Contents

1 Introduction

The Cahn-Hilliard equation is an equation of mathematical physics which describes the process of phase separation, by which the two components of a binary fluid spontaneously separate and form domains pure in each component.

In this thesis we study numerical approximation of the Cahn-Hilliard equation. We consider both the original equation and the equation perturbed by noise. The stochastic Cahn-Hilliard equation also called the Cahn-Hilliard-Cook equation. This work involves several mathematical topics:

- Semigroup theory
- Cahn-Hilliard equation
- Stochastic analysis in Hilbert space
- Finite element method
- A posteriori error analysis based on the calculus of variations

In the following we give a brief survey of these topics and finally a summary of the appended papers.

2 Semigroup approach

Semigroup theory is the abstract study of first order ordinary differential equations with values in Banach space, driven by linear, but possibly unbounded operators. This approach has a wide applications in different branches of analysis, such as harmonic analysis, approximation theory and many other subjects. In this section we outline the basics of the theory, without proof. For more complete and advanced details of the theory and its applications the partial differential equations, one may refer to Evans [8] and Pazy [17].

Definition 2.1 (Semigroup)**.** A family $\{E(t)\}_{t\geq 0}$ of bounded linear operators from Banach space X to X is called a *semigroup of bounded linear operators* if

1. $E(0) = I$, (identity operator)
2. $E(t+s) = E(t)E(s), \quad \forall s,t \geq 0.$ (semigroup property)

The semigroup is called *strongly continuous* if

$$\lim_{t\to 0^+} E(t)x = x \quad \forall x \in X.$$

The *infinitesimal generator* of the semigroup is the linear operator G defined by

$$Gx = \lim_{t \to 0^+} \frac{E(t)x - x}{t},$$

its domain of definition $D(G)$ being the space of all $x \in X$ for which the limit exists. The semigroup can be denoted by $E(t) = \mathrm{e}^{tG}$.

A strongly continuous semigroups of bounded linear operators on X is often called a C_0 semigroup. If, moreover, $\|E(t)\| \leq 1$ for $t \geq 0$, it is called *a semigroup of contractions.*

In this work we consider $-\Delta$ with the homogeneous Neumann boundary condition as an unbounded linear operator on $L_2 = L_2(\mathcal{D})$ with standard scalar product $\langle \cdot, \cdot \rangle$ and norm $\|\cdot\|$. It has eigenvalues $\{\lambda_j\}_{j=0}^{\infty}$ with

$$0 = \lambda_0 < \lambda_1 \leq \cdots \leq \lambda_j \leq \cdots \quad \leq \lambda_j \to \infty,$$

and corresponding orthonormal eigenfunctions $\{\varphi_j\}_{j=0}^{\infty}$. The first eigenfunction φ_0 is constant. Also we let $\dot{H}$ be the subspace of H, which is orthogonal to the constants,

$$\dot{H} = \Big\{ v \in L_2 : \langle v, 1 \rangle = 0 \Big\},$$

and let P be the orthogonal projection of H onto $\dot{H}$. Define the linear operator $A = -\Delta$ with domain of definition

$$D(A) = \Big\{ v \in H^2 \cap H : \frac{\partial v}{\partial n} = 0 \text{ on } \partial\mathcal{D} \Big\}.$$

By spectral theory we define $\dot{H}^s = \mathcal{D}(A^{s/2})$ with norms $|v|_s = \|A^{s/2} v\|$ for real $s \geq 0$. Then the semigroup e^{-tA^2} generated by $G = -A^2$ can be written as

$$\mathrm{e}^{-tA^2} v = \sum_{j=0}^{\infty} \mathrm{e}^{-t\lambda_j^2} \langle v, \varphi_j \rangle \varphi_j.$$

This is a strongly continuous semigroup. Moreover, it is analytic, meaning that e^{-tA^2} can be extended to be a holomorphic function of t. This leads to the important properties in the following lemma. For the proof and more details about properties of semigroups we refer to [17].

Lemma 2.2. *If $\{\mathrm{e}^{-tA^2}\}_{t \geq 0}$ is the semigroup generated by $-A^2$, then the following hold:*

$$\|A^{\beta} \mathrm{e}^{-tA^2} v\| \leq C t^{-\beta/2} \|v\|, \quad t > 0, \ \beta \geq 0,$$
$$\int_0^t \|A \mathrm{e}^{-sA^2} v\|^2 \, \mathrm{d}s \leq C \|v\|^2.$$

3 Cahn-Hilliard equation

When a homogeneous molten binary alloy is rapidly cooled, the resulting solid is usually found to be not homogeneous, but instead has a fine grained structure consisting of just two materials, which differs only in the mass fraction of the components of the alloy. The development of a fine grained structure from a homogeneous state is referred to as spinodal decomposition.

In 1958, J. Cahn and J. Hilliard [4] derived an expression for the free energy of a sample V of binary alloy with concentration field $c(x)$ of one of two species. They assumed that the free energy density depends not only on $c(x)$ but also on the derivative of c. The expression for the total free energy has the form,

$$\mathcal{E} = N_V \int_V \left(F(c) + \kappa|\nabla c|^2\right) \, \mathrm{d}V, \tag{3.1}$$

where N_V is the number of molecules per unit volume, F is the free energy per molecule of an alloy of uniform composition, and κ is a material constant which is typically very small. The function F has two wells with minima located at the two coexistent concentration states, labeled c_α and $c_\beta > c_\alpha$.

With the given average concentration τ, the equilibrium configurations satisfy the Cahn-Hilliard equation

$$2\kappa\Delta c - F'(c) = \lambda \quad \text{in } V, \tag{3.2}$$

$$\frac{\partial c}{\partial n} = 0 \quad \text{on } \partial V, \tag{3.3}$$

where Δ is the Laplacian, λ is a Lagrange multiplier associated with the constraint τ, and n is the normal to ∂V. In [4], equations (3.2), (3.3) together with the constraint are used to predict the profile and thickness of one-dimensional transitions between concentration phases c_α and c_β.

The general equation governing the evolution of a non-equilibrium state $c(x,t)$ is put forth in [3] and this is what is now referred to as the Cahn-Hilliard equation

$$\frac{\partial c}{\partial t} = \nabla \cdot \{M\nabla(F'(c) - 2\kappa\Delta c)\} \quad \text{in } V, \tag{3.4}$$

with the boundary conditions

$$\frac{\partial c}{\partial n} = \frac{\partial \Delta c}{\partial n} = 0 \quad \text{on } \partial V. \tag{3.5}$$

The positive quantity M is related to the mobility of the two atomic species which comprise the alloy.

In the this thesis we consider the Cahn-Hilliard equation in the form

$$\begin{aligned} u_t - \epsilon\Delta w \,\mathrm{d}t &= 0 \quad && \text{in } \mathcal{D}\times[0,T], \\ w + \Delta u - f(u) &= 0 \quad && \text{in } \mathcal{D}\times[0,T], \\ \frac{\partial u}{\partial n} = \frac{\partial w}{\partial n} &= 0 \quad && \text{on } \partial\mathcal{D}\times[0,T], \\ u(0) &= u_0 \quad && \text{in } \mathcal{D}, \end{aligned} \tag{3.6}$$

where $u_t = \partial u/\partial t$. The equation perturbed by noise is

$$\begin{aligned} \mathrm{d}u - \epsilon\Delta w \,\mathrm{d}t &= \mathrm{d}W \quad && \text{in } \mathcal{D}\times[0,T], \\ w + \Delta u - f(u) &= 0 \quad && \text{in } \mathcal{D}\times[0,T], \\ \frac{\partial u}{\partial n} = \frac{\partial w}{\partial n} &= 0 \quad && \text{on } \partial\mathcal{D}\times[0,T], \\ u(0) &= u_0 \quad && \text{in } \mathcal{D}, \end{aligned} \tag{3.7}$$

where $\mathcal{D}$ is a bounded domain in $\mathbf{R}^d$, $d = 1, 2, 3$ and $f(s) = s^3 - s$.

In the sequel we will write the equation (3.6) in operator form. By definition of $D(A)$ and H, equation (3.6) can be written as

$$\begin{aligned} &u_t + A^2 u = -Af(u), \quad t > 0, \\ &u(0) = u_0, \end{aligned} \tag{3.8}$$

which is equivalent to the fixed point equation

$$u(t) = \mathrm{e}^{-tA^2} u_0 - \int_0^t \mathrm{e}^{-(t-s)A^2} A f(u(s)) \,\mathrm{d}s.$$

The generator $-A^2$ is the infinitesimal generator of an analytic semigroup e^{-tA^2} on H so that

$$\begin{aligned} \mathrm{e}^{-tA^2} v &= \sum_{j=0}^{\infty} \mathrm{e}^{-t\lambda_j^2} \langle v, \varphi_j\rangle \varphi_j = \sum_{j=1}^{\infty} \mathrm{e}^{-t\lambda_j^2} \langle v, \varphi_j\rangle \varphi_j + \langle v, \varphi_0\rangle \varphi_0 \\ &= \mathrm{e}^{-tA^2} P v + (I - P) v. \end{aligned}$$

4 Stochastic analysis in Hilbert space

In this thesis we use the stochastic integrals and its properties frequently, so in this section we recall some definitions and theorems about stochastic integrals without proof. For more details one may refer to Prévôt and Röckner [20], Da Prato and Zabczyk [7], Klebaner [13] and Grigoriu [12].

4.1 Wiener process

Let Q be a selfadjoint, positive semidefinite, bounded linear operator on the Hilbert space U with $\mathrm{Tr}(Q) < \infty$. Let U and H be separable Hilbert spaces and assume that $\{W(t)\}_{t\in[0,T]}$ is a U-valued Q-Wiener process on a probability space $(\Omega, \mathcal{F}, P)$ with respect to the normal filtration $\{\mathcal{F}_t\}_{t\in[0,T]}$, where $T > 0$ is fixed.

Definition 4.1. A U-valued stochastic process $\{W(t)\}_{t\geq 0}$ is called a Q-Wiener process if

- $W(0) = 0$,
- $\{W(t)\}_{t\geq 0}$ has continuous paths almost surely,
- $\{W(t)\}_{t\geq 0}$ has independent increments,
- The increments have a Gaussian law, that is,

$$P \circ (W(t) - W(s))^{-1} = N(0, (t-s)Q), \quad 0 \leq s < t.$$

Let $\{e_k\}_{k=1}^{\infty}$ be an orthonormal eigenbasis for Q with corresponding eigenvalues $\{\gamma_k\}_{k=1}^{\infty}$. Then we define

$$W(t) = \sum_{k=1}^{\infty} \gamma_k^{\frac{1}{2}} \beta_k(t) e_k,$$

where the β_k are real valued independent Brownian motions. The series converges in $L_2(\Omega, H)$.

4.2 Stochastic integral

Definition 4.2. Let $L(U,H)$ denote the space of bounded linear operators $U \to H$. An $L(U,H)$-valued process $\{\Phi(t)\}_{t\in[0,T]}$ is called elementary if there exist $0 = t_0 < t_1 < \cdots < t_N = T$, $N \in \mathbf{N}$, such that

$$\Phi(t) = \sum_{m=0}^{N-1} \Phi_m 1_{(t_m, t_{m+1}]}(t), \quad t \in [0,T],$$

where

- $\Phi_m \colon (\Omega, \mathcal{F}) \to L(U,H)$ is strongly $\mathcal{F}_{t_m}$ measurable,
- Φ_m takes only a finite number of values in $L(U,H)$.

We denote the (linear) space of elementary process by $\mathcal{E}$.

Definition 4.3 (Itô integral)**.** For $\Phi \in \mathcal{E}$, we define the stochastic integral by

$$\int_0^t \Phi \, \mathrm{d}W := \sum_{n=0}^{N-1} \Phi_n(\Delta W_n(t)), \quad t \in [0, T],$$

where

$$\Delta W_n(t) = W(t_{n+1} \wedge t) - W(t_n \wedge t) \quad t \wedge s = \min(t, s).$$

Definition 4.4 (Hilbert-Schmidt operators)**.** An operator $T \in L(U, H)$ is Hilbert-Schmidt if $\sum_{k=1}^{\infty} \|Te_k\|^2 < \infty$ for an orthonormal basis $\{e_k\}_{k\in\mathbf{N}}$ in U. The Hilbert-Schmidt operators form a linear space denoted by $\mathcal{L}_2(U, H)$ which becomes a Hilbert space with scalar product and norm

$$\langle T, S\rangle_{\mathrm{HS}} = \sum_{k=1}^{\infty} \langle Te_k, Se_k\rangle_H, \ \|T\|_{\mathrm{HS}} = \Big(\sum_{k=1}^{\infty} \|Te_k\|_H^2\Big)^{\frac{1}{2}}.$$

We recall that the trace of a linear operator T is

$$\mathrm{Tr}(T) = \sum_{k=1}^{\infty} \langle Te_k, e_k\rangle.$$

Consider the covariance operator $Q: U \to U$, selfadjoint, positive semidefinite, bounded and linear. Also assume that $W(t)$ is Q-Wiener process. If

$$\mathbf{E} \int_0^t \|T(s)Q^{1/2}\|_{\mathrm{HS}}^2 \, \mathrm{d}s < \infty,$$

we can define the stochastic integral $\int_0^t T(s) \, \mathrm{d}W(s)$ as a limit in $L_2(\Omega, H)$ of integrals of elementary processes.

One important property the stochastic integral is the *isometry property*:

Proposition 4.5 (Isometry property)**.**

$$\mathbf{E}\Big\|\int_0^t T(s) \, \mathrm{d}W(s)\Big\|^2 = \mathbf{E} \int_0^t \|T(s)Q^{1/2}\|_{\mathrm{HS}}^2 \, \mathrm{d}s. \tag{4.1}$$

4.3 Stochastic ordinary differential equation

Stochastic differential equations arise naturally in various engineering problems, where the effects of random *noise* perturbations to a system are being considered. For example in the problem of tracking a satellite, we know that it's motion will obey Newton's law to a very high degree of accuracy, so in theory we can integrate the trajectories from the initial points. However in practice there are rather random effects which perturb the motions.

For more details one can refer to Kuo [14], Klebaner [13] and Chung and Williams [6] The variety of SDE to be considered here describes a *diffusion* process and has the form

$$\mathrm{d}X_t = b(t, X_t) + \sigma(t, X_t)\,\mathrm{d}B_t, \tag{4.2}$$

where $b_i(x,t)$ and $\sigma_{ij}(t,x)$ for $1 \le i \le d$ and $1 \le j \le r$ are Borel measurable functions.

Definition 4.6 (Strong solution)**.** A strong solution of the SDE (4.2) on the given probability space $(\Omega, \mathcal{F}, P)$ with initial condition ξ is a process $\{X_t\}_{t\ge 0}$ which has continuous sample paths such that

- X_t is adapted to the augmented filtration generated by the Brownian motion B and initial condition ξ, which is denoted $\mathcal{F}_t$.
- $P(X_0 = \xi) = 1$.
- For every $0 \le t < \infty$ and for each $1 \le i \le d$ and $1 \le j \le r$, then the following hold almost surely

$$\int_0^t |b_i(s, X_s)| + \sigma_{ij}^2(s, X_s)\mathrm{d}s < \infty.$$

- Almost surely the following holds

$$X_t = \xi + \int_0^t b(s, X_s)\,\mathrm{d}s + \int_0^t \sigma(s, X_s)\,\mathrm{d}B_s.$$

4.4 Stochastic partial differential equation

A stochastic partial differential equation (SPDE) is a partial differential equation containing a random (noise) term. The study of SPDEs is an exciting topic which brings together techniques from probability theory, functional analysis, and the theory of partial differential equations.

Stochastic partial differential equations appear in several different applications: study of random evolution of systems with a spatial extension (random interface growth, random evolution of surfaces, fluids subject to random forcing), study of stochastic models where the state variable is infinite dimensional (for example, a curve or surface), see Carmona [5], Musiela [16], Goldys et al. [11], Goldys and Maslowski [10], Peszat and Zabczyk [19], [18]. The solution to a stochastic partial differential equations may be viewed in several manners. One can view a solution as a random field (set of random variables indexed by a multidimensional parameter). In the case where the SPDE is an evolution equation, the infinite dimensional point of view consists in viewing the solution at a given time as a random element in a function space and thus view the SPDE as a stochastic evolution equation in an infinite dimensional space. In the pathwise point of view, one tries to give a meaning to the solution for (almost) every realization of the noise and then view the solution as a random variable on the set of (infinite dimensional) paths thus defined.

In this section we have a short introduction to the stochastic partial differential equations. For more details and proofs we refer to Frieler and Knoche [9], Da Prato and Zabczyk [7] and Prévôt and Röckner [20].

Definition 4.7. Let $\{W(t)\}_{t\in[0,T]}$ be a U-valued Q-Wiener process on the probability space $(\Omega, \mathcal{F}, P)$, adapted to a normal filtration $\{\mathcal{F}_t\}_{t\in[0,T]}$. The stochastic partial differential equation (SPDE) is of the form

$$\begin{aligned} &\mathrm{d}X(t) = (AX(t) + f(t))\,\mathrm{d}t + \mathrm{d}W(t), \quad 0 < t < T, \\ &X(0) = \xi, \end{aligned} \tag{4.3}$$

where the following assumptions hold:

1. A is a linear operator, generating a strongly continuous semigroup (C_0-semigroup) of bounded linear operators $\{E(t)\}_{t\geq 0}$,
2. $B \in L(U, H)$,
3. $\{f(t)\}_{t\in[0,T]}$ is a predictable H-valued process with Bochner integrable trajectories,
4. ξ is an $\mathcal{F}_0$-measurable H-valued random variable.

Definition 4.8 (Weak solution)**.** An H-valued process $\{X(t)\}_{t\in[0,T]}$ is a weak solution of (4.3) if $\{X(t)\}_{t\in[0,T]}$ is H-predictable, $\{X(t)\}_{t\in[0,T]}$ has Bochner integrable trajectories P-almost surely and

$$\begin{aligned} \langle X(t), \eta\rangle = \langle \xi, \eta\rangle &+ \int_0^t (\langle X(s), A^*\eta\rangle + \langle f(s), \eta\rangle)\,\mathrm{d}s \\ &+ \int_0^t B\,\mathrm{d}W(s), \quad P\text{-a.s.}, \ \forall \eta \in D(A),\, t \in [0,T]. \end{aligned}$$

Definition 4.9 (Mild solution)**.** An U-valued predictable process $X(t)$, $t \in [0,T]$, is called a *mild solution* of problem (4.3) if

$$X(t) = E(t)\xi + \int_0^t E(t-s)f(s)\,\mathrm{d}s + \int_0^t E(t-s)B(X(s))\,\mathrm{d}W(s)$$

P-a.s. for each $t \in [0,T]$. In particular, the appearing integrals have to be well defined.

Definition 4.10 (Strong solution)**.** An H-valued process $\{X(t)\}_{t\in[0,T]}$ is a strong solution of (4.3) if $\{X(t)\}_{t\in[0,T]}$ is H-predictable, $X(t,\omega)\in\mathcal{D}(A)$ P_T-almost surely, $\int_0^T \|AX(t)\|\,\mathrm{d}t<\infty$ P-almost surely, and, for all $t\in[0,T]$,

$$X(t)=\xi+\int_0^t (AX(s)+f(s))\,\mathrm{d}s+\int_0^t B\,\mathrm{d}W(s),\quad P\text{-a.s.}$$

Recall that the integral $\int_0^t B\,\mathrm{d}W(s)$ is defined if and only if $\|B\|_{\mathrm{HS}}^2=\mathrm{Tr}(BQB^*)<\infty$.

In a special case we have the stochastic Cahn-Hilliard equation as

$$\begin{aligned}&\mathrm{d}X(t)+A^2X(t)\,\mathrm{d}t+Af(X(t))\,\mathrm{d}t=\mathrm{d}W(t),\quad t>0,\\&X(0)=X_0,\end{aligned}\tag{4.4}$$

where $A=-\Delta$, P is the orthogonal projection of L_2 onto $\dot{H}$. By using the semigroup approach we can write the mild solution to the equation (4.4) as

$$X(t)=E(t)X_0-\int_0^t E(t-s)Af(X(s))\,\mathrm{d}s+\int_0^t E(t-s)\,\mathrm{d}W(s),\tag{4.5}$$

where $\{E(t)\}_{t\geq0}=\{\mathrm{e}^{-tA^2}\}_{t\geq0}$ is the semigroup generated by $-A^2$. In this thesis we study the equation (4.4) in linear, $f\equiv0$, and nonlinear cases.

4.5 Stochastic convolution

The last term in (4.5) is a stochastic convolution

$$\begin{aligned}W_A(t)&=\int_0^t \mathrm{e}^{-(t-s)A^2}\,\mathrm{d}W(s)\\&=\int_0^t \mathrm{e}^{-(t-s)A^2}\,P\mathrm{d}W(s)+\int_0^t\langle \mathrm{d}W(s),\varphi_0\rangle\varphi_0\\&=\int_0^t \mathrm{e}^{-(t-s)A^2}\,P\mathrm{d}W(s)+\langle W(t),\varphi_0\rangle\varphi_0.\\&=\int_0^t \mathrm{e}^{-(t-s)A^2}\,P\mathrm{d}W(s)+(I-P)W(t).\end{aligned}\tag{4.6}$$

5 Finite element method

The finite element method (FEM) is a numerical technique for finding approximate solutions of partial differential equations (PDE). In solving PDEs, the primary challenge is to create an equation that approximates the equation to be studied, but is numerically stable, meaning that errors in the input data and intermediate calculations do not accumulate and cause the resulting output to be meaningless. There are many ways of doing this, all with advantages and disadvantages. The finite element method is a good choice for solving partial differential equations over complicated domains. For more details one can refer to Larsson and Thomée [15] and Thomée [21].

In this section we study the FEM for the Cahn-Hilliard equation in deterministic and stochastic cases.

Let $\{\mathcal{T}_h\}_{h>0}$ denote a family of regular triangulations of $\mathcal{D}$ with maximal mesh size h. Let S_h the space of continuous functions on $\mathcal{D}$, which are piecewise polynomials of degree ≤ 1 with respect to $\mathcal{T}_h$. Hence $S_h \subset H^1$. We also define $\dot{S}_h = PS_h$, that is,

$$\dot{S}_h = \Big\{ v_h \in S_h : \int_{\mathcal{D}} v_h \,\mathrm{d}x = 0 \Big\}.$$

The space $\dot{S}_h$ is only used for the purpose of theory but not for computation. Now we define the "discrete Laplacian" $A_h \colon S_h \to \dot{S}_h$ by

$$\langle A_h v_h, w_h \rangle = \langle \nabla v_h, \nabla w_h \rangle, \quad \forall v_h \in S_h,\, w_h \in \dot{S}_h. \tag{5.1}$$

The operator A_h is selfadjoint, positive definite on $\dot{S}_h$ and A_h has an orthonormal eigenbasis $\{\varphi_{h,j}\}_{j=0}^{N_h}$ with corresponding eigenvalues $\{\lambda_{h,j}\}_{j=0}^{N_h}$. We have

$$0 = \lambda_{h,0} < \lambda_{h,1} < \cdots \leq \lambda_{h,j} \leq \lambda_{h,N_h},$$

and $\varphi_{h,0} = \varphi_0 = |\mathcal{D}|^{-\frac{1}{2}}$. Moreover we define $\mathrm{e}^{-tA_h^2} \colon S_h \to S_h$ by

$$\mathrm{e}^{-tA_h^2} v_h = \sum_{j=0}^{N_h} \mathrm{e}^{-t\lambda_{h,j}^2} \langle v_h, \varphi_{h,j} \rangle \varphi_{h,j} = \sum_{j=1}^{N_h} \mathrm{e}^{-t\lambda_{h,j}^2} \langle v_h, \varphi_{h,j} \rangle \varphi_{h,j} + \langle v_h, \varphi_0 \rangle \varphi_0,$$

and the orthogonal projector $P_h \colon H \to S_h$ by

$$\langle P_h v, w_h \rangle = \langle v, w_h \rangle \quad \forall v \in H,\, w_h \in S_h. \tag{5.2}$$

Clearly $P_h \colon \dot{H} \to \dot{S}_h$ and

$$\mathrm{e}^{-tA_h^2} P_h v = \mathrm{e}^{-tA_h^2} P_h P v + (I - P) v.$$

5.1 FEM for the deterministic Cahn-Hilliard equation

Consider the Cahn-Hilliard equation (3.6) with $\epsilon = 1$

$$\begin{aligned} u_t - \Delta w &= 0, && x \in \mathcal{D},\ t > 0, \\ w + \Delta u - f(u) &= 0, && x \in \mathcal{D},\ t > 0, \\ \frac{\partial u}{\partial n} = 0,\ \frac{\partial v}{\partial n} &= 0, && x \in \partial\mathcal{D},\ t > 0, \\ u(x,0) &= u_0(x), && x \in \mathcal{D}. \end{aligned} \tag{5.3}$$

Multiply the first and the second equation of (5.3) by $\phi = \phi(x) \in H^1(\mathcal{D}) = H^1$ and integrate over $\mathcal{D}$. Using Green's formula gives

$$\begin{aligned} &\langle u_t, \phi\rangle + \langle \nabla w, \nabla \phi\rangle = 0 && \forall \phi \in H^1, \\ &\langle w, \phi\rangle = \langle \nabla u, \nabla \phi\rangle + \langle f(u), \phi\rangle && \forall \phi \in H^1. \end{aligned} \tag{5.4}$$

So the variational formulation is: Find $u(t), w(t) \in H^1$ such that (5.4) holds and such that $u(x, 0) = u_0(x)$ for $x \in \mathcal{D}$.

Let $\mathcal{T}_h = \{K\}$ denote a triangulation of $\mathcal{D}$ and let S_h denote the continuous piecewise polynomial functions on $\mathcal{T}_h$. So the finite element problem is: Find $u_h(t), w_h(t) \in S_h$ such that

$$\begin{aligned} &\langle u_{h,t}, \chi\rangle + \langle \nabla w_h, \nabla \chi\rangle = 0 && \forall \chi \in S_h,\ t > 0, \\ &\langle w_h, \chi\rangle = \langle \nabla u_h, \nabla \chi\rangle + \langle f(u_h), \chi\rangle && \forall \chi \in S_h,\ t > 0, \\ &u_h(0) = u_{h,0}. \end{aligned} \tag{5.5}$$

Then we can write the equation (5.5) as

$$\begin{aligned} &u_{h,t} + A_h^2 u_h + A_h P_h f(u_h) = 0, \quad t > 0, \\ &u_h(0) = u_{0,h}, \end{aligned} \tag{5.6}$$

which is equivalent to the fixed point equation

$$u_h(t) = \mathrm{e}^{-tA_h^2} u_{0,h} - \int_0^t \mathrm{e}^{-(t-s)A_h^2} A_h P_h f(u_h(s))\, \mathrm{d}s,$$

where

$$\mathrm{e}^{-tA_h^2} v = \sum_{j=0}^{\infty} \mathrm{e}^{-t\lambda_{h,j}^2} \langle v, \varphi_{h,j}\rangle \varphi_{h,j},$$

where $(\lambda_{h,j}, \varphi_{h,j})$ are the eigenpairs of A_h.

5.2 FEM for the stochastic Cahn-Hilliard equation

Consider the equation (4.4) and assume that $\{\mathcal{T}_h\}_{0<h<1}$ is a triangulation with mesh size h and $\{S_h\}_{0<h<1}$ is the set of continuous piecewise linear functions where $S_h \subset H^1(\mathcal{D})$. Also let A_h and P_h be the same as in (5.1) and (5.2). The finite element problem for (4.4) is:

Find $X_h(t) \in \dot{S}_h$ such that

$$\begin{aligned} &\mathrm{d}X_h(t) + A_h^2 X_h(t)\, \mathrm{d}t + A_h P_h f(X_h(s))\, \mathrm{d}t = P_h\, \mathrm{d}W(t), \\ &X_h(0) = P_h X_0, \end{aligned} \tag{5.7}$$

where $P_hW(t)$ is Q_h-Wiener process on S_h with $Q_h = P_hQP_h$. The mild solution is given by the equation

$$X_h(t) = E_h(t)P_hX_0 - \int_0^t E_h(t-s)A_hP_hf(X_h(s))\,\mathrm{d}s + \int_0^t E(t-s)P_h\,\mathrm{d}W(s),$$

where $E_h(t) = \mathrm{e}^{-tA_h^2}$. In the linear case, the finite element problem is

$$\begin{aligned}&\mathrm{d}X_h(t) + A_h^2X_h(t)\,\mathrm{d}t = P_h\,\mathrm{d}W(t),\\ &X_h(0) = P_hX_0,\end{aligned} \tag{5.8}$$

with mild solution

$$X_h(t) = E(t)P_hX_0 + \int_0^t E(t-s)P_h\,\mathrm{d}W(s).$$

Now define the stochastic convolution

$$\begin{aligned}W_{A_h}(t) &= \int_0^t \mathrm{e}^{-(t-s)A_h^2}\,P_h\mathrm{d}W(s)\\ &= \int_0^t \mathrm{e}^{-(t-s)A_h^2}\,P_hP\mathrm{d}W(s) + \langle W(t),\varphi_0\rangle\varphi_0\\ &= \int_0^t \mathrm{e}^{-(t-s)A_h^2}\,P_hP\mathrm{d}W(s) + (I-P)W(t).\end{aligned}$$

Hence, in view of (4.6),

$$W_{A_h}(t) - W_A(t) = \int_0^t \left(\mathrm{e}^{-(t-s)A_h^2}P_h - \mathrm{e}^{-(t-s)A^2}\right)P\,\mathrm{d}W(s),$$

so that the error can be analyzed in the spaces $\dot{H}$ and $\dot{S}_h$.

Let $k = \Delta t_n$, $t_n = nk$ and $\Delta W_n = W(t_n) - W(t_{n-1})$. Also consider $\Delta X_{h,n} = X_{h,n} - X_{h,n-1}$ and apply the backward Euler method to (5.8) to get

$$\begin{aligned}&X_{h,n} \in S_h,\\ &\Delta X_{h,n} + A_h^2X_{h,n}\Delta t_n = P_h\Delta W_n,\\ &X_{h,0} = P_hX_0.\end{aligned} \tag{5.9}$$

This implies

$$X_{h,n} - X_{h,n-1} + kA_h^2X_{h,n} = P_h\Delta W_n.$$

If we set $E_{k,h} = (I + kA_h^2)^{-1}$ we get

$$(I + kA_h^2)X_{h,n} = P_h\Delta W_n + X_{h,n-1}.$$

So

$$X_{h,n} = E_{k,h} P_h \Delta W_n + E_{k,h} X_{h,n-1}.$$

We repeat it to get

$$X_{h,n} = E_{k,h}^n P_h X_0 + \sum_{j=1}^{n} E_{k,h}^{n-j+1} P_h \Delta W_j. \tag{5.10}$$

6 A posteriori error estimate

In this section we recall some theorems and techniques for a posteriori error estimates for the Galerkin approximation of nonlinear variational problems. For more details and proofs, we refer to Bangerth and Rannacher [1] and Becker and Rannacher [2].

Let $A(u, \cdot)$ be a semi-linear form and $J(\cdot)$ an output functional, not necessarily linear, defined on some function space V. Consider the variational problem: Find $u \in V$ such that

$$A(u; \psi) = 0 \quad \forall \psi \in V, \tag{6.1}$$

and the corresponding finite element problem: Find $u_h \in V_h \subset V$ such that

$$A(u_h; \psi_h) = 0 \quad \forall \psi_h \in V_h. \tag{6.2}$$

Suppose that the directional derivatives of A and J up to order three exist and denoted by

$$A'(u; \varphi, \cdot),\ A''(u; \psi, \varphi, \cdot),\ A'''(u; \xi, \psi, \varphi, \cdot),$$

and

$$J'(u; \varphi),\ J''(u; \psi, \varphi),\ A''(u; \xi, \psi, \varphi),$$

respectively for increments φ, ψ, $\xi \in V$. We want to estimate $J(u) - J(u_h)$. Introduce dual variable $z \in V$ and define the Lagrangian functional

$$\mathcal{L}(u; z) := J(u) - J(u_h),$$

and seek for the stationary points $\{u, z\} \in V \times V$ of $\mathcal{L}(\cdot, \cdot)$. i.e. for all $\psi, \varphi \in V$

$$\mathcal{L}'(u; z, \varphi, \psi) = J'(u; \varphi) - A'(u; z, \varphi) - A(u; \psi) = 0.$$

We quote three lemmas from [1].

Lemma 6.1. *Let $L(\cdot)$ be a three times differentiable functional defined on a (real or complex) vector space X which has a stationary point $x \in X$, i.e.*

$$L'(x; y) = 0, \quad \forall y \in X,$$

Suppose that on a finite dimensional subspace $X_h \subset X$ the corresponding Galerkin approximation

$$L'(x_h; y_h) = 0 \quad \forall y_h \in X_h.$$

has a solution, $x_h \in X_h$. Then there holds the error representation

$$L(x) - L(x_h) = \frac{1}{2} L'(x_h; x - y_h) + \mathcal{R}_h \quad \forall y_h \in X_h,$$

with a remainder term $\mathcal{R}_h$, which is cubic in the error $e := x - x_h$,

$$\mathcal{R}_h := \frac{1}{2} \int_0^1 L'''(x_h + se; e, e, e) s(s-1)\, \mathrm{d}s.$$

From Lemma 6.1 we obtain the following result for the Galerkin approximation of the variational equation.

Lemma 6.2. *For any solutions of equations (6.1) and (6.2) we have the error representation*

$$J(u) - J(u_h) = \frac{1}{2} \rho(u_h; e_z) + \frac{1}{2} \rho^*(u_h; z_h, e_u) + \mathcal{R}_h^{(3)},$$

where

$$\rho(u_h; e_z) = -A'(u_h; z_h, e_z),$$
$$\rho^*(u_h; z_h, e_u) = J'(u_h; e_u) - A'(u_h; z_h, e_u),$$

with $e_u = u - u_h$, $e_z = z - z_h$ and

$$\mathcal{R}_h^{(3)} = \frac{1}{2} \int_0^1 \Big(J'''(u_h + se_u; e_u, e_u, e_u) - A'''(u_h + se_u; z_h + se_z, e_u, e_u, e_u) - 3A''(u_h + se_u; e_u, e_u, e_z) \Big) s(s-1)\, \mathrm{d}s$$

The forms $\rho(\cdot,\cdot), \rho^*(\cdot;\cdot,\cdot)$ are the residuals of (6.1) and the linearized adjoint equation, respectively. The remainder $\mathcal{R}_h^{(3)}$ is cubic in the error. The following lemma shows that the residuals are equal up to a quadratic remainder.

Lemma 6.3. *With the notation from above, for any φ_h, $\psi_h \in V_h$ there holds*

$$\rho^*(u_h; z_h, u-\varphi_h) = \rho(u_h; z-\psi_h) + \Delta\rho \quad \forall \varphi_h, \psi_h \in V_h,$$

with

$$\Delta\rho = \int_0^1 \Big(A''(u_h+se_u; e_u, e_u, z_h+se_z) - J''(u_h+se_u; e_u, e_u) \Big)\,\mathrm{d}s.$$

Moreover, we we have the simplified error representation

$$J(u) - J(u_h) = \rho(u_h, z-\varphi_h) + \mathcal{R}_h^{(2)} \quad \forall \varphi_h \in V_h,$$

with quadratic remainder

$$\mathcal{R}_h^{(2)} = \int_0^1 \Big(A''(u_h+se_u, e_u, e_u, z) - J''(u_h+se_u; e_u, e_u) \Big)\,\mathrm{d}s.$$

In Paper III we apply this methodology to a space and time discretization of the deterministic Cahn-Hilliard equation.

7 Summary of appended papers

7.1 Paper I

In this paper we prove error bounds for the linear Cahn-Hilliard-Cook equation; that is, (3.7) with $f(u)=0$. The main result is a mean square error estimate for the finite element approximation defined in (5.8):

$$\begin{aligned}\|X_h(t) - &X(t)\|_{L_2(\Omega,H)} \\ &\le Ch^\beta(\|X_0\|_{L_2(\Omega,\dot H^\beta)} + |\log h| \|A^{\frac{\beta-2}{2}} Q^{\frac12}\|_{\mathrm{HS}}).\end{aligned}$$

The proof is essentially based on applying the isometry (4.5) to

$$W_{A_h}(t) - W_A(t) = \int_0^t \Big(\mathrm{e}^{-(t-s)A_h^2} P_h - \mathrm{e}^{-(t-s)A^2} \Big) P\,\mathrm{d}W(s).$$

The proof is then reduced to proving bounds for the error operator $F_h(t) = E_h(t)P_hP - E(t)P$ for the corresponding linear deterministic problem. For this problem we show the following error bounds with optimal dependence on the regularity of the initial value v:

$$\begin{aligned} &\|F_h(t)v\| \le Ch^\beta |v|_\beta, && v \in \dot H^\beta, \\ &\Big(\int_0^t \|F_h(\tau)v\|^2\,\mathrm{d}\tau\Big)^{\frac12} \le C|\log h| h^\beta |v|_{\beta-2}, && v \in \dot H^{\beta-2}, \end{aligned}$$

for $1 \leq \beta \leq r$, where $r \geq 2$ is the order of the finite element method.

The same program is carried out for the the backward Euler method in (5.9). The result is the error bound

$$\begin{aligned} &\|X_{h,n}(t) - X(t_n)\|_{L_2(\Omega,H)} \\ &\qquad \leq \left(C|\log h|h^{\beta} + C_{\beta,k}k^{\frac{\beta}{4}}\right)\left(\|X_0\|_{L_2(\Omega,\dot{H}^{\beta})} + \|A^{\frac{\beta-2}{2}}Q^{\frac{1}{2}}\|_{\mathrm{HS}}\right), \end{aligned}$$

where where $C_{\beta,k} = \frac{C}{4-\beta}$ for $\beta < 4$ and $C_{\beta,k} = C|\log k|$ for $\beta = 4$.

7.2 Paper II

We study the nonlinear stochastic Cahn-Hilliard equation driven by additive colored noise (3.7). Using the framework of [7] we write this as an abstract evolution equation of the form

$$\mathrm{d}X + (A^2X + Af(X))\,\mathrm{d}t = \mathrm{d}W, \quad t > 0; \quad X(0) = X_0, \tag{7.1}$$

Our goal is to study the convergence properties of the spatially semidiscrete finite element approximation X_h of X, which is defined by an equation of the form

$$\mathrm{d}X_h + (A_h^2X + A_hP_hf(X))\,\mathrm{d}t = P_h\mathrm{d}W, \quad t > 0; \quad X(0) = P_hX_0.$$

In order to do so, we need to prove existence and regularity for solutions of (7.1).

Following the semigroup framework of [7] we write the equation (7.1) as the integral equation (mild solution)

$$X(t) = \mathrm{e}^{-tA^2}X_0 - \int_0^t \mathrm{e}^{-(t-s)A^2}Af(X(s))\,\mathrm{d}s + \int_0^t \mathrm{e}^{-(t-s)A^2}\,\mathrm{d}W(s).$$

This naturally splits the solution as $X = Y + W_A$, where $W_A(t)$ is the stochastic convolution that was studied in Paper I. The remaining part, Y, satisfies an evolution equation without noise, but with a random coefficient,

$$\dot{Y} + A^2Y + Af(X) = 0, \quad t > 0; \quad Y(0) = X_0.$$

The regularity and error analysis can now be performed on this equation.

An important step is to bound the functional

$$J(u) = \frac{1}{2}\|\nabla u\|^2 + \int_{\mathcal{D}} F(u)\,\mathrm{d}x,$$

where $F(s)$ is a primitive function to $f(s)$. For the deterministic equation this is a Lyapunov functional, which means that it does not increase along solution paths. For the equation which is perturbed by noise we show that

$$\mathbf{E}[J(X(t))] \leq C(t),$$

where $C(t)$ grows quadratically in t. The same result holds for $X_h(t)$. By means of Chebyshev's inequality we may then show that for each $T > 0$ and $\epsilon \in (0,1)$ there are K_T and $\Omega_\epsilon \subset \Omega$ with $\mathbf{P}(\Omega_\epsilon) \geq 1 - \epsilon$ and such that

$$\|X(t)\|_{H^1}^2 + \|X_h(t)\|_{H^1}^2 \leq \epsilon^{-1} K_T \quad \text{on } \Omega_\epsilon,\ t \in [0,T].$$

These bounds are then used to control the random term $f(X)$ and we show the necessary regularity and the error estimate

$$\|X_h(t) - X(t)\| \leq C(\epsilon^{-1}K_T, T)h^2|\log(h)| \quad \text{on } \Omega_\epsilon,\ t \in [0,T].$$

We thus have optimal rate of convergence on sets of probability arbitrarily close 1, but the constant increases rapidly when $\epsilon \to 0$. Nevertheless, we show that this implies strong convergence but without known rate:

$$\max_{t\in[0,T]} \big(\mathbf{E}[\|X_h(t) - X(t)\|^2]\big)^{\frac{1}{2}} \to 0 \quad \text{as } h \to 0.$$

7.3 Paper III

In this paper we consider the deterministic Cahn-Hilliard equation (3.6) and we discretize it by a Galerkin finite element method, which is based on continuous piecewise linear functions with respect to x and discontinuous piecewise constant functions with respect to t. The numerical method is the same as the implicit Euler time stepping combined with spatial discretization by a standard finite element method.

We perform an a posteriori error analysis within the framework of dual weighted residuals as in section 6. If $J(u)$ is a given goal functional, this results in an error estimate essentially of the form

$$|J(u) - J(U)| \leq \sum_{n=1}^{N} \sum_{K\in\mathbf{T}_n} \Big\{\rho_{u,K}\omega_{u,K} + \rho_{w,K}\omega_{w,K}\Big\} + \mathcal{R},$$

where U denotes the numerical solution and $\mathbf{T}_n$ is the spatial mesh at time level t_n. The terms $\rho_{u,K}, \rho_{w,K}$ are local residuals from the first and second equations in (3.6), respectively. The weights $\omega_{u,K}, \omega_{w,K}$ are derived from the solution of the linearized adjoint problem. The remainder $\mathcal{R}$ is quadratic in the error.

We also derive a variant of this, where the weights are replaced by stability constants, which are obtained by proving a priori estimates for the solution of the linearized adjoint problem.

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