

An A_r threesome: Matrix models, $2d$ conformal field theories, and $4d$ $\mathcal{N}=2$ gauge theories

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We explore the connections between three classes of theories: A_r quiver matrix models, $d=2$ conformal A_r Toda field theories, and $d=4$ $\mathcal{N}=2$ supersymmetric conformal A_r quiver gauge theories. In particular, we analyze the quiver matrix models recently introduced by Dijkgraaf and Vafa (unpublished) and make detailed comparisons with the corresponding quantities in the Toda field theories and the $\mathcal{N}=2$ quiver gauge theories. We also make a speculative proposal for how the matrix models should be modified in order for them to reproduce the instanton partition functions in quiver gauge theories in five dimensions. © 2010 American Institute of Physics. [doi:10.1063/1.3449328]

I. INTRODUCTION

The AGT (Alday–Gaiotto–Tachikawa) relation,¹ which is a relation between Nekrasov partition functions² in (conformal) $d=4$ $\mathcal{N}=2$ quiver gauge theories and correlation functions in conformal field theories (CFTs) in two dimensions, has been studied in several papers over the past couple of months. The original conjecture¹ involves a relation between Nekrasov partition functions in $d=4$ $SU(2)$ (or A_1) quiver gauge theories³ and correlation functions in the Liouville $d=2$ conformal field theory. It was subsequently extended⁴ to a relation between the A_r quiver gauge theories³ and the $d=2$ conformal A_r Toda field theories. The proposals in Refs. 1 and 4 have passed many nontrivial checks, see, e.g., Refs. 5–7. In Ref. 8 the A_1 AGT relation was extended by the inclusion of surface, Wilson and 't Hooft operators in the A_1 quiver gauge theories and proposals were made for the corresponding quantities in the Liouville theory. Another line of investigation concerns the extension to nonconformal $SU(2)$ gauge theories.⁹ A suggestion for how this approach should be modified to capture the instanton partition function for pure $SU(2)$ gauge theory in five dimensions was presented in Ref. 10. In Ref. 11, another relation between four-dimensional and two-dimensional theories was uncovered which is similar in spirit to the AGT relation.

Recently, Dijkgraaf and Vafa¹² presented an argument explaining the AGT relations. (Another argument, using M-theory, for the validity of the AGT relation was presented in Ref. 13; see also the followup paper.¹⁴) The argument involves relating the relevant quantities in the two theories via an intermediate matrix model. The first step is to realize the gauge theories in string theory using geometric engineering¹⁵ and then use the relation to matrix models via a large N duality, together with the relation between matrix models and conformal field theories^{16,17} to recover the AGT relation. This chain of arguments removes some of the mystery of the AGT relation. More importantly, it also implies that there are now *three different ways* to compute the same quantities: using the $4d$ quiver gauge theories, using the $2d$ Toda theories, or using the $0d$ quiver matrix models. In all three cases a Riemann surface plays a crucial role: in the gauge theory the Riemann

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surface is related to the Seiberg–Witten curve, in the Toda theory the Riemann surface is the manifold on which the theory is defined, and in the matrix model the Riemann surface is the spectral curve arising from the loop equations in the large N limit.

The goal of this paper is to develop and exemplify how calculations are performed in the matrix model framework. We will rederive several known results in quiver gauge theories and Toda field theories from the matrix model integrals. We also make a speculative proposal on how the matrix models should be modified to reproduce the Nekrasov partition function for quiver gauge theories in five dimensions.

In Sec. II we review the AGT relation for the case of the A_r theories, and in Sec. III we describe the A_r quiver matrix models introduced in Ref. 12. In the two subsequent sections we then perform several matrix model calculations and compare the results with the Toda theories and the quiver gauge theories. In Sec. IV we treat the A_1 model and in Sec. V we discuss the A_r models for general r . Finally, in Sec. VI we describe our proposal on how the matrix models should be modified in order to describe the Nekrasov partition function for quiver gauge theories in five dimensions. In Appendix some technical details are collected.

II. THE A_r AGT relation

In this section we review the AGT proposal for the class of theories based on the A_r Lie algebras. We start with a brief recap of the A_r Toda field theories, followed by a summary of the A_r quiver gauge theories, and then describe the AGT relation connecting the two classes of theories.

A. The A_r Toda field theories

The A_r Toda field theories are defined by the action

$$S = \int d^2\sigma \sqrt{g} \left[\frac{1}{8\pi} g^{ad} \langle \partial_a \phi, \partial_d \phi \rangle + \mu \sum_{i=1}^{N-1} e^{b \langle e_i, \phi \rangle} + \frac{\langle Q \rho, \phi \rangle}{4\pi} R \phi \right], \quad (2.1)$$

where g_{ad} ($a, d=1, 2$) is the metric on the two-dimensional worldsheet and R is the worldsheet curvature. The e_i are the simple roots of the A_r Lie algebra, $\langle \cdot, \cdot \rangle$ denotes the scalar product on the root space, ρ is the Weyl vector (half the sum of all positive roots), and the r -dimensional vector of fields ϕ can be expanded as $\phi = \sum_i \phi_i e_i$. The A_r Toda theory is conformal provided Q and b are related via

$$Q = \left(b + \frac{1}{b} \right). \quad (2.2)$$

The central charge is¹⁸

$$c = r + 12Q^2 \langle \rho, \rho \rangle = r \left(1 + (r+1)(r+2) \left(b + \frac{1}{b} \right)^2 \right). \quad (2.3)$$

The general form of a three-point correlation function in a $2d$ conformal field theory is¹⁹

$$\langle V_{\alpha_1}(z_1, \bar{z}_1) V_{\alpha_2}(z_2, \bar{z}_2) V_{\alpha_3}(z_3, \bar{z}_3) \rangle = \frac{C(\alpha_1, \alpha_2, \alpha_3)}{|z_{12}|^{2(\Delta_1 + \Delta_2 - \Delta_3)} |z_{13}|^{2(\Delta_1 + \Delta_3 - \Delta_2)} |z_{23}|^{2(\Delta_2 + \Delta_3 - \Delta_1)}}. \quad (2.4)$$

The Liouville theory is identical to the A_1 Toda field theory and has a set of primary fields,

$$V_\alpha = e^{2\alpha\phi}. \quad (2.5)$$

The correlation function of three primary fields in the Liouville theory is²⁰ (see also Ref. 21)

$$C(\alpha_1, \alpha_2, \alpha_3) = [\pi\mu\gamma(b^2)b^{2-2b^2}]^{(Q-\alpha_1-\alpha_2-\alpha_3)/b} \times \frac{Y(b)Y(2\alpha_1)Y(2\alpha_2)Y(2\alpha_3)}{Y(\alpha_1 + \alpha_2 + \alpha_3 - Q)Y(-\alpha_1 + \alpha_2 + \alpha_3)Y(\alpha_1 - \alpha_2 + \alpha_3)Y(\alpha_1 + \alpha_2 - \alpha_3)}, \tag{2.6}$$

with $\gamma(b^2) = \Gamma(b^2)/\Gamma(1-b^2)$ and $Y(x) = 1/[\Gamma_2(x|b, b^{-1})\Gamma_2(Q-x|b, b^{-1})]$, where $\Gamma_2(z|\epsilon_1, \epsilon_2)$ is the Barnes double gamma function.²²

In the Toda theories with $r > 1$ (\mathcal{W}) primary fields can be defined in analogy with the Liouville case via

$$V_\alpha = e^{\langle \alpha, \phi \rangle}. \tag{2.7}$$

Recently it was shown^{23,24} that in the special case when one of the α 's takes one of the two special values,

$$\chi = \varkappa\Lambda_1 \quad \text{or} \quad \chi = \varkappa\Lambda_r, \tag{2.8}$$

where Λ_1 (Λ_r) is the highest weight of the fundamental (antifundamental) representation of the A_r Lie algebra and \varkappa is a complex number, the three-point function is given by (2.4) with

$$C(\alpha_1, \alpha_2, \chi) = [\pi\mu\gamma(b^2)b^{2-2b^2}]^{(2Q\rho-\alpha_1-\alpha_2-\chi, \rho)/b} \times \frac{(Y(b))^r Y(\varkappa) \prod_{e>0} Y(\langle Q\rho - \alpha_1, e \rangle) Y(\langle Q\rho - \alpha_2, e \rangle)}{\prod_{ij} Y\left(\frac{\varkappa}{r+1} + \langle \alpha_1 - Q\rho, h_i \rangle + \langle \alpha_2 - Q\rho, h_j \rangle\right)}, \tag{2.9}$$

where the product in the numerator is over all positive roots and in the denominator the h_i are the weights of the representation with highest weight Λ_1 , cf. (A2). (The result for $\chi = \varkappa\Lambda_r$ is obtained by replacing h_i by $h'_i = -h_{r+2-i}$.)

Higher-point correlation functions in any CFT can be related to the three-point function of primary fields, which therefore determines the entire theory.¹⁹ Note that when $r > 1$ knowledge of the three-point function of \mathcal{W} primary fields (2.9) does not determine all higher-point correlation functions, see, e.g., Refs. 4 and 25.

As an example, consider a four-point function. It is convenient to fix three points to $0, 1, \infty$ and use a bra-ket notation which has the property $\langle \alpha | \alpha \rangle = 1$, and is such that

$$\langle \alpha_1 | V_{\alpha_2}(1) V_{\alpha_3}(z) | \alpha_4 \rangle = \langle V_{Q-\alpha_1}(0) V_{\alpha_2}(1) V_{\alpha_3}(z) V_{\alpha_4}(\infty) \rangle. \tag{2.10}$$

Inserting a complete set of states we find

$$\langle \alpha_1 | V_{\alpha_2}(1) V_{\alpha_3}(z) | \alpha_4 \rangle = \int d\sigma \sum_{\mathbf{k}, \mathbf{k}'} \frac{\langle \alpha_1 | V_{\alpha_2}(1) | \psi_{\mathbf{k}}(\sigma) \rangle \langle \psi_{\mathbf{k}'}(\sigma) | V_{\alpha_3}(z) | \alpha_4 \rangle}{\langle \psi_{-\mathbf{k}}(\sigma) | \psi_{-\mathbf{k}'}(\sigma) \rangle}. \tag{2.11}$$

Here the intermediate states $|\psi_{\mathbf{k}}(\sigma)\rangle$ are descendants of the primary state labeled by σ . (Throughout this paper we will label the internal momenta by σ reserving the symbol α for the external momenta.)

In the Liouville case it can be shown that $\langle \psi_{\mathbf{k}'}(\sigma) | V_{\alpha_3}(z) | \alpha_4 \rangle$ is proportional to $\langle \sigma | V_{\alpha_3}(z) | \alpha_4 \rangle$ (Ref. 19) and hence (2.11) can be calculated perturbatively. The ratio

$$\frac{\sum_{\mathbf{k}, \mathbf{k}'} \langle \alpha_1 | V_{\alpha_2}(1) | \psi_{\mathbf{k}}(\sigma) \rangle \langle \psi_{-\mathbf{k}}(\sigma) | \psi_{-\mathbf{k}'}(\sigma) \rangle^{-1} \langle \psi_{\mathbf{k}'}(\sigma) | V_{\alpha_3}(z) | \alpha_4 \rangle}{\langle \alpha_1 | V_{\alpha_2}(1) | \sigma \rangle \langle \sigma | V_{\alpha_3}(z) | \alpha_4 \rangle} \tag{2.12}$$

is called a conformal block. General n -point functions can be dealt with in an analogous manner. They depend on $(n-3)$ cross ratios.

In the $r > 1$ case the situation is a little more involved, see Ref. 4, for a discussion.

B. The A_r quiver gauge theories and Nekrasov partition functions

In Ref. 3 a class of conformal $4d \mathcal{N}=2$ generalized A_r quiver gauge theories were introduced. This class of theories was denoted $\mathcal{T}_{(n,g)}(A_r)$. The simplest example in this class of theories is the theory with a single $SU(r+1)$ gauge factor with $2(r+1)$ matter hypermultiplets in the fundamental representation of the gauge group. But the $\mathcal{T}_{(n,g)}(A_r)$ class of theories includes many more theories, not all of which are conventional weakly coupled gauge theories. The $\mathcal{T}_{(n,g)}(A_r)$ theories can be viewed as arising from the six-dimensional $A_r(2,0)$ theory²⁶ compactified on $C \times \mathbb{R}^4$, where C is a genus g Riemann surface with n punctures. The genus of the Riemann surface depends on the number of loops in the (generalized) quiver diagram. The punctures are due to codimension 2 defects filling \mathbb{R}^4 and intersecting C at points, and were argued in Ref. 3 to be classified by partitions of $r+1$ (which can be represented graphically in terms of Young tableaux). One can therefore associate a Young tableau with each puncture. In the case of the A_1 theories there is only one kind of nontrivial puncture. In the above example [a $\mathcal{T}_{4,0}(A_r)$ theory] there are two kinds of punctures. These are associated with the factors in the $U(1)^2 SU(r+1)^2$ subgroup of the flavor symmetry group. The punctures associated with the $SU(r+1)$ factors are called full punctures and involve r mass parameters each and the punctures associated with the $U(1)$ factors are called basic punctures and involve one mass parameter each. We refer to Ref. 3, for further details.

A fundamental object in an $\mathcal{N}=2$ gauge theory is the Nekrasov partition function (from which the prepotential can be obtained). The partition function factorizes into two parts as

$$Z = Z_{\text{pert}} Z_{\text{inst}}, \tag{2.13}$$

where Z_{pert} is the contribution from perturbative calculations (because of supersymmetry there are contributions only at tree and one-loop level), and Z_{inst} is the contribution from instantons. The most efficient method to obtain Z_{inst} is via the instanton counting method of Nekrasov.² This approach involves deforming the $\mathcal{N}=2$ gauge theory with two parameters ϵ_1 and ϵ_2 which belong to an $SO(2) \times SO(2)$ subgroup of the $SO(4)$ Lorentz symmetry. We should stress that one needs the theory to be weakly coupled to be able to apply the instanton counting method.

As an example, the instanton partition function in the $SU(r+1)$ theory with $2(r+1)$ fundamentals can be written as² (see also Ref. 27)

$$Z_{\text{inst}} = \sum_{\vec{Y}} y^{|\vec{Y}|} \prod_{m,n=1}^{r+1} \prod_{s \in Y_m} \frac{P(\hat{a}_m, Y_m, s)}{E(\hat{a}_m - \hat{a}_n, Y_m, Y_n, s)(E(\hat{a}_m - \hat{a}_n, Y_m, Y_n, s) - \epsilon)}, \tag{2.14}$$

where $y = e^{2\pi i \tau}$ and the sum is over the $(r+1)$ -dimensional vector of Young tableaux, $\vec{Y} = (Y_1, Y_2, \dots, Y_{r+1})$, and $|\vec{Y}|$ (the instanton number) is the total number of boxes in all the Y_m 's. The \hat{a}_m parametrize the Coulomb branch of the theory and satisfy $\sum_{i=1}^{r+1} \hat{a}_i = 0$. It is convenient to write $\hat{a} = \sum_{i=1}^r a_i e_i$, where e_i are the simple roots of the A_r Lie algebra. In the particular case of $SU(2)$ this translates into $\hat{a} = (a, -a)$. In (2.14) $\epsilon \equiv \epsilon_1 + \epsilon_2$ and

$$E(x, Y_m, Y_n, s) = x - \epsilon_1 L_{Y_n}(s) + \epsilon_2 (A_{Y_m}(s) + 1), \tag{2.15}$$

where $s = (i, j)$ and i refers to the vertical position and j to the horizontal position of the box. Furthermore, $L_{Y_n} = k_{n,i} - j$ and $A_{Y_m} = k_{m,j}^T - i$, where $k_{n,i}$ is the length of the i th row of Y_n and $k_{m,j}^T$ is the height of the j th column of Y_m . Finally,

$$P(x, Y_i, s) = \prod_{f=1}^{2r+2} (x - (j-1)\epsilon_1 - (i-1)\epsilon_2 - m_f), \tag{2.16}$$

where the m_f are the masses of the matter fields (suitably defined).

The perturbative (one-loop) piece in (2.13), Z_{pert} , is a product of various factors. For $SU(r+1)$ the gauge field contributes a factor

$$\prod_{i < j}^r \frac{1}{\Gamma_2(\hat{a}_i - \hat{a}_j - \epsilon_2 | \epsilon_1, \epsilon_2) \Gamma_2(\hat{a}_i - \hat{a}_j - \epsilon_1 | \epsilon_1, \epsilon_2)}, \quad (2.17)$$

where $\Gamma_2(x | \epsilon_1, \epsilon_2)$ is the Barnes double gamma function,²² and each of the massive hypermultiplets transforming in the fundamental representation of the gauge group contributes a factor

$$\prod_{i=1}^r \Gamma_2(a_i - m_f + \epsilon | \epsilon_1, \epsilon_2). \quad (2.18)$$

C. The AGT relation

The AGT relation is a relation between the two classes of theories discussed in Secs. II A and II B. Up to an overall factor, the instanton partition function of a $\mathcal{T}_{(n,g)}(A_r)$ theory is (conjectured to be) equal to a chiral block of an n -point correlation function in the A_r Toda field theory formulated on a genus g surface. In other words, the punctures correspond to insertions of vertex operators in the Toda theory. For example, the instanton partition function in the $SU(r+1)$ theory with $2r+2$ fundamentals is equal to the chiral block (in a specific channel) of the four-point function in the A_r Toda theory on the sphere. The momenta of the vertex operators, α_i , are mapped to the masses m_i in the gauge theory. This relation is linear (the exact form depends on conventions for the gauge theory masses). Furthermore, the internal momenta in the chiral block, σ_k , are linearly related to the a_k Coulomb moduli. Finally, the parameters ϵ_1 and ϵ_2 in the instanton partition function are related to the parameter b in the Toda theory via

$$b = \sqrt{\frac{\epsilon_1}{\epsilon_2}}, \quad \frac{1}{b} = \sqrt{\frac{\epsilon_2}{\epsilon_1}}. \quad (2.19)$$

The most common choice is to set $\epsilon_1 = b$, $\epsilon_2 = 1/b$. There is also a slight extension of the above result where the full partition function including the perturbative piece is related to the full correlation function including the three-point pieces. For further details about the AGT relation, see Refs. 1 and 4.

III. THE MATRIX MODEL APPROACH OF DIJKGRAAF AND Vafa

The first step in the analysis of Ref. 12 is to realize the relevant quiver gauge theories in string theory using geometric engineering¹⁵ and then use the results in Ref. 28 to relate the topological string partition function (which is equal to the Nekrasov partition function) to a matrix model calculation. This argument works provided $\epsilon_1 = -\epsilon_2 = g_s$. How to deal with the case of general $\epsilon_1, 2$ was also proposed in Ref. 12 and will be discussed later in this section.

In the case of interest to us the relevant geometries are A_r singularities

$$uv - x^{r+1} = 0. \quad (3.1)$$

(There is also a decoupled C factor parametrized by z .) The matrix model corresponding to this geometry is the so-called A_r quiver matrix model.^{16,29} It involves r Hermitian $N_i \times N_i$ matrices Φ_i ($i=1, \dots, r$) as well as the $N_i \times N_j$ matrices B_{ij} . The matrix model partition function is (proportional to)

$$\int \prod_i d\Phi_i \prod_{i,j} dB_{ij} \exp \left[-\frac{1}{g_s} \prod_{i < j} \text{tr}[(2\delta_{ij} - A_{ij})(B_{ij}\Phi_j B_{ji} - B_{ji}\Phi_i B_{ij})] \right], \quad (3.2)$$

where A_{ij} is the Cartan matrix of the A_r Lie algebra, i.e., $A_{ij} = \langle e_i, e_j \rangle$. In terms of the eigenvalues of Φ_i , λ_i^l ($l=1, \dots, N_i$), partition function (3.2) becomes, after integrating out the B_{ij} ,

$$\int \prod_{iI} d\lambda_i^I \prod_{(i,I) < (j,J)} |\lambda_i^I - \lambda_j^J|^{A_{ij}}. \tag{3.3}$$

Here $(i,I) < (j,J)$ if $i < j$ or $i = j, I < J$.

There is a close relationship between the A_r matrix models and conformal field theory.^{16,17} In this relation r free bosons, φ_i , in a $2d$ conformal field theory are related to matrix model quantities in the following way:

$$\partial\varphi_i(z) = \frac{1}{g_s} \text{tr} \left(\frac{1}{z - \Phi_i} \right). \tag{3.4}$$

(This relation holds when the matrix model potential is zero which is the case we are interested in. The factor of g_s is unconventional but convenient for our purposes.) Because of relation (3.4) the free boson CFT vertex operator $\hat{V}_\alpha(z) = e^{\alpha\varphi_i(z)}$ translates into

$$\prod_i \det(z - \Phi_i)^{\alpha_i/g_s}. \tag{3.5}$$

Here $\alpha^i = \langle e_i, \alpha \rangle$, where $\alpha = \sum_i \alpha^i \Lambda_i$ (see Appendix, Sec. 1 for a summary of the Lie algebra terminology). Correlation functions of (free boson) CFT vertex operators can therefore be calculated using matrix model technology. The correlation function,

$$\langle \hat{V}_{\alpha_1/g_s}(z_1) \cdots \hat{V}_{\alpha_k/g_s}(z_k) \rangle_{\alpha_0, N}, \tag{3.6}$$

translates into

$$\int \prod_i d\Phi_i \prod_{i,j} dB_{ij} \prod_i \det(z_1 - \Phi_i)^{\alpha_1^i/g_s} \cdots \prod_i \det(z_k - \Phi_i)^{\alpha_k^i/g_s} \times \exp \left[-\frac{1}{g_s} \prod_{i < j} \text{tr}((\delta_{ij} - A_{ij})(B^{ij}\Phi^j\tilde{B}^{ji} - B^{ji}\Phi^i\tilde{B}^{ij})) \right]. \tag{3.7}$$

(In this paper we restrict our attention to correlation functions on genus 0 surfaces.) In (3.6) the subscript α_0 refers to the fact that there is an extra α_0 charge at infinity. This means that the correlation function really involves $k+1$ vertex operators. Furthermore, in our conventions,

$$A_{ij}N_j = \alpha_0^i/g_s - \sum_{n=1}^k \alpha_n^i/g_s. \tag{3.8}$$

To analyze matrix model expression (3.7), one can use the following relation: $\det(z - \Phi)^{\alpha/g_s} = \exp((\alpha/g_s)\text{tr} \log(z - \Phi))$. The insertions therefore effectively induce the matrix model potentials (of multi Penner type),

$$W_i(\Phi_i) = \text{tr} \sum_{a=1}^k \alpha_a^i \log(z_a - \Phi_i), \tag{3.9}$$

and the matrix models can therefore be analyzed using standard techniques.

In terms of the eigenvalues the matrix model correlation function is

$$\int \prod_{iI} d\lambda_i^I \prod_{(i,I) < (j,J)} |\lambda_i^I - \lambda_j^J|^{A_{ij}} \prod_{i,I} (z_1 - \lambda_i^I)^{\alpha_1^i/g_s} \cdots \prod_{i,I} (z_k - \lambda_i^I)^{\alpha_k^i/g_s}. \tag{3.10}$$

In this expression we have left the integration contour unspecified. The choice of integration contour turns out to be quite subtle and will be discussed in later sections. In previous applications of matrix models to supersymmetric gauge theories,²⁸ the matrix models should properly be

thought of as holomorphic matrix models with a choice of contour, see, e.g., Ref. 30, for a discussion. In the present case, there are also additional subtleties since the potentials are logarithmic.

The proposal of Dijkgraaf and Vafa¹² is as follows: to connect the chiral correlation functions of the vertex operators \hat{V} involving the free scalar fields φ_i to the correlation functions of the chiral vertex operators V involving the fields ϕ_i in the A_r Toda field theory, one should take the large N limit and identify the α_a 's (including α_0) with the external momenta of the Toda theory vertex operators. Furthermore, matrix model potential (3.9) has stationary points which in the large N limit expand into cuts. The corresponding filling fractions $g_s N_i^m$ (subject to the constraint $\sum_{m=1}^{k-2} N_i^m = N_i$) are related to the internal momenta σ_m in the Toda theory (m label the internal momenta and i label the components of each of the σ_m).

This proposal shares many similarities with the earlier work,²⁸ but we should stress that in Ref. 12, the number of terms in the potential is related to the number of nodes in the gauge theory quiver, whereas the number of matrices is related to the rank of the gauge group; in Ref. 28 the roles were reversed.

In Ref. 12 there is also a discussion of the AGT relation using brane probes; this approach will not be used in this paper.

The analysis so far only involves g_s , i.e., $\epsilon_1 = -\epsilon_2$. As mentioned above there is a further refinement of the matrix model that is needed to treat the case with general $\epsilon_{1,2}$. In Ref. 12 it was suggested that the required modification is the so-called β deformation (or β ensemble).³¹ This deformation changes (3.10) to

$$\int \prod_{i,l} d\lambda_i^l \prod_{(i,l) < (j,l)} |\lambda_i^l - \lambda_j^l|^{\beta A_{ij}} \prod_{i,l} (z_1 - \lambda_i^l)^{\sqrt{\beta} \alpha_i^l / g_s} \dots \prod_{i,l} (z_k - \lambda_i^l)^{\sqrt{\beta} \alpha_i^l / g_s}, \tag{3.11}$$

with the identification $\beta = -\epsilon_2 / \epsilon_1$ and $g_s = \sqrt{-\epsilon_1 \epsilon_2}$. Also, (3.8) changes to

$$\beta A_{ij} N_j = \frac{\sqrt{\beta}}{g_s} \alpha_0^j - \frac{\sqrt{\beta}}{g_s} \sum_{n=1}^k \alpha_n^j. \tag{3.12}$$

We should stress that for general β the above integral (3.11) can no longer be viewed as arising from an integral over matrices in any reasonable way. Therefore, strictly speaking, we are no longer dealing with a matrix model. Sometimes the model for general β is called a generalized matrix model, but we will by a slight abuse of terminology continue to call it a matrix model.

In Sec. IV we analyze various aspects of the above matrix model for the case of the A_1 theory and make detailed calculations and comparisons with the corresponding expressions in the $4d$ A_1 quiver gauge theories and the $2d$ Liouville theory. In Sec. V a similar analysis will be performed for the A_r theories with $r > 1$.

IV. THE A_1 MATRIX MODEL

In this section we perform several calculations in the A_1 matrix model. The resulting expressions are compared to the corresponding expressions in the Liouville theory and the A_1 quiver gauge theories.

A. The three-point function

Our first example is the matrix model three-point function,

$$\frac{1}{(2\pi)^N N!} \int \prod_l d\lambda^l \prod_{l < j} |\lambda^l - \lambda^j|^{2\beta} \prod_l (\lambda^l)^{2\alpha_1/\epsilon_1} (1 - \lambda^l)^{2\alpha_2/\epsilon_1}. \tag{4.1}$$

Note that $1/\epsilon_1 = \sqrt{\beta}/g_s$. (When referring to matrix model quantities we will use the notation ϵ_i rather than α_i since it will turn out that our conventions are such that $\epsilon_i = -\alpha_i$.) This complicated looking integral is the so-called Selberg integral³² which can be evaluated exactly with the result

$$\frac{1}{(2\pi)^N} \prod_{I=1}^N \frac{\Gamma(2\alpha_1/\varepsilon_1 + 1 + (I-1)\beta)\Gamma(2\alpha_2/\varepsilon_1 + 1 + (I-1)\beta)\Gamma(I\beta)}{\Gamma(2\alpha_1/\varepsilon_1 + 2\alpha_2/\varepsilon_1 + 2 + (I+N-2)\beta)\Gamma(\beta)}. \tag{4.2}$$

(In Appendix, Sec. 2 we present an alternative derivation of this result when $\beta=1$ using orthogonal polynomials.) In evaluating the above integral we assumed that the choice of integration contour is such that the λ^I 's are integrated over the interval $[0,1]$. (It is possible to perform changes of variables in the above integral to obtain other integration ranges; see, e.g., Ref. 31, Sec. 17.5.) Using the result

$$\Gamma(z) = \sqrt{2\pi}(-\varepsilon_1)^{1/2-z} \frac{\Gamma_2(-z\varepsilon_1 | -\varepsilon_1, -\varepsilon_2)}{\Gamma_2(-z\varepsilon_1 - \varepsilon_2 | -\varepsilon_1, -\varepsilon_2)}, \tag{4.3}$$

where $\Gamma_2(x | -\varepsilon_1, -\varepsilon_2)$ is the Barnes double gamma function,²² together with $\beta = -\varepsilon_2/\varepsilon_1$ and

$$\prod_{I=1}^N \Gamma(z + (I-1)\beta) = (2\pi)^{N/2} (-\varepsilon_1)^{N/2 - Nz} \frac{\Gamma_2(-z\varepsilon_1 + (N-1)\varepsilon_2 | -\varepsilon_1, -\varepsilon_2)}{\Gamma_2(-z\varepsilon_1 - \varepsilon_2 | -\varepsilon_1, -\varepsilon_2)}, \tag{4.4}$$

we find that (4.2) equals [here $\Gamma_2(x)$ is short hand for $\Gamma_2(x | -\varepsilon_1, -\varepsilon_2)$ and $\varepsilon \equiv \varepsilon_1 + \varepsilon_2$]

$$\left(\frac{\varepsilon_1^{N\beta-2\beta+1}}{\Gamma(\beta)}\right)^N \times \frac{\Gamma_2(-2\alpha_1 + N\varepsilon_2 - \varepsilon)\Gamma_2(-2\alpha_2 + N\varepsilon_2 - \varepsilon)\Gamma_2(N\varepsilon_2)\Gamma_2(-2\alpha_1 - 2\alpha_2 + N\varepsilon_2 - 2\varepsilon)}{\Gamma_2(-2\alpha_1 - \varepsilon)\Gamma_2(-2\alpha_2 - \varepsilon)\Gamma_2(0)\Gamma_2(-2\alpha_1 - 2\alpha_2 + 2N\varepsilon_2 - 2\varepsilon)}. \tag{4.5}$$

Finally, using $N\varepsilon_2 = -N\beta\varepsilon_1 = (-\alpha_0 + \alpha_1 + \alpha_2)$ we obtain

$$\left(\frac{\varepsilon_1^{N\beta-2\beta+1}}{\Gamma(\beta)}\right)^N \times \frac{\Gamma_2(-\alpha_0 - \alpha_1 + \alpha_2 - \varepsilon)\Gamma_2(-\alpha_0 + \alpha_1 - \alpha_2 - \varepsilon)\Gamma_2(-\alpha_0 + \alpha_1 + \alpha_2)\Gamma_2(-\alpha_0 - \alpha_1 - \alpha_2 - 2\varepsilon)}{\Gamma_2(-2\alpha_1 - \varepsilon)\Gamma_2(-2\alpha_2 - \varepsilon)\Gamma_2(0)\Gamma_2(-2\alpha_0 - 2\varepsilon)}. \tag{4.6}$$

In general, the three-point function in a $2d$ conformal field theory does not factorize into holomorphic and antiholomorphic parts so there is no unambiguous meaning to a ‘‘chiral three-point function.’’ However, after suitably rescaling the vertex operators with multiplicative factors depending on their momenta, Liouville three-point function (2.6) can be written as (recall that $\alpha_i^* = Q - \alpha_i$)

$$\begin{aligned} & [Y(\alpha_1 + \alpha_2 + \alpha_3 - Q)Y(-\alpha_1 + \alpha_2 + \alpha_3)Y(\alpha_1 - \alpha_2 + \alpha_3)Y(\alpha_1 + \alpha_2 - \alpha_3)]^{-1} \\ & = |\Gamma_b(2Q - \alpha_1 - \alpha_2 - \alpha_3)\Gamma_b(Q + \alpha_1 - \alpha_2 - \alpha_3)\Gamma_2(Q - \alpha_1 + \alpha_2 - \alpha_3)\Gamma_2(\alpha_1 + \alpha_2 - \alpha_3)|^2, \end{aligned} \tag{4.7}$$

where $\Gamma_b(x)$ is short hand for $\Gamma_2(x | b, b^{-1})$. Therefore, for the Liouville theory there is a natural definition of a chiral three-point function as the ‘‘square root’’ of (4.7). After a suitable redefinition of the matrix model vertex operators, we see that matrix model three-point function (4.6) precisely captures the chiral part of the Liouville three-point function, provided that $\alpha_0 \rightarrow \alpha_3$, and we identify

$$\varepsilon_1 = -b, \quad \varepsilon_2 = -1/b. \tag{4.8}$$

The matrix model expression can also be compared with (the perturbative part of) the Nekrasov partition function of the corresponding gauge theory, which in the present case is the so-called T_2 (or $\mathcal{T}_{3,0}(A_1)$) theory—a theory of four free hypermultiplets.³ Redefining the matrix model vertex operators as above, we are left with the four Γ_2 factors in the numerator of (4.6). These are of

precisely the right form to reproduce (the perturbative part of) the T_2 theory, cf. (2.18) (with a suitable definition of the four masses). (Possibly one can also make sense of (4.6) within the framework in Ref. 33.)

We should also mention that the derivation in Ref. 20 of the full three-point function in the Liouville theory was based on an argument which involved a complex version of the above integral. Although the equations are similar the logic was different.

B. Higher-point functions

One can also consider higher-point functions in the matrix model. A tractable example is a four-point function, where one of the α_i , α_3 say, is equal to $\varepsilon_1/2$ or $\varepsilon_2/2$, i.e., the integral

$$\int \prod_I d\lambda^I \prod_{I<J} |\lambda^I - \lambda^J|^{2\beta} \prod_I (z - \lambda^I)^{2\alpha_3/\varepsilon_1} (\lambda^I)^{2\alpha_1/\varepsilon_1} (1 - \lambda^I)^{2\alpha_2/\varepsilon_1}, \quad (4.9)$$

with α_3 equal to $\varepsilon_1/2$ or $\varepsilon_2/2$. In Ref. 34 it was shown that the above integral satisfies the hypergeometric differential equation,

$$z(1-z) \frac{d^2 F(z)}{dz^2} + [C - (A+B+1)z] \frac{dF(z)}{dz} - ABF(z) = 0, \quad (4.10)$$

where, if $\alpha_3 = \varepsilon_1/2$,

$$A = -N, \quad B = \frac{1}{\beta} (2\alpha_1/\varepsilon_1 + 2\alpha_2/\varepsilon_1 + 2) + N - 1, \quad C = \frac{1}{\beta} (2\alpha_1/\varepsilon_1 + 1), \quad (4.11)$$

and, if $\alpha_3 = \varepsilon_2/2$,

$$A = \beta N, \quad B = -(2\alpha_1/\varepsilon_1 + 2\alpha_2/\varepsilon_1 + 1) + \beta(2 - N), \quad C = -2\alpha_1/\varepsilon_1 + \beta. \quad (4.12)$$

On the other hand, it is known that in the Liouville theory the four-point function,

$$\langle V_{-b/2}(z) V_{\alpha_1}(0) V_{\alpha_2}(1) V_{\alpha_0}(\infty) \rangle, \quad (4.13)$$

satisfies the equation

$$\left[\partial_z^2 - b^2 \frac{2z-1}{z(z-1)} \partial_z + b^2 \frac{\Delta(\alpha_1)}{z^2} + b^2 \frac{\Delta(\alpha_2)}{(z-1)^2} - b^2 \frac{\Delta(-b/2) + \Delta(\alpha_1) + \Delta(\alpha_2) - \Delta(\alpha_0)}{z(z-1)} \right] H(z) = 0, \quad (4.14)$$

where $\Delta(\alpha) = \alpha(Q - \alpha)$. After writing $H(z) = z^{b\alpha_1} (1-z)^{b\alpha_2} F(z)$ the above equation reduces to hypergeometric equation (4.10) with

$$A = b(\alpha_1 + \alpha_2 - \alpha_0 - b/2), \quad B = b \left(\alpha_1 + \alpha_2 + \alpha_0 - \frac{3}{2}b - \frac{1}{b} \right), \quad C = b(2\alpha_1 - b). \quad (4.15)$$

Using $\beta = -\varepsilon_2/\varepsilon_1$ and $-\varepsilon_2 N - \alpha_0 + \alpha_1 + \alpha_2 + \varepsilon_1/2 = 0$ (compared to the corresponding expression for the three-point function there is now an extra term coming from $\alpha_3 = \varepsilon_1/2$), we see that (4.11) and (4.15) agree provided we use identifications (4.8). This analysis shows that matrix model integral (4.9) with the above identifications is proportional to the chiral block in the Liouville CFT. Note that for special correlation function (4.13) the internal momentum is restricted to two discrete values corresponding to the two solutions to the hypergeometric equation. The case with an insertion of the vertex operator $V_{-1/b}(z)$ can be treated analogously and compared to (4.12). The choice of signs in (4.8) differs from the usual convention and means that $\varepsilon_i = -\varepsilon_i$, but note that both $\varepsilon_1 \varepsilon_2$ and $\varepsilon_2/\varepsilon_1$ are unaffected. We could make sign changes elsewhere to restore the usual rule, but this would clutter some of the above formulas.

One can also check that the above expression agrees with the corresponding Nekrasov partition function. This was anticipated in Ref. 4 and derived in detail in Ref. 5. We briefly recall the argument here. The instanton partition function is as in (4.17) with $m, n=1, 2$ and $(\hat{a}_1, \hat{a}_2)=(a, -a)$. If we tune the Coulomb modulus a to fulfill $P(a)=0$ by setting $a=m_1$, then only terms in the sum with only Y_2 nonempty give nonvanishing contributions. If we furthermore set $m_2=-a-\epsilon_1$ then only those Y_2 tableaux that have boxes only in the first column survive (in other words $k_{2,j}^T$ is only nonzero for $j=1$, so that $i=1, \dots, k_{2,1}^T$ and $k_{2,i}=1$). Next using the AGT relation in the form

$$m_1 = -\frac{\epsilon}{2} + \alpha_1 + \alpha_3, \quad m_2 = \frac{\epsilon}{2} - \alpha_1 + \alpha_3, \quad m_3 = -\frac{\epsilon}{2} + \alpha_4 + \alpha_2, \quad m_4 = \frac{\epsilon}{2} - \alpha_4 + \alpha_2, \tag{4.16}$$

one finds $\alpha_3 = -\epsilon_1/2$ and

$$Z_{\text{inst}} = \sum_{\ell=0}^{\infty} \frac{(A)_\ell (B)_\ell y^\ell}{(C)_\ell \ell!}, \tag{4.17}$$

where $(X)_n = X(X+1)\cdots(X+n-1)$ is the Pochhammer symbol and

$$A = \left(\alpha_1 + \alpha_2 - \alpha_4 - \frac{\epsilon_1}{2} \right) / \epsilon_2, \quad B = \left(\alpha_1 + \alpha_2 + \alpha_4 - \frac{3}{2} \epsilon_1 - \epsilon_2 \right) / \epsilon_2, \quad C = -(2\alpha_1 - \epsilon_1) / \epsilon_2, \tag{4.18}$$

which agrees with (4.15) provided $\alpha_4 \rightarrow \alpha_0$, $\epsilon_1 = b$, and $\epsilon_2 = 1/b$. Together with the fact that expression (4.17) is precisely the series expansion of the hypergeometric function ${}_2F_1(A, B; C; y)$, which solves differential equation (4.10) with $y=z$, this shows that the instanton partition function agrees with the chiral block (up to an overall factor).

The matrix integral corresponding to a correlation function with k insertions of $V_{-b/2}$, i.e.,

$$\int \prod_I d\lambda^I \prod_{I < J} |\lambda^I - \lambda^J|^{2\beta} \prod_I (\lambda^I)^{2\alpha_1/\epsilon_1} (1 - \lambda^I)^{2\alpha_2/\epsilon_1} \prod_{a=1}^k (z_a - \lambda^I), \tag{4.19}$$

was also calculated exactly in Ref. 34. The result is (proportional to) the generalized hypergeometric function,

$${}_2F_1^\beta \left(-N, \frac{1}{\beta} ([2\alpha_1 + 2\alpha_2]/\epsilon_1 + k + 1) + N - 1; \frac{1}{\beta} (2\alpha_1/\epsilon_1 + k); z_1, \dots, z_k \right). \tag{4.20}$$

[Note that it is also possible treat the cases with $\hat{V}_{-\ell b/2}$ insertions by setting $z_1 = \dots = z_\ell$ in (4.19) and (4.20).] The function ${}_2F_1^\beta(A_1, A_2; B_1; z_1, \dots, z_k)$ is defined as

$${}_2F_1^\beta(A, B; C; z_1, \dots, z_k) = \sum_{\xi} \frac{[A]_{\xi}^{\beta} [B]_{\xi}^{\beta} P_{\xi}^{\beta}(z_1, \dots, z_k)}{[C]_{\xi}^{\beta} |\xi|!}, \tag{4.21}$$

where the sum is over all partitions ξ with at most k parts,

$$[X]_{\xi}^{\beta} = \prod_i \left(X - \frac{1}{\beta} (i - 1) \right)_{\xi_i}, \tag{4.22}$$

and $P_{\xi}^{\beta}(z_1, \dots, z_k)$ is a (properly normalized) Jack polynomial. See Ref. 34, for further details about the notation. It should be possible to show that the corresponding CFT and gauge theory calculations lead to the same result.

We should also point out that the treatment of gauge theory surface and line operators in the Liouville language discussed in Ref. 8 involves the insertion of vertex operators with $\alpha=-b/2$. This is precisely the class of operator insertions we have discussed above.

To go beyond the restricted set of correlation functions discussed above, one possible approach would be to try to mimic the CFT method and determine the correlation functions in a perturbative expansion in the z_i .

An alternative approach is to use matrix model perturbation theory. For the four-point function one can (at least when $\beta=1$) use the method in Refs. 35 and 36; for the multicut and A_r quiver extensions, see, e.g., Ref. 37. One can also obtain a perturbative expansion within the framework of Secs. IV C and IV D; see Sec. IV E below for some sample calculations. On the CFT side the matrix model perturbation theory is a somewhat peculiar expansion and it is not clear what its relevance is. A drawback of the perturbative matrix model approach is that solving for the stationary points leads to complicated expressions, but a clear advantage is that one can handle arbitrary punctures (i.e., arbitrary vertex operator momenta) using this method (both for the A_1 and A_r theories). Let us also mention that in Ref. 37 another, less direct, matrix model approach to $\mathcal{N}=2$ gauge theories was discussed; it might be interesting to try to connect it to the present approach.

C. The curve: One-cut solutions

In this section we analyze the one-cut matrix model spectral curves, focusing on the matrix model corresponding to the three-point function discussed in Sec. IV A. We start by reviewing some well-known results. See, e.g., Ref. 38, for further details. In the standard diagonal gauge (eigenvalue basis) the one-matrix model partition function is

$$Z = \frac{1}{(2\pi)^N N!} \int \prod_{I=1}^N d\lambda^I \prod_{I<J} (\lambda^I - \lambda^J)^2 \exp\left(-\frac{1}{g_s} \sum_{I=1}^N W(\lambda^I)\right) = \frac{1}{(2\pi)^N N!} \int \sum_{I=1}^N d\lambda^I \exp(N^2 S_{\text{eff}}[\rho(\lambda)]), \quad (4.23)$$

where

$$S_{\text{eff}}[\rho(\lambda)] = -\frac{1}{t} \int_{\mathcal{C}} d\lambda \rho(\lambda) W(\lambda) + \int \int_{\mathcal{C} \times \mathcal{C}} d\lambda d\lambda' \rho(\lambda) \rho(\lambda') \log|\lambda - \lambda'|. \quad (4.24)$$

(To conform with standard matrix model conventions W in this section and in Secs. IV D and IV E corresponds to $-W$ elsewhere in the paper. This also implies that the α_i differ by a sign.) Here t is the 't Hooft coupling $t=Ng_s$ and we have introduced the eigenvalue density $\rho(\lambda) = (1/N) \sum_{I=1}^N \delta(\lambda - \lambda^I)$, normalized as $\int_{\mathcal{C}} d\lambda \rho(\lambda) = 1$. In this expression one still needs to specify the geometrical nature of the cut, \mathcal{C} . In the most general case $\rho(\lambda)$ has compact support, with \mathcal{C} a multicut region with s cuts. For the moment we shall focus on the one-cut case, with $\mathcal{C}=[a, b]$. If one now considers the Riemann surface which corresponds to a double-sheet covering of the complex plane, \mathbb{C} , with precisely the above cut, it is natural to define the A -cycle as the cycle around the cut. In this case, the B -cycle goes from the end point of the cut to infinity on one of the two sheets and back again on the other.

The generator of single-trace correlation functions is given by the resolvent

$$\omega(z) = \frac{1}{N} \left\langle \text{tr} \frac{1}{z - \Phi} \right\rangle = \frac{1}{N} \sum_{k=0}^{+\infty} \frac{1}{z^{k+1}} \langle \text{tr} \Phi^k \rangle, \quad (4.25)$$

which has the standard expansion $\omega(z) = \sum_{g=0}^{+\infty} g_s^{2g} \omega_g(z)$ with

$$\omega_0(z) = \int d\lambda \frac{\rho(\lambda)}{z - \lambda}. \quad (4.26)$$

The normalization of the eigenvalue density then implies that $\omega_0(z) \sim 1/z$ as $z \rightarrow +\infty$. Also, observe that $\omega_0(z)$ is singular for $z \in \mathcal{C}$, while it is analytic for $z \notin \mathcal{C}$. One may compute $\omega_0(z)$ by making use of the large N saddle-point equations of motion of the matrix model,

$$\omega_0(z + i\epsilon) + \omega_0(z - i\epsilon) = \frac{1}{t} W'(z) = 2\text{PV} \int_{\mathcal{C}} d\lambda \frac{\rho(\lambda)}{z - \lambda}. \quad (4.27)$$

In a similar fashion, $\omega_0(z)$ is related to the eigenvalue density as

$$\rho(z) = -\frac{1}{2\pi i} (\omega_0(z + i\epsilon) - \omega_0(z - i\epsilon)) = -\frac{1}{\pi} \text{Im} \omega_0(z). \quad (4.28)$$

For a generic one-cut solution, the large N resolvent is given by the ansatz

$$\omega_0(z) = \frac{1}{2t} \oint_{\mathcal{C}} \frac{dw}{2\pi i} \frac{W'(w)}{z - w} \sqrt{\frac{(z - a)(z - b)}{(w - a)(w - b)}}, \quad (4.29)$$

where one still needs to specify the end points of the cut, $\{a, b\}$. An equivalent way to describe the matrix model geometry is via the corresponding spectral curve, $y(z)$, which basically describes the geometry of the Riemann surface we mentioned above. One may write

$$y(z) = W'(z) - 2t\omega_0(z) \equiv M(z) \sqrt{(z - a)(z - b)}, \quad (4.30)$$

with (this particular expression only holds for polynomial potentials)

$$M(z) = \oint_{(0)} \frac{dw}{2\pi i} \frac{W'(1/w)}{1 - wz} \frac{1}{\sqrt{(1 - aw)(1 - bw)}}, \quad (4.31)$$

where, again, one needs to specify the end points of the cut, $\{a, b\}$. The aforementioned large z asymptotics of the resolvent immediately yield two conditions for these two unknowns. They are

$$\oint_{\mathcal{C}} \frac{dw}{2\pi i} \frac{w^n W'(w)}{\sqrt{(w - a)(w - b)}} = 2t\delta_{ns}, \quad (4.32)$$

for $n=0, 1$, fully determining the end points of the single cut $s=1$.

It is also useful to define the holomorphic effective potential as $V'_{\text{h,eff}}(z) = y(z)$. The effective potential is then given by the real part of the holomorphic effective potential, in such a way that

$$V_{\text{eff}}(\lambda) = \text{Re} \int_a^\lambda dz y(z). \quad (4.33)$$

The real part of the spectral curve therefore corresponds to the force exerted on a given eigenvalue. The imaginary part of the spectral curve, on the other hand, is related to the eigenvalue density via

$$\rho(z) = \frac{1}{2\pi t} \text{Im} y(z). \quad (4.34)$$

Finally, it turns out that one may also write the 't Hooft parameter in terms of the spectral geometry as

$$t = \frac{1}{4\pi i} \oint_A dz y(z). \quad (4.35)$$

We now turn to our main point and consider the large N expansion of the matrix model with potential

$$W(z) = \sum_{i=1}^k 2\alpha_i \log(z - z_i) \quad (4.36)$$

where k , $\{\alpha_i\}_{i=1}^k$, and $\{z_i\}_{i=1}^k$ are parameters we shall keep unspecified for the moment. It follows from (4.36) that

$$W'(z) = \sum_{i=1}^k \frac{2\alpha_i}{z - z_i}, \quad (4.37)$$

implying that the logarithmic terms in (4.36) will not be terribly problematic—one only needs to take into account extra poles, when moving the contours of integration around the complex plane. We begin by focusing on the one-cut solution, for which the large N resolvent is given by the ansatz

$$\omega_0(z) = \frac{1}{2t} \oint_C \frac{dw}{2\pi i} \frac{W'(w)}{z - w} \sqrt{\frac{(z-a)(z-b)}{(w-a)(w-b)}}, \quad (4.38)$$

where the integrand now has poles at the locations $\{z_i\}$ but, because the potential is purely logarithmic, any pole of the integrand at infinity is gone. In this case, a straightforward deformation of the integration contour reduces the integral along the cut to a sum of simple poles as

$$\omega_0(z) = \frac{1}{2t} \left(W'(z) - \sum_{i=1}^k \frac{2\alpha_i}{(z - z_i) \sqrt{(z_i - a)(z_i - b)}} \sqrt{(z - a)(z - b)} \right). \quad (4.39)$$

The large z asymptotics, $\omega_0(z) \sim 1/z + \dots$ as $z \rightarrow \infty$, immediately implies that the end points of the cut $\mathcal{C} = [a, b]$ are determined by the system

$$\sum_{i=1}^k \frac{2\alpha_i}{\sqrt{(z_i - a)(z_i - b)}} = 0, \quad (4.40)$$

$$\sum_{i=1}^k \left(2\alpha_i - \frac{2\alpha_i z_i}{\sqrt{(z_i - a)(z_i - b)}} \right) = 2t, \quad (4.41)$$

and the single-cut spectral geometry is then described by the curve

$$y(z) = \sum_{i=1}^k \frac{2\alpha_i}{(z - z_i) \sqrt{(z_i - a)(z_i - b)}} \sqrt{(z - a)(z - b)}. \quad (4.42)$$

One may also compute the holomorphic effective potential in a simple manner. We obtain

$$\begin{aligned} V_{\text{h,eff}}(z) = & -2 \left(2t - \sum_{i=1}^k 2\alpha_i \right) \log[2(\sqrt{z-a} + \sqrt{z-b})] + \sum_{i=1}^k \left(2\alpha_i \log \left[1 - \frac{\sqrt{z-a}\sqrt{z_i-b}}{\sqrt{z-b}\sqrt{z_i-a}} \right] \right. \\ & \left. - 2\alpha_i \log \left[1 + \frac{\sqrt{z-a}\sqrt{z_i-b}}{\sqrt{z-b}\sqrt{z_i-a}} \right] \right), \end{aligned} \quad (4.43)$$

with a and b determined by the system above. The structure of Stokes lines for this effective potential will be more complicated than in the usual polynomial cases.

Let us now specialize to the matrix model corresponding to the three-point function, with potential

$$W_{3\text{pf}}(z) = 2\alpha_1 \log z + 2\alpha_2 \log(z-1), \quad (4.44)$$

and the further constraint

$$t = \alpha_0 + \sum_{i=1}^k \alpha_i. \quad (4.45)$$

Next we turn to the study of the spectral geometry associated with matrix model potential (4.44) beginning with the “classical” geometry. The critical points are located at the points z_* , such that

$$W'_{3\text{pf}}(z_*) = \frac{2\alpha_1}{z_*} + \frac{2\alpha_2}{z_* - 1} = 0. \quad (4.46)$$

In this case, the general solution is

$$z_* = \frac{\alpha_1}{\alpha_1 + \alpha_2}. \quad (4.47)$$

In the classical limit where the 't Hooft coupling vanishes (i.e., where we choose vertex operators such that $\alpha_0 + \alpha_1 + \alpha_2 = 0$), one simply has $y(z) = W'_{3\text{pf}}(z)$ and the cut collapses to the critical point of the potential, $[a, b] \rightarrow z_*$. In this particular case,

$$W_{3\text{pf}}(z_*) = 2\alpha_1 \log \frac{\alpha_1}{\alpha_1 + \alpha_2} + 2\alpha_2 \log \frac{\alpha_2}{\alpha_1 + \alpha_2}. \quad (4.48)$$

Now, the full spectral geometry will be such that the critical point z_* opens up into a branch cut, of size t (and not touching the marked points associated with the vertex operator insertions at $\{0, 1, \infty\}$). This blown-up geometry of the spectral curve will have its shape determined by the parameters α_0 , α_1 , and α_2 . Clearly, because there is a single critical point, the spectral geometry will correspondingly have a single cut—the situation we studied above. The spectral geometry associated with the matrix model with potential (4.44) is therefore a genus zero one-cut Riemann surface. From our previous general results it follows that the spectral curve is

$$y(z) = \left(\frac{2\alpha_1}{z\sqrt{ab}} + \frac{2\alpha_2}{(z-1)\sqrt{(1-a)(1-b)}} \right) \sqrt{(z-a)(z-b)}, \quad (4.49)$$

where the end points a and b are obtained from the solution to the system

$$\begin{aligned} \frac{2\alpha_1}{\sqrt{ab}} + \frac{2\alpha_2}{\sqrt{(1-a)(1-b)}} &= 0, \\ \frac{2\alpha_2}{\sqrt{(1-a)(1-b)}} &= -2\alpha_0, \end{aligned} \quad (4.50)$$

with (partial) solution

$$\sqrt{ab} = \frac{\alpha_1}{\alpha_0}, \quad (4.51)$$

$$\sqrt{(1-a)(1-b)} = -\frac{\alpha_2}{\alpha_0}. \quad (4.52)$$

This immediately simplifies the spectral curve to

$$y(z) = \frac{2\alpha_0}{z(1-z)} \sqrt{(z-a)(z-b)}. \tag{4.53}$$

For completeness we also give the explicit solution to (4.50),

$$a = \frac{(\alpha_0^2 + \alpha_1^2 - \alpha_2^2 - \sqrt{(\alpha_0 - \alpha_1 - \alpha_2)(\alpha_0 + \alpha_1 - \alpha_2)(\alpha_0 - \alpha_1 + \alpha_2)(\alpha_0 + \alpha_1 + \alpha_2)})}{2\alpha_0^2},$$

$$b = \frac{(\alpha_0^2 + \alpha_1^2 - \alpha_2^2 + \sqrt{(\alpha_0 - \alpha_1 - \alpha_2)(\alpha_0 + \alpha_1 - \alpha_2)(\alpha_0 - \alpha_1 + \alpha_2)(\alpha_0 + \alpha_1 + \alpha_2)})}{2\alpha_0^2}. \tag{4.54}$$

Spectral curve (4.53) can be written as

$$y^2 = \frac{P_2(z)}{z^2(1-z)^2}, \tag{4.55}$$

where $P_2(z)$ is a polynomial of degree of 2. This expression was also written in Ref. 12; here we have also explicitly determined the coefficients in $P_2(z)$ in terms of the three α_i 's.

D. The curve: Multicut solutions

Let us return to the geometrical nature of the cut, \mathcal{C} , where \mathcal{C} is now a multicut region with s cuts. There are two cases: when s is smaller or equal to the number of minima of the potential $V(\lambda)$, a typical situation in the standard matrix model context; or when s is equal to the number of nondegenerate extrema of the potential $V(\lambda)$, the situation which arises when dealing with topological strings which is the case relevant to us. More precisely,

$$\mathcal{C} = \bigcup_{l=1}^s A^l, \tag{4.56}$$

where $A^l = [x_{2l-1}, x_{2l}]$ are the s cuts and $x_1 < x_2 < \dots < x_{2s}$. If one now considers the hyperelliptic Riemann surface which corresponds to a double-sheet covering of the complex plane, \mathbb{C} , with precisely the same cuts as above, A^l , it is then natural to define the A^l -cycle as the cycle around the A^l cut, with the B_l -cycle following via $B_l \cap A^l = \delta_l^j$. In this case, the B_l -cycle goes from the end point of the A^l cut to infinity on one of the two sheets and back again on the other.

For a generic multicut solution, the large N resolvent is given by the ansatz

$$\omega_0(z) = \frac{1}{2t} \oint_{\mathcal{C}} \frac{dw}{2\pi i} \frac{W'(w)}{z-w} \sqrt{\prod_{k=1}^{2s} \frac{z-x_k}{w-x_k}}, \tag{4.57}$$

where one still needs to specify the end points of the s cuts, $\{x_k\}$. An equivalent way to describe the matrix model geometry is via the corresponding spectral curve, $y(z)$, which basically describes the hyperelliptic geometry of the Riemann surface we mentioned above. One may write

$$y(z) = W'(z) - 2t\omega_0(z) \equiv M(z)\sqrt{\mathfrak{S}_s(z)}, \tag{4.58}$$

where

$$\mathfrak{S}_s(z) \equiv \prod_{k=1}^{2s} (z-x_k) \tag{4.59}$$

and (this particular expression only holds for polynomial potentials)

$$M(z) = \oint_{(0)} \frac{dw}{2\pi i} \frac{W'(1/w)}{1-wz} \frac{w^{s-1}}{\sqrt{\prod_{k=1}^{2s} (1-x_k w)}}, \tag{4.60}$$

and where, again, one still needs to specify the end points of the s cuts, $\{x_k\}$. The aforementioned large z asymptotics of the resolvent immediately yield $s+1$ conditions for these $2s$ unknowns. They are

$$\oint_C \frac{dw}{2\pi i} \frac{w^n W'(w)}{\sqrt{\prod_{k=1}^{2s} (w-x_k)}} = 2t \delta_{ns} \tag{4.61}$$

for $n=0, 1, \dots, s$. In order to fully solve the problem, one still requires $s-1$ extra conditions for the full set of $\{x_k\}$. (Observe that no further conditions were required in the previous one-cut case, where $C=[x_1, x_2] \equiv [a, b]$. Indeed, in that situation the large z asymptotics fully determined the end points of the single cut.) These extra conditions depend on whether one wants to consider the standard matrix model or the topological string case. In the first option one considers all the different cuts at equipotential lines, where this condition may be written as

$$\int_{x_{2l}}^{x_{2l+1}} dz y(z) = 0. \tag{4.62}$$

A physical understanding of this expression says that there is no force moving eigenvalues from one cut to another. In contrast, the topological string option (which is the case relevant to our analysis) generically corresponds to an unstable situation from a purely matrix model point of view. In this case one considers the filling fractions,

$$\eta^I \equiv \frac{N_I}{N} \equiv \int_{A^I} d\lambda \rho(\lambda), \quad I = 1, 2, \dots, s, \tag{4.63}$$

as parameters, or moduli, of the problem under consideration. Observe that here $\sum_{I=1}^s \eta^I = 1$, making it an actual total of $s-1$ extra parameters, precisely the number required. By rewriting the eigenvalue density in terms of the resolvent, and the resolvent in terms of the spectral curve, one is led to the equivalent definition,

$$\eta^I = \frac{1}{4\pi i t} \oint_{A^I} dz y(z). \tag{4.64}$$

One may also use as moduli the partial 't Hooft couplings $t^I = t \eta^I = g_s N_I$. In this case

$$t^I = \frac{1}{4\pi i} \oint_{A^I} dz y(z), \tag{4.65}$$

with $\sum_{I=1}^s t^I = t$, making a total of $s-1$ moduli.

Let us now return to the large N expansion of the matrix model with potential (4.36) and associated derivative (4.37) and briefly discuss how one may use standard saddle-point techniques to address multicut solutions. Again the logarithmic terms are not a problem; as we have seen before one only needs to take into account extra poles, when moving around the complex plane. When addressing multicut solutions, with s cuts, the large N resolvent is given by ansatz (4.57), where the integrand now has poles at the locations z_i and any pole of the integrand at infinity is gone. As in the single-cut case, a straightforward deformation of the integration contour reduces the integral along the cut to a sum of simple poles as

$$\omega_0(z) = \frac{1}{2t} \left(W'(z) - \sum_{i=1}^k \frac{2\alpha_i}{(z-z_i)\sqrt{\prod_{n=1}^{2s}(z_i-x_n)}} \sqrt{\prod_{n=1}^{2s}(z-x_n)} \right). \quad (4.66)$$

The large z asymptotics, $\omega_0(z) \sim 1/z + \dots$ as $z \rightarrow \infty$, immediately yield $s+1$ conditions on the $2s$ end points of the cuts via the system

$$\sum_{i=1}^k \oint_C \frac{dw}{2\pi i} \frac{2\alpha_i w^m}{(w-z_i)\sqrt{\prod_{n=1}^{2s}(w-x_n)}} = 2t \delta_{ms} \quad (4.67)$$

for $m=0, 1, \dots, s$. This may be written more explicitly as (notice that now the integrand does have a pole at infinity, when $m=s$)

$$\sum_{i=1}^k \frac{2\alpha_i}{\sqrt{\prod_{n=1}^{2s}(z_i-x_n)}} = 0,$$

$$\sum_{i=1}^k \frac{2\alpha_i z_i}{\sqrt{\prod_{n=1}^{2s}(z_i-x_n)}} = 0,$$

...

$$\sum_{i=1}^k \frac{2\alpha_i z_i^{s-1}}{\sqrt{\prod_{n=1}^{2s}(z_i-x_n)}} = 0,$$

$$\sum_{i=1}^k \left(2\alpha_i - \frac{2\alpha_i z_i^s}{\sqrt{\prod_{n=1}^{2s}(z_i-x_n)}} \right) = 2t. \quad (4.68)$$

Of course in order to fully solve the problem, one still requires $s-1$ extra conditions for the full set of end points $\{x_k\}$. Finally, the spectral geometry is described by the hyperelliptic spectral curve,

$$y(z) = \sum_{i=1}^k \frac{2\alpha_i}{(z-z_i)\sqrt{\prod_{n=1}^{2s}(z_i-x_n)}} \sqrt{\prod_{n=1}^{2s}(z-x_n)}. \quad (4.69)$$

Having understood the multicut spectral geometry, we now focus on the case corresponding to the chiral four-point function in the Liouville theory, namely, the matrix model with potential

$$W_{4\text{pf}}(z) = 2\alpha_1 \log z + 2\alpha_2 \log(z-1) + 2\alpha_3 \log(z-\zeta). \quad (4.70)$$

(Here and in Sec. IV E for clarity we use $\zeta \equiv z_1$ to denote the location of the vertex operator insertion; in other sections z is used.) We begin with the ‘‘classical’’ geometry of the potential. The critical points are located at the points z_* such that

$$W'_{4\text{pf}}(z_*) = \frac{2\alpha_1}{z_*} + \frac{2\alpha_2}{z_*-1} + \frac{2\alpha_3}{z_*-\zeta} = 0. \quad (4.71)$$

In this case, the general solutions are

$$z_{*,1} = \frac{(1+\zeta)\alpha_1 + \zeta\alpha_2 + \alpha_3 - \sqrt{((1+\zeta)\alpha_1 + \zeta\alpha_2 + \alpha_3)^2 - 4\alpha_1\zeta(\alpha_1 + \alpha_2 + \alpha_3)}}{2(\alpha_1 + \alpha_2 + \alpha_3)},$$

$$z_{*,2} = \frac{(1 + \zeta)\alpha_1 + \zeta\alpha_2 + \alpha_3 + \sqrt{((1 + \zeta)\alpha_1 + \zeta\alpha_2 + \alpha_3)^2 - 4\alpha_1\zeta(\alpha_1 + \alpha_2 + \alpha_3)}}{2(\alpha_1 + \alpha_2 + \alpha_3)}. \quad (4.72)$$

Generically, the critical points $z_{*,1}$ and $z_{*,2}$ will open up into branch cuts, of sizes t_1 and t_2 (and not touching the marked points associated with the vertex operator insertions at $\{0, 1, \zeta, \infty\}$). Because of the two critical points, the most general spectral geometry will correspondingly have two cuts and the spectral geometry associated with the chiral Liouville four-point function is a genus one two-cut (elliptic) Riemann surface. This two-cut blown-up geometry of the spectral curve will have its shape determined by α_0 , α_1 , α_2 , and α_3 . To be more precise, from the large z asymptotics of the genus zero resolvent one obtains three conditions on the end points of the two cuts; the remaining required condition arising from the partial 't Hooft moduli, t_1 or t_2 (where $t_1 + t_2 = t$). In Ref. 12 this modulus is actually traded for $a = t_2 - t_1$, the Coulomb modulus in the gauge theory, and we shall use this notation henceforth.

Of course there are particular points in the moduli space of the elliptic spectral curve where the geometry simplifies. One is the degenerate case where both partial 't Hooft couplings vanish. Another special point occurs when only one of the critical points opens up into a branch cut, in which case one is dealing with a one-cut pinched spectral geometry, the pinch at the location of the critical point that remains "closed." Let us consider this special case, where

$$t_1 = -\frac{1}{2}a + \frac{1}{2}\sum_{i=0}^3 \alpha_i = 0, \quad (4.73)$$

$$t_2 = \frac{1}{2}a + \frac{1}{2}\sum_{i=0}^3 \alpha_i = t. \quad (4.74)$$

From our previous (single-cut) result (4.42) we have

$$y(z) = \sqrt{(z-a)(z-b)} \times \left[\frac{2\alpha_1}{z\sqrt{ab}} + \frac{2\alpha_2}{(z-1)\sqrt{(1-a)(1-b)}} + \frac{2\alpha_3}{(z-\zeta)\sqrt{(\zeta-a)(\zeta-b)}} \right], \quad (4.75)$$

where the end points a and b are a solution to the system

$$\frac{2\alpha_1}{\sqrt{ab}} + \frac{2\alpha_2}{\sqrt{(1-a)(1-b)}} + \frac{2\alpha_3}{\sqrt{(\zeta-a)(\zeta-b)}} = 0, \quad (4.76)$$

$$\frac{2\alpha_2}{\sqrt{(1-a)(1-b)}} + \frac{\zeta 2\alpha_3}{\sqrt{(\zeta-a)(\zeta-b)}} = -2\alpha_0. \quad (4.77)$$

This may be equivalently written as

$$\frac{2\alpha_1}{\sqrt{ab}} = 2\alpha_0 + \frac{(\zeta-1)2\alpha_3}{\sqrt{(\zeta-a)(\zeta-b)}}, \quad (4.78)$$

$$\frac{2\alpha_2}{\sqrt{(1-a)(1-b)}} = -2\alpha_0 - \frac{\zeta 2\alpha_3}{\sqrt{(\zeta-a)(\zeta-b)}}, \quad (4.79)$$

which slightly simplifies the spectral curve to

$$y(z) = \left(\frac{2\alpha_0}{z(1-z)} + \frac{\zeta(\zeta-1)}{z(z-1)(z-\zeta)} \frac{2\alpha_3}{\sqrt{(\zeta-a)(\zeta-b)}} \right) \sqrt{(z-a)(z-b)}. \quad (4.80)$$

Generically, however, we are dealing with a system of quartic equations and, although it can be solved algebraically, its exact solution is not terribly illuminating. In the following, we therefore choose a different route and solve this system perturbatively in ζ . This is motivated by the expansion on the CFT side and simplifies the problem considerably. To first order, we obtain the solution

$$\begin{aligned} \sqrt{ab} &= \frac{\alpha_1 + \alpha_3}{\alpha_0} - \frac{((\alpha_1 + \alpha_3)^2 + \alpha_2^2 - \alpha_0^2)\alpha_3\zeta}{2\alpha_0(\alpha_1 + \alpha_3)^2} + \mathcal{O}(\zeta^{3/2}), \\ \sqrt{(1-a)(1-b)} &= -\frac{\alpha_2}{\alpha_0} \left(1 - \frac{\alpha_3\zeta}{\alpha_1 + \alpha_3} + \mathcal{O}(\zeta^2) \right), \end{aligned} \quad (4.81)$$

or, equivalently,

$$\begin{aligned} a &= \frac{\alpha_0^2 + (\alpha_1 + \alpha_3)^2 - \alpha_2^2 - \sqrt{\Omega}}{2\alpha_0^2} + \frac{\alpha_0^2 - (\alpha_1 + \alpha_3)^2 + \alpha_2^2 + \sqrt{\Omega}}{2\alpha_0^2(\alpha_1 + \alpha_3)} \alpha_3\zeta + \mathcal{O}(\zeta^2), \\ b &= \frac{\alpha_0^2 + (\alpha_1 + \alpha_3)^2 - \alpha_2^2 + \sqrt{\Omega}}{2\alpha_0^2} + \frac{\alpha_0^2 - (\alpha_1 + \alpha_3)^2 + \alpha_2^2 - \sqrt{\Omega}}{2\alpha_0^2(\alpha_1 + \alpha_3)} \alpha_3\zeta + \mathcal{O}(\zeta^2), \end{aligned} \quad (4.82)$$

where we have defined

$$\Omega \equiv (\alpha_0 + \alpha_1 + \alpha_2 + \alpha_3)(\alpha_0 - \alpha_1 + \alpha_2 - \alpha_3)(\alpha_0 + \alpha_1 - \alpha_2 + \alpha_3)(\alpha_0 - \alpha_1 - \alpha_2 - \alpha_3). \quad (4.83)$$

This solution will be important in the perturbative calculations in Sec. IV E below.

From having worked out some degenerate cases of the matrix model associated with the Liouville four-point function, we have acquired some intuition about what to expect as we move on to the case with arbitrary vertex operators with conformal dimensions such that, generically, one will find a two-cut geometry. Let us briefly comment on this geometry. We have previously computed the spectral curve for a general multicut case and, for the two-cut ansatz associated to the Liouville four-point function, this is

$$\begin{aligned} y(z) &= \left(\frac{2\alpha_1}{z\sqrt{x_1x_2x_3x_4}} + \frac{2\alpha_2}{(z-1)\sqrt{(1-x_1)(1-x_2)(1-x_3)(1-x_4)}} \right. \\ &\quad \left. + \frac{2\alpha_3}{(z-\zeta)\sqrt{(\zeta-x_1)(\zeta-x_2)(\zeta-x_3)(\zeta-x_4)}} \right) \sqrt{\mathfrak{s}(z)}, \end{aligned} \quad (4.84)$$

with

$$\mathfrak{s}(z) = (z-x_1)(z-x_2)(z-x_3)(z-x_4). \quad (4.85)$$

It is simple to see that the end points of the two cuts, $\{x_i\}_{i=1}^4$, are now a solution to the system

$$\sqrt{x_1x_2x_3x_4} = -\frac{\zeta\alpha_1}{\alpha_0}, \quad (4.86)$$

$$\sqrt{(1-x_1)(1-x_2)(1-x_3)(1-x_4)} = \frac{(\zeta-1)\alpha_2}{\alpha_0}, \quad (4.87)$$

$$\sqrt{(\zeta - x_1)(\zeta - x_2)(\zeta - x_3)(\zeta - x_4)} = -\frac{(\zeta - 1)\zeta\alpha_3}{\alpha_0}, \quad (4.88)$$

which immediately simplifies the two-cut spectral curve as

$$y(z) = -\frac{2\alpha_0}{z(z-1)(z-\zeta)}\sqrt{s(z)}. \quad (4.89)$$

Further notice, as we have discussed before, that above we have three equations for four unknowns, and there is still one further moduli to consider; either t_1 or t_2 (they are not independent as $t_1 + t_2 = t$),

$$t_1 = -\frac{1}{2}a + \frac{1}{2}\sum_{i=0}^3 \alpha_i, \quad (4.90)$$

$$t_2 = \frac{1}{2}a + \frac{1}{2}\sum_{i=0}^3 \alpha_i. \quad (4.91)$$

E. Some perturbative calculations

We shall now discuss some perturbative calculations in the matrix models considered above. By perturbative, we mean a 't Hooft expansion in g_s . As a warm up we focus on matrix model (4.44). One can compute the partition function exactly (see Appendix, Sec. 2); the result can be written

$$Z = \frac{G_2(N+1)}{(2\pi)^N} \frac{G_2(-N-2\alpha_0/g_s+1)}{G_2(-2\alpha_0/g_s+1)} \frac{G_2(N+2\alpha_1/g_s+1)}{G_2(2\alpha_1/g_s+1)} \frac{G_2(N+2\alpha_2/g_s+1)}{G_2(2\alpha_2/g_s+1)}, \quad (4.92)$$

where $G_2(z)$ is the Barnes function. The Barnes function has the asymptotic expansion

$$\log G_2(N+1) = \frac{1}{2}N^2 \log N + \frac{1}{2}N \log 2\pi - \frac{3}{4}N^2 - \frac{1}{12} \log N + \zeta'(-1) + \sum_{g=2}^{+\infty} \frac{1}{N^{2g-2}} \frac{B_{2g}}{2g(2g-2)}, \quad (4.93)$$

with B_{2g} the Bernoulli numbers. However, (4.93) only deals with the $N \rightarrow +\infty$, or $g_s \rightarrow 0^+$, asymptotic region. Depending on the sign of the finite parameters α_1 and α_2 , as $g_s \rightarrow 0^+$ one will have $-\alpha_i/g_s$ either going to $+\infty$ or to $-\infty$, and one thus needs to also understand the asymptotics of the logarithm of the Barnes function in the region $N \rightarrow -\infty$. However, it turns out that, from the point of view of the perturbative expansion, this sign difference is not very relevant. To clarify this issue, first notice the relation

$$\log G_2(1-N) = \log G_2(1+N) - N \log 2\pi + \int_0^N dx \pi x \cot \pi x. \quad (4.94)$$

Explicitly evaluating the integral we find

$$\log G_2(1-N) = \log G_2(1+N) - N \log 2\pi + \frac{i\pi}{12}(1-6N^2) + N \log(1-e^{2\pi i N}) - \frac{i}{2\pi} \text{Li}_2(e^{2\pi i N}), \quad (4.95)$$

where $\text{Li}_2(z)$ is the dilogarithm. The logarithmic and dilogarithmic contributions can be expanded as

$$N \log(1 - e^{2\pi i N}) - \frac{i}{2\pi} \text{Li}_2(e^{2\pi i N}) = - \sum_{m=1}^{+\infty} \left(\frac{N}{m} - \frac{1}{2\pi i m^2} \right) e^{2\pi i N m}. \quad (4.96)$$

As explained in Ref. 39, this is actually the instanton contribution to the Barnes function, also describable in terms of Stokes phenomena (across the $+\pi/2$ Stokes line). In other words, this contribution is purely nonperturbative, and we shall neglect it at this stage, i.e., from a *purely perturbative* point of view we may use following result:

$$\log G_2(1 - N) \simeq \log G_2(1 + N) - N \log 2\pi + \frac{i\pi}{12}(1 - 6N^2). \quad (4.97)$$

From the above discussion it follows that the logarithm of Z , in the 't Hooft limit, has the expansion (as usual, we are only considering the real part of the free energy)

$$F \equiv \log Z = \sum_g g_s^{2g-2} F_g, \quad (4.98)$$

where, if we define $\mathcal{F} = F - \frac{1}{2}N^2 \log g_s - \frac{1}{2}N \log 2\pi + \log G_2(N+1)$ (essentially amounting to the Gaussian normalization of the free energy) and use relation (4.45) with $k=2$, we find

$$\begin{aligned} \mathcal{F}_0^{\text{3pf}} &= \frac{1}{2}(-\alpha_0 + \alpha_1 + \alpha_2)^2 \left(\log[-\alpha_0 + \alpha_1 + \alpha_2] - \frac{3}{2} \right) - 2\alpha_0^2 \left(\log[2\alpha_0] - \frac{3}{2} \right) \\ &\quad + \frac{1}{2}(\alpha_0 - \alpha_1 + \alpha_2)^2 \left(\log[\alpha_0 - \alpha_1 + \alpha_2] - \frac{3}{2} \right) - 2\alpha_1^2 \left(\log[2\alpha_1] - \frac{3}{2} \right) \\ &\quad + \frac{1}{2}(\alpha_0 + \alpha_1 - \alpha_2)^2 \left(\log[\alpha_0 + \alpha_1 - \alpha_2] - \frac{3}{2} \right) - 2\alpha_2^2 \left(\log[2\alpha_2] - \frac{3}{2} \right) \end{aligned} \quad (4.99)$$

for $g=0$,

$$\mathcal{F}_1^{\text{3pf}} = -\frac{1}{12} \log \frac{-\alpha_0 + \alpha_1 + \alpha_2}{2\alpha_0} - \frac{1}{12} \log \frac{\alpha_0 - \alpha_1 + \alpha_2}{2\alpha_1} - \frac{1}{12} \log \frac{\alpha_0 + \alpha_1 - \alpha_2}{2\alpha_2} \quad (4.100)$$

for $g=1$; and

$$\begin{aligned} \mathcal{F}_g^{\text{3pf}} &= \frac{B_{2g}}{2g(2g-2)} [(-\alpha_0 + \alpha_1 + \alpha_2)^{2-2g} - (2\alpha_0)^{2-2g} + (\alpha_0 - \alpha_1 + \alpha_2)^{2-2g} - (2\alpha_1)^{2-2g} \\ &\quad + (\alpha_0 + \alpha_1 - \alpha_2)^{2-2g} - (2\alpha_2)^{2-2g}] \end{aligned} \quad (4.101)$$

for $g \geq 2$. As alluded to above, it is also rather straightforward to compute the full nonperturbative contribution to this result. Because this is far from our present discussion we refer the reader to Ref. 39, for details, but the main idea essentially follows from the application of

$$\text{disc } \log G_2(N+1) = i \sum_{m=1}^{+\infty} \left(\frac{|N|}{m} + \frac{1}{2\pi m^2} \right) e^{-2\pi |N| m} \quad (4.102)$$

to the expression for the free energy (where the discontinuity of the free energy will yield the full tower of multi-instanton corrections). In this case one simply obtains

$$\begin{aligned} \text{disc } \mathcal{F}_{\text{3pf}} &= \frac{i}{2\pi \bar{g}_s} \sum_{m=1}^{+\infty} \left(\frac{2\pi(-\alpha_0 + \alpha_1 + \alpha_2)}{m} + \frac{\bar{g}_s}{m^2} \right) e^{-2\pi(-\alpha_0 + \alpha_1 + \alpha_2)m/\bar{g}_s} + \frac{i}{2\pi \bar{g}_s} \sum_{m=1}^{+\infty} \left(\frac{2\pi(\alpha_0 - \alpha_1 + \alpha_2)}{m} \right. \\ &\quad \left. + \frac{\bar{g}_s}{m^2} \right) e^{-2\pi(\alpha_0 - \alpha_1 + \alpha_2)m/\bar{g}_s} + \frac{i}{2\pi \bar{g}_s} \sum_{m=1}^{+\infty} \left(\frac{2\pi(\alpha_0 + \alpha_1 - \alpha_2)}{m} + \frac{\bar{g}_s}{m^2} \right) e^{-2\pi(\alpha_0 + \alpha_1 - \alpha_2)m/\bar{g}_s} \\ &\quad - \frac{i}{2\pi \bar{g}_s} \sum_{m=1}^{+\infty} \left(\frac{4\pi\alpha_0}{m} + \frac{\bar{g}_s}{m^2} \right) e^{-4\pi\alpha_0 m/\bar{g}_s} - \frac{i}{2\pi \bar{g}_s} \sum_{m=1}^{+\infty} \left(\frac{4\pi\alpha_1}{m} + \frac{\bar{g}_s}{m^2} \right) e^{-4\pi\alpha_1 m/\bar{g}_s} \end{aligned}$$

$$-\frac{i}{2\pi\bar{g}_s} \sum_{m=1}^{+\infty} \left(\frac{4\pi\alpha_2}{m} + \frac{\bar{g}_s}{m^2} \right) e^{-4\pi\alpha_2 m/\bar{g}_s}. \quad (4.103)$$

Notice that this is an exact result, to all loops and including all instanton numbers.

Even though we have (for this case) an exact perturbative expression, it is useful to also compute F_1 using a method that generalizes to more complicated situations where exact results are unavailable. There is a universal formula for F_1 which takes the form⁴⁰

$$F_1 = -\frac{1}{24} \log(M(a)M(b)(a-b)^4), \quad (4.104)$$

where [this follows from (4.30) and (4.54)]

$$M(a)M(b) = 4 \frac{\alpha_0^6}{\alpha_1^2 \alpha_2^2}. \quad (4.105)$$

(The universal expression was derived for polynomial potentials, but our results indicate that it also holds for the multi-Penner-type potentials considered in Ref. 12.) It immediately follows, using (4.54), that

$$F_1 = -\frac{1}{12} \log(2\alpha_0 + 2\alpha_1 + 2\alpha_2) - \frac{1}{12} \log \frac{-\alpha_0 + \alpha_1 + \alpha_2}{\alpha_0} - \frac{1}{12} \log \frac{\alpha_0 - \alpha_1 + \alpha_2}{\alpha_1} - \frac{1}{12} \log \frac{\alpha_0 + \alpha_1 - \alpha_2}{\alpha_2}, \quad (4.106)$$

reproducing the result we have previously obtained, up to some irrelevant numerical terms (this expression explicitly includes the Gaussian contribution $-\frac{1}{12} \log t$).

Next we turn to the case corresponding to the chiral four-point function in the Liouville theory, namely, the matrix model with potential (4.70). In this case we do not have an exact solution, but as discussed in Sec. IV D we can get tractable expressions if we work order by order in ζ .

As an example we consider the special case discussed at the end of Sec. IV D and focus on F_1 , which may be computed in a straightforward fashion from universal result (4.104). We have

$$\begin{aligned} M(z) &= \frac{2\alpha_1}{z\sqrt{ab}} + \frac{2\alpha_2}{(z-1)\sqrt{(1-a)(1-b)}} + \frac{2\alpha_3}{(z-\zeta)\sqrt{(\zeta-a)(\zeta-b)}} \simeq \frac{2\alpha_1 + 2\alpha_3}{z\sqrt{ab}} \\ &+ \frac{2\alpha_2}{(z-1)\sqrt{(1-a)(1-b)}} + \frac{2\alpha_3\zeta}{z^2\sqrt{ab}} + \frac{2\alpha_3(a+b)\zeta}{2z(ab)^{3/2}} + \mathcal{O}(\zeta^2) = \frac{2\alpha_0}{z(1-z)} \left(1 + \frac{\alpha_3\zeta}{z(\alpha_1 + \alpha_3)} \right. \\ &\left. + \mathcal{O}(\zeta^2) \right), \end{aligned} \quad (4.107)$$

leading to

$$M(a)M(b) = \frac{4\alpha_0^2}{ab(1-a)(1-b)} \left(1 + \frac{a+b}{ab} \frac{\alpha_3\zeta}{\alpha_1 + \alpha_3} + \mathcal{O}(\zeta^2) \right), \quad (4.108)$$

and further computing

$$b-a = \frac{\sqrt{\Omega}}{\alpha_0^2} \left(1 - \frac{\alpha_3\zeta}{\alpha_1 + \alpha_3} + \mathcal{O}(\zeta^2) \right), \quad (4.109)$$

where Ω was defined in (4.83), and

$$b + a = \frac{\alpha_0^2 + (\alpha_1 + \alpha_3)^2 - \alpha_2^2}{\alpha_0^2} + \frac{\alpha_0^2 - (\alpha_1 + \alpha_3)^2 + \alpha_2^2}{\alpha_0^2(\alpha_1 + \alpha_3)} \alpha_3 \zeta + \mathcal{O}(\zeta^2) \quad (4.110)$$

it finally follows, after putting all the above expressions together,

$$M(a)M(b) = \frac{4\alpha_0^6}{\alpha_2^2(\alpha_1 + \alpha_3)^2} \left(1 + \frac{4\alpha_3\zeta}{\alpha_1 + \alpha_3} + \mathcal{O}(\zeta^2) \right), \quad (4.111)$$

$$(b - a)^4 = \frac{\Omega^2}{\alpha_0^8} \left(1 - \frac{4\alpha_3\zeta}{2\alpha_1 + 2\alpha_3} + \mathcal{O}(\zeta^2) \right), \quad (4.112)$$

and (here we also included the ζ^2 terms)

$$F_1 = -\frac{1}{12} \log \frac{2\Omega}{\alpha_0(\alpha_1 + \alpha_3)\alpha_2} - \frac{\alpha_1\alpha_3(-\alpha_0 + \alpha_1 - \alpha_2 + \alpha_3)(\alpha_0 + \alpha_1 - \alpha_2 + \alpha_3)(-\alpha_0 + \alpha_1 + \alpha_2 + \alpha_3)(\alpha_0 + \alpha_1 + \alpha_2 + \alpha_3)}{32(\alpha_1 + \alpha_3)^6} \zeta^2. \quad (4.113)$$

Notice that, up to numerical constants that we drop, and to first nontrivial order in ζ , this result is exactly the same as the result in (4.106), except for the shift $\alpha_1 \rightarrow \alpha_1 + \alpha_3$. Note also that the order $\mathcal{O}(\zeta)$ contribution to F_1 vanishes.

The above result can be compared to the corresponding result for the Nekrasov instanton partition function of the $SU(2)$ theory with four fundamental hypermultiplets. In this case one easily obtains

$$F_1^{\text{inst}} = \frac{-a^6\sigma_1(m^2) + 2a^4\sigma_2(m^2) - 3a^2\sigma_3(m^2) + 4\sigma_4(m^2)}{128a^8} y^2 + \mathcal{O}(y^3), \quad (4.114)$$

where $\sigma_k(m^2) = \sum_{i_1 < \dots < i_k} m_{i_1}^2 \cdots m_{i_k}^2$. Note that the $\mathcal{O}(y)$ term vanishes in agreement with (4.113) identifying y with ζ . Furthermore, after implementing the relations,

$$m_1 = \alpha_1 + \alpha_3, \quad m_2 = \alpha_1 - \alpha_3, \quad m_3 = \alpha_0 + \alpha_2, \quad m_4 = \alpha_0 - \alpha_2, \quad (4.115)$$

together with $a = m_1$, we see that also the second order terms in (4.113) and (4.114) agree perfectly. [The definition of a in the Nekrasov expression differs from the one in (4.73).] This result is consistent with the analysis in Sec. IV B and supports the approach in Ref. 12.

Computing \mathcal{F}_g for $g \geq 2$ would require heavier machinery, see, e.g., Ref. 41, and will not be attempted here. Similarly, an explicit example involving the full-fledged two-cut geometry would take us too far afield. The integrals of the periods associated with the 't Hooft moduli are generically hard to evaluate exactly (although it is possible to do so in the present situation) and even harder to invert in order to find explicit solutions for the end points of the two cuts. A possible way out is to resort to perturbation theory, along the lines of Refs. 35 and 36, but we shall leave this question for future work.

Let us close with a comment about the relation to the chiral four-point function in the Liouville theory to leading order in ζ . In order to reconstruct the full Liouville three-point function at this order perturbative calculations are not enough, one would also need to add the full set of nonperturbative corrections and, in general, also let $\beta \neq 1$, in order to obtain the desired result. It seems plausible that a matrix model perturbation theory in ζ exists (for general β) which is exact in g_s order by order, but we leave this question for future work.

V. THE A_r MATRIX MODELS

In this section we perform several calculations in the A_r quiver matrix models. The resulting expressions are compared to the corresponding expressions in the A_r Toda theories and the A_r quiver gauge theories.

A. The three-point function

The above analysis of the A_1 matrix model three-point function (see Sec. IV A) can be extended to the A_r theory for any r . The relevant integral is

$$S_r(\alpha_1, \alpha_2, \beta) = \int \prod_{i,l} d\lambda_i^l \prod_{(i,l) < (j,l)} |\lambda_i^l - \lambda_j^l|^{\beta A_{ij}} \prod_{i,l} (\lambda_i^l)^{\alpha_1^i / \varepsilon_1} \prod_{i,l} (1 - \lambda_i^l)^{\alpha_2^i / \varepsilon_1}. \quad (5.1)$$

Here $\alpha_1^i = \langle \alpha_1, e_i \rangle$ with $\alpha_1 = \alpha_1^i \Lambda_i$ and similarly for α_2 (see Appendix, Sec. 1, for our Lie algebra conventions). Integral (5.1) can be explicitly evaluated provided one imposes the restriction $\alpha_1^i = \varkappa \delta_{ir}$ (i.e., $\alpha_1 = \varkappa \Lambda_r$); the result is⁴²

$$\begin{aligned} & \prod_{1 \leq i \leq j \leq r} \prod_{l=1}^{N_i - N_{i-1}} \frac{\Gamma([\alpha_2^i + \cdots + \alpha_2^j] / \varepsilon_1 + j - i + 1 + (I - 1 + i - j)\beta)}{\Gamma([\alpha_2^i + \cdots + \alpha_2^j + \alpha_1^i] / \varepsilon_1 + j - i + 2 + (I - 2 + i - j + N_j - N_{j+1})\beta)} \\ & \times \prod_{i=1}^r \prod_{l=1}^{N_i} \frac{\Gamma(\alpha_1^i / \varepsilon_1 + 1 + (I - N_{i+1} - 1)\beta) \Gamma(I\beta)}{\Gamma(\beta)}, \end{aligned} \quad (5.2)$$

where $N_0 = N_{r+1} = 0$. Result (5.2) depends on a very particular choice of integration contour, which is quite subtle and will not be discussed here; see Ref. 42, for further details.

Using various Lie algebra results (see Appendix, Sec. 1, for further details) one can show that the relation

$$\alpha_1^i + \alpha_2^i - \alpha_0^i - \varepsilon_2 A_{ij} N_j = 0, \quad (5.3)$$

where $\alpha_1 = \varkappa \Lambda_r$ (i.e., $\alpha_1^i = \varkappa \delta_{ir}$), implies that

$$-\varepsilon_2(N_i - N_{i+1}) = -A_{ir}^{-1} \varkappa + A_{i+1,r}^{-1} \varkappa + \langle \alpha_2, h_{i+1} \rangle - \langle \alpha_0, h_{i+1} \rangle. \quad (5.4)$$

(Note the special case, $\varepsilon_2 N_1 = \varkappa / (r+1) + \langle \alpha_2, h_1 \rangle - \langle \alpha_0, h_1 \rangle$.) Furthermore,

$$\alpha_1^i - A_{ir}^{-1} \varkappa + A_{i+1,r}^{-1} \varkappa = \frac{\varkappa}{r+1}. \quad (5.5)$$

Using these results together with

$$\alpha_2^i + \cdots + \alpha_2^j = -\langle \alpha_2, u_{j+1} - u_i \rangle, \quad i - j - 1 = -\langle \rho, u_i - u_{j+1} \rangle, \quad (5.6)$$

and (4.4), it follows that expression (5.2) can be written [we suppress an unimportant prefactor and $\Gamma_2(x)$ is short hand for $\Gamma_2(x | -\varepsilon_1, -\varepsilon_2)$]

$$\begin{aligned}
& \prod_{1 \leq i < j \leq r+1} \frac{\Gamma_2\left(\langle \alpha_2 + \rho \varepsilon, h_j \rangle - \langle \alpha_0 + \rho \varepsilon, h_i \rangle + \frac{\varkappa}{r+1}\right) \Gamma_2\left(-\varepsilon - \langle \alpha_2 + \rho \varepsilon, h_i \rangle + \langle \alpha_0 + \rho \varepsilon, h_j \rangle - \frac{\varkappa}{r+1}\right)}{\Gamma_2(-\langle \alpha_2 + \rho \varepsilon, e_{ij} \rangle) \Gamma_2(-\varepsilon - \langle \alpha_0 + \rho \varepsilon, e_{ij} \rangle)} \\
& \times \frac{\Gamma_2\left(-\varepsilon - \langle \alpha_2 + \rho \varepsilon, h_1 \rangle + \langle \alpha_0 + \rho \varepsilon, h_1 \rangle - \frac{\varkappa}{r+1}\right)}{\Gamma_2(-\varepsilon - \varkappa)} \\
& \times \prod_{i=2}^{r+1} \Gamma_2\left(\langle \alpha_2 + \rho \varepsilon, h_i \rangle - \langle \alpha_0 + \rho \varepsilon, h_i \rangle + \frac{\varkappa}{r+1}\right) \times \prod_{i=2}^r \frac{\Gamma_2(N_i \varepsilon_2)}{\Gamma_2(-\varepsilon - N_i \varepsilon_2)}. \tag{5.7}
\end{aligned}$$

The first thing to note is that the factor on the last line is a phase. As in the A_1 case discussed in Sec. IV A the above expression can be compared to the three-point function in the A_r Toda theory. After suitably rescaling the vertex operators with multiplicative factors depending on their momenta, the A_r Toda theory three-point function (2.9) can be written

$$\begin{aligned}
& \prod_{i,j=1}^{r+1} \left[Y\left(\frac{\varkappa}{r+1} + \langle \alpha_1 - Q\rho, h_i \rangle + \langle \alpha_2 - Q\rho, h_j \rangle\right) \right]^{-1} \\
& = \prod_{1 \leq i < j \leq r+1} \left| \Gamma_b\left(\frac{\varkappa}{r+1} + \langle \alpha_1 - Q\rho, h_i \rangle + \langle \alpha_2 - Q\rho, h_j \rangle\right) \right|^2 \\
& \times \prod_{1 \leq i < j \leq r+1} \left| \Gamma_b\left(Q - \frac{\varkappa}{r+1} - \langle \alpha_1 - Q\rho, h_j \rangle - \langle \alpha_2 - Q\rho, h_i \rangle\right) \right|^2 \\
& \times \left| \Gamma_b\left(Q - \frac{\varkappa}{r+1} - \langle \alpha_1 - Q\rho, h_1 \rangle - \langle \alpha_2 - Q\rho, h_1 \rangle\right) \right|^2 \\
& \times \prod_{i=2}^{r+1} \left| \Gamma_b\left(\frac{\varkappa}{r+1} + \langle \alpha_1 - Q\rho, h_i \rangle + \langle \alpha_2 - Q\rho, h_i \rangle\right) \right|^2, \tag{5.8}
\end{aligned}$$

where $\Gamma_b(x)$ is short hand for $\Gamma_2(x|b, 1/b)$. By similarly rescaling of the matrix model vertex operators, we are left with the factors in the numerators on the first two lines of (5.7). After using identification (4.8) together with $Q\rho - \alpha_0 \rightarrow -(Q\rho - \alpha_1)$, we find complete agreement with the ‘‘square root’’ of (5.8). Expression (5.7) should probably also be related to (the perturbative piece in) the $T_{r+1} = \mathcal{T}_{3,0}(A_r)$ theory where one of the masses satisfies a restriction inherited from $\alpha_1 = \varkappa \Lambda_r$ via the AGT relation.

We note that in Refs. 23 and 24 a complex version of the above integral was used to derive the three-point function in the A_r Toda theory when one of the momenta takes the special value $\varkappa \Lambda_r$.

In the above evaluation of integral (5.1) the condition $\alpha_1 = \varkappa \Lambda_r$ was imposed with α_2 left arbitrary. In Ref. 43 the above integral was evaluated for the rank 2 case with the alternative restriction: $\alpha_2 = \varkappa \Lambda_1 - (\varkappa + \varepsilon) \Lambda_2$, with α_1 left arbitrary.

To understand why there is more than one possible choice which allows for an explicit evaluation of the above integral, we recall that in the A_2 Toda theory the condition $\alpha = \varkappa \Lambda_r$ translates into the fact that the corresponding \mathcal{W} primary state satisfies (see, e.g., Refs. 4 and 24 and references therein)

$$\left(W_{-1} - \frac{3w(\alpha)}{2\Delta(\alpha)} L_{-1} \right) |\alpha\rangle = 0, \tag{5.9}$$

which implies

$$\Delta(\alpha)^2 \left(\frac{32}{22+5c} \left[\Delta(\alpha) + \frac{1}{5} \right] - \frac{1}{5} \right) - \frac{9}{2} w(\alpha)^2 = 0, \quad (5.10)$$

where

$$\Delta(\alpha) = \frac{\langle 2Q\rho - \alpha, \alpha \rangle}{2} \quad (5.11)$$

and

$$w(\alpha) = i \sqrt{\frac{48}{22+5c}} \langle \alpha - Q\rho, h_1 \rangle \langle \alpha - Q\rho, h_2 \rangle \langle \alpha - Q\rho, h_3 \rangle. \quad (5.12)$$

In (5.12) the h_i are the weights of the fundamental representation of the A_2 Lie algebra, cf. (A2). If we write $\alpha = \alpha^1 \Lambda_1 + \alpha^2 \Lambda_2$, then it turns out that there are six one parameter solutions to (5.10),

$$\begin{aligned} \alpha = \varkappa \Lambda_1, \quad \alpha = \varkappa \Lambda_1 + 2Q\Lambda_2, \quad \alpha = \varkappa \Lambda_2, \quad \alpha = 2Q\Lambda_1 + \varkappa \Lambda_2, \\ \alpha = \varkappa \Lambda_1 - (\varkappa - Q)\Lambda_2, \quad \alpha = \varkappa \Lambda_1 - (\varkappa - 3Q)\Lambda_2 \end{aligned} \quad (5.13)$$

Using that in our conventions $\varepsilon = -Q$, we see that both conditions (2.8) as well as the condition used in Ref. 43 belong to set (5.13). In addition to these three solutions there are three more which differ from the other ones only when $Q \neq 0$. It is known that in the A_2 Toda theory these six possibilities do not correspond to distinct states, rather they are related via the so-called shifted Weyl group acting on the momenta which changes the corresponding vertex operators by the so called reflection amplitudes (we thank Yuji Tachikawa for clarifying this point). Therefore, there should also be a simple relation between the corresponding matrix integrals; in particular, if one can be explicitly evaluated then that should also be the case for the others.

B. Higher-point functions

As in the A_1 case we can analyze a certain class of four-point correlation functions exactly. We need the following result:⁴²

$$\begin{aligned} \int \prod_{i\ell} d\lambda_i^\ell e_\ell(\lambda_1) \prod_{(i,\ell) < (j,\ell)} |\lambda_i^\ell - \lambda_j^\ell|^{\beta A_{ij}} \prod_{i,\ell} (\lambda_i^\ell)^{\alpha_1^\ell / \varepsilon_1} (1 - \lambda_i^\ell)^{\alpha_2^\ell / \varepsilon_1} = S_r(\alpha_1, \alpha_2, \beta) \\ \times \binom{N_1}{\ell} \prod_{i=1}^r \prod_{\ell=1}^\ell \frac{[\alpha_2^1 + \dots + \alpha_2^i] / \varepsilon_1 + i + (N_1 - I - i + 1)\beta}{[\alpha_2^1 + \dots + \alpha_2^i + \alpha_1^i] / \varepsilon + i + 1 + (N_1 + N_i - N_{i+1} - I - i)\beta}, \end{aligned} \quad (5.14)$$

where the A_r Selberg integral $S_r(\alpha_1, \alpha_2, \beta)$ was defined in (5.1), $\alpha_1^i = \varkappa \delta_{ir}$, and $e_\ell(\lambda_1)$ is the ℓ th elementary symmetric polynomial defined as

$$e_\ell(\lambda_1) = \sum_{I_1 < \dots < I_\ell} \lambda_{I_1}^{I_1} \dots \lambda_{I_\ell}^{I_\ell}. \quad (5.15)$$

From result (5.14) it follows that

$$\begin{aligned}
 S_r(\alpha_1, \alpha_2, \beta; z) &= \int \prod_{iI} d\lambda_i^I \prod_{(i,I) < (j,J)} |\lambda_i^I - \lambda_j^J|^{\beta A_{ij}} \prod_{i,I} (\lambda_i^I)^{\alpha_i^I/\varepsilon_1} (1 - \lambda_i^I)^{\alpha_2^I/\varepsilon_1} \prod_I (z - \lambda_i^I) \\
 &= S_r(\alpha_1, \alpha_2, \beta) \sum_{\ell=0}^{N_1} z^\ell (-1)^{N_1-\ell} \binom{N_1}{\ell} \\
 &\quad \times \prod_{i=1}^r \prod_{I=1}^{N_1-\ell} \frac{[\alpha_2^1 + \dots + \alpha_2^i]/\varepsilon_1 + i + (N_1 - I - i + 1)\beta}{[\alpha_2^1 + \dots + \alpha_2^i + \alpha_1^i]/\varepsilon_1 + i + 1 + (N_1 + N_i - N_{i+1} - I - i)\beta} \\
 &= S_r(\alpha_1, \alpha_2, \beta; 0) \sum_{\ell=0}^{N_1} \frac{z^\ell}{\ell!} (-1)^\ell \frac{N_1!}{(N_1 - \ell)!} \\
 &\quad \times \prod_{i=1}^r \prod_{I=0}^{\ell-1} \frac{\frac{1}{\varepsilon_2}(\alpha_2^1 + \dots + \alpha_2^i + \alpha_1^i + i\varepsilon_1 + \varepsilon_1) + I + N_i - N_{i+1} - i}{-\frac{1}{\varepsilon_2}(\alpha_2^1 + \dots + \alpha_2^i + i\varepsilon_1) + I - i + 1}. \tag{5.16}
 \end{aligned}$$

As above one can show using various Lie algebra results (see Appendix, Sec. 1 for further details) that

$$-\varepsilon_2(N_i - N_{i+1}) = -A_{i1}^{-1}\varepsilon_1 + A_{i+1,1}^{-1}\varepsilon_1 - A_{ir}^{-1}\varkappa + A_{i+1,r}^{-1}\varkappa + \langle \alpha_2, h_{i+1} \rangle - \langle \alpha_0, h_{i+1} \rangle, \tag{5.17}$$

where we have used $\alpha_1 = \varkappa \Lambda_r$ and $\alpha_3 = \varepsilon_1 \Lambda_1$. Furthermore,

$$-A_{i1}^{-1} + A_{i+1,1}^{-1} = -\frac{1}{r+1}, \quad \alpha_1^i - A_{ir}^{-1}\varkappa + A_{i+1,r}^{-1}\varkappa = \frac{\varkappa}{r+1}. \tag{5.18}$$

Using these results together with

$$\alpha_2^1 + \dots + \alpha_2^i = -\langle \alpha_2, h_{i+1} - h_1 \rangle, \quad i = -\langle \rho, h_{i+1} - h_1 \rangle, \tag{5.19}$$

it follows that

$$\begin{aligned}
 S_r(\alpha_1, \alpha_2, \beta; z) &= S_r(\alpha_1, \alpha_2, \beta; 0) \sum_{\ell=0}^{N_1} \frac{z^\ell (A_1)_\ell \dots (A_{r+1})_\ell}{\ell! (B_1)_\ell \dots (B_r)_\ell} \\
 &= S_r(\alpha_1, \alpha_2, \beta; 0) {}_{r+1}F_r(A_1, \dots, A_{r+1}; B_1, \dots, B_r; z), \tag{5.20}
 \end{aligned}$$

where $(X)_n = X(X+1)\dots(X+n-1)$ is the Pochhammer symbol,

$$B_i = -\frac{1}{\varepsilon_2}(\alpha_2^1 + \dots + \alpha_2^i + i\varepsilon_1) - i + 1 = \frac{1}{\varepsilon_2}\langle \alpha_2 + \varepsilon\rho, h_{i+1} - h_1 \rangle + 1 \tag{5.21}$$

and

$$A_i = -\frac{1}{\varepsilon_2} \left(\frac{\varkappa}{r+1} + \varepsilon_1 \frac{r}{r+1} + \langle \alpha_2 + \varepsilon\rho, h_1 \rangle - \langle \alpha_0 + \varepsilon\rho, h_i \rangle \right). \tag{5.22}$$

Note that $(-1)^\ell N_1! / (N_1 - \ell)! = (A_1)_\ell$ and that the sum over ℓ in (5.20) can be extended to ∞ since $(A_1)_\ell = 0$ for $\ell > N_1$.

To compare result (5.20) with the corresponding result in the A_r Toda theory we recall that in Refs. 23 and 24 it was shown that the correlator

$$\langle V_{-b\Lambda_1}(z)V_{\alpha_1}(0)V_{\alpha_2}(\infty)V_{\kappa\Lambda_r}(1) \rangle \tag{5.23}$$

satisfies a differential equation of hypergeometric type whose solutions involves ${}_{r+1}F_r(A_1, \dots, A_{r+1}; B_1, \dots, B_r; z)$, where

$$A_i = b \left(\frac{\kappa}{r+1} - b \frac{r}{r+1} + \langle \alpha_1 - Q\rho, h_1 \rangle + \langle \alpha_2 - Q\rho, h_i \rangle \right),$$

$$B_i = 1 + b \langle \alpha_1 - Q\rho, h_1 - h_{i+1} \rangle. \tag{5.24}$$

Replacing $\alpha_1 \rightarrow \alpha_2$, $\alpha_2 - Q\rho \rightarrow -(\alpha_0 - Q\rho)$ and using rule (4.8) we see that (5.24) agrees perfectly with (5.21) and (5.22).

One can also show that the Nekrasov partition function leads to the same result, see Ref. 6, for a discussion. It should also be possible to analyze the correlation functions involving the insertion of $V_{-\Lambda_1/b}$ as well as the case with several insertions of $V_{-b\Lambda_1}$.

To analyze general correlation functions without imposing the above restrictions on the momenta is much more difficult. However, we stress that matrix model perturbation theory can handle any correlation function, although this method appears to be somewhat cumbersome and it is not clear what the meaning of the resulting expansion is in the $2d$ CFT.

C. The curve

We now turn to the discussion of the loop equations and the large N matrix model curve. In Ref. 12 and in Secs. IV C and IV D some examples of curves in the A_1 case were presented. Here we mainly focus on the A_2 case. We call the two matrices Φ and $\tilde{\Phi}$ and the associated potentials W and \tilde{W} . The (nonhyperelliptic) curve is known to be of the form^{16,29}

$$x^3 = r(z)x + s(z), \tag{5.25}$$

where

$$r(z) = \frac{1}{3} [W'(z)^2 + \tilde{W}'(z)^2 + W'(z)\tilde{W}'(z)] - g_s \left\langle \text{tr} \left(\frac{W'(z) - W'(\Phi)}{z - \Phi} \right) \right\rangle - g_s \left\langle \text{tr} \left(\frac{\tilde{W}'(z) - \tilde{W}'(\tilde{\Phi})}{z - \tilde{\Phi}} \right) \right\rangle \tag{5.26}$$

and⁴⁴

$$s(z) = -\frac{1}{27} [W'(z) + 2\tilde{W}'(z)][2W'(z) + \tilde{W}'(z)][W'(z) - \tilde{W}'(z)] + g_s \tilde{\omega}_r(z) \left\langle \text{tr} \left(\frac{W'(z) - W'(\Phi)}{z - \Phi} \right) \right\rangle$$

$$- g_s \omega_r(z) \left\langle \text{tr} \left(\frac{\tilde{W}'(z) - \tilde{W}'(\tilde{\Phi})}{z - \tilde{\Phi}} \right) \right\rangle - g_s^2 \left\langle \text{tr} \left[\frac{d}{d\Phi} \left(\frac{W'(z) - W'(\Phi)}{z - \Phi} \right) \right] \right\rangle$$

$$+ g_s^2 \left\langle \text{tr} \left[\frac{d}{d\tilde{\Phi}} \left(\frac{\tilde{W}'(z) - \tilde{W}'(\tilde{\Phi})}{z - \tilde{\Phi}} \right) \right] \right\rangle + g_s \left\langle \text{tr} \left(\frac{W'(z) - W'(\Phi)}{z - \Phi} \right) \right\rangle$$

$$- g_s \left\langle \text{tr} \left(\frac{\tilde{W}'(z) - \tilde{W}'(\tilde{\Phi})}{z - \tilde{\Phi}} \right) \right\rangle, \tag{5.27}$$

where

$$\omega_r(z) = \frac{1}{3}(2W'(z) + \tilde{W}'(z)), \quad \tilde{\omega}_r(z) = \frac{1}{3}(W'(z) + 2\tilde{W}'(z)). \quad (5.28)$$

[The expressions on the last two lines in (5.27) can be simplified in the eigenvalue basis by using the saddle-point equations, but we will not need the resulting expression here.]

Inserting the explicit expressions for W and \tilde{W} as sums of logarithms

$$W(\Phi) = \sum_{i=1}^p m_i \log(z_i - \Phi), \quad \tilde{W}(\tilde{\Phi}) = \sum_{i=1}^p \tilde{m}_i \log(z_i - \tilde{\Phi}), \quad (5.29)$$

and using the above expressions for $r(z)$ and $s(z)$, we find

$$x^3 = \frac{P_{2p-2}^{(2)}(z)}{\prod_{i=1}^p (z - z_i)^2} x + \frac{P_{3p-3}^{(3)}(z)}{\prod_{i=1}^p (z - z_i)^3}, \quad (5.30)$$

where $P_{2p-2}^{(2)}(z)$ and $P_{3p-3}^{(3)}(z)$ are polynomials of degree $2p-2$ and $3p-3$, respectively. (Since the curve is derived in the limit of vanishing ε the AGT relation is $\alpha_i = m_i$ using a suitable definition of the masses.) This curve is of precisely the right form to agree with the expression in Ref. 3 (which was obtained by starting from the earlier result⁴⁵). However, there is a further property that should be checked.

Recall that, in the A_2 case, there are two types of punctures, one full and one basic. For the basic (or special) puncture there is a relation between m_i and \tilde{m}_i . In our conventions that relation is $\tilde{m}_i = 0$. Now for a special puncture at $z = z_i$, there is a relation between $P^{(2)}(z)$ and $P^{(3)}(z)$ that has to be satisfied at that location, see (3.25) in Ref. 3. To check that this condition holds for the matrix model curve, it is sufficient to focus on a single special puncture which we can take to be located at $z_i = 0$, i.e., $W(\Phi) = m \log(\Phi)$ and $\tilde{W}(\tilde{\Phi}) = 0$. We need to check that $4r(z)^3 - 27s(z)^2$ scales like $1/z^4$ (naively it would scale like $1/z^6$). In other words we need

$$4\left(\frac{1}{3}[W'(z)^2 + \tilde{W}'(z)^2 + W'(z)\tilde{W}'(z)]\right)^3 - 27\left(-\frac{1}{27}[W'(z) + 2\tilde{W}'(z)][2W'(z) + \tilde{W}'(z)][W'(z) - \tilde{W}'(z)]\right)^2 = 0 \quad (5.31)$$

and

$$12\left(\frac{1}{3}[W'(z)^2 + \tilde{W}'(z)^2 + W'(z)\tilde{W}'(z)]\right)^2(f(z) + \tilde{f}(z)) + 2[W'(z) + 2\tilde{W}'(z)][2W'(z) + \tilde{W}'(z)][W'(z) - \tilde{W}'(z)](\omega_r(z)\tilde{f}(z) - \tilde{\omega}_r(z)f(z)) = 0, \quad (5.32)$$

where

$$f(z) = -g_s \left\langle \text{tr} \left(\frac{W'(z) - W'(\Phi)}{z - \Phi} \right) \right\rangle, \quad \tilde{f}(z) = -g_s \left\langle \text{tr} \left(\frac{\tilde{W}'(z) - \tilde{W}'(\tilde{\Phi})}{z - \tilde{\Phi}} \right) \right\rangle. \quad (5.33)$$

Both the above equations are easily shown to hold for the special puncture with $\tilde{m}_i = 0$, thereby establishing the equivalence with the results in Ref. 3. [There is also another solution to (5.31) and (5.32), viz. $m_i = \tilde{m}_i$. This solution is precisely the alternative solution discussed at the end of Sec. V A (note that the curve is derived for $\varepsilon = 0$).]

For higher rank curves the general structure of the matrix model curve is also known; see Ref. 46 for the state-of-the-art knowledge. The matrix model curves can be analyzed and compared to the gauge theory curve as above.

VI. 5d GAUGE THEORIES AND q -DEFORMED MATRIX MODELS

Nekrasov partition functions can also be defined for supersymmetric gauge theories in five dimensions formulated on $\mathbb{R}^4 \times S^1$.² As an example, in the $SU(2)$ theory with four matter hypermultiplets in the fundamental representation, the instanton partition function is

$$\mathcal{Z}_{\text{inst}} = \sum_{\vec{Y}} y^{|\vec{Y}|} \prod_{m,n=1}^2 \prod_{s \in Y_m} \frac{\mathcal{P}(\hat{a}_m, Y_m, s)}{\mathcal{E}(\hat{a}_m - \hat{a}_n, Y_m, Y_n, s) (\mathcal{E}(\hat{a}_m - \hat{a}_n, Y_m, Y_n, s) - \varepsilon)}, \tag{6.1}$$

where $(\hat{a}_1, \hat{a}_2) = (a, -a)$ and

$$\begin{aligned} \mathcal{E}(x, Y_m, Y_n, s) &= \sinh(R(x - \varepsilon_1 L_{Y_n}(s) + \varepsilon_2 (A_{Y_m}(s) + 1))), \\ \mathcal{P}(\hat{a}_m, Y_i, s) &= \prod_{f=1}^4 \sinh(R(\hat{a}_m - (j-1)\varepsilon_1 - (i-1)\varepsilon_2 - m_f)), \end{aligned} \tag{6.2}$$

with R the radius of the S^1 . See Sec. II B, for more details about the notation. Expression (6.1) can also be obtained from topological string considerations, see, e.g., Refs. 47 and 48). The partition function also has a perturbative piece whose explicit expression can be found, e.g., in Refs. 2 and 49.

The question we would like to address in this section is: Are there matrix model and CFT descriptions of partition functions of type (6.1)?

In the recent paper¹⁰ a proposal was made for the CFT description of the pure SU(2) theory in five dimensions. This proposal involved the so called q -deformed Virasoro algebra.⁵⁰ It is very natural to expect that there is a CFT description of $5d$ (conformal) quiver gauge theories which involves q -deformed Virasoro⁵⁰ and q -deformed \mathcal{W} algebras.⁵¹ Unfortunately the representation theory of these algebras is not very well developed. Analogs of the primary fields and their quantum numbers are known, but the analog of, e.g., the relation

$$[L_m, V_\alpha] = z^m [(m+1)\Delta(\alpha)V_\alpha + z(L_{-1}V_\alpha)] = z^m (m\Delta(\alpha)V_\alpha + [L_0, V_\alpha]), \tag{6.3}$$

is not known (as far as we know). This fact complicates the analysis and makes direct calculations of chiral blocks difficult.

Instead, we try to obtain a matrix model description. A natural starting point is to look for a generalization of the A_1 three-point function (4.1).

There is a known q -deformation of (4.1) in the literature, which can be written

$$\int \prod_{I=1}^N d_q \lambda^I \prod_{I=1}^N (\lambda_I)^{2\alpha_I/\varepsilon_1} \frac{(q^{-\alpha_2/\varepsilon_1} \lambda^I; q)_\infty}{(q^{\alpha_2/\varepsilon_1} \lambda_I; q)_\infty} \prod_{I < J} (\lambda^J)^{2\beta} \frac{(q^{-\beta} \lambda^I / \lambda^J; q)_\infty}{(q^\beta \lambda^I / \lambda^J)_\infty}, \tag{6.4}$$

where $0 < q < 1$ and $\int d x_q$ is the so-called q -integral (or Jackson integral) defined via

$$\int_0^1 d_q x f(x) = (1-q) \sum_{k=0}^{\infty} f(q^k) q^k. \tag{6.5}$$

In the limit $q \rightarrow 1^-$ this expression converges to the Riemann integral $\int_0^1 f(x) dx$. Furthermore, $(a; q)_\infty = \prod_{k=0}^{\infty} (1 - aq^k)$, and we also use the notation

$$(a; q)_\ell = (1-a)(1-aq) \cdots (1-aq^{\ell-1}) = \frac{(a; q)_\infty}{(q^\ell a; q)_\infty}. \tag{6.6}$$

Based on the above expression we tentatively propose the rule

$$\hat{V}_\alpha(z) \equiv \prod_I (z - \lambda^I)^{2\alpha/\varepsilon_1} \rightarrow \prod_I z^{2\alpha/\varepsilon_1} \frac{(q^{-\alpha/\varepsilon_1} \lambda^I / z; q)_\infty}{(q^{\alpha/\varepsilon_1} \lambda^I / z, q)_\infty} \equiv \hat{V}_\alpha^q(z). \tag{6.7}$$

In the limit $q \rightarrow 1^-$, $\hat{V}_\alpha^q(z) \rightarrow \hat{V}_\alpha(z)$. In the special case $z=0$ we assume that $\hat{V}_\alpha^q(0) = \hat{V}_\alpha(0) = \prod_I (\lambda^I)^{2\alpha/\varepsilon_1}$. Note that when $\alpha = \varepsilon_1/2$, $V_\alpha^q(z) = \prod_I z [(q^{-1/2} \lambda^I / z; q)_\infty / (q^{1/2} \lambda^I / z, q)_\infty]$ reduces to $\prod_I (z - q^{-1/2} \lambda^I)$.

The above three-point function can be evaluated (at least when β is an integer) and leads to a product of q -gamma functions, but since we are not aware of a q -analog of (4.4) we will not discuss the result here.

Instead we turn to the four-point function with $\alpha_3 = \varepsilon_1/2$. Using the result in Ref. 52 the resulting expression can be explicitly evaluated,

$$S^q(\alpha_1, \alpha_2, \beta; z) = \int \prod_{I=1}^N d_q \lambda^I \left(z - \frac{\lambda^I}{q^{1/2}} \right) (\lambda^I)^{2\alpha_1/\varepsilon_1} \frac{(q^{-\alpha_2/\varepsilon_1} \lambda^I; q)_\infty}{(q^{\alpha_2/\varepsilon_1} \lambda^I; q)_\infty} \prod_{I < J} (\lambda^I)^{2\beta} \frac{\left(q^{-\beta} \frac{\lambda^I}{\lambda^J}; q \right)_\infty}{\left(q^\beta \frac{\lambda^I}{\lambda^J} \right)_\infty} \tag{6.8}$$

$$= S^q(\alpha_1, \alpha_2, \beta; 0) {}_2\phi_1(q^{-\beta N}, q^{2\alpha_1/\varepsilon_1 + 2\alpha_2/\varepsilon_1 + 2 + \beta N - \beta}; q^{2\alpha_1/\varepsilon_1 + 1}; q^\beta, \tilde{z}),$$

where $\tilde{z} = zq^{-2\alpha_2/\varepsilon_1 - 1/2}$ and

$${}_2\phi_1(A, B; C; q, z) = \sum_{k=0}^{\infty} \frac{(A; q)_k (B; q)_k}{(C; q)_k (q; q)_k} z^k. \tag{6.9}$$

(Actually, since N is an integer ${}_2\phi_1$ in the above expression reduces to a so-called little q -Jacobi polynomial.)

Integral (6.8) was also discussed in Ref. 53, albeit in a somewhat different guise. In that paper it was shown that the integral satisfies a certain difference equation. This is a q -analog of the result in Ref. 34 (cf. the discussion in Sec. IV B). However, the analysis in Ref. 52 is more transparent, although the case corresponding to multiple insertions of $\hat{V}_{\varepsilon_1/2}^q$ is not discussed in Ref. 52.

The above result (6.8) can be rewritten

$${}_2\phi_1\left(q^{-\beta N}, q^{(2\alpha_1/\varepsilon_1 + 2\alpha_2/\varepsilon_1 + 2) + \beta N - \beta}; q^{(2\alpha_1/\varepsilon_1 + 1)}; q^\beta, \frac{z}{q^{2\alpha_2/\varepsilon_1 - 1/2}}\right) \\ = \sum_{\ell=0}^{\infty} \prod_{k=0}^{\ell-1} \frac{\sinh(R[-N+k]) \sinh\left(R\left[\beta^{-1}\left(\frac{2\alpha_1}{\varepsilon_1} + \frac{2\alpha_2}{\varepsilon_1} + 2\right) + N - 1 + k\right]\right)}{\sinh(R[\beta^{-1}(2\alpha_1/\varepsilon_1 + 1) + k]) \sinh(R[1+k])} (zq^{1-\beta})^\ell, \tag{6.10}$$

where we have used $q = e^{-2R/\beta}$.

To compare this expression with the one arising from the Nekrasov partition function, we go through the same steps as in Sec. IV B and make the choices $m_1 = a$ and $m_2 = -a + \varepsilon_1$ which implies that (6.1) reduces to

$$\sum_{\ell=0}^{\infty} y^\ell \prod_{k=0}^{\ell-1} \frac{\sinh\left(R\left[\left(\alpha_1 + \alpha_2 + \alpha_4 - \frac{\varepsilon_1}{2}\right)/\varepsilon_2 + k\right]\right) \sinh\left(R\left[\left(\alpha_1 + \alpha_2 + \alpha_4 - \frac{3}{2}\varepsilon_1 - \varepsilon_2\right)/\varepsilon_2 + k\right]\right)}{\sinh(R[(\varepsilon_1 - 2\alpha_1)/\varepsilon_2 + k]) \sinh(R[1+k])}, \tag{6.11}$$

where we have also used AGT relation (4.16). This expression is readily seen to agree with (6.10) using $y = zq^{1-\beta}$ and the same arguments as in Sec. IV B, cf. the discussion after Eq. (4.18).

In Ref. 53 there is also an extension of the above result to the case with multiple insertions of $\hat{V}_{\varepsilon_1/2}^q$. In this case the result involves the function ${}_2\Phi_1^{(q,t)}(A, B; C; z)$ which is a q analog of (4.21) and involves Macdonald polynomials rather than Jack polynomials, see Ref. 53, for further details. So far we have only considered the A_1 case; it should also be possible to consider the A_r case using the results in Ref. 42.

We close this section with a few words of caution. There are, in general, several possible q -deformations and the one above may not be the right one. Also, we should mention that in Ref.

54 another deformation of the matrix model was shown to be related to Nekrasov partition functions for five-dimensional gauge theories. The deformation in Ref. 54 replaces the Vandermonde determinant with $\prod_{I < J} \sinh(\lambda^I - \lambda^J)$. This possibility was also mentioned in Ref. 12.

VII. DISCUSSION AND OUTLOOK

In this paper we have studied the A_r quiver matrix models which were introduced in Ref. 12 and argued to capture correlation functions (chiral blocks) in the $2d$ A_r Toda field theories and Nekrasov partition functions (instanton partition functions) in the $4d$ A_r quiver gauge theories.

From the point of view of the matrix model the expansion in z_i (the locations of the vertex operators) is somewhat awkward, but we have shown that several known results can be rederived from the matrix model; some of our checks are quite nontrivial. It would be interesting to develop the matrix model technology further, and, for instance, to clarify the choice of integration contour and to develop a perturbation theory in z_i .

We also made a proposal for an extension of the matrix model to capture the Nekrasov partition function of $5d$ quiver gauge theories. This speculative proposal passed a nontrivial check, but deserves further study.

One open problem is to extend the analysis to the other ADE Lie algebras. Let us make a comment about the D_r case. The matrix model curve for the D_r model can be extracted from Ref. 46, Eq. (3.51), and can be seen (after some changes of notation) to be of the same general form as the curves in Ref. 55. It would be interesting to study this in more detail.

Note added: After this paper was finished, Refs. 56 and 57 appeared which have some overlap with some parts of this paper.

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APPENDIX: TECHNICAL DETAILS

1. A_r roots and weights

Here we collect some standard results for the A_r Lie algebras. The root/weight space of the A_r Lie algebra can be viewed as a r -dimensional subspace of \mathbb{R}^{r+1} . The unit vectors of \mathbb{R}^{r+1} will be denoted u_i ($i=1, \dots, r+1$) and satisfy $\langle u_i, u_j \rangle = \delta_{ij}$. The simple roots are $e_i = u_i - u_{i+1}$ ($i=1, \dots, r$) and the positive roots are $e_{ij} = u_i - u_j$ (with $1 \leq i < j \leq r+1$). The Cartan matrix is $A_{ij} = \langle e_i, e_j \rangle$, and its inverse is $A_{ij}^{-1} = \frac{1}{r+1} \min(i, j) [r+1 - \max(i, j)]$. The Weyl vector ρ is half the sum of the positive roots; hence $\rho = \frac{1}{2} \sum_{i=1}^{r+1} (r-2i+2) u_i$. The fundamental weights Λ_i are defined as

$$\Lambda_i = u_1 + \dots + u_i - \frac{i}{r+1} \sum_{j=1}^{r+1} u_j \quad (i=1, \dots, r) \quad (\text{A1})$$

and satisfy $\langle \Lambda_i, e_j \rangle = \delta_{ij}$. Note that $\sum_{i=1}^r \Lambda_i = \rho$. Finally, the weights of the fundamental representation can be chosen as

$$h_i = u_i - \frac{1}{r+1} \sum_j u_j = \Lambda_1 - \sum_{j=1}^{i-1} e_j \quad (i=1, \dots, r+1). \quad (\text{A2})$$

Note that $h_1 = \Lambda_1$ and $\sum_j h_j = 0$.

2. Orthogonal polynomials and the A_1 three-point function

Here we present an alternative evaluation method for integral (4.1) in the case $\beta=1$ using orthogonal polynomials. Consider the one-matrix model partition function,

$$Z = \frac{1}{(2\pi)^N N!} \int \prod_{l=1}^N d\lambda^l \prod_{l < j} (\lambda^l - \lambda^j)^2 e^{1/g_s \sum_{i=1}^N W(\lambda^i)}, \tag{A3}$$

and introduce orthogonal polynomials, $\{p_n(z)\}$, with respect to the measure

$$d\mu(z) = e^{1/g_s W(z)} dz \tag{A4}$$

as

$$\int_{\mathbb{R}} d\mu(z) p_n(z) p_m(z) = h_n \delta_{nm}, \quad n \geq 0, \tag{A5}$$

where one further normalizes $p_n(z)$, such that $p_n(z) = z^n + \dots$. Noticing that the Vandermonde determinant $\prod_{l < j} (\lambda^l - \lambda^j)^2$ equals $\det p_{j-1}(\lambda^l)$, the one-matrix model partition function may be computed as

$$Z = \frac{1}{(2\pi)^N} \prod_{n=0}^{N-1} h_n. \tag{A6}$$

In the case of interest to us, the potential is

$$W(z) = \text{tr} \sum_{a=1}^k 2\alpha_a \log(z_a - z), \tag{A7}$$

which, in principle, forbids the use of standard orthogonal polynomial techniques. However, the fact that the nonpolynomial structure is logarithmic actually allows us to get around this issue when $k=2$, as we shall see now. Indeed, in this case (setting $z_1=0$ and $z_2=1$) the measure associated with (A7) becomes

$$d\mu(z) = (1-z)^{2\alpha_2} z^{2\alpha_1} dz, \tag{A8}$$

and is immediately related to the orthogonal polynomial family of Jacobi polynomials. The combination

$$J_n^{(\alpha, \gamma)}(z) \equiv \frac{n! \Gamma(n + \alpha + \gamma + 1)}{\Gamma(2n + \alpha + \gamma + 1)} P_n^{(\alpha, \gamma)}(2z - 1), \tag{A9}$$

where $P_n^{(\alpha, \gamma)}(z)$ is a Jacobi polynomial, is normalized such that $J_n^{(\alpha, \gamma)}(z) = z^n + \dots$ and satisfies

$$\int_0^1 dz (1-z)^{\alpha_2} z^{\alpha_1} J_n^{(\alpha, \gamma)}(z) J_m^{(\alpha, \gamma)}(z) = h_n \delta_{nm}, \tag{A10}$$

with

$$h_n = n! \frac{\Gamma(n + \alpha + 1) \Gamma(n + \gamma + 1) \Gamma(n + \alpha + \gamma + 1)}{\Gamma(2n + \alpha + \gamma + 2) \Gamma(2n + \alpha + \gamma + 1)}. \tag{A11}$$

Using (A6) this immediately leads to the *exact* result,

$$\begin{aligned} Z &= \frac{1}{(2\pi)^N} \prod_{n=0}^{N-1} n! \frac{\Gamma(n + 2\alpha_2/g_s + 1) \Gamma(n + \alpha_1/g_s + 1) \Gamma(n + \alpha_2 + \alpha_1 + 1)}{\Gamma(2n + \alpha_2 + \alpha_1 + 2) \Gamma(2n + \alpha_2 + \alpha_1 + 1)} \\ &= \frac{1}{(2\pi)^N} \prod_{n=0}^{N-1} n! \frac{\Gamma(n + \alpha_2 + 1) \Gamma(n + \alpha_1 + 1)}{\Gamma(N + n + 1 + \alpha_2 + \alpha_1)}. \end{aligned} \tag{A12}$$

which agrees with (4.2) (when $\beta=1$).

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