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ASYMPTOTIC ERROR EXPANSIONS FOR THE FINITE ELEMENT METHOD FOR SECOND ORDER ELLIPTIC PROBLEMS IN R^N, $N \geq 2$. I: LOCAL INTERIOR EXPANSIONS*

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Abstract. Our aim here is to give sufficient conditions on the finite element spaces near a point so that the error in the finite element method for the function and its derivatives at the point have exact asymptotic expansions in terms of the mesh parameter h, valid for h sufficiently small. Such expansions are obtained from the so-called asymptotic expansion inequalities valid in $\mathbf{R}^N$ for $N \geq 2$, studies by Schatz in [*Math. Comp.*, 67 (1998), pp. 877–899] and [*SIAM J. Numer. Anal.*, 38 (2000), pp. 1269–1293].

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1. Introduction and statement of main results.

This paper is devoted to the study of local interior asymptotic error expansions for the finite element method for second order elliptic problems in $\mathbf{R}^N$, $N \geq 2$. The aim here is to give sufficient conditions on the finite element spaces near a point so that the error in the finite element method for the function and its derivatives at the point have exact asymptotic expansions in terms of the mesh size parameter h, valid for h sufficiently small. To our knowledge, what has been proved in the literature is in the piecewise linear case and piecewise quadratics in two dimensions. This work takes place in $\mathbf{R}^N$ for more general finite element spaces than found in the literature, and our results are of a more local character. This study relies on the so-called *asymptotic error expansion inequalities* at a point for the finite element method which were derived in Schatz [29] and [31]. More specifically, our main tools in deriving asymptotic error expansions are the results from [31] that are stated in the Lemmas 2.1 and 2.2 below. The asymptotic error expansion inequalities, developed in [29] and [31], have several interesting applications, e.g., as in [30], to improve superconvergence error estimates and in the paper [1], where Richardson extrapolation was justified using asymptotic error expansion inequalities. Hoffmann et al. [14] used the estimates in [31] to prove that a class of recovered gradient estimators are asymptotically exact on each element underlying mesh, provided certain conditions are fulfilled. Chen [7] and Guzman [13] are using the results in [31] for finite element approximations of the solution of the Stokes problem. In the present work we shall use asymptotic expansion inequalities to derive asymptotic expansions that can be used in the traditional approach to justify Richardson extrapolations.

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Let us first describe the problem more precisely and then state the results. Let Ω be a domain in $\mathbf{R}^N$, $N \geq 2$, and for $d > 0$ let $B_d(x_0)$ be a ball of radius d centered at x_0. We shall assume that at the point x_0, that we are interested in, there exists a $d > 0$ such that $B_d = B_d(x_0) \subset\subset \Omega$.

Consider the second order elliptic equation of the form

$$Lu(x) = -\sum_{i,j=1}^{N} a_{ij} \frac{\partial^2 u(x)}{\partial x_i \partial x_j} = f(x), \quad \text{in } B_d(x_0), \qquad a_{ij} = \text{constants}. \tag{1.1}$$

The local weak formulation of (1) is

$$A(u,v) = \int_{B_d(x_0)} fv\,dx \qquad \text{for all } v \in \mathring{W}_2^1(B_d(x_0)), \tag{1.2}$$

where

$$A(u,v) = \int_{B_d(x_0)} \left(\sum_{i,j=1}^{N} a_{ij} \frac{\partial u}{\partial x_i} \frac{\partial v}{\partial x_j} \right) dx, \qquad a_{ij} = \text{constants}, \tag{1.3}$$

$$\text{with} \qquad \sum_{i,j=1}^{N} a_{ij}\xi_i\xi_j \geq C\,|\xi|^2 \qquad \text{for all} \quad \xi \in \mathbf{R}^N.$$

Consider now the finite element approximations u_h of u. For this purpose for each $0 < h < 1$, let $S_r^h(B_d) \subset W_\infty^1(B_d)$ be finite element space satisfying the Assumptions A.1–A.4 given in Appendix A. These assumptions are satisfied by many types of commonly used finite elements and are taken from the related assumptions given by Schatz and Wahlbin in [32]–[34]. For our purpose here, think of them as any variety of spaces of continuous functions on $B_d(x_0)$, which on each set τ_h of a quasi-uniform partition (of roughly size h) which covers $B_d(x_0)$, contain all polynomials of degree strictly less than r where $r \geq 2$. For any set $G \subset B_d$, $S_r^h(G)$ is the restriction of $S_r^h(B_d)$ to G and $\mathring{S}_r^h(G)$ are the subspaces of $S_r^h(G)$ of functions supported in G.

Now the finite element approximation $u_h \in S_r^h(B_d)$ of u is assumed to satisfy

$$A(u - u_h, \varphi) = 0 \quad \text{for all} \quad \varphi \in \mathring{S}_r^h(B_d(x_0)). \tag{1.4}$$

Let us begin with discussing asymptotic expansions for the values of functions at the point x_0.

Definition 1.1. *Let $\gamma \geq r+1$ be an integer.* A γ term asymptotic expansion *for the function values at a point x_0 is an expression of the form*

$$u(x_0) = u_h(x_0) + \sum_{r \leq k \leq \gamma - 1} h^k E_k(x_0, u) + \mathcal{R}_\gamma(h; x_0, u), \tag{1.5}$$

where the error coefficients E_k, with k being an integer, are independent of h and the remainder term $\mathcal{R}_\gamma$ is $\mathcal{O}(h^\gamma)$.

Our main result will be concerned with deriving $2r-2$ term asymptotic expansions for equations of the form (1.1) for $r \geq 3$.

The general case of variable coefficients and lower order terms

$$\mathcal{L}u = -\sum_{i,j=1}^{N} \frac{\partial}{\partial x_j}\left(a_{ij}(x)\frac{\partial u}{\partial x_i}\right) + \sum_{i=1}^{N} b_i(x)\frac{\partial u}{\partial x_i} + c(x)u = f(x) \quad \text{in } B_d(x_0) \tag{1.6}$$

can also be treated by the methods given in [30] and [1]. However, in this case, which is not treated here, we are only able to prove a *one term* asymptotic expansion.

In order to find expansions of the form (1.5) we shall impose so-called *self-similarity conditions* on the finite element spaces in a neighborhood of the point x_0.

Expansions of this type have been known for some time for some finite difference methods (cf., e.g., Böhmer [6]). The main contributions to the finite element literature are on plane domains and are due to Lin and coworkers. More specifically, in the case of finite elements with $r = 2$ (in particular piecewise linear or bilinear elements), such expansions were first derived at points x which are the vertices of a uniform triangulation of the plane domain in Lin and Zhu [21], Lin and Lü [17], and Lin and Wang [18]. Improvements and extensions of these expansions were then given in the papers by Blum [3] and Blum, Lin, and Rannacher [4], where the recent paper contains an excellent presentation of the derivation of the exact asymptotic expansion. Further results of this kind can be found in the literature: Lin and Wang [19] improve the results in [18], Lin and Xie [20] address superconvergence under natural assumptions, Rannacher [25] studies mixed finite elements for the shell problem, Wang [36] derives L^∞-estimates for mixed finite elements, Chen and Lin [8] consider the case of rectangular domains, Ding and Lin [11] consider the variable coefficient case, and Chen and Rannacher [9] study the streamline diffusion method, where other references can be found. Blum [2] and Blum and Rannacher [5] study the particular problem of an asymptotic expansion near a corner. For a related study of the extrapolation of the energy functional see Rüde [27] and [28]. More on corner domains can be found in Huang, Han, and Zhou, e.g., [15], where they assume sufficiently smooth initial data and rectangular domains. Lin [16] considers an extrapolation technique for the eigenvalues on nonconvex domains. We recommend the survey articles by Rannacher [26] and Blum [2]. For studies in unstructured grids, see Scott [35] and Xu and Zhang [37].

An outline of the remaining part of this paper is as follows: In section 1.1 we give assumptions on the finite element spaces, in particular the *self-similarity* property, and we state the main results of the paper: Theorems 1.1 and 1.2 which are concerned with the accuracy of pointwise error estimates for the function and its first order partial derivatives, i.e., $(u-u_h)(x_0)$ and $\partial(u-u_h)(x_0)/\partial x_i$, $i = 1, \dots, N$, respectively, at certain (similarity) points x_0 of the grid. Section 2 contains preliminaries and (versions) of the results from [31], which are used in the proofs of Theorems 1.1 and 1.2. Section 3 is devoted to the proofs of the main results, where we develop a framework and reduce the problem to show convergence of a Cauchy sequence in Lemma 3.2. Finally in Appendix A we recall the usual finite element assumptions used throughout the paper. Below we denote by C a general constant independent of the parameters involved in the estimates unless otherwise explicitly stated or clear from the context.

1.1. Some assumptions on the subspaces and statement of the main theorems. The requirements we shall make on the subspaces, near a point x_0, are motivated by looking at the meshes which are systematically refined in a neighborhood of x_0; for example, as in so-called nested spaces constructed to be used in the multigrid methods. In order to do so it will be more convenient to work with a sequence of subspaces $S_r^{h_j}(B_1(x_0))$, where for some sufficiently small h and fixed $K > 1$,

$$h_j = \frac{h}{K^j}, \qquad \text{(for example, } K = 2, \quad j = 1, 2, \dots, M, \dots,). \tag{1.7}$$

Now we state the most important assumption, Assumption A.5, on the finite element spaces (the usual properties A.1–A.4 of the finite element spaces are listed in Appendix A).

DEFINITION 1.2. *Two subspaces* $S_r^{h_j}(B_{h_j}(x_0))$ *and* $S_r^{h_i}(B_{h_i}(x_0))$ *are said to be* similar near x_0 *if the mapping (a scaling about* x_0*)*

$$(T\varphi)(x) = \varphi\left(x_0 + \frac{h_j}{h_i}(x - x_0)\right) \tag{1.8}$$

is a one-to-one mapping of $S_r^{h_j}(B_{h_j}(x_0))$ *onto* $S_r^{h_i}(B_{h_i}(x_0))$.

Examples of subspaces satisfying this definition are given in [1].

Assumption A.5 (on the mesh). We shall now assume that given x_0 there exists a $d > 0$ and an integer k_0 such that for any pair of integers j and k with $j > k > k_0$ (i.e., $h_j < h_k < h_{k_0}$), the scaling $S_r^{h_k}(B_{dh_k/h_j}(x_0))$ is similar, near x_0, to $S_r^{h_j}(B_{h_j}(x_0))$. This just says that from some mesh size h_{k_0} on the mesh on a disk of radius d is constantly uniformly refined about x_0, resulting in self-similar subspaces about x_0.

In what follows we shall use the following notation: For $m \geq 0$ an integer, $1 \leq p \leq \infty$ and $G \subseteq \Omega$, $W_p^m(G)$ denotes the usual Sobolev space of functions with distributional derivatives of order up to m which are in $L_p(G)$. Define the seminorms

$$|u|_{W_p^j(G)} = \begin{cases} \left(\sum_{|\alpha|=j} \|D^\alpha u\|_{L_p(G)}^p\right)^{1/p} & \text{if} \quad 1 \leq p < \infty, \\ \sum_{|\alpha|=j} \|D^\alpha u\|_{L_\infty(G)} & \text{if} \quad p = \infty, \end{cases}$$

and the norms

$$\|u\|_{W_p^m(G)} = \begin{cases} \left(\sum_{j=0}^{m} |u|_{W_p^j(G)}^p\right)^{1/p} & \text{if} \quad 1 \leq p < \infty, \\ \sum_{j=0}^{m} |u|_{W_\infty^j(G)} & \text{if} \quad p = \infty. \end{cases}$$

If $m \geq 0$, $W_p^{-m}(G)$ is the completion of $C_0^\infty(G)$ under the norm

$$\|u\|_{W_p^{-m}(G)} = \sup_{\substack{\psi \in C_0^\infty(G) \\ \|\psi\|_{W_q^m(G)}=1}} \int_G u\psi \, dx, \qquad \frac{1}{p} + \frac{1}{q} = 1.$$

Our first result is as follows.

THEOREM 1.1. *Let* $r \geq 3$, $r + 1 \leq \gamma \leq 2r - 2$, *and suppose that Assumption* A.5 *is satisfied in a ball of radius* $d \geq Ch$ *centered at* x_0, *and suppose* $u - u_h$ *satisfies* (1.4). *Further suppose that* $u \in W_\infty^\gamma(B_d(x_0))$. *Then*

$$u(x_0) = u_h(x_0) + \left(\sum_{r \leq |\alpha| \leq \gamma-1} C_\alpha D^\alpha u(x_0) h^{|\alpha|}\right) + \mathcal{R}_\gamma, \tag{1.9}$$

where

$$|\mathcal{R}_\gamma| \leq C\left(h^\gamma \|u\|_{W_\infty^\gamma(B_d(x_0))} + d^{-t-N/p} \|u - u_h\|_{W_p^{-t}(B_d(x_0))}\right). \tag{1.10}$$

Remark 1.1. It is unreasonable to expect that an estimate like (1.9) holds at a point if the "shape of the mesh" changes at the point as h changes. This is suggested by the fact that the interpolation error at a point depends heavily on the shape of the domain in which x_0 is located; hence our need for the Assumption A.5 of the similarity near the point x_0. However, if the mesh is quasiuniform as $h \to 0$, we can show that there is a subsequence of these expansion coefficients which converges in a limit to an asymptotic expansion at a point which does not require a similarity condition to hold. But, this does not seem to be a useful result.

We now give an asymptotic expansion for first derivatives. Note, however, if x_0 is a point of the mesh where $\partial u_h/\partial x_i$ is discontinuous, then we define

$$\frac{\partial \widetilde{u}_h}{\partial x_i}(x_0, \beta) = \lim_{s \to 0+} \frac{\partial u_h}{\partial x_i}(x_0 + s\beta), \tag{1.11}$$

where $\beta = (\beta_1, \dots, \beta_N)$ is any unit vector chosen so that for s sufficiently small, say, $0 < s \le s_0$, $\partial u_h/\partial x_i$ exists and has a limit as $s \to 0$. There may be many possible choices of $\frac{\partial \widetilde{u}_h}{\partial x_i}(x_0, \beta)$. Obviously, $\frac{\partial \widetilde{u}_h}{\partial x_i}(x_0, \beta) = \frac{\partial u_h}{\partial x_i}(x_0)$ at points x_0, where $\frac{\partial u_h}{\partial x_i}$ is continuous.

Now we state our second result.

THEOREM 1.2. *Suppose the assumptions of Theorem* 1.1 *hold, but with* $r \ge 2$*; then for* $i = 1, 2, \dots, N$ *and* $u \in W_\infty^{\gamma+1}(B_d)$,

$$\frac{\partial u(x_0)}{\partial x_i} = \frac{\partial \widetilde{u}_h(x_0, \beta)}{\partial x_i} + \left(\sum_{r \le |\alpha| \le \gamma - 1} C_\alpha D^\alpha u(x_0) h^{|\alpha| - 1} \right) + \mathcal{R}'_\gamma, \tag{1.12}$$

where $h = h_j,\ j = k, k+1, \dots,$ *and*

$$\left| \mathcal{R}'_\gamma \right| \le C \Big(h^\gamma \left\| u \right\|_{W_\infty^{\gamma+1}(B_d)} + d^{-1-t-N/p} \left\| u - u_h \right\|_{W_p^{-t}(B_d)} \Big). \tag{1.13}$$

2. Preliminaries. Our starting point in proving Theorems 1.1 and 1.2 will be results from Schatz in [31], so-called asymptotic expansion inequalities which we state for equations of the form (1.4).

LEMMA 2.1. *For* $r \ge 3$, $r + 1 \le \gamma \le 2r - 2$, γ *integer, let* $u \in W_\infty^\gamma(B_d(x_0))$ *and suppose* $d \ge \hat{C}h$ *for some* $\hat{C}$ *chosen sufficiently large. Then the following "asymptotic expansion inequality" holds:*

$$\begin{aligned} |(u - u_h)(x_0)| \le C \left(\ln \frac{d}{h} \right)^{\overline{\gamma}} \Bigg[h^r \sum_{|\alpha| = r} |D^\alpha u(x_0)| + \cdots \\ + h^{\gamma - 1} \sum_{|\alpha| = \gamma - 1} |D^\alpha u(x_0)| + h^\gamma \|u\|_{W_\infty^\gamma(B_d(x_0))} \Bigg] \\ + d^{-t-N/p} \|u - u_h\|_{W_p^{-t}(B_d(x_0))}. \end{aligned} \tag{2.1}$$

Here, $\overline{\gamma} = 1$ *if* $\gamma = 2r - 2$, *and* $\overline{\gamma} = 0$ *otherwise.*

Remark 2.1. The estimate (2.1) is valid on irregular meshes.

Remark 2.2. The case $r = 2$ is excluded from (2.1). The reason is that, for $r = 2$, there are no asymptotic expansions available for the function itself, and the expansions that are derived are only for the first order partial derivatives. This negative result is confirmed in [10] for a one-dimensional problem.

Let us state the corresponding result for the first derivatives, which includes also $r = 2$ and is as follows.

LEMMA 2.2. *Suppose that* $r \geq 2$ *and Assumptions* A.1–A.4 *(given in Appendix A) are satisfied. Let* t *be a nonnegative integer,* $1 \leq p \leq \infty$, *and* γ *an integer,* $r < \gamma \leq 2r - 2$. *Let* $x \in \Omega_0$ *and* $d \geq kh$ *for some* k *sufficiently large, and let* $u \in W_\infty^{\gamma+1}(B_d(x_0))$ *and* $u_h \in S_r^h(B_d(x_0))$ *satisfy* (1.4). *Then*

$$\|u - u_h\|_{W_\infty^1(B_h(x_0))} \leq C \left(\ln \frac{d}{h}\right)^{\tilde{\gamma}} \left(h^{r-1} \sum_{|\alpha|=r} |D^\alpha u(x_0)| + \cdots \right. \tag{2.2}$$
$$\left. + h^{\gamma-1} \sum_{|\alpha|=\gamma} |D^\alpha u(x_0)| + h^\gamma \|u\|_{W_\infty^{\gamma+1}(B_d(x_0))}\right)$$
$$+ C \left(d^{-1-t-N/p} \|u - u_h\|_{W_p^{-t}(B_d(x_0))}\right).$$

Here $\tilde{\gamma} = 1$ *if* $\gamma = 2r - 2$, *and* $\tilde{\gamma} = 0$ *if* $r \leq \gamma < 2r - 2$.

3. Proofs of Theorems 1.1 and 1.2.

Case 1. Let us start with the case where $d = 1$ and where without loss of generality we may take $x_0 = 0$. It would be convenient to have $A(u, u)$ coercive on $W_2^1(B_1(0))$; that is, we would like there to be a constant $C_0 > 0$ such that

$$C_0 \|u\|^2_{W_2^1(B_1(0))} \leq A(u, u). \tag{3.1}$$

This can be easily done by changing $A(\cdot, \cdot)$ to $\widetilde{A}(\cdot, \cdot)$, a bilinear form of the form

$$\widetilde{A}(u, v) = \int_{B_1} \left(\sum_{i,j=1}^N a_{ij} \frac{\partial u}{\partial x_i} \frac{\partial v}{\partial x_j} + \sigma\, uv\right) dx, \tag{3.2}$$

where $\sigma(x) \in C^\infty$ with

$$\sigma(x) = \begin{cases} 0 & \text{for} \quad |x| \leq \frac{1}{2}, \\ 0 \leq \sigma \leq 1 & \text{for} \quad \frac{1}{2} \leq |x| \leq \frac{3}{4}, \\ 1 & \text{for} \quad \frac{3}{4} \leq |x| \leq 1. \end{cases}$$

Remark 3.1. We leave it to the reader to see that (3.1) is satisfied.

Remark 3.2. Obviously $\widetilde{A}(u - u_h, \varphi) = a(u - u_h, \varphi)$ for all $\varphi \in \mathring{W}_2^1(B_{1/2})$. This information will be enough to establish an asymptotic expansion at $x = 0$.

For each multi-index α and for each monomial x^α set

$$w^\alpha = x^\alpha \tag{3.3}$$

(we do this in order to avoid some confusion with notation that may occur later on), and let $w_h^\alpha \in S_r^h(B_1)$ be the projection of w^α defined by

$$\widetilde{A}(w^\alpha - w_h^\alpha, \varphi) = 0 \qquad \text{for all} \quad \varphi \in S_r^h(B_1). \tag{3.4}$$

Then the function

$$\rho(x) = u(x) - \sum_{r \leq |\alpha| \leq \gamma-1} \frac{D^\alpha u(0)}{\alpha!} w^\alpha(x) \tag{3.5}$$

and $\rho_h(x) \in S_r^h(B_1)$ defined by

$$\rho_h(x) = u_h(x) - \sum_{r\le|\alpha|\gamma-1} \frac{D^\alpha u(0)}{\alpha!} w_h^\alpha(x), \tag{3.6}$$

satisfy the local error equation

$$A(\rho - \rho_h \varphi) = 0 \qquad \text{for all} \quad \varphi \in S_r^h(B_{1/2}). \tag{3.7}$$

Remark 3.3. The reason that the sums in (3.5)and (3.6) start at r is due to the fact that the Galerkin approximation is exact for polynomials up to degree $r-1$, and the approximation makes sense only if $|\alpha| \ge r$, otherwise $w^\alpha = w_h^\alpha$.

The proof of Theorem 1.1 is based on the following two auxiliary lemmas, whose proofs we postpone for a little while.

LEMMA 3.1. *Let α be any fixed multi-integer, and let $w^\alpha - w_h^\alpha$ satisfy* (3.4)*; then there exists a constant C such that*

$$\|w^\alpha - w_h^\alpha\|_{W_2^{2-r}(B_1)} \le Ch^{2r-2}. \tag{3.8}$$

LEMMA 3.2. *Let $r \le |\alpha| \le 2r-3$. Then $\{w_{h_j}^\alpha / h_j^{|\alpha|}\}$ is a Cauchy sequence; in particular there exists an integer k such that for all $j > k$,*

$$\left| \frac{w_{h_k}^\alpha(0)}{h_k^{|\alpha|}} - \frac{w_{h_j}^\alpha(0)}{h_j^{|\alpha|}} \right| \le Ch_k^{2r-2-|\alpha|}. \tag{3.9}$$

Thus

$$\lim_{h_j\to\infty} \frac{w_{h_j}^\alpha(0)}{h_j^{|\alpha|}} = C_\alpha \quad \textit{exists and} \quad \left| \frac{w_{h_k}^\alpha(0)}{h_k^{|\alpha|}} - C_\alpha \right| \le Ch^{2r-2-|\alpha|}. \tag{3.10}$$

Granting for a moment the validity of Lemmas 3.1 and 3.2, let us complete the proof of theorem in the case $d = 1$.

In view of (3.7), we may apply Lemma 2.1 to the function $\rho - \rho_h$. Noticing that $\rho(0) = u(0)$ and using the fact that $|D^\alpha \rho|(0) = 0$ for all $r \le |\alpha| \le \gamma - 1$, we obtain

$$u(0) = u_h(0) - \sum_{r\le|\alpha|\le\gamma-1} \frac{D^\alpha u(0)}{\alpha!} w_h^\alpha(0) + \mathcal{R}_\gamma(\rho)$$

or

$$u(0) = u_{h_j}(0) - \sum_{r\le|\alpha|\le\gamma-1} \frac{D^\alpha u(0)}{\alpha!} \left(C_\alpha + \frac{w_{h_j}^\alpha(0)}{h_j^{|\alpha|}} - C_\alpha \right) h_j^{|\alpha|} + \mathcal{R}_\gamma(\rho).$$

In view of (3.10)

$$u(0) = u_{h_j}(0) - \sum_{r\le|\alpha|\le\gamma-1} C_\alpha \frac{D^\alpha u(0)}{\alpha!} h_j^{|\alpha|} + \mathcal{R}_\gamma(u) + \mathcal{R}_\gamma(\rho). \tag{3.11}$$

To complete the proof for $d = 1$ it remains to estimate $\mathcal{R}_\gamma(\rho)$.

In order to simplify notation, from now on we shall drop the subscript j and set $h_j = h$. Thus by definition

$$(\rho - \rho_h)(x) = u(x) - u_h(x) - \sum_{r \le |\alpha| \le \gamma-1} \frac{D^\alpha u(0)}{\alpha!}(w^\alpha - w_h^\alpha) \quad \text{for } x \in B_{1/2}. \tag{3.12}$$

In view of Lemmas 3.1 and 3.2

$$\|\rho - \rho_h\|_{W_2^{2-r}(B_{1/2})} = \|u - u_h\|_{W_2^{2-r}(B_{1/2})} + \sum_{r \le |\alpha| \le \gamma-1} \left| \frac{D^\alpha u(0)}{\alpha!} \right| h^{2r-2}. \tag{3.13}$$

Using this in (3.11) will complete the proof of Theorem 1.1 in the case $d = 1$, once we have proved Lemmas 3.1 and 3.2.

We now turn to the proofs of Lemmas 3.1 and 3.2.

Proof of Lemma 3.1. By definition of the negative norm,

$$\|w^\alpha - w_h^\alpha\|_{W_2^{2-r}(B_1)} = \sup_{\varphi \in \mathring{W}_2^{r-2}(B_1)} \int_{B_1} (w^\alpha - w_h^\alpha)\varphi \, dx. \tag{3.14}$$

Using an Aubin–Nitsche duality, let $v \in W_2^r(B_1)$ be the unique solution of

$$\widetilde{A}(\eta, v) = \int_{B_1} \eta\varphi \, dx \qquad \text{for all} \quad \eta \in W_2^1(B_1).$$

Then

$$\begin{aligned} \|w^\alpha - w_h^\alpha\|_{W_2^{2-r}(B_1)} &= A(w^\alpha - w_h^\alpha, v - v_h) \\ &\le \|w^\alpha - w_h^\alpha\|_{W_2^1(B_1)} \|v - v_h\|_{W_2^1(B_1)} . \end{aligned} \tag{3.15}$$

Now in general, the boundary of B_1 does not coincide with a mesh domain, but by assumption there is a mesh domain that extends beyond B_1 and is contained in B_{1+Ch}, where w^α may be interpolated. Obviously

$$\begin{aligned} \|w^\alpha - w_h^\alpha\|_{W_2^1(B_1)} &\le \|w^\alpha - w_I^\alpha\|_{W_2^1(B_{1+Ch})} \le Ch^{r-1} \|w^\alpha\|_{W_2^r(B_{1+Ch})} \\ &\le Ch^{r-1} \|w^\alpha\|_{W_2^r(B_1)} , \end{aligned} \tag{3.16}$$

where w_I^α is an interpolant of w^α on B_{1+Ch}. In order to treat the term involving $v - v_h$ we first use the Calderon extension theorem [23] to extend v to B_{1+Ch} continuously in the norm $W_2^r(B_1)$. Then proceeding as in (3.16), this time with $v - v_h$ we obtain

$$\|v - v_h\|_{W_2^1(B_1)} \le Ch^{r-1} \|v\|_{W_2^r(B_{1+Ch})} \le Ch^{r-1} \|v\|_{W_2^r(B_1)} . \tag{3.17}$$

Using (3.16) and (3.17) in (3.15) completes the proof of Lemma 3.1 in the case $d = 1$. ∎

Case 2. $Ch_j < d < 1$, d fixed. Let $A(u - u_h, \varphi) = 0$ for all $\varphi \in \mathring{S}_r^{h_j}(B_d)$. Then under the change of variable $y = x/d$, $\widetilde{u}(x) := u(dy)$ and $\widetilde{u}_h(x) := u_h(dy)$ satisfy

$$A(\widetilde{u} - \widetilde{u}_h, \psi) = 0 \quad \text{for all } \psi \in \mathring{S}_r^{h_j/d}(B_1).$$

Therefore we may apply the asymptotic expansion to obtain on unit size domain

$$\widetilde{u}(0) - \widetilde{u}_h(0) = - \sum_{r \le |\alpha| \le \gamma-1} \left(\frac{C_\alpha}{\alpha!} \right) D_y^\alpha \widetilde{u}(0) \cdot \left(\frac{h}{d} \right)^{|\alpha|} + \widetilde{\mathcal{R}}_\gamma, \tag{3.18}$$

where

$$\left|\widetilde{\mathcal{R}}_\gamma\right| \le C\left(\frac{h}{d}\right)^\gamma \|\widetilde{u}\|_{W^\gamma_\infty(B_{1/2})} + C\,\|\widetilde{u}-\widetilde{u}_h\|_{W_p^{-t}(B_1)}. \tag{3.19}$$

Changing variables back again we get the main result for Case 2:

$$u(0)-u_h(0) = -\sum_{r\le|\alpha|\le\gamma-1}\left(\frac{C_\alpha}{\alpha!}\right) D_x^\alpha u(0) h^{|\alpha|} + \mathcal{R}_{(\gamma,d)}, \tag{3.20}$$

where

$$\left|\mathcal{R}_{(\gamma,d)}\right| \le C\Big(h^\gamma \|u\|_{W^\gamma_\infty(B_d)} + d^{-t-N/p}\,\|u-u_h\|_{W_p^{-t}(B_d)}\Big). \tag{3.21}$$

Proof of Lemma 3.2. We start with a fixed h_j for j sufficiently large so that the estimate (2.1) holds (h_j sufficiently small). By Assumption A.5 each $S_r^{h_k}(B_{h_k/h_j}(0))$ is similar to $S_r^{h_j}(B_1(0))$ for all $k \ge j$. Set $h_k/h_j = \lambda$ and note that $w^\alpha_{h_k}(\lambda x)/(\lambda^{|\alpha|}) \in S_r^{h_j}(B_1(0))$ ($\lambda < 1$ is scaling factor). Now the proof is separated into some technical steps as follows: First we shall show that

$$A\left(x^\alpha - \frac{1}{\lambda^{|\alpha|}} w^\alpha_{h_k}(\lambda x), \varphi(x)\right) = 0 \qquad \text{for all } \varphi \in \mathring{S}_r^{h_j}\Big(B_{1/2}(0)\Big). \tag{3.22}$$

Since the a_{ij} are constants, we have by the change of variable $x = y/\lambda$ that the left side of (3.22) is equal to

$$\begin{aligned}
&\int_{B_{1/2}(0)} \sum_{i,j=1}^N a_{ij}\frac{\partial}{\partial x_i}\left(x^\alpha - \frac{1}{\lambda^{|\alpha|}} w^\alpha_{h_k}(\lambda x)\right)\cdot\frac{\partial\varphi(x)}{\partial x_j}\,dx\\
&\qquad = \lambda^{1-N-|\alpha|}\int_{B_{1/2}(0)} \sum_{i,j=1}^N a_{ij}\frac{\partial}{\partial y_i}\Big(y^\alpha - w^\alpha_{h_k}(y)\Big)\cdot\frac{\partial}{\partial y_j}\varphi\Big(\frac{y}{\lambda}\Big)\,dy = 0,
\end{aligned}$$

where in the last step we used (3.4). This proves (3.22).

Because of (3.22) and (3.4), recalling that $x^\alpha \equiv w^\alpha(x)$, it follows that the scaling error $w^\alpha_{h_j}(x) - \frac{1}{\lambda^{|\alpha|}} w^\alpha_{h_k}(\lambda x) \in S_r^{h_j}(B_1(0))$ and satisfies

$$A\left(w^\alpha_{h_j}(x) - \frac{1}{\lambda^{|\alpha|}} w^\alpha_{h_k}(\lambda x), \varphi(x)\right) = 0 \qquad \text{for all } \varphi \in S_r^{h_j}(B_{1/2}(0)). \tag{3.23}$$

Thus the difference in (3.23) is a discrete A harmonic function in $B_{1/2}(0)$, and it follows from (2.1) (first proved by Schatz and Wahlbin [33]) with $u = 0$ and $u_h = w^\alpha_{h_j}(x) - \frac{1}{\lambda^{|\alpha|}} w^\alpha_{h_k}(\lambda x)$ that

$$\begin{aligned}
\left|w^\alpha_{h_j}(0) - \frac{1}{\lambda^{|\alpha|}} w^\alpha_{h_k}(0)\right| &\le \left\|w^\alpha_{h_j} - \frac{1}{\lambda^{|\alpha|}} w^\alpha_{h_k}\right\|_{W_2^{2-r}(B_{1/2}(0))}\\
&\le \left\|w^\alpha - w^\alpha_{h_j}\right\|_{W_2^{2-r}(B_{1/2}(0))} + \left\|w^\alpha - \frac{1}{\lambda^{|\alpha|}} w^\alpha_{h_k}\right\|_{W_2^{2-r}(B_{1/2}(0))}\\
&:= J_1 + J_2.
\end{aligned} \tag{3.24}$$

We shall now estimate J_1 and J_2. An estimate for J_1 is given in Lemma 3.1:

$$J_1 \le C h_j^{2r-2}. \tag{3.25}$$

The estimate of J_2 is rather lengthy. We begin by using a duality argument where, setting $x^\alpha - w_{h_k}^\alpha := E_k^\alpha$, for the scaling $\widetilde{E}_k^\alpha := x^\alpha - \frac{1}{\lambda^{|\alpha|}} w_{h_k}^\alpha(\lambda x)$ we have

$$J_2 = \sup_{\substack{\psi \in \mathring{W}_2^{r-2}(B_{1/2}(0)) \\ |\psi|_{W_2^{r-2}(B_{1/2}(0))}=1}} (\widetilde{E}_k^\alpha, \psi), \tag{3.26}$$

where $|\psi|_{W_2^{r-2}(B_{1/2}(0))}$ is the seminorm and we have used the obvious fact that $|\psi|_{W_2^{r-2}(B_{1/2}(0))} \le \|\psi\|_{W_2^{r-2}(B_{1/2}(0))}$. We again make the change of variable $x = y/\lambda$ and obtain

$$\begin{aligned}(\widetilde{E}_k^\alpha, \psi) &= \int_{B_{1/2}(0)} \widetilde{E}_k^\alpha(x)\psi(x)\,dx = \int_{B_{\lambda/2}(0)} \widetilde{E}_k^\alpha\Big(\frac{y}{\lambda}\Big)\psi\Big(\frac{y}{\lambda}\Big)\,\frac{dy}{\lambda^N} \\ &= \frac{1}{\lambda^{N+|\alpha|}}\int_{B_{\lambda/2}(0)} \Big(y^\alpha - w_{h_k}^\alpha(y)\Big)\widetilde{\psi}(y)\,dy,\end{aligned}$$

where $\widetilde{\psi}(y) = \psi(\frac{y}{\lambda}) \in \mathring{W}_2^{r-2}(B_{\lambda/2}(0))$ and

$$\begin{aligned}|\psi|_{W_2^{r-2}(B_{1/2}(0))} &= \left(\sum_{|\beta|=r-2}\int \left|D_x^\beta \psi(x)\right|^2\,dx\right)^{1/2} \\ &= \lambda^{r-2-N/2}\left(\sum_{|\beta|=r-2}\int \left|D_y^\beta \widetilde{\psi}(y)\right|^2\,dy\right)^{1/2}.\end{aligned}$$

Inserting these into (3.26) yields

$$J_2 \le \lambda^{2-r-|\alpha|-N/2} \sup_{\widetilde{\psi}\in \mathring{W}_2^{r-2}(B_{\lambda/2}(0))} \frac{(y^\alpha - w_{h_k}^\alpha(y), \widetilde{\psi}(y))}{\left|\widetilde{\psi}\right|_{W_2^{r-2}(B_{\lambda/2}(0))}}.$$

Using a duality argument, for each such $\widetilde{\psi}$ let v be the unique solution of

$$\widetilde{A}(\eta, v) = (\eta, \widetilde{\psi}) \qquad \text{for all } \eta \in W_2^r(B_1(0)). \tag{3.27}$$

Let now v_I be the interpolant of v; then it follows that for each $\widetilde{\psi}$,

$$\begin{aligned}\lambda^{2-r-|\alpha|-N/2}\Big(y^\alpha - w_{h_k}^\alpha, \widetilde{\psi}\Big) &= \lambda^{2-r-|\alpha|-N/2} A(y^\alpha - w_{h_k}^\alpha, v - v_I) \\ &\le \lambda^{2-r-|\alpha|-N/2}\Big(\left\|(|y|+\lambda)^{r-|\alpha|}\nabla E_k^\alpha(y)\right\|_{L_\infty(B_{1/2}(0))} \\ &\qquad \times \left\|(|y|+\lambda)^{|\alpha|-r}\nabla(v-v_I)(y)\right\|_{L_1(B_{1/2}(0))} \\ &\qquad + \|E_k^\alpha\|_{W_2^1(B_1\setminus B_{1/2})}\,\|v-v_I\|_{W_2^1(B_1\setminus B_{1/2})}\Big) \\ &\equiv \lambda^{2-r-|\alpha|-N/2}\Big(J_{2a}J_{2b} + J_{2c}J_{2d}\Big).\end{aligned} \tag{3.28}$$

Notice that $\frac{\lambda}{2} < \frac{1}{4}$, if $\lambda < \frac{1}{2}$, and also notice the return from $\widetilde{E}_k^\alpha$ to E_k^α in (3.28). Note that we can always decompose the unit disk into two parts as an inner and an outer region so that v satisfies a homogeneous differential equation on the outer region. Estimating the above J_2 terms in the reverse order we obtain for J_{2d},

$$J_{2d} = \|v - v_I\|_{W_2^1(B_1 \setminus B_{1/2})} \le C\lambda^{r-1} h_j^{r-1} \|v\|_{W_2^r(B_1 \setminus B_{7/16})}. \tag{3.29}$$

As in Lemma 3.1, we use the Calderon extension theorem to extend v continuously in W_2^r to B_{1+Ch}. By well-known a priori error estimates (see Lions and Magenes [22, Chapter 2.6]),

$$\|v\|_{W_2^r(B_1 \setminus B_{7/16})} \le \|v\|_{W_2^{2-\delta'}(B_1)} \qquad \text{for all } \delta' > \frac{N}{2} + 1.$$

Thus, for v the solution, we have evidently (see [12, Chapter 6B], e.g., (6.25), also shift theorems in [24] for $H^{-s}(\Omega)$, $s \in \mathbf{R}$, $s > 0$) that

$$\|v\|_{W_2^{2-\delta'}(B_1)} \le \left\|\widetilde{\psi}\right\|_{W_2^{-\delta'}(B_1)}.$$

Now with $\widetilde{\psi}$ vanishing at outer region of $B_1 \setminus B_{7/16}$, we can apply Poincare's inequality to obtain

$$J_{2d} \le \lambda^{r-1} h_j^{r-1} \left\|\widetilde{\psi}\right\|_{W_2^{-\delta'}(B_1)} \le \lambda^{2r-3+N/2} h_j^{r-1} \left|\widetilde{\psi}\right|_{W_2^{r-2}(B_1)}, \tag{3.30}$$

where we have used the fact that the measure of $B_{\lambda/2}$ is proportional to λ^N, since

$$\begin{aligned}\left\|\widetilde{\psi}\right\|_{W_2^{-\delta'}(B_1)} &= \sup_{\|\varphi\|_{W_2^{\delta'}(B_1)}=1} (\widetilde{\psi}, \varphi) \le \sup_{\|\varphi\|_{W_2^{\delta'}(B_1)}=1} \left\|\widetilde{\psi}\right\|_{L_1(B_{\lambda/2})} \|\varphi\|_{L_\infty} \\ &\le \sup_{\|\varphi\|_{W_2^{\delta'}(B_1)}=1} \lambda^{N/2} \left|\widetilde{\psi}\right|_{L_2(B_{\lambda/2})} \|\varphi\|_{L_\infty} \le C\lambda^{r-2+N/2} \left|\widetilde{\psi}\right|_{W_2^{r-2}(B_1)},\end{aligned}$$

where we have used the Sobolev inequality to obtain the bound $\|\varphi\|_{L_\infty} \le C$. As for J_{2c}, we have

$$J_{2c} = \|E_k^\alpha\|_{W_2^1(B_1 \setminus B_{1/2})} \le \|E_k^\alpha\|_{W_2^1(B_1)} \le \lambda^{r-1} h_j^{r-1}. \tag{3.31}$$

Taken together (3.30) and (3.31) yield

$$\begin{aligned}\lambda^{2-r-|\alpha|-N/2} J_{2c} J_{2d} &\le C\lambda^{2-r-|\alpha|-N/2} h_j^{r-1} \lambda^{3r-4+N/2} h_j^{r-1} \left|\widetilde{\psi}\right|_{W_2^{r-2}(B_1)} \\ &\le C\lambda^{2r-2-|\alpha|} h_j^{2r-2} \left|\widetilde{\psi}\right|_{W_2^{r-2}(B_1)}.\end{aligned} \tag{3.32}$$

This is half of the estimate for J_2. We shall now estimate J_{2b} by writing

$$\begin{aligned}J_{2b} \le C&\left\|(|y| + \lambda)^{|\alpha|-r} \nabla(v - v_I)\right\|_{L_1(B_{4\lambda})} \\ &+ \sum_{4\lambda \le \rho_j \le \frac{1}{2}} \left\|(|y| + \lambda)^{|\alpha|-r} \nabla(v - v_I)\right\|_{L_1(\Omega_j)},\end{aligned} \tag{3.33}$$

where Ω_ℓ are the annuli

$$\Omega_\ell = \{x \in B_1 : \ \rho_{\ell+1} \le |x| \le \rho_\ell\}, \quad \rho_\ell = 2^{-\ell}, \quad \ell = 1, 2, \ldots.$$

Then by approximation theory and the Cauchy–Schwarz inequality, (3.33) yields

$$J_{2b} \le C\lambda^{|\alpha|-1+N/2} h_j^{r-1} \, |v|_{W_2^r(B_{5\lambda})} + \sum_{1\le \ell \le \bar\ell} \rho_\ell^{|\alpha|-r+N/2} \lambda^{r-1} h_j^{r-1} \, |v|_{W_2^r(\Omega'_\ell)} \,, \tag{3.34}$$

where $\bar\ell = [(\ln \frac{1}{4\lambda})/\ln 2]$, with $[\tau]$ denoting the integer part of τ and $\Omega'_\ell = \cup_{m=\ell-1}^{\ell+1}\Omega_m$ is the union of Ω_ℓ and its closest adjacent neighbors. Further, using local estimates for the continuous problem, with $\Omega''_\ell = \cup_{m=\ell-1}^{\ell+1}\Omega'_m$ and the fact that $3 \le 2r - |\alpha| \le r$,

$$|v|_{W_2^r(\Omega'_\ell)} \le C\rho_\ell^{r-|\alpha|-N/2} \, |v|_{W_1^{2r-|\alpha|}(\Omega''_\ell)} \,. \tag{3.35}$$

Hence the last term on the right-hand side of (3.34) can be estimated as

$$\sum_{1\le \ell \le \bar\ell} \rho_\ell^{|\alpha|-r+N/2} \lambda^{r-1} h_j^{r-1} \, |v|_{W_2^r(\Omega'_\ell)} \le C\lambda^{r-1} h_j^{r-1} \, |v|_{W_1^{2r-|\alpha|}(B_1\setminus B_{2\lambda})} \,. \tag{3.36}$$

The last term will be estimated using the Green's function. For any multi-index β, with $|\beta| = 2r - |\alpha|$ we have, with $\widetilde\psi$ and v satisfying (3.27),

$$\begin{aligned} \int_{|x|\ge 2\lambda} \left| D^\beta v(x) \right| \, dx &= \int_{|x|\ge 2\lambda} \left(\int_{|y|\le \lambda/2} \left| \widetilde\psi(y) \right| \left| D_x^\beta G(x,y) \right| \, dy \right) dx \\ &\le \int_{|y|\le \lambda/2} \left| \widetilde\psi(y) \right| \left(\int_{|x|\ge 2\lambda} \frac{1}{|x-y|^{N-2+2r-|\alpha|}} \, dx \right) dy. \end{aligned} \tag{3.37}$$

For each y let $R = |x-y|$; then in spherical coordinates

$$\begin{aligned} \int_{|x|\ge 2\lambda} \frac{1}{|x-y|^{N-2+2r-|\alpha|}} \, dx &\le C \int_{\lambda\le R\le 1} \frac{R^{N-1}}{R^{N-2+2r-|\alpha|}} \, dR \\ &= C \int_{\lambda\le R\le 1} R^{|\alpha|+1-2r} \, dR \le C\lambda^{|\alpha|+2-2r}, \end{aligned} \tag{3.38}$$

where we use the inequality $-r+1 \le |\alpha| + 1 - 2r \le -2$. Therefore using (3.38) in (3.37) and the Poincaré inequality on $\widetilde\psi$ we get for $\widetilde\psi \in \mathring{W}_2^{r-2}(B_{\lambda/2})$,

$$\int_{|x|\ge 2\lambda} \left| D^\beta v(x) \right| \, dx \le C\lambda^{|\alpha|+2-2r} \left\| \widetilde\psi \right\|_{L_1(B_{\lambda/2})} \le C\lambda^{|\alpha|-r+N/2} \left| \widetilde\psi \right|_{W_2^{r-2}(B_{\lambda/2})} \,,$$

and from using (3.35)–(3.38) in (3.34) we end up with

$$J_{2b} \le C\lambda^{|\alpha|-1+N/2} h_j^{r-1} \left| \widetilde\psi \right|_{W_2^{r-2}} \,. \tag{3.39}$$

Finally, it remains, to estimate J_{2a}. In order to do this we shall need a simple variant of a result proved in Schatz [31] for

$$J_{2a} = \left\| \left(\frac{1}{|y|+\lambda} \right)^{|\alpha|-r} \nabla E_k^\alpha(y) \right\|_{L_\infty(B_{1/2})} . \tag{3.40}$$

The result in [31] is as follows: For any $0 \le s \le r-1$,

$$|\nabla(u-u_{h_k})(y)| \le C \left\| \left(\frac{h_k}{|y-z|+h_k} \right)^s \nabla(u-\chi)(z) \right\|_{L_\infty(B_{1/4})} + \|e\|_{W_2^{2-r}(B_{1/4})} \,. \tag{3.41}$$

Applying (3.41) to $u = w^\alpha$, $u_h = w^\alpha_{h_k}$ we obtain for any $z \in B_{1/2}(0)$,

$$\begin{aligned}\left|\left(\frac{1}{|y|+\lambda}\right)^{|\alpha|-r} \nabla E^\alpha_{h_k}(y)\right| &\\ \le \left\|\left(\frac{1}{|y|+\lambda}\right)^{|\alpha|-r} \left(\frac{h_k}{|y-z|+h_k}\right)^{|\alpha|-r} \nabla(u-\chi)(z)\right\|_{L_\infty(B_{3/4})} &\\ + \left(\frac{1}{|y|+\lambda}\right)^{|\alpha|-r} \left\|\nabla E^\alpha_{h_k}\right\|_{W_2^{2-r}(B_1)}. &\end{aligned} \tag{3.42}$$

Now, since $\lambda = h_k/h_j < 1$, thus $h_k/(|y|+\lambda) \approx h_j < 1$, and hence

$$\begin{aligned}\left(\frac{1}{|y|+\lambda}\cdot\frac{h_k}{|y-z|+h_k}\right)\left(\frac{|z|+\lambda}{|z|+\lambda}\right) &\le \frac{1}{|z|+\lambda}\left(\frac{|y-z|+|y|+\lambda}{(|y-z|+h_k)(|y|+\lambda)}\right)\times h_k\\ &\le \frac{1}{|z|+\lambda}\left(\frac{h_k}{|y|+\lambda}+\frac{h_k}{|y-z|+h_k}\right).\end{aligned}$$

Therefore from the above inequality and (3.42), taking the supremum over all $y \in B_{1/2}$ we get

$$J_{2a} \le C\left\|\left(\frac{2}{|z|+\lambda}\right)^{|\alpha|-r} \nabla(x^\alpha - x^\alpha_I)(z)\right\|_{L_\infty(B_{3/4})} + C\lambda^{r-|\alpha|}\left\|\nabla E^\alpha_k\right\|_{W_2^{2-r}(B_1)}. \tag{3.43}$$

Now, in view of (3.31),

$$\begin{aligned}\lambda^{r-|\alpha|}\left\|\nabla E^\alpha_k\right\|_{W_2^{2-r}(B_1)} &\le \lambda^{r-|\alpha|}(\lambda h_j)^{r-1}\left\|\nabla E^\alpha_k\right\|_{W_2^{r}(B_1)}\\ &\le \lambda^{r-|\alpha|}\lambda^{2r-2}h_j^{2r-2} \le \lambda^{3r-|\alpha|-2}h_j^{2r-2}.\end{aligned} \tag{3.44}$$

To estimate the first term on the right-hand side of (3.43) we use the same dyadic decomposition as before and write $B_{3/4} = B_{\lambda h_j} \cup_\ell B_{\Omega_\ell}$, where $\lambda h_j \le \rho_1$, Then on $B_{\lambda h_j}$,

$$\left(\frac{1}{|z|+\lambda}\right)^{|\alpha|-r} \le \lambda^{r-|\alpha|},$$

and

$$|\nabla(x^\alpha - x^\alpha_I)| \le \lambda^{r-1}h_j^{r-1}\left\|x^\alpha\right\|_{W^r_\infty} \le C\lambda^{r-1}h_j^{r-1}\left|\lambda h_j\right|^{|\alpha|-r}.$$

Similarly, on Ω_ℓ,

$$\sup_\ell \left\|\left(\frac{1}{|z|+\lambda}\right)^{|\alpha|-r} \nabla(x^\alpha - x^\alpha_I)(z)\right\|_{L_\infty(\Omega_\ell)} \le \left(\frac{1}{\rho_\ell}\right)^{|\alpha|-r}\lambda^{r-1}h_j^{r-1}\rho_\ell^{|\alpha|-r}.$$

Summing up we get

$$\begin{aligned}\left\|\left(\frac{1}{|z|+\lambda}\right)^{|\alpha|-r} \nabla(x^\alpha - x^\alpha_I)(z)\right\|_{L_\infty(B_{3/4})} &\\ \le C\left(\lambda^{r-|\alpha|}\lambda^{r-1}h_j^{r-1}\lambda^{|\alpha|-r}h_j^{|\alpha|-r} + \sup_{1\le\ell\le\bar\ell}\rho_\ell^{r-|\alpha|}\rho_\ell^{|\alpha|-r}\lambda^{r-1}h_j^{r-1}\right) &\\ \le C\left(\lambda^{r-1}h_j^{r-1}\right). &\end{aligned} \tag{3.45}$$

Taken together these last two inequalities (3.44) and (3.45) yield

$$J_{2a} \leq C\Big(\lambda^{r-1}h_j^{r-1} + \lambda^{3r-|\alpha|-2}h_j^{2r-2}\Big).$$

Now since $r-1 < r+1 = 3r-2r+3-2 \leq 3r-|\alpha|-2 \leq 2r-2$, thus

$$J_{2a} \leq C\Big(\lambda^{r-1}h_j^{r-1}\Big). \tag{3.46}$$

Combining (3.46), (3.39), and (3.32) we have in view of (3.28) and finally (3.25) and (3.24) that

$$\begin{aligned}\left|w_{h_j}^{\alpha}(0) - \frac{1}{\lambda^{|\alpha|}}w_{h_k}^{\alpha}(0)\right| &\leq Ch_j^{2r-2} + C\lambda^{2r-2-|\alpha|}h_j^{2r-2} \\ &\quad + \lambda^{2-r-|\alpha|-N/2}\Big(\lambda^{|\alpha|-1+N/2}\lambda^{r-1}h_j^{2r-2}\Big) \\ &\leq C\Big(h_j^{2r-2}\Big)\Big(1+\lambda^{2r-2-|\alpha|}+1\Big) \\ &\leq Ch_j^{2r-2} \qquad \text{for all} \quad h_k.\end{aligned}$$

Consequently we have the final answer for the case $d=1$, viz.,

$$\left|\frac{w_{h_j}^{\alpha}(0)}{h_j^{|\alpha|}} - \frac{w_{h_k}^{\alpha}(0)}{h_k^{|\alpha|}}\right| \leq Ch_j^{2r-2-|\alpha|} \qquad \text{for all} \quad h_k < h_j \;\; (k>j). \tag{3.47}$$

Therefore for each α, $r \leq |\alpha| \leq 2r-3$, there exists a constant $C_\alpha > 0$ such that

$$\lim_{j\to\infty} \frac{w_{h_j}^{\alpha}(0)}{h_j^{|\alpha|}} = C_\alpha,$$

and the proof is complete. ▯

Now it remains to prove Theorem 1.2

Proof of Theorem 1.2.

Case 1. $d=1$. We shall give a sketch of the proof of Theorem 1.2, mentioning only the differences between the two proofs for Theorems 1.1 and 1.2. For $r \geq 2$, i.e., piecewise linear as above, consider

$$u(0) - u_h(0) - \sum_{r\leq|\alpha|\leq\gamma-1}\left(\frac{D^\alpha u(0)}{\alpha!}x^{|\alpha|} - C_\alpha D^\alpha u(0) w_h^\alpha(x)\right). \tag{3.48}$$

The result we shall need from Schatz and Wahlbin [33] is the Lemma 2.2. Applying Lemma 2.2 to calculate the error at 0 for the difference $\frac{\partial u(0)}{\partial x_i} - \frac{\partial u_h(0)}{\partial x_i}$, we have for $i = 1,2,\ldots,N$, on unit size domain

$$\frac{\partial u(0)}{\partial x_i} - \frac{\partial u_h(0)}{\partial x_i} - \sum_{|\alpha|}\frac{D^\alpha u(0)}{\alpha!}\cdot\frac{\partial w_{h_j}^{\alpha}(0)}{\partial x_i} = R'_\gamma,$$

where $r \leq \gamma \leq 2r-1$ and

$$\left|R'_\gamma\right| \leq C\Big(h_j^\gamma \,\|u\|_{W_\infty^{\gamma+1}(B_1(0))} + \|u-u_h\|_{W_p^{-t}(B_1(0))}\Big).$$

Thus in this case we are led to showing that

$$M = \left| \frac{\frac{\partial w^{\alpha}_{h_j}(0)}{\partial x_i}}{h_j^{|\alpha|-1}} - \frac{\frac{\partial w^{\alpha}_{h_k}(0)}{\partial x_i}}{h_k^{|\alpha|-1}} \right| \leq C h_j^{2r-1-|\alpha|}.$$

To this end we note that in view of the fact that $w^{\alpha}_{h_j} - \frac{1}{\lambda^{|\alpha|}} w^{\alpha}_{h_l}$ with $\lambda = h_k/h_j$, as proved before, is discrete harmonic and thus, in view of (3.47), satisfies

$$M \leq \frac{1}{h_j^{|\alpha|-1}} \left\| w^{\alpha}_{h_j} - \frac{1}{\lambda^{|\alpha|}} w^{\alpha}_{h_k} \right\|_{W_2^{2r-2}} \leq h_j^{2r-1-|\alpha|},$$

which proves the expansion on unit sized domain. Here, the same procedure as in the proof of Theorem 1.1 and Lemma 3.1 gives the main result for the derivatives: for $i = 1, 2, \ldots, N$,

$$\frac{\partial u(0)}{\partial x_i} - \frac{\partial u_h(0)}{\partial x_i} = - \sum_{r \leq |\alpha| \leq \gamma - 1} C'_{\alpha} D^{\alpha}_{u}(0) h^{|\alpha|-1} + \mathcal{R}'_{\gamma}, \tag{3.49}$$

where $h = h_j,\ j = k, k+1, \ldots,\ \gamma \leq 2r - 1$, and

$$\left|\mathcal{R}'_{\gamma}\right| \leq \Big(C h^{\gamma-1} \|u\|_{W^{\gamma}_{\infty}(B_1(0))} + \|u - u_h\|_{W_p^{-t}(B_1(0))} \Big). \tag{3.50}$$

Case 2. The asymptotic expansion for $d < 1$.

Following the scaling argument given in the proof of the Theorem 1.1, in the corresponding case, $d < 1$, the proof is an exercise to the reader. The estimate for the remainder can be written as

$$\left|\mathcal{R}'_{\gamma}\right| \leq C\Big(h^{\gamma-1} \|u\|_{W^{\gamma}_{\infty}(B_d)} + d^{-1-t-N/p} \|u - u_h\|_{W_p^{-t}(B_d)} \Big).$$

This completes the proof of Theorem 1.2. ◻

Appendix A. Properties of the finite element subspaces. Here we shall state our assumptions on the finite element subspaces used in this paper. They are basically the same as those given in Schatz and Wahlbin [34] and [33]. The precise statements here are versions of the assumptions that are taken from [33].

For $0 < h < 1$ a parameter and $r \geq 2$ an integer, $S^h_r(\Omega)$ will denote a family of finite-dimensional subspaces of $W^1_{\infty}(\Omega)$. If $D \subseteq \Omega$, then $S^h_r(D)$ will denote the restriction of functions in $S^h_r(\Omega)$ to D and $\mathring{S}^h_r(D)$ is the subspace of $S^h_r(D)$ consisting of functions whose support is contained in D. In what follows $D_1 \subset\subset D_2 \subset\subset D_3 \subset\subset D_4$, etc., denote concentric open balls which are contained in Ω. Assume that there exists a constant k such that if $\operatorname{dist}(D_j, \partial D_{j+1}) \geq kh$, $j = 1, 2, 3$. Then the following hold.

Assumption A.1 (approximation). If $t = 0, 1$, $1 \leq \ell \leq r$, $1 \leq p \leq \infty$, then for each $v \in W^{\ell}_p(D_2)$ there exists a $\chi \in S^h_r(D_2)$ such that

$$\|v - \chi\|_{W^t_p(D_1)} \leq C h^{\ell - t} |v|_{W^{\ell}_p(D_2)}.$$

In addition there exists a χ such that

$$\|v - \chi\|_{W^1_{\infty}(D_1)} \leq C h^{r-1-N/p} \, |v|_{W^r_p(D_2)}\,.$$

Here

$$|v|_{W_p^\ell} = \begin{cases} \Big(\sum_{|\alpha|=\ell} \|D^\alpha v\|_{L_p}^p \Big)^{1/p} & \text{if} \quad 1 \le p < \infty, \\ \sum_{|\alpha|=\ell} \|D^\alpha v\|_{L_\infty} & \text{if} \quad p = \infty. \end{cases}$$

Furthermore, if $v \in \mathring{W}_p^\ell(D_1)$, then $\chi \in \mathring{S}_r^h(D_2)$. Here C is independent of h, v, χ, and D_j, $j = 1, 2$.

Assumption A.2 (inverse properties). If $\chi \in S_r^h(D_2)$, then for $t = 0, 1$,

$$\|\chi\|_{W_\infty^t(D_1)} \le Ch^{-N/2-t} \|\chi\|_{L_2(D_2)},$$

and for $\ell = 0, 1$,

$$\|\chi\|_{W_2^t(D_1)} \le Ch^{\ell-t} \|\chi\|_{W_2^{-\ell}(D_2)}.$$

Here C is independent of h, χ, D_1, and D_2.

Assumption A.3 (superapproximation). Let $\omega \in C_0^\infty(D_3)$; then for each $\chi \in S_r^h(D_4)$ there exists an $\eta \in \mathring{S}_r^h(D_4)$ such that for some integer $\gamma > 0$

$$\|\omega\chi - \eta\|_{W_2^1(D_4)} \le Ch \|\omega\|_{W_\infty^\gamma(D_3)} \|\chi\|_{W_2^1(D_4)}.$$

Furthermore, if $\omega \equiv 1$ on D_2 and with $\operatorname{dist}(D_1, \partial D_2) \ge k$, then $\eta = \chi$ on D_1, and

$$\|\omega\chi - \eta\|_{W_2^1(D_4)} \le Ch \|\omega\|_{W_\infty^\gamma(D_3)} \|\chi\|_{W_2^1(D_4 \setminus D_1)}.$$

Here C is independent of ω, χ, η, h, D_j, $j = 1, 2, 3, 4$.

Assumption A.4 (scaling). Let $x_0 \in \overline{\Omega}$ and $d \ge kh$. The linear transformation $y = x_0 + (x - x_0)/d$ takes $B_d(x_0) = \{x : |x - x_0| < d\} \cap \Omega$ into a new domain $\widehat{B}_1(x_0)$ and $S_r^h(B_d(x_0))$ into a new function space $\widehat{S}_r^{h/d}(\widehat{B}_1(x_0))$. The $\widehat{S}_r^{h/d}(\widehat{B}_1(x_0))$ satisfies Assumptions A.1, A.2, and A.3 with h replaced by h/d. The constants occurring in Assumptions A.1, A.2, and A.3 remain unchanged, in particular independent of d.

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