Thesis for the Degree of Licentiate of Philosophy

Enumeration on words, complexes and polytopes

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ABSTRACT

This thesis presents four papers, studying enumerative problems on combinatorial structures.

The first paper studies Forman's discrete Morse theory in the case where a group acts on the underlying complex. We generalize the notion of a Morse matching, and obtain a theory that can be used to simplify the description of the G-homotopy type of a simplicial complex. The main motivation is the case where some group acts transitively on the vertex set of the complex, and G is some large subgroup of this group. In particular we are interested in complexes of graph properties. As an application, we determine the $C_2 \times \mathfrak{S}_{n-2}$-homotopy type of the complex of non-connected graphs on n nodes.

The motivation behind the second paper is Gil Kalai's conjecture from 1989, that a centrally symmetric d-polytope must have at least 3^d non-empty faces. Looking for examples that are close to achieving the lower bound, we study the centrally symmetric Hansen polytopes, associated to perfect graphs. In particular, we study Hansen polytopes of split graphs. Among them, we find an infinite family of polytopes with $3^d + 16$ faces. We also prove that a Hansen polytope of a split graph has at least 3^d non-empty faces.

The third paper studies the problem of packing a pattern as densely as possible into compositions. We are able to find the packing density for some classes of generalized patterns and all the three letter binary patterns.

In the fourth paper, we enumerate derangements with descents in prescribed positions. A generating function was given by Guo-Niu Han and Guoce Xin in 2007. We give a combinatorial proof of this result, and derive several explicit formulas. To this end, we consider fixed point λ-coloured permutations, which are easily enumerated. Several formulae regarding these numbers are given. We also prove that except in a trivial special case, the event that π has descents in a set S of positions is positively correlated with the event that π is a derangement, if π is chosen uniformly in $\mathfrak{S}_n$.

Keywords: Simplicial complex, discrete Morse theory, Hansen polytope, 3^d-conjecture, split graph, derangement, pattern containment, pattern packing, composition.

Preface

This thesis consists of the following papers.

- ▷ **Ragnar Freij**,
  "Equivariant discrete Morse theory",
  in *Discrete Mathematics* **309** (2009), 3821–3829.
- ▷ **Ragnar Freij**, Matthias Henze, Günter M. Ziegler,
  "Hansen polytopes of split graphs", (2010).
- ▷ **Ragnar Freij**, Toufik Mansour,
  "Packing a binary pattern in compositions",
  submitted to *Journal of Combinatorics*, (2010).
- ▷ Niklas Eriksen, **Ragnar Freij**, Johan Wästlund,
  "Enumeration of derangements with descents in prescribed positions",
  in *Electronic Journal of Combinatorics* **16** (2009), #R32.

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Paper I was originally prepared in my master's thesis, written under the supervision of Jan Alve Svensson. I am very thankful for the effort he has given to that paper, and to my career.

Most of Paper II was prepared at the DocCourse in Combinatorics and Geometry 2009, at CRM in Barcelona. To the organizers of that event, as well as to my fellow students there, I owe a word of thank you, for bringing back my love for mathematics.

To the graduate students, professors and administrative staff whose paths have crossed mine at the department: Thank you all for the assistance you have given me in various matters, and for creating such a nice work environment.

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Ragnar Freij
Göteborg, March 2010

Till morfar

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Part I

INTRODUCTION

1
Introduction

The present thesis consists of four different papers. Roughly spoken, they treat two different research areas: (generalizations of) pattern containment in words and combinatorial geometric structures occuring in graph theory. A common theme is the enumeration of combinatorial structures, in one way or another. Papers II, III and IV can also be said to have a common flavour of extremal combinatorics, where we prove that certain "extremal structures" have some natural "standard form". These "extremal structures" are as diverse as

- Hansen polytopes of split graphs, that have few faces,
- compositions that densely pack a given pattern, and
- compositions that are the descent sets of many non-derangements.

In the following, we will discuss these and other aspects of the thesis more closely.

1.1 Geometries from graph theory

We will consider geometric invariants occurring in two different contexts in graph theory. All graphs considered will be assumed to be finite. To avoid overloading words, a *vertex* will always mean a vertex of some geometric object, such as a simplicial complex or a polytope. The sites of graphs will be called *nodes*. For general questions about polytopes, we refer to [19]. We will start by a few standard definitions, to fix notation.

Definition 1 (Simplicial complexes) *An abstract simplicial complex is a collection Σ of finite sets, such that if $\sigma \in \Sigma$ and $\tau \subseteq \sigma$, then $\tau \in \Sigma$.*

An abstract simplicial complex has a geometric realization $|\Sigma|$, obtained by embedding the points of $\cup_{\sigma\in\Sigma}\sigma$ in general position in Euclidean space, denoting by $|\sigma|$ the convex hull of the points in σ, and constructing $|\Sigma| = \cup_{\sigma\in\Sigma}|\sigma|$.

Definition 2 (Polytopes) *A polytope is the convex hull of finitely many points in Euclidean space.*

Equivalently, a polytope is the bounded intersection of finitely many half-spaces in Euclidean space.

The dimension of a polytope is the dimension of its affine hull.

Definition 3 (Faces) *A face of a polytope P is*

$$\{\boldsymbol{x} \in P \mid \langle \boldsymbol{f}, \boldsymbol{x} \rangle = c\},$$

where $\langle \boldsymbol{f}, \boldsymbol{x} \rangle \leq c$ is an inequality that holds for every $\boldsymbol{x} \in P$.

It is easy to prove (see [19]) that every polytope has finitely many faces, that every face is a polytope itself, and that the faces, ordered by inclusion, form a lattice. If $\dim(P) = d$, a maximal face $F \subsetneq P$

has dimension $d-1$ and is called a *facet*. We denote the number of i-dimensional faces of P by $f_i(P)$. The empty set is considered to be a (-1)-dimensional face.

Finally, the following notion of duality will be crucial in Paper II.

Definition 4 (Polar polytope) *Let $P \in \mathbb{R}^d$ be a polytope with the origin in its interior (so in particular P has dimension d). Define the polar of P to be the polytope*

$$P^* = \{\boldsymbol{x} \in \mathbb{R}^d \mid \forall \boldsymbol{v} \in P : \langle \boldsymbol{v}, \boldsymbol{x} \rangle \leq 1\}.$$

We will use the words "polar" and "dual" interchangeably. The face lattice of P^* is isomorphic to the inverted face lattice of P, so in particular we have $f_i(P^*) = f_{d-i-1}(P)$ for $i = -1, \dots, d$.

1.1.1 Simplicial complexes of graph properties

In Paper I we develop a method suitable for studying simplicial complexes of graph properties, although most of the theory is developed in a more general context. A graph property is a property defined on graphs on a fixed set of nodes, which is invariant under permutations of the nodes. Properties such as "the nodes labelled one and two being connected" are hence not graph properties. Examples of graph properties are being connected, being planar, being cycle-free, and so on.

Let $\mathcal{P}$ be a graph property, which we think about as the set of graphs having this property. Assume that, whenever $G \in \mathcal{P}$ and $H \subseteq G$ is obtained by deleting some edges from G (but keeping all the nodes), then $H \in \mathcal{P}$. Then we say that $\mathcal{P}$ is *monotonic*.

A (non-trivial) monotonic graph property $\mathcal{P}$, defined on graphs with n nodes, can be viewed as an abstract simplicial complex $\Sigma_\mathcal{P}$. Indeed, the vertices of $\Sigma_\mathcal{P}$ are indexed by the edge set $\binom{[n]}{2}$ of the complete graph K_n. A set $\sigma \subseteq \binom{[n]}{2}$ is in Σ if the graph with the corresponding edge set has the property $\mathcal{P}$. Clearly, the simplicial complex $\Sigma_\mathcal{P}$ contains all information about $\mathcal{P}$. A general reference on complexes of graph properties is [11].

When studying graph properties, a natural invariant to consider is the *homology* of $\Sigma_{\mathcal{P}}$, which is a topological invariant of $|\Sigma_{\mathcal{P}}|$. A recent and widely used method to calculate the homotopy type—and hence also homology—of simplicial complexes is Forman's discrete Morse theory [5]. In short, discrete Morse theory describes how certain matchings on a (typically large) simplicial complex Σ induce a deformation of Σ onto a (typically much smaller) CW-complex, whose homology can hopefully be calculated more easily.

But since a graph property is invariant under permutations of the nodes, there is a natural $\mathfrak{S}_n$-action on it. This action induces an $\mathfrak{S}_n$-module structure on the simplicial homology $H_*(\Sigma_{\mathcal{P}})$. To study this structure, we need to operate on $\Sigma_{\mathcal{P}}$ in a way that respects the group action, and unfortunately discrete Morse theory is ill suited for this. In [6], we generalized the basic notions of discrete Morse theory, to suit the equivariant case. However, for practical purposes, one can often not consider the full $\mathfrak{S}_n$ action on $\Sigma_{\mathcal{P}}$, but must restrict attention to some subgroup.

Paper I is essentially a rewritten version of [6], and contains a calculation of $H_*(\Sigma_{\mathcal{P}})$, where $\mathcal{P}$ is the collection of non-connected graphs on n nodes. The homology groups are considered as $C_2 \times \mathfrak{S}_{n-2}$–modules, where $C_2 \times \mathfrak{S}_{n-2} \cong \mathfrak{S}_2 \times \mathfrak{S}_{n-2} \subseteq \mathfrak{S}_n$ acts by permuting the nodes $\{1,2\}$ and the nodes $\{3,\ldots,n\}$ independently. The homology groups, without the group action, are well known, and were calculated in [18].

1.1.2 Independence complexes and Hansen polytopes

While Paper I studies simplicial complexes originating from graph properties, Paper II considers geometric invariants of particular graphs. Recall that a set $I \subseteq G$ of nodes is called independent in G, if there is no edge between two elements of I. The dual notion is that of a clique $C \subseteq G$, where every pair of nodes in C have an edge between them.

A graph G on n nodes gives an abstract simplicial complex on n vertices, whose simplices are the independent sets in G. This complex is called the *independence complex* of G, and is denoted Σ_G. It is worth observing that the 1-skeleton of the independence complex of G is just the complement graph $\overline{G}$. In this respect, it would be more natural to look at the clique complex of G, whose 1-skeleton is G itself, but we stick to the independence complex for historical reasons.

For a finite abstract simplicial complex Σ, with vertex set $[n]$, one can define the dual simplicial complex

$$\overline{\Sigma} = \{\tau \subseteq [n] \mid \forall \sigma \in \Sigma : |\sigma \cap \tau| \leq 1\}.$$

For example, the dual of the independence complex of a graph is the clique complex of the same graph. It follows from the definition that $\Sigma \subseteq \overline{\overline{\Sigma}}$. The inclusion can be strict, as is seen in the following example.

EXAMPLE 1 *Let* $\Sigma = \{\emptyset,\{1\},\{2\},\{3\},\{1,2\},\{2,3\},\{3,1\}\}$ *be the complex whose geometric realisation is the empty triangle. Then* $\overline{\Sigma}$ *is the three point set, and* $\overline{\overline{\Sigma}} = \Sigma \cup \{1,2,3\}$ *is the filled triangle.*

To any finite abstract simplicial complex Σ with n vertices, Hansen [10] associates a polytope in $n+1$ dimensions. This is constructed as

$$\mathrm{Hans}\,(\Sigma) = \mathrm{conv}\left\{\pm\left(\mathbf{e}_0 + \sum_{i\in\sigma}\mathbf{e}_i\right) \mid \sigma \in \Sigma\right\},$$

and so has two vertices for every simplex in Σ. It is a *centrally symmetric* polytope, which just means that $P = -P$. If Σ_G is the independence complex of G, we will abuse notation slightly, and write Hans(G) rather than Hans(Σ_G). It is elementary to see that Hans(K_n) is affinely equivalent to a cross-polytope, and Hans($\overline{K_n}$) is affinely equivalent to a cube, where K_n is the complete graph on n nodes.

It follows from the definition of $\overline{\Sigma}$ that, for any $C \in \overline{\Sigma}$, and any $x \in \mathrm{Hans}(\Sigma)$, we have $-1 \leq -x_0 + 2\sum_{i\in C} x_i \leq 1$. It is easily seen that there are no redundancies among these inequalities. It is natural to ask whether these conditions are sufficient, i.e. whether we have

$$\mathrm{Hans}(\Sigma) = \left\{ x \in \mathbb{R}^{n+1} \mid \forall C \in \overline{\Sigma} : -1 \leq -x_0 + 2 \sum_{i \in C} x_i \leq 1 \right\}. \quad (1.1)$$

In [10] it is proven that this is the case if and only if Σ is the independence complex of a so called *perfect graph*.

DEFINITION 5 (PERFECT GRAPHS) *A graph G is perfect if, for every induced subgraph $H \subseteq G$, the chromatic number χ_H equals the size of the largest clique in H.*

Equivalently, by the strong perfect graph theorem [4], G is perfect if it contains no odd cycle C_{2k+1} or complement of an odd cycle $\overline{C_{2k+1}}$ as an induced subgraph, for $k \geq 2$.

A reason to study Hansen polytopes of perfect graphs, is their combinatorial simplicity, which makes their face lattice relatively easy to understand. Also, they often turn out to have "few faces" in one way or another. More precisely, in [16], certain Hansen polytopes show up as counterexamples to the so-called B- and C-conjectures of Kalai, posed in [13]. These were stronger versions of Conjecture 1, which is still open. Before stating Conjecture 1, we need one more definition.

DEFINITION 6 (HANNER POLYTOPES) *A line segment is a Hanner polytope. A d-polytope P with $d > 1$ is a Hanner polytope if it can be written as the cartesian product of two Hanner polytopes or as the polar of a Hanner polytope.*

A line segment has three non-empty faces, namely its two endpoints and the segment itself. The face number is preserved when taking polars, and is multiplicative when taking products. Hence any d-dimensional Hanner polytope will have 3^d non-empty faces. This is conjectured by Kalai [13] to be minimal among all centrally symmetric polytopes.

Conjecture 1 (Kalai, 1989) *Any centrally symmetric d-polytope P has at least 3^d non-empty faces. Equality holds if and only if P is combinatorially equivalent to a Hanner polytope.*

This conjecture can be placed in a tradition of trying to determine the “least round” centrally symmetric convex body, where the notorious Mahler conjecture [15] may be the most famous.

Conjecture 2 (Mahler, 1939) *For any centrally symmetric convex body $P \in \mathbb{R}^d$, let P^* be its polar body. Then $Vol(P) \cdot Vol(P^*) \geq 4^d/d!$. Equality holds if and only if P is affinely equivalent to a Hanner polytope.*

Notice that the product $Vol(P) \cdot Vol(P^*)$ is an affine invariant for centrally symmetric bodies, because scaling P with a factor λ along one axis, scales P^* with a factor λ^{-1} along the same axis.

We are now looking for possible counterexamples to Conjectures 1 and 2, and in particular to the first one, with its combinatorial flavour. Geometric intuition suggests that a vertex that is situated between two parallell facets, would typically increase the face number. Hence it should not be a big restriction to only look at the following class of polytopes.

Definition 7 (Weakly Hanner polytopes) *A polytope is weakly Hanner if it is centrally symmetric, and each facet contains exactly half of the vertices.*

It is not hard to show that—as the names suggest—every Hanner polytope is weakly Hanner. A weakly Hanner polytope is clearly the twisted prism over any of its faces Q, so we get $P \cong \text{conv}(\{-1\} \times -Q, \{1\} \times Q)$. Again, the intuition that a minimal polytope should not live in too many different hyperplanes suggests that we should focus on subpolytopes of a cube. So we assume $Q \subseteq C^{d-1}$, where C^d is the d-dimensional cube.

Very heuristically, “pushing Q to one corner of the cube” should add structure to the polytope, and decrease the risk of getting unnecessary faces. This means we should let Q be spanned by the indicator vectors

$\{\sum_{i\in\sigma} \mathbf{e}_i \mid \sigma \in \Sigma\}$ of a simplicial complex Σ, so our twisted prism P becomes a Hansen polytope.

But in [10], it is proven that (1.1) is equivalent not only to Σ being the independence complex of a perfect graph, but also to $\mathrm{Hans}(\Sigma)$ being a weakly Hanner polytope. So there are vague, heuristic, reasons to believe that we do not lose any counterexamples to the 3^d–conjecture in the chain of restrictions

$$\begin{aligned} \text{C.s. polytopes } &\supseteq \\ \text{0-1–polytopes } &\supseteq \\ \text{Hansen polytopes } &\supseteq \\ \mathrm{Hans}(G) \text{ for perfect graphs } G.& \end{aligned}$$

In Paper II, we consider Hansen polytopes of *split graphs*, which are graphs whose node set can be partitioned into one clique and one independent set. It is easy to see that all split graphs are perfect. Moreover, computer simulations using `polymake` [12] suggest that Hansen polytopes of split graphs have remarkably few faces. However, we show that if S is a split graph on $d-1$ nodes, then $\mathrm{Hans}(S)$ has at least 3^d faces. Equality holds only if $\mathrm{Hans}(S)$ is indeed combinatorially equivalent to a Hanner polytope, and otherwise the difference is at least 16.

We also consider the following, very natural operation on split graphs: Add a new node, and connect it to every node in the clique of S. We then get a new split graph S' (where the new node can be considered to be an element of either the clique or the independent set). We prove that $s(\mathrm{Hans}(S)) - 3^d = s(\mathrm{Hans}(S')) - 3^{d+1}$, so the number of "additional faces" is invariant under this construction.

Finally, we look at the special case where S can be obtained by applying the $S \mapsto S'$ operation repeatedly to a four-path. This graph gives exactly 16 "additional faces". In this case, we also get some experimental results related to the Mahler conjecture.

1.2 Pattern containments in words

Papers III and IV concern enumerative problems related to pattern containments. Let $\pi = \pi_1 \cdots \pi_m$ and $\tau = \tau_1 \cdots \tau_\ell$ be two words on an ordered alphabet. Usually we consider words of positive integers, so in particular we have a linear order of the letters. An *occurrence* of τ in π is a subsequence $1 \le i_1 < i_2 < \cdots < i_\ell \le m$ such that $\pi_{i_1}, \ldots, \pi_{i_\ell}$ is order-isomorphic to τ. In such a context, τ is usually called a *pattern*.

Patterns are usually studied on the special case of permutations, i.e. words that are reduced and have no repeated letters. They also behave reasonably well with respect to "permutation structure", for example an occurence of τ in π gives an occurence of τ^{-1} in π^{-1}. However, the definitions and most questions regarding pattern containment are just as naturally stated for words over an ordered alphabet in general.

For a word $\tau = \tau_1 \cdots \tau_\ell$ over the totally ordered alphabet $[n] = \{1, \ldots, n\}$, we define its *reversal* $\overline{\tau}$ to be the word $\tau_\ell \cdots \tau_1$. We also define the *complement* of τ with respect to the alphabet $[n]$ to be the word $\tau^c = (n+1-\tau_1) \cdots (n+1-\tau_\ell)$. It is easy to see that an occurence of τ in π gives an occurence of $\overline{\tau}$ in $\overline{\pi}$, and of τ^c in π^c.

In [1], Babson and Steingrímsson introduced a notion of *generalized patterns*. A generalized pattern is a word τ with dashes - between some letters. An occurrence of τ in π is an occurrence of τ as an ordinary pattern, where the letters corresponding to τ_i and τ_j must be consecutive, unless there is a dash between τ_i and τ_j. For example, the subsequence 243 in 2413 is an occurrence of 13-2, but is not an occurrence of 1-32. Note that among generalized patterns, there is no inversion operation, while the reversal and complement operations remain.

Much of the work on permutation patterns has been about fixing a pattern τ, and enumerating the number of m letter permutations with exactly k occurences of τ, especially when $k = 0$. A nice account on some of the results in this direction are given in [2]. The work in Paper III is in another dirction, following [3]: Fix a pattern τ, and study the maximal number $\mu(\tau,n)$ of times that τ can occur in a word π of given size n. It is fairly easy to see that this number scales like $\binom{n}{\ell}$, where ℓ is the number of maximal dash-free subwords in τ. Therefore one can define the *packing density* $\delta(\tau)$ as $\lim_{n\to\infty} \mu(\tau,n)/\binom{n}{\ell}$, and calculate this for certain cases of patterns. When τ is a classical pattern, ℓ is just the length of τ.

The most straightforward meaning of "size" of π would be to just fix the number of letters n, as is done in [3]. This can be generalized to assigning different weights to different letters, and it may be interesting to study how the packing density changes when adjusting these weights. The first step in this direction is assigning weight i to the letter i, so the size of π is $n = \|\pi\| = \sum_i \pi_i$. Looking at words (over $\mathbb{Z}_+$) of fixed size n is thus equivalent to looking at integer compositions of n. In Paper III, we determine the packing densities into compositions of all patterns of length 3, except 1-2-3 and 1-3-2 and their reversals, and prove some more general results for patterns of special kinds.

Among the most elementary patterns are descents 21 and inversions 2-1. Their distributions on permutations are well known. In Paper IV, we take a closer look at the descent statistic and study its joint distribution with the fixed point statistic.

Specifically, compose $[n]$ in blocks of length a_i, with $a_1 + \cdots + a_k = n$, and consider the set $\mathfrak{S}_{\boldsymbol{a}}$ of permutations that descend within each of these blocks. It is clear that $|\mathfrak{S}_{\boldsymbol{a}}| = \binom{n}{a_1,\ldots,a_k}$, since a permutation in $\mathfrak{S}_{\boldsymbol{a}}$ is determined by which letters go in which block. For example, the 6 permutations in $\mathfrak{S}_{(2,2)}$ are

$$21|43\ ,\ 31|42\ ,\ 41|32\ ,\ 32|41\ ,\ 42|31 \text{ and } 43|21.$$

Paper IV enumerates the *derangements* in $\mathfrak{S}_{\boldsymbol{a}}$, i.e. the permutations with no fixed points. We denote the set of derangements in $\mathfrak{S}_{\boldsymbol{a}}$ by $D_{\boldsymbol{a}}$. For example, we have

$$D_{(2,2)} = \{21|43, 31|42, 43|21\}.$$

It is well known, and easy to prove by an inclusion-exclusion argument, that the number of derangements in $\mathfrak{S}_n$ is the integer closest to $n!/\mathrm{e}$, for every $n \geq 1$.

If having descents in specified positions and being fixed-point free were almost independent events, we would hence have

$$|D_{\boldsymbol{a}}| \cdot \prod_i a_i! \approx \frac{n!}{\mathrm{e}},$$

with the squig $\approx$ interpreted properly—maybe even meaning that the difference tends to zero. But the events can be pretty far from independent. To see this, consider the one block composition $\boldsymbol{a} = (n)$. There is only one permutation in $\mathfrak{S}_{(n)}$, namely the strictly decreasing one. This one is a derangement if and only if n is even.

Before further describing the results of Paper IV, we define the number of *fixed point λ-coloured permutations* to be $f_\lambda(m) = \sum_{\pi \in \mathfrak{S}_m} \mathrm{fix}(\pi)^\lambda$. It follows directly that $f_1(m) = m!$, and that $f_0(m) = |D_{(1,\ldots,1)}|$ is just the number of derangements of $[m]$.

The generating function of $|D_{\boldsymbol{a}}|$ is

$$\sum_{\boldsymbol{a}} |D_{\boldsymbol{a}}| x_1^{a_1} \cdots x_k^{a_k} = \frac{1}{(1+x_1)\cdots(1+x_k)(1-x_1-\cdots-x_k)},$$

where the sum is taken over all compositions $\boldsymbol{a}$ with at most k blocks, allowing some of the blocks to be empty. The generating function was first given by Han and Xin in [8], using symmetric function methods. We reprove their theorem with a simple recursive method.

From the generating function, we derive a closed formula for $D_{\boldsymbol{a}}$, namely

$$|D_{\boldsymbol{a}}| = \frac{1}{\prod_i a_i!} \sum_{\boldsymbol{0} \leq \boldsymbol{b} \leq \boldsymbol{a}} (-1)^{\sum b_j} \Big(n - \sum b_j\Big)! \prod_i \binom{a_i}{b_i} b_i!. \tag{1.2}$$

Curiously, (1.2) still holds if we replace every occurence of $m!$ by $f_\lambda(m)$

in the summation, for any λ.

The independence of λ in (1.2) is proven in two ways, by differentiation and by using bijections. The most interesting application of this independence is writing

$$|D_{\boldsymbol{a}}| = \frac{1}{\prod_i a_i!} \sum_{\mathbf{0} \le \boldsymbol{b} \le \boldsymbol{a}} (-1)^{\sum b_j} f_0\left(n - \sum b_j\right) \prod_i \binom{a_i}{b_i} f_0(b_i). \quad (1.3)$$

Noting that $f_0(1) = 0$, we reduce the number of non-zero terms in the summation quite remarkably.

Finally, we look back at the question of dependencies between the events of being in $\mathfrak{S}_a$ and being a derangement. We prove that, except in the trivial case where $\boldsymbol{a} = (n)$ and n is odd, the two events are always positively correlated. In other words, we always have

$$|D_{\boldsymbol{a}}| \ge \frac{1}{\prod_i a_i!} f_0(n).$$

Noting that the right hand side corresponds to the term with $\boldsymbol{b} = \mathbf{0}$ in (1.3), we consider all other terms as "correlation terms", and conclude that their sum is negative.

This correlation result is actually proven via a stronger monotonicity result. Consider $h(\boldsymbol{a}) = |D_{\boldsymbol{a}}| \prod_i a_i!$, which is the number of permutations in $\mathfrak{S}_n$, that become derangements when sorted decreasingly within the blocks. Suppose that $\boldsymbol{a}'$ is not a single block of odd length, and is constructed from $\boldsymbol{a}$ by moving a position from a smaller block to a larger one. Then we prove that $h(\boldsymbol{a}') \ge h(\boldsymbol{a})$. In particular, this implies that h is monotonic with respect to the natural "containment order" on compositions of n.

Since the publication of Paper IV, there has been some further progress in the same direction. In [17], Steinhardt studies the more general concept of $(\boldsymbol{a},S)$-permutations. As before, $\boldsymbol{a} = (a_1, \ldots, a_k)$ is a composition of n, and we let $S \subseteq [k]$ be a subset of the blocks. An $(\boldsymbol{a},S)$-permutation is a permutation that descends in the blocks indexed by S, and that ascends within each of the other blocks. With this notation, $\mathfrak{S}_{\boldsymbol{a}}$ is just the set of $(\boldsymbol{a}, [k])$-permutations.

In [17], a bijection that goes back to [7] is used, to study $(\boldsymbol{a},S)$-permutations according to their cycle structure. In particular, it is shown combinatorially that for any conjugacy class $\mathcal{C}$, and any permutation $\sigma \in \mathfrak{S}_k$, the $(a_1,\ldots,a_k,S)$-permutations in $\mathcal{C}$ are in bijection with the $(a_{\sigma(1)},\ldots,a_{\sigma(k)},\sigma^{-1}(S))$-permutations in $\mathcal{C}$. Since the class of derangements is just the union of all conjugacy classes with no 1-cycle, this answers the open Problem 4 posed in Paper IV.

The fact that the function

$$\frac{1}{\prod_i a_i!} \sum_{\mathbf{0} \leq \boldsymbol{b} \leq \boldsymbol{a}} (-1)^{\sum b_j} f_\lambda \left(n - \sum b_j\right) \prod_i \binom{a_i}{b_i} f_\lambda(b_i)$$

is constant in λ is also given a neat combinatorial proof in [17]. Moreover, the generating function from Paper IV, and the closed formula that follows from it, are generalized to the case of $(\boldsymbol{a}, S)$-derangements. Indeed, it is shown that the generating function for $(\boldsymbol{a}, S)$-derangements is

$$\frac{\prod_{i \notin S}(1 - x_i)}{(1 - x_1 - \cdots - x_k) \prod_{i \in S}(1 + x_i)}$$

The bijection from [7] is also recycled to prove the following enumeration of $(\boldsymbol{a}, S)$-derangements: Let $c_{(\boldsymbol{a},S)}(\pi) = 0$ if π has any odd length cycle contained in a block of $\boldsymbol{a}$, or if it has *any* cycle contained in an *ascending* block of $\boldsymbol{a}$. Otherwise, let $c_{(\boldsymbol{a},S)}(\pi) = 2^m$, where m is the number of (even length) cycles contained in a (descending) block of $\boldsymbol{a}$. Then the number of $(\boldsymbol{a}, S)$-derangements is

$$\frac{1}{\prod a_i!} \sum_{\pi \in \mathfrak{S}_n} c_{(\boldsymbol{a},S)}(\pi).$$

Essentially the same results on $(\boldsymbol{a}, S)$-derangements were also obtained by Kim and Seo [14], independently of Steinhardt.

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