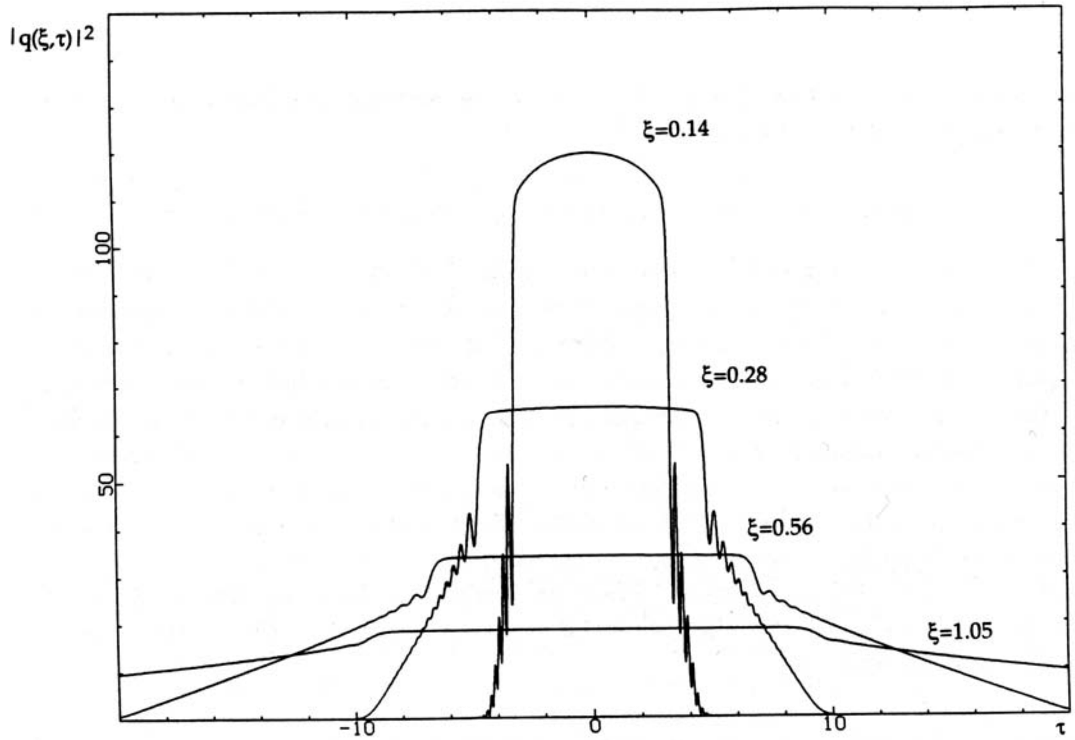

Figure 6.1: *Simulation of Eq. (6.5) with* $q(0,\tau) = 20\, sech(\tau)$.

$$q(\xi = 0, \tau) = A\, sech(\tau) \tag{6.7}$$

in Eq. (6.5), could be divided into two stages. The first stage is dominated by SPM, and results in reshaping and broadening of the pulse into a roughly square-shaped pulse with an almost linear chirp. During this stage the spectrum is broadened. The second stage is essentially a linear dispersive spreading of the pulse, since the amplitude has decreased and the nonlinear effects are weak. This dispersion-dominated stage has an almost constant evolution of the spectral intesity, which indicates the linear character of this stage of propagation. Asymptotically, for large ξ, the ringings gradually disappear, and the temporal intensity distribution approaches a trapezium, see Fig. (6.1).

The transition between these different stages is characterized by strong oscillations in the pulse wings and the development of spectral sidebands, a phenomenon called *optical wave breaking* [18, 19], because of the similarities with the breaking of water waves. The wave breaking arises in the transition region between the initial SPM-dominated and the later dispersion dominated regions of evolution, and we will define the "wave breaking distance" as the point along ξ where the oscillations first appear, i.e. where the pulse envelope becomes nonmonotonic. The wave breaking phenomenon was experimentally verified by Rothenberg and Grischkowsky 1989 [20, 21]. An analytical theory for wave breaking was given recently by Anderson et al. [22], in which expressions for the wave breaking distance for different pulse shapes were found. This theory also pointed out that different pulse shapes may have different wave breaking distances. It is noteworthy that the early simulations [15]-[18] only considered sech-shaped pulses as inital conditions in Eq. (6.5). The shape of the input pulse is in fact an important parameter that may qualitatively alter the wave breaking dynamics [21]. It is shown in Fig (6.2) that the wave breaking is less severe for Gaussian pulses, and almost absent for parabolic pulses. The latter is the main point of paper F, discussed below. Another analytical approach to wave breaking was recently suggested by Shvartsburg [23], and is based on changing the dependent and independent variables of the wave breaking equations, thus obtaining a set of linear equations which can be solved by standard methods. However, there are some difficulties associated with formulating the initial conditions using this approach [24].

There is a rather limited number of analytical investigations of nonlinear pulse broadening for normal dispersion, mainly due to the difficulties of modeling the reshaping of the pulse. Meinel [25] used the inverse scattering theory and the invariants of the NLS equation to relate the initial pulse to the width and chirp of the square pulse at the wavebreaking distance. Another approach is to use the variational method, originally suggested for pulse propagation in the anomalous regime [26]. In this method the pulse width and chirp are modelled as

$$A(\xi) f(\frac{\tau}{a(\xi)}) \exp[i\tau^2 b(\xi) + i\phi(\xi)] \tag{6.8}$$

where $f(\rho)$ is an appropriate shape, e.g. a Gaussian- or sech-function, and $a(\xi)$ and $b(\xi)$ are the width and chirp, respectively, of the pulse. These functions can be calculated in a straightforward manner [26]. Obviously, this method does not take into account the change of shape that takes place. Nevertheless it provides a good overall description of how the width and chirp evolves along ξ. An exact way to obtain some information of the asymptotic behaviour is to consider the invariants of the NLS equation. For Eq. (6.5), the third invariant is (cf. next section)

$$I_3 = \int_{-\infty}^{+\infty} |\frac{\partial q}{\partial \tau}|^2 + |q|^4 d\tau = \int_{-\infty}^{+\infty} \omega^2 |\tilde{q}(\omega)|^2 d\omega + \int_{-\infty}^{+\infty} |q|^4 d\tau. \tag{6.9}$$

From this invariant we find the asymptotic spectral broadening of the input pulse $q(0,\tau) = Asech(\tau)$ as

$$\frac{\Delta\omega^2(\xi \to \infty)}{\Delta\omega^2(\xi = 0)} \equiv \frac{\int_{-\infty}^{+\infty} \omega^2 |\tilde{q}(\xi \to \infty, \omega)|^2 d\omega}{\int_{-\infty}^{+\infty} \omega^2 |\tilde{q}(\xi = 0, \omega)|^2 d\omega} = 1 + 2A^2 \tag{6.10}$$

were we have used the fact that the integral over $|q|^4$ vanishes as $\xi \to \infty$, see [27]. The result of Eq. (6.10) gives an estimation of the achievable compression ratio in an optimum compressor, since the temporal duration of the transform-limited pulse is approximately inversely proportional to its spectral width.

6.2.2 Wave-breaking-free pulses (paper F)

The theory of wavebreaking of Ref. [22] suggested that form-invariant, i.e. *wave-breaking-free* pulses might exist. They could be found by the requirement that the chirp $\partial(\arg q)/\partial\tau$ should be constant across the pulse. If not, then different parts of the pulse can overtake other parts, with pulse deformation as a result. In **paper F** we consider the equation

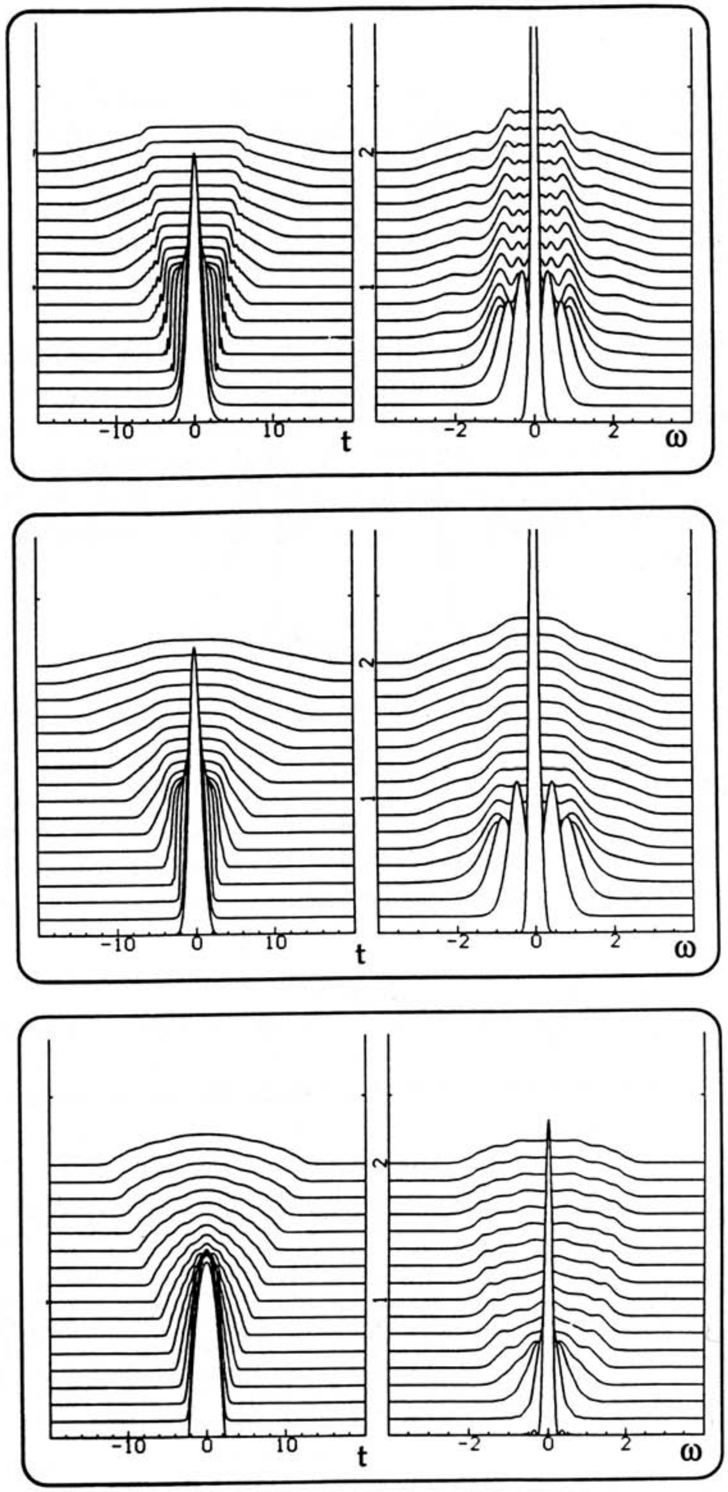

Figure 6.2: *Evolution of (from top) sech, Gaussian and parabolic pulses showing that the wave breaking oscillations are much smaller for the Gaussian, and almost absent for the parabola. The inital amplitude correspond to A=5.*

$$i\frac{\partial q}{\partial \xi} = \alpha\frac{\partial^2 q}{\partial \tau^2} + \sigma|q|^2 q. \tag{6.11}$$

We show that if the chirp of the pulse shall be constant, then $\partial^2(\arg q)/\partial\tau^2 = 0$, and the initial pulse modulus $A(t) = |q(0,t)|$ should be governed by

$$\alpha\frac{1}{A^2}\frac{\partial^2 A}{\partial \tau^2} + \sigma A^2 = c_0 + c_1\tau^2 \tag{6.12}$$

where c_0, c_1 are arbitrary constants and we have assumed the initial pulse to be chirp-free. From equation (6.12) we can recover several well-known limits. In the linear limit, $\sigma = 0$, the wavebreaking-free pulse is the Gaussian. Similarly, in the case of anomalous/normal dispersion we find the wave-breaking-free pulses to be the bright/dark solitons. The limit we then consider is the strongly nonlinear limit for normal dispersion. Then, we can neglect the term proportional to α in eq. (6.12) and find

$$\sigma A^2 = c_0 + c_1\tau^2 \tag{6.13}$$

which implies that a parabolic pulse is approximately wave-breaking free. We also manage to analytically describe the evolution of an initially parabolic pulse, and this description is shown to agree very well with numerical results. Finally, we point out that a useful application for such wave-breaking-free pulses should be in optical fiber-grating compressors, which would obtain much better compression properties using pulses of an invariant shape.

6.2.3 Pulse compression

As stated above, the analyses of nonlinear pulse propagation in normal dispersion are closely related to the theory for optical pulse compressors. There are many important applications for ultrashort ($< 1ps$) optical pulses, e.g. in spectroscopy and ultrafast measurements, and it has therefore been a strong incentive in the development of optical pulse compressors. The most common optical compression scheme is the *fiber-grating compressor*, which consists of two parts. Firstly, the pulses are spectrally broadened by propagation through a nonlinear medium with normal dispersion, and secondly, the acquired linear chirp is removed by propagation through an anomalously dispersive delay line. This method have been used in chirp radar systems since the early sixties, and it was suggested for optics by Fisher et al. 1969 [28]. Numerical simulations of nonlinear propagation in CS_2 were carried out rather early [29], and shock formation and strong asymmetry of the pulse was observed. The nonlinearity in CS_2 has a rather long response time ($\sim 2ps$), which give rise to this asymmetry. It was realized in 1981 that optical fibers in the normal dispersion region would be a useful medium, since it does not have the delayed nonlinear response of e.g. CS_2 [15]. For the second stage, a linear delay line, anomalous dispersion is required, and it is not obvious how this could be accomplished in a simple manner. It was shown by Treacy 1969 [30], that a grating pair provides anomalous dispersion, and his expressions for the delay function were later generalized to include higher-order dispersive effects [31]. Other, proposed schemes for the delay line include Gires-Tournois interferometers [32] and prism pairs [33]. In fact, it has been generally shown [34] that all types of refraction, irrespective of whether it occurs in gratings or prisms, will give rise to anomalous dispersion. This effect is somtimes called “angular dispersion”, since the refraction angle is frequency dependent. In the earliest experiments, however, grating pairs were used, and the compression scheme is therefore commonly denoted “fiber-grating compression”.

Following the experiment of Nakatsuka et al. [15], several other groups presented experimental results, which in a competition-like manner demonstrated shorter and shorter femtosecond pulse generation [35], down to 8 fs [36]. At the same time, Tomlinson et al. [37] published a comprehensive theoretical analysis on fiber-grating compression, based on numerical simulations. This showed that an optimum fiber length exists, which produces the shortest pulse for a given initial pulse. An extension to initially chirped pulses showed only quantitative differences from these results [38]. For longer fibers, the onset of wavebreaking degrades the performance of the compressor, and we point out in **paper F** that the optimum fiber length is nearly twice the wavebreaking distance, indicating that the pulse can take a certain amount of wavebreaking before the compression starts to deteriorate. An important limiting factor for fiber-grating compressors was identified by Tomlinson and Knox [39] to be the third-order dispersion (3OD) in the delay line. Indeed, in an experiment utilizing a delay line with no 3OD, compressed pulses as short as 6 fs were obtained [40]. Such short pulses comprise only three optical cycles, and this is still the record for short optical pulses.

It is also important to discuss the higher-order effects of the fiber when studying the propagation of high bandwidth-pulses. For instance, already a weak amount of third order dispersion may create asymmetric pulses [41], similar to those observed in experiments [21]. A particular feature of the experimentally obtained spectra after propagating pulses through fibers is the spectral asymmetry [36]. One reason for this is that experiments use wavelength scales, which asymmetrizes a symmetric frequency plot [39, 42]. This make significant differences when studying high-bandwidth pulses. This alone cannot explain the amount of asymmetry observed, however. It was suggested that third-order dispersion together with the nonlinear shock term could explain the asymmetry [42]. However, the experimental values of 3OD is not large enough to account for the observed pulse shapes [42]. Asymmetry of the initial pulse has also been suggested to account for the spectral asymmetry [21, 43, 44]. A more likely explanation, however, is the Raman effect, which was neglected in Refs. [42, 43]. The Raman contribution to the nonlinearity is important to include when studying high-bandwidth pulses like this [21, 45]. Models including 3OD and the Raman nonlinearity have been found to give excellent agreement with experiments [45, 46]. Noise have been incorporated in some simulations, but it does not seem to qualitatively influence the compression properties [47].

Finally, we shall for completeness also mention the “soliton-compression” scheme [48] that have been proposed for optical pulse compression. This method utilizes an optical fiber in the anomalous dispersion regime, in which higher-order solitons (see below) are generated. These solitons are very peaked at a certain point in their propagation cycle. However, short pulses generated by this method suffer from being placed on a broad pedestal, which means that the compressed pulse is not of very good quality. For further discussions around soliton-, and fiber-grating compression schemes see e.g. the book by Agrawal [49], and references therein.

6.3 Pulse propagation in the anomalous dispersion regime

6.3.1 Properties of the nonlinear Schrödinger equation

In the anomalous dispersion regime, the nonlinearity and the dispersion counteract each other. For pulse durations down to ~ 1 ps the evolution equation for the pulse envelope is to a good description given by the NLS equation (6.3). Due to the integrability of this equation, we can obtain directly from the inital condition the properties of the solitons that will emerge asymptotically. This is done from a linear scattering problem, the so called *Zakharov-Shabat eigenvalue problem* [4]:

$$i\frac{\partial \psi_1}{\partial \tau} + q(\xi,\tau)\psi_2 = \zeta\psi_1 \tag{6.14}$$

$$i\frac{\partial \psi_2}{\partial \tau} + q^*(\xi,\tau)\psi_1 = -\zeta\psi_2. \tag{6.15}$$

where $\psi_{1,2}$ are complex functions, the so called *Jost functions*, and ζ is a complex eigenvalue. Zakharov and Shabat showed that the N eigenvalues ζ of this eigenvalue problem remain constant if $q(\xi,\tau)$ evolves according to the NLS equation. Furthermore, the real and imaginary parts of each eigenvalue correspond to the amplitude and the velocity of a soliton (6.4) with $A = 2Im(\zeta)$ and $V = 2Re(\zeta)$. The phase and the absolute position in time of the soliton, as well as the dispersive (non-soliton) part of q are also obtainable from the data of the scattering problem, but we disregard this for the moment. The important implication of the IST is that we can already from the initial condition $q(0,\tau)$, derive the eigenvalues from Eqs. (6.14,6.15) and from these conclude what kind of soliton will emerge for large values of ξ.

The Zakharov-Shabat eigenvalue problem can be solved exactly in some simple cases, which was done by Satsuma and Yajima 1974 [50]. For the initial condition $q(0,\tau) = A\,sech(\tau)$, the emerging soliton is an N:th order soliton, where $A = N + \epsilon$, N is an integer and $|\epsilon| < 1/2$. In the case the initial pulse is given by $(1+\epsilon)\,sech(\tau)$, the emerging soliton is

$$q(\xi,\tau) = (1+2\epsilon)\,sech((1+2\epsilon)\tau)\exp[-i\xi\frac{(1+2\epsilon)^2}{2}] \tag{6.16}$$

For an arbitrarily shaped real initial pulse, it was shown by Kivshar [51] that the condition for soliton creation is that

$$\int_{-\infty}^{+\infty} q(0,\tau)d\tau \geq \pi/2. \tag{6.17}$$

which means that there is a critical area for soliton creation, and not a critical power level. Therefore, solitons having all power levels $|q(0,0)|^2$, and energies $\int |q|^2 d\tau$ exist. For a given pulse duration and shape, however, the condition for creation gives the necessary power to obtain a soliton, and this can be viewed as a critical power level.

The eigenvalue problem has also been investigated numerically, and analytically using the WKB-method, to yield the soliton content of initially chirped pulses [52, 53]. These are initial pulses of the form $q(0,\tau) = A\ sech(\tau)\exp[ib\tau^2]$. The soliton content is found to decrease with increasing chirp b, and above a critical value of b, no soliton is created. A similar result is found when phase noise is added to the inital conditions [54]. Recently a variational approach to the solution of the Z-S scattering problem was suggested [55], which demonstrated reasonable agreement with numerical results for e.g. chirped initial pulses.

It follows from its integrability [4] that the NLS equation has an infinite hierarchy of invariants, i.e. quantities that are independent of ξ. The first three invariants read

$$E = \int_{-\infty}^{+\infty} |q|^2 d\tau \tag{6.18}$$

$$M = \int_{-\infty}^{+\infty} (q\frac{\partial q^*}{\partial \tau} - q^*\frac{\partial q}{\partial \tau})d\tau \tag{6.19}$$

$$H = \int_{\infty}^{+\infty} (|\frac{\partial q}{\partial \tau}|^2 - |q|^4)d\tau \tag{6.20}$$

which can be physically interpreted as the invariance of energy, momentum, and Hamiltoninan. Note that the momentum conservation is the conservation of the spectral "center-of-mass", i.e. $M = -2i\int \omega|\tilde{q}|^2 d\omega$. Below, we will investigate soliton dynamics in a convenient way by using the invariants, and in particular it is valuable to see how the invariants are affected by additional terms to the NLS equation. Thus, if we consider an equation of the form

$$i\frac{\partial q(\xi,\tau)}{\partial \xi} = \frac{1}{2}\frac{\partial^2 q}{\partial \tau^2} + |q|^2 q + \hat{P} \tag{6.21}$$

where $\hat{P}$ is an arbitrary additional term of the NLS equation, the above invariants can be written

$$\frac{dE}{d\xi} = i\int_{-\infty}^{+\infty} (\hat{P}^* q - \hat{P} q^*)d\tau \tag{6.22}$$

$$\frac{dM}{d\xi} = -i2\int_{-\infty}^{+\infty} (\hat{P}^*\frac{\partial q}{\partial \tau} + \hat{P}\frac{\partial q^*}{\partial \tau})d\tau \tag{6.23}$$

$$\frac{dH}{d\xi} = i\int_{-\infty}^{+\infty} \frac{\partial q}{\partial \tau}\frac{\partial \hat{P}^*}{\partial \tau} - \frac{\partial q^*}{\partial \tau}\frac{\partial \hat{P}}{\partial \tau} - |q|^2(\hat{P}^* q - \hat{P} q^*)d\tau. \tag{6.24}$$

Before leaving this section, we will give a brief discussion of the experimental parameters of optical solitons. Optical solitons became an experimental reality after the pioneering experiment by Mollenauer et al. [6], which also proved the validity of the NLS equation as a model for pulse propagation. This was further strengthened in a later experiment which verified the periodic evolution of the higher-order solitons [56]. From the soliton solution of Eq. (6.21) with $\hat{P} = 0$, and the transformation of Eq. (6.2), we can write the fundamental soliton of duration t_0 as

$$u_{sol}(z,t) = \sqrt{\frac{|\beta_0''|}{\sigma t_0^2}}\, sech(\frac{t}{t_0}) \exp[-izk_{sol}]. \tag{6.25}$$

where the soliton wavenumber $k_{sol} = |\beta_0''|/2t_0^2 = (2L_d)^{-1}$. It was shown in the early works [50, 56] that the higher-order solitons oscillate with the *soliton period* z_0, which is commonly used in the literature as a measure of the characteristic nonlinear length. The soliton period is related to the soliton wavenumber via $4k_{sol} = \pi z_0$ so that the fundamental soliton "wavelength" $\lambda_{sol} \equiv 2\pi/k_{sol} = 8z_0$. From Eq. (6.25) we can express the soliton peak power P *(W)* and energy E *(J)* as

$$P = |u_{sol}(z,0)|^2 = \frac{|\beta_0''|}{\sigma t_0^2} = \frac{|\beta_0''| A_{eff} c}{N_2 \omega_0 t_0^2} \tag{6.26}$$

$$E = 2Pt_0 = \frac{2|\beta_0''|}{\sigma t_0} = \frac{2|\beta_0''| A_{eff} c}{N_2 \omega_0 t_0}. \tag{6.27}$$

Typical parameters of a dispersion shifted fiber at the carrier wavelength $\lambda_0 = 1.55(\mu m)$ are

$$\begin{aligned} N_2 = 3.2 \times 10^{-8}(W^{-1}\mu m^2) \qquad & A_{eff} = 20(\mu m^2) \\ \beta_0'' = -1.1(ps^2 km^{-1}) \qquad & \omega_0 = \frac{2\pi c}{\lambda_0} = 1.2 \times 10^{15}(rad\ s^{-1}) \end{aligned} \tag{6.28}$$

so that typical soliton parameters become

$$P = 0.12(W) \quad E = 0.24(pJ) \quad z_0 = 1.6(km) \tag{6.29}$$

which shows that optical solitons are by no means an unrealistic concept.

In the optical soliton research of today, there is a strong consensus of the validity of the NLS equation in fibers, and the main research efforts are aimed at investigations of higher-order effects on the solitons. Below, we will review the effects of a few of the most important perturbations of solitons in fibers. We will use the normalized units $q(\xi, \tau)$ instead of the physical units $u(z,t)$, thus avoiding obscuringly complicated expressions. We will also briefly review the properties and limitations of optical soliton-based communication systems.

6.3.2 Solitons in presence of amplification and loss

The most important property that has been neglected in the derivation of the NLS equation for optical pulses is the effect of loss. However, since the invention of the low-loss fiber [57], loss is significant only in very long fibers, of the order of kilometers. This means that for devices using short fibers, e.g. fiber amplifiers, fiber switches, optical loop mirrors etc., the effects of fiber loss can often be neglected. In the most common application, however, when fibers are used for long-distance transmission, loss is always a limiting factor. Any long-distance transmission using fibers is operated at a point where losses are significant. Therefore, the effect of loss on solitons is an important issue.

Mathematically, the effect of linear loss (or gain) is modelled in the normalized NLS equation as

$$i\frac{\partial q(\xi,\tau)}{\partial \xi} = \frac{1}{2}\frac{\partial^2 q}{\partial \tau^2} + |q|^2 q + i\Gamma q \tag{6.30}$$

where the normalized loss coefficient $\Gamma = \gamma L_d$ is negative (or positive). This implies that the energy of the pulse is no longer invariant, and from Eq. (6.22) we find

$$\frac{dE}{d\xi} = 2\Gamma E. \tag{6.31}$$

We will first restrict this discussion to loss, i.e. Γ negative. Assuming that the loss is weak, we introduce the fundamental soliton $q_{sol} = A\, sech(A\tau)\exp[-i\xi A^2/2]$ in equation (6.31), and we find that the amplitude and width are adiabatically modified according to

$$A(\xi) = A_0 \exp[2\Gamma\xi]. \tag{6.32}$$

where A_0 is the initial amplitude. The total longitudinal phase-shift ψ_{tot} of the perturbed soliton is the accumulated value of every instantaneous $k_{sol}(\xi) = A^2/2$, which means that according to WKB theory we can write

$$\psi_{tot} = \int_0^{\xi} \frac{A(\xi')^2}{2} d\xi' = \frac{A_0^2}{8\Gamma}(\exp[4\Gamma\xi] - 1) \tag{6.33}$$

and the total adiabatically evolving lossy "soliton" becomes

$$q(\xi,\tau) = A_0 \exp[2\Gamma\xi]\, sech(A_0 \exp[2\Gamma\xi]\tau) \exp[-i\psi_{tot}]. \tag{6.34}$$

This is not a soliton in the rigorous theoretical meaning of the concept, but it is a very good approximation of an adiabatically evolving soliton in presence of loss. This model was devised by e.g. Lamb [58] and Hasegawa et al. [59], using a perturbative method based on the IST-technique. Later Blow et al. [54] suggested the simpler WKB-approach used above. There is one obvious problem with this model, however. It is clear that the width of the pulse increases exponentially, in other words for large ξ increase *faster* than linear dispersive broadening. Moreover, the amplitude decays as $\sim \exp[2\Gamma\xi]$, whereas the amplitude of linear pulses decays slower, as $\sim \exp[\Gamma\xi]$. Yet we have a nonlinearity that should counteract the linear spreading. This apparent paradox is resolved by realizing that the above model is only valid for sufficiently small values of ξ, for which the "exponential" spreading is in fact slower than the "linear" spreading. For longer distances, the "adiabatic soliton" (6.34) is no longer a valid approximation. Numerical investigations show [60] that asymptotically, the pulse width broadens linearly, with a spreading rate lower than that of a linearly spreading pulse. This rate is called the "asymptotic dispersion" of nonlinear pulses [60, 61], and its value depends on the normalized loss coefficient Γ. The dependence can be found approximately [61] using the variational approach [26, 61] to describe nonlinear pulse propagation. In particular, using this method the equation governing the normalized pulse width $y(\xi)$ is found as [61]

$$\frac{d^2 y}{d\xi^2} = \frac{1}{y^3} - \frac{f \exp[2\Gamma\xi]}{y^2} \tag{6.35}$$

where f is a positive constant depending on the initial amplitude. In fact, this equation describes very well the pulse dynamics in presence of either loss or amplification. Obviously, for $\Gamma < 0$ (loss) the influence of the term $\sim y^{-2}$ is small for high ξ, and we have approximately linear broadening. In the other case, when $\Gamma > 0$ the same term will dominate and cause compression at a rate $\sim \exp[\Gamma\xi]$ accompanied by oscillations of the pulse width [62]. These oscillations are connected with the oscillations of the higher-order solitons. It is noteworthy that only a slight amount of radiation is found in simulations of this kind of soliton amplification [62], provided Γ is small. It was also observed in this work that there is a critical value of the gain coefficient $\Gamma \sim 0.6$, below which the first-order soliton absorbs nearly all energy, and above which higher-order solitons are created. For instance, using $\Gamma = 0.2$ the fundamental soliton consists of 95% of the total energy, whereas the rest is lost as radiation. Adiabatic amplification with a weak gain could therefore be a simple and efficient means by which to compress optical solitons.

In soliton communication systems the effect of fiber loss must be compensated by periodically spaced amplifiers. As was shown in the previous chapter on instabilities, solitons will radiate if the amplification period L is close to, or higher than the soliton "wavelength" $2\pi/k_{sol} = 8z_0$ [63]. However, in the limit $L \ll z_0$ it is possible to rigorously prove that the governing equation for pulses in fibers is the exact, lossless, renormalised NLS equation [64, 65] and consequently, that optical solitons in such systems are very stable entities. On the other hand, if the amplification period is close to $8z_0$, the soliton will emit dispersive radiation, thereby increasing its width and its soliton period z_0 to a point where $L \ll z_0$, and stable propagation can occur. This is again an example of the remarkable kind of self-stabilization that solitons may undertake, and it resembles the spectral-recoil scenario in presence of third-order dispersion (see the discussion around the "spectral-recoil effect" in chapter 5.4.1).

6.3.3 Solitons in presence of higher-order dispersion

The dispersion in optical fibers has two main contributions in *material* and *waveguide* dispersion. Both contributions can be modified in the fiber drawing process, although the waveguide dispersion is more easy to modify by changing the index profile of the fiber. For instance, the zero-dispersion wavelength can be pushed to higher wavelengths by modifying the index profile as is done in so called *dispersion-shifted* fibers [66]. It is also possible to manufacture fibers in which the two contributions nearly cancel each other over a wide spectral range [67]. For the present treatment, there is no point in separating these different contributions, and we will use a total dispersion relation for the fundamental fiber mode which contains both the material and the waveguide contributions. Note, however, that for high pulse powers ($\geq 10kW$) in the fiber, the waveguide dispersion will change due to self-focusing effects, (see chapter 4.4.3). This is also the origin of the difficulties associated with the function S (see Eq. (3.23)) in the derivation of the NLS-equation. A proper description of femtosecond solitons which can have peak powers above kW must take this effect into account. However, to my knowledge no theoretical investigation including the nonlinear contribution to the modal dispersion has been done, although some approaches have been suggested [68].

The significance of third-order dispersion (3OD) depends on the amount of soliton energy that lies in the normal-dispersion regime, which in practice depends on the pulse width and the carrier wavelength. The effects from 3OD are particularly important near the zero-dispersion wavelength, and the governing equation is then

$$i\frac{\partial q(\xi,\tau)}{\partial \xi} = \frac{1}{2}\frac{\partial^2 q}{\partial \tau^2} + |q|^2 q + i\epsilon\frac{\partial^3 q}{\partial \tau^3}. \tag{6.36}$$

where $\epsilon = \beta_0'''/|\beta_0''|t_0$. It is straightforward to show that the energy and momentum invariants are still conserved with this additional term, and that the invariant Hamiltonian can be rewritten as

$$H = \int_{\infty}^{+\infty} |\frac{\partial q}{\partial \tau}|^2 - |q|^4 + i2\epsilon q\frac{\partial^3 q^*}{\partial \tau^3} d\tau \tag{6.37}$$

so that equation (6.36) has at least three conserved quantities. Using perturbation theory, it is possible to find the lowest-order (in ϵ) corrections to the soliton $q = A\, sech(A\tau)\exp[-i\xi A^2/2]$ as [59, 69]

$$q_{sol}(\xi,\tau) \cong A\, sech(Ay)\exp[-i\frac{A^2}{2}\xi + i\epsilon(2A^2\tau - 3\tanh(Ay))] \tag{6.38}$$

where $y = (\tau + \epsilon A^2\xi)$. Thus, the first-order corrections only affect the phase and the velocity of the soliton, leaving its amplitude, width, and shape unperturbed. Furthermore, it has been shown numerically [70] that the condition for stationary pulses to exist in presence of 3OD is $A\epsilon < 0.04$, which in physical terms means that the soliton should lie mainly in the anomalous dispersion regime. The perturbation result (6.38) suggests that the soliton acquires a velocity-shift, and consequently that it is spectrally shifted away from the zero-GVD-frequency to a frequency with a new group-velocity. These features were also found in the earliest numerical calculation of soliton dynamics near the zero-dispersion frequency [71]. However, this picture cannot be complete. Only a spectral shift of the soliton would counteract the conservation of the spectral center-of-mass, in particular as the perturbed soliton does not change its energy to lowest order in ϵ. This paradox was resolved in the numerical simulations by Wai et al. 1986 [72], in which the solitons were shown to emit radiation. The physics behind this emission process was described in chapter 5.4.1. The radiation, being an unstable sideband in the normal-dispersion regime, will exactly counterbalance the spectral shift of the soliton, so that the spectral center-of-mass is indeed conserved. The problem remains why the solitons energy is not affected to lowest order in ϵ - it obviously emits radiation. This is due to the fact that the amplitude of the radiation is exponentially small, and cannot be found by conventional perturbation methods. Instead, cumbersome mathematical perturbation methods "beyond all orders" have been applied [69, 73] to find the radiation amplitude. If a pulse is launched very close to the zero-dispersion frequency, the scenario outlined above is more pronounced. The pulse emits radiation and recoils into the anomalous dispersion regime, where it eventually forms a soliton. The limit case of launching a pulse *at* the zero-dispersion frequency gives rise to a splitting of the pulse into two parts - one dispersive part in the normal dispersion regime, and one soliton-like part in the anomalous dispersion regime [70, 74]. This is found also for randomly modulated initial pulses [75]. For a given fiber length, it is possible to optimize the input pulse width and carrier wavelength in order to obtain the minimum output pulse width [76, 77]. Although the problem of radiation due to 3OD has been investigated extensively [78, 79], the numerical findings have not yet recieved full analytical explanation. The variational method [80], has been used to verify a numerically found relation [70] between the width and amplitude of the emerging soliton. In particular, the problem of what kind of soliton that emerges from an arbitrary pulse launched close to the zero-dispersion frequency is yet unsolved.

It was also found in the simulations that higher-order solitons will break up into individual soliton pulses due to the presence of a weak 3OD [72, 81]. The reason for this is easily understood from the fact that an N-soliton can consist of N solitons of different amplitudes. The perturbative result (6.38) suggests that a soliton of amplitude A will move with the velocity ϵA^2. Consequently, an N=2 soliton in which the individual solitons have different amplitudes will break up because the individual solitons obtain different velocities from the 3OD. For a recent review on solitons perturbed by 3OD, see [82].

6.3.4 Solitons under fourth-order dispersion (Papers G and H)

Solitons perturbed by fourth-order dispersion (4OD) has recieved very little attention. The main reason is that the influence of 4OD is rather weak for experimentally convenient (picosecond) pulse durations in the fibers of today. Future systems, however, are likely to utilize shorter pulses, and for these the onset of 4OD might be an important effect. In order to purify the effects of 4OD we assume that 3OD is absent. This is physically possible by selecting the carrier frequency so as to correspond to maximum/minimum group-velocity dispersion, see fig (6.3). For examples of fibers with 4OD, see Refs. [66, 83].

The equation for the pulse envelope in fibers with 4OD is

$$i\frac{\partial q(\xi,\tau)}{\partial \xi} = \frac{1}{2}\frac{\partial^2 q}{\partial \tau^2} + |q|^2 q + \epsilon\frac{\partial^4 q}{\partial \tau^4}. \tag{6.39}$$

where $\epsilon = \beta_0''''/(24|\beta_0''|t_0^2)$. Numerically, 4OD was briefly analyzed in Ref. [62], and qualitative differences of the pulse propagation dynamics was observed depending on the sign of ϵ. The reason for this is related to the dispersive properties of the fiber, described by the dispersion relation

Figure 6.3: *Positive (short-dashed) and negative (solid) 4OD in fibers correspond to a curvature of the group-velocity dispersion.*

$$k_{lin}(\Omega) = -\frac{\Omega^2}{2} + \epsilon\frac{\Omega^4}{24}. \tag{6.40}$$

The case of $\epsilon > 0$ (*positive 4OD*), is studied in **paper G**. In such a fiber, we have anomalous dispersion in the regime $\Omega^2 < 2/\epsilon$, and normal dispersion for frequencies outside this interval, see fig (6.3). This means that a soliton launched in the spectral "well" of anomalous dispersion will have tails extending into the normal dispersion regime, and obviously radiate. In **paper G** we point out that the mechanism for this radiation is similar to the radiation induced by 3OD, and we identify the unstable frequencies as

$$\Omega_{uns} \cong \pm\frac{1}{\sqrt{2\epsilon}} \tag{6.41}$$

which is found to agree with numerical results. In fact, the expression (6.41) is the lowest-order approximation of the exact resonance condition given by (see chapter 5.4.1) $k_{sol} = k_{lin}$. A soliton governed by Eq. (6.39) will therefore radiate symmetrically at the frequencies (6.41) but since power is lost from the soliton, it will contract spectrally and stabilize itself. It will not exhibit any spectral recoil because of the symmetry of the problem, but remain trapped in the anomalous-dispersion well.

We also point out a remarkable fact obtained in **paper G**, namely that there exists an exact, stationary, two-humped solution of eq. (6.39). However, this solution is unstable and will decay into radiation when perturbed.

The case of $\epsilon < 0$ (*negative 4OD*), is considered in **paper H**. In this kind of fiber we have anomalous dispersion for all frequencies, see fig (6.3). Pulses in such media will not necessarily be unstable, but stationary, stable soliton-like states could exist. In **paper H** we show that this is indeed the case. We derive an approximate variational description for an entire family of stationary solutions. One particular member of this family has a $sech^2(\tau)$ - shape, and can be given an exact analytic expression. We also demonstrate the stability of these states, in the sense that an arbitrary input pulse evolves into the stationary states in a similar way as a pulse evolves into the soliton-states of the unperturbed NLS equation. When these pulses collide, we note that small amounts of dispersive radiation emerges, and this indicates the non-integrability of Eq. (6.39). For nonmoving pulses, i.e. pulses spectrally centered symmetrically on the dispersion maximum, we have found some interesting features, e.g. breather-like solutions. Later research has also shown that novel, stationary solitons having oscillating-decaying tails are also possible solutions of eq. (6.39) [84]. Such soliton-like pulses could be useful for communication purposes, because two pulses could be asymptotically matched and joined together as a couple having non-jittering features [84, 85].

6.3.5 Solitons in presence of the Raman nonlinearity

Stimulated Raman scattering (SRS), was one of the earliest investigated nonlinear optical effects [86]. Phenomenologically, it acts on intense optical waves in glasses by transferring power from high to low frequencies [87]. It is an effect having nonlinear characteristics, i.e. it requires high powers to be observed. Moreover, silica fibers have a characteristic *Raman gain spectrum*, which have been measured by e.g. Stolen et al. [45, 88, 89]. The width of the gain spectrum is 13.2 Thz, which sets the scale over which the frequency conversion takes place [89]. It is thus two criteria that have to be met in order to observe (SRS): (i) The pump wave has to be intense enough, and (ii) the signal must lie within 13.2 THz from the pump. Consequently, a short intense pulse having a bandwith comparable to 13.2 Thz will exhibit gain on the red side and loss on the blue side of the spectum. Too long pulses are not spectrally wide enough to experience the Raman gain, and consequently the effect is not observed for picosecond pulses. For subpicosecond pulses, however, the Raman effect will give rise to observable spectral downshifts, an effect that was observed by Mitschke and Mollenauer [90], and explained by Gordon [91] 1986. The effect was demonstrated with respect to optical solitons and was coined "the soliton self-frequency shift". The term "intra-pulse Raman scattering" (IPRS) is also used to emphasize that the frequency conversion takes place within the bandwidth of a single pulse.

In the time domain, the frequency downshift of a pulse gives rise to a uniform acceleration, because the carrier frequency shifts linearly, thereby linearly increasing the group velocity of the pulse. Mathematically we model the effect of IPRS with an additional nonlinear term in the NLS equation. Using equation (6.21) this can be expressed as [91]

$$\hat{P} = -\eta q \frac{\partial |q|^2}{\partial \tau} \tag{6.42}$$

where $\eta = \tau_R/t_0$. The characteristic time scale for the Raman gain is given by the slope of the Raman gain curve near zero, and a commonly used value in the literature is $\tau_R = 6$

fs [92, 93]. In order to study the effects of this additional nonlinear term on soliton propagation, the most simple way is to use the invariants of the NLS equation. We use the soliton (6.4) which is described by an amplitude A and a frequency shift V. Note that a positive V in eq. (6.4) corresponds to a downshift in frequency. We can now obtain the invariants E and M from eqs. (6.18, 6.19) as

$$E = 2A \qquad M = i4AV. \tag{6.43}$$

Assuming the Raman term to be of perturbative character, which is true for pulse durations down to 100 fs, we use the equations of motion for the invariants (6.22,6.23) and find

$$\frac{dA}{d\xi} = 0 \tag{6.44}$$

$$\frac{dV}{d\xi} = \eta \frac{8}{15} A^4 \tag{6.45}$$

so that the shifting rate of the frequency downshift V of the soliton is proportional to the soliton amplitude (or inverse width) to fourth power. This result was obtained originally by Gordon 1986 [91], and Kodama and Hasegawa 1987 [94]. The fourth power of the soliton amplitude can be physically understood from the fact that the shift must scale with the pulse intensity A^2 and with the spectral width A. In addition the Raman gain coefficient also scales with the spectral width A [95], which gives the fourth-power scaling of the downshift. It is possible to derive the first-order (in η) perturbation to the NLS-soliton, by considering the symmetries and the transformation properties of the governing equation [96]. This correction shows a slight asymmetry of shape of the down-shifting pulse.

An important consequence of the downshift formula (6.45) is that higher-order solitons will break up due to the Raman effect. This is observed in the experiments [90, 97] as a spectral splitting of a high-intensity input pulse into its constituent solitons. Theoretically, this was explained by Tai et al. 1988 [98], and the effect is also observed in the simulations of Afanasyev et al. [93]. For instance, the N=2 soliton $q(0, \tau) = 2\, sech(\tau)$ corresponds to two solitons with amplitudes 1 and 3 respectively. These will downshift according to the rule above, so that the high-amplitude soliton will accelerate faster than the low-amplitude soliton. It is not possible to directly use the shifting rule (6.44) to study the separation between the two solitons, because of the momentum conservation. Due to the Raman effect, the momentum is not conserved, but it is *nearly* conserved, since the Raman term is perturbative. This means that the high-amplitude soliton shifts away roughly according to the A^4-rule, but the low-amplitude soliton will recoil *upwards* in frequency due to momentum conservation [98] and eventually start its shifting from the recoiled position. The separation between the two solitons is therefore larger than one would expect from a naive use of the shifting formula for each of the separate solitons. An analytic theory that accounts for this is yet to be seen. Finally, it should be mentioned that a similar splitting is observed in soliton-interaction experiments [99, 100]. Two initially equal-amplitude, well-separated pulses interact and emerge as two pulses with different amplitudes after the interaction. This is related to the fact that the initial conditon in the Z-S scattering problem corresponds to two different eigenvalues, i.e. two solitons with different amplitudes that will separate from each other due to the self-frequency-shift.

A potentially useful application of the Raman effect could be to use it together with the modulational instability for the generation of pulse-trains [92]. It has also been shown experimentally [101] and theoretically [102] that the Raman downshift is suppressed by bandwidth-limited amplification.

6.3.6 Instabilities with 4OD and Raman downshift (Paper I)

Obviously, when ultrashort pulse-propagation is considered in fibers, it is important to take the Raman contribution to the nonlinearity into acount. In conventional fibers, this frequency dowshift corresponds to a wavelength upshift, which shifts the pulse further into the anomalous dispersion regime. This causes the solitons to broaden, since the value of the GVD, i.e. $|\beta_0''|$, increases with the frequency downshift. However, it is possible to draw fibers which have a second zero-dispersion wavelength above the conventional one at $1.3\mu m$ [66]. Raman downshift in such fibers would cause solitons to compress adiabatically, because the GVD now decreases with the downshift. In **paper I** we investigate this effect numerically and analytically. We find that the compression (and the Raman downshift) is limited as the pulse comes close to the zero-dispersion frequency. This limitation is caused by the generation of dispersive waves in the normal dispersion regime on the other side of the zero-dispersion frequency. This process can be viewed as the conventional resonant instability that arises for pulses close to the zero-dispersion wavelength, see chapter 5.4.1.

Experimentally, this compression effect was recently observed by Mamyshev et al. [103]. It was found that a 95 fs soliton was adiabatically compressed to 55 fs over a fiber length of 65 m. This agrees very well with our numerical findings. We also point out that the compression effect is most efficient for shallow dispersion curves, so that the GVD is slowly changing. However, then the fiber length recquired to obtain the necessary downshift will be very long, and fiber loss will limit the process. We predict optimal operation for 0.1-0.3 ps pulses, and typical compression factors of 2-3 can be expected.

6.3.7 Optical solitons in communication systems

The most important application for optical soliton pulses is in high bit-rate communication systems, in which solitons are the natural fundamental information bit. However there are several system aspects that will influence and degrade the propagation of trains of solitons in communication systems. These effects are be briefly reviewed below.

Obviously a long-distance system needs periodically spaced amplifiers. This will put limits on the soliton period, which must be larger than the amplifier spacing in order to avoid *resonant instabilities* (see ch. 5.4.2). Moreover, the noise of each amplifier will give rise to a small jitter in the carrier frequency of the solitons, thereby changing the group velocity and severely affecting the arrival time of each pulse [104, 105]. This is known as the *Gordon-Haus effect* after the authors of Ref. [104], and it is a crucial effect that has to be accounted for in long-distance systems. Experiments carried out in 1991 showed that Gordon-Haus jitter is a limiting factor over long distances, but soliton systems were still able to produce 2.5 GBit/s over 14 Mm, [106]. A straightforward way to remedy this jitter is to install modulators along the transmission line, that periodically reshapes and retimes the pulse train. Using this technique, Nakazawa et al. [107] accomplished 10 Gbit/s over one million km. However, this kind of active reshaping of the pulses suffers from the the same drawbacks as the conventional electronic regeneration, i.e. incompatibility with WDM (see below), complexity and high cost. *Wavelength division multiplexing* (WDM) is another way of incresing the bit-rate of soliton systems. It means that several frequency bands are used for solitons tranmission, and this technique was pioneered by Olsson et al. [108] 1991.

Solitons, being nonlinear pulses, will *interact* with adjacent pulses in the pulse train, as pointed out by e.g. Gordon [109], and Hermanson et al. [110]. It turns out that solitons in (out of) phase will attract (repel) each other. The interaction force decreases exponentially with the spacing between the pulses. However, a large spacing obviously limits the available bit-rate. A useful way of decreasing the interaction is to give adjacent solitons a slight difference in amplitude [111], and thereby different wavenumbers. The consequence will be that the relative phase between the solitons rotates during propagation, with a strongly reduced interaction as the result. For a recent review on soliton interactions, see Ref. [111].

Bandwidth-limited gain has been suggested [102, 112] as a remedy to many of the problems mentioned above. It turns out that many of the effects that give rise to timing jitter (e.g. Gordon-Haus jitter, soliton interaction jitter, Raman downshift) correspond to a jitter in the carrier frequency of the pulse. *Bandwidth limited amplification*, or filtering after the amplification, tends to stabilize jittering of the carrier frequency, and it will therefore counteract the Gordon-Haus effect [113] as well as soliton interactions [114]. An alternative way, although not as effective as filtering, in reducing the Gordon-Haus jitter is by dispersion compensation after the transmission [115]. A problem with bandwidth-limited gain is that there will be *excess gain* over a finite frequency interval. This will cause noise in this interval to grow and interact with the soliton. In order to avoid this problem, it has been suggested that the center frequency should move spectrally during propagation [116]. Technically, this is simply realized by letting the mid-frequency of the filters slide with propagation distance. Numerical simulations [116, 117] and experiments [118] with such *sliding filters* have demonstrated 10 Gbit/s over 20 000 kilometers. Recent theoretical studies indicate that 30 Gbit/s over transoceanic distances lies within reach [119].

The different polarization states of a fiber is in most cases not taken into account, and the scalar approximation for the electric field envelope is used. This would not be a problem if it was not for imperfections of the fiber, which completely scrambles the polarization state of linear waves. In particular, random perturbations in the birefringence of the fiber will lead to walk-off between the different polarizations, and a subsequent pulse broadening. This effect is known as *polarization mode dispersion* (PMD) [120], and it is by several researchers considered as a fundamental limit for linear pulse propagation [121]. Soliton pulses, however, are found to be rather insensitive to PMD [121, 122]. The reason for this is that the nonlinear induced refractive index keeps both polarization components of the pulse together as one entity. This property may be an important advantage for solitons over linear pulses in long-distance fiber-transmission.

Although solitons have been mostly considered for ultra-long distance propagation, they can also be utilized for high bit-rates over shorter distances. As an example, it was recently demonstrated by Nakazawa et al. [123], that 80 Gbit/s over 500 km is possible. The limiting factors in these systems is the resonant amplifier instability together with soliton interaction. In fact, the superiority of soliton systems is not self-evident for high bit-rate/short-distance systems, because linear WDM systems at 17×20Gbit/s $\cong 340$ Gbit/s have been demonstrated over 150km [124].

6.3.8 Femtosecond pulse propagation - general considerations

Finally we will make a brief discussion of effects which are important for subpicosecond pulse propagations in fibers. Obviously, as we go down in pulse duration the effects of higher-order linear dispersion become significant. Especially near the zero-dispersion wavelengths 3OD can not be neglected. The effects of 4OD are important for pulses which have so high a bandwidth that the curvature of the GVD versus frequency is significant over the pulse. The significance of 4OD is therefore strongly dependent of the dispersion properties of the fiber at the carrier wavelength.

The higher-order nonlinear effects that must be taken into account is firstly the Raman nonlinearity. As a first approximation, for pulse durations above ~ 100 fs, the first term in the expansion of the Raman gain, i.e. eq. (6.42) is a fairly good approximation. For shorter pulses, the entire Raman gain spectrum must be included, as was done e.g. in Refs. [45, 68, 125, 126] and **Paper I** of this thesis.

For even shorter pulses, below 50 fs, the last term of Eq. (6.1) is important. This term gives rise to *self-steepening*, or an *optical shock-front* of the pulse [127]. The self-steepening, although predicted early [127], has not yet been experimentally observed. This is mainly because of the extremely short pulse durations that is required for an observation. Furthermore, it is not known whether a shock-front can be observed in the presence of higher-order dispersion. For a recent treatment of optical self-steepening, see Ref. [128] and references therein. It is, however doubtful, whether a treatment of self-steepening without including the Raman nonlinearity is physically relevant in fibers.

Moreover, for high powers and subpicosecond pulses, the nonlinear contribution to the modal dispersion must be included, as outlined above. Also backscattering of waves, i.e. the inclusion of second derivatives in z, may be relevant for high powers. Stationary soliton-like solutions for the NLS equation including several higher-order terms (e.g. 3OD and self-steepening) have been found [129]. However, these are possible only for one particular pulse duration, similarly to the exact solutions found in papers G and H. Since the solution of paper H is one particular member of an entire family, one could expect that similar families exist for the solutions of Ref. [129]. However this has not yet been investigated in detail.

We end this section by noting that the theory for subpicosecond pulse propagation in optical fibers is essentially based on extensions of the NLS equation. The most important higher-order terms have been identified as higher-order dispersion and the Raman nonlinearity, but for pulses substantially shorter than 100 fs, the above mentioned effects must also be included. Which one of these that are of greatest importance can only be settled by comparisons with experiments. However, in experiments it is difficult to isolate only one effect and it is most likely that combinations of several higher-order effects will determine the pulse dynamics.

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Chapter 7

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Paper A

"Dynamic effects of Kerr nonlinearity and spatial diffraction on self-phase modulation of optical pulses"

by

M. Karlsson, D. Anderson, M. Desaix and M. Lisak

Optics Letters **16**, *1373 (1991)*

Paper B

"Dynamics of self-focusing and self-phase modulation in a parabolic index optical fiber"

by

M. Karlsson, D. Anderson, and M. Desaix

Optics Letters **17**, *22 (1992)*

Paper C

"Super-Gaussian approximation of the fundamental radial mode in nonlinear parabolic-index optical fibers"

by

M. Karlsson and D. Anderson

Journal of the Optical Society of America B **9**, *1558 (1992)*

Paper D

"Optical beams in saturable self-focusing media"

by

M. Karlsson

Physical Review A **46**, *2726 (1992)*

Paper E

“Modulational instability dynamics in a spatial focusing and temporal defocusing medium”

by

D. Anderson, M. Karlsson, M. Lisak, and A. Sergeev

Physical Review E **47**, *3617 (1993)*

Paper F

"Wave-breaking-free pulses in nonlinear-optical fibers"

by

D. Anderson, M. Desaix, M. Karlsson, M. Lisak and M. L. Quiroga-Teixeiro

Journal of the Optical Society of America B **10**, *1185 (1993)*

Paper G

"Ultrashort solitons at the minimum-dispersion wavelength:
effects of fourth-order dispersion"

by

A. Höök and M. Karlsson

Optics Letters **18**, *1388 (1993)*

Paper H

" Soliton-like pulses governed by fourth order dispersion in optical fibers"

by

M. Karlsson and A. Höök

Optics Communications **104**, *303 (1993)*

Paper I

"Soliton Instabilities and Pulse Compression in Minimum Dispersion Fibers"

by

A. Höök and M. Karlsson

IEEE Journal of Quantum Electronics **30**, *1831 (1994)*