Reference Governor for Constrained Piecewise Affine Systems. A Vehicle Dynamics Control Application

Paolo Falcone, Francesco Borrelli, Jaroslav Pekar and Greg Stewart

***Abstract*— In this paper we present a novel approach to the design of reference tracking controllers for constrained, discrete-time piecewise affine systems and apply it to a vehicle yaw control problem. The paper is divided in two parts. In the first part, we present a methodology for designing reference tracking controllers for constrained, discrete-time piecewise affine systems. The approach follows the idea of reference governor techniques where the desired set-point is filtered by a system called the "reference governor". Based on the system current state, set-point, and prescribed constraints, the reference governor computes a new set-point for a low-level controller so that the state and input constraints are satisfied and convergence to the original set-point is guaranteed.**

In the second part of the paper, we apply the proposed approach to a vehicle dynamics control problem where the vehicle yaw rate have to be controlled through an Active Front Steering (AFS) system.

I. Introduction

Different methods for the analysis and design of controllers for hybrid systems have emerged over the last few years [33], [24], [3]. Among them, the class of optimal controllers is one of the most studied. The existing approaches differ greatly in the hybrid models adopted, in the formulation of the optimal control problem and in the method used to solve it. In this work we will focus on discrete-time piecewise affine (PWA) models. Discrete-time PWA models can describe a large number of processes, such as: discrete-time linear systems with static piecewise-linearities; discrete-time linear systems with discrete states and inputs; switching systems where the dynamic behavior is described by a finite number of discrete-time linear models together with a set of logic rules for switching among these models; approximation of nonlinear discrete-time dynamics, e.g., via multiple linearizations at different operating points.

This work deals with the design of reference tracking state-feedback controllers for constrained PWA systems. Our interest stems from industrial practice where, for control synthesis purposes, nonlinear plants are often approximated by partitioning the space spanned by the inputs, state, and exogenous signals into a finite number of regions (also called "modes"). Each region is then assigned an affine model and the nonlinear system is thus approximated by a PWA system. A standard gain scheduling strategy consists of designing a linear controller for each region along with an appropriate strategy for switching between them. It should be noted that the resulting closed-loop system contains reference signals for the controller which are exogenous to the closed-loop. In order to satisfy the state and input constraints, the control designer has to explicitly considered the case where a change in the reference signal results in a system transition between two or more regions.

In principle one could solve an optimal tacking problem for the constrained PWA systems by using the approach presented in [14]. There the authors have characterized the state-feedback solution to optimal control problems for PWA systems with performance criteria based on quadratic and linear norms. They have shown that the solution is a time-varying piecewise affine feedback control law, possibly defined over non-convex regions and proposed an algorithm that solves the Hamilton-Jacobi-Bellman equation by using a simple multiparametric solver. However, the implementation of the explicit controller might require significant computation infrastructure which might not be available on processes with fast sampling time and limited computational resources. For example, controllers designed for use in production automotive applications must be implemented on electronic control units (ECUs) that typically have a processor speed less than 60MHz and less than 3MB of flash memory. The sample time of a controller function depends on its purpose. For instance, the sample time of powertrain air handling controllers is typically faster than 100ms, while the one for a traction control or an ABS (AntiLock Braking) system is in the order of 20ms. One must also consider that a single ECU usually runs several functions in addition to any function under development. This means that the implementation footprint of any controller must be kept as small as possible. As an example, in this paper we consider a vehicle dynamics control problem, where the vehicle yaw rate has to be controlled through an AFS system. In particular, we consider a scenario where the yaw dynamic of a vehicle has to be controlled in a constant radius curve on a low friction surface through the use of the front steering angle. A typical sampling time for such vehicle dynamics application is 20 ms.

In the first part of this paper, we present an approach to the design of reference tracking controllers for constrained, discrete-time piecewise affine systems based on the concept of "reference governor" [5], [6], [7], [2], [1], [34], [23], [17], [18]. The idea underlying reference governor is to add a

Paolo Falcone is with Department of Signals and Systems, Chalmers University of Technology, Göteborg, SE-412 96, Sweden. E-mail: falcone@chalmers.se

Francesco Borrelli is with Department of Mechanical Engineering, University of California, Berkeley, 94720-1740, USA. E-mail: fborrelli@me.berkeley.edu

J. Pekar is with Honeywell Prague Laboratory, Prague, Czech Republic. E-mail: jaroslav.pekar@honeywell.com

Greg Stewart is with Honeywell Automation and Control Solutions, North Vancouver, BC V7J 3S4, Canada. E-mail: greg.stewart@honeywell.com

nonlinear device to a controlled system. Such device is called reference governor (RG) and its operation is based on the current state, set-point, and prescribed constraints. Typically the RG selects at any time a virtual reference sequence among a family of linearly parameterized sequences, by solving a convex constrained quadratic optimization problem, and feeds the controlled system according to a receding horizon control philosophy [5]. The overall system is proved to fulfill the constraints, be asymptotically stable, and exhibit an offset-free tracking behavior, provided that an admissibility condition on the initial state is satisfied [5].

This works shows how to design a reference governor for constrained piecewise-affine systems by using polyhedral invariant sets, reachable sets, multiparametric programming and dynamic programming. Compared to the infinite time optimal solution [14], [13] the approach presented in this paper will be less computational demanding at the price of suboptimality and smaller region of attraction. The main steps of the method are briefly summarized next. First, local tracking controllers are designed for each mode i of the PWA system and the invariant sets $\mathcal{O}^i$, in the state and reference space, are computed for the corresponding closed loop systems. Secondly, for any pair of modes (i, j), transition controllers [32] are designed for steering the current state in mode i to the invariant set $\mathcal{O}^j$ in mode j. Lastly, for any pair of modes (i, j), an optimal sequence of transitions is computed from the mode i to the mode j as the shortest path on a weighted graph. The graph weights are functions of the "transition cost" between any two modes. The online reference governor algorithm solves a simple constrained Quadratic Programming (QP) problem in order to modify the reference and move to the next mode according to the determined shortest path.

In the second part of the paper we apply the proposed approach to a yaw rate control problem in a passenger vehicle. We consider a scenario where the yaw motion of a vehicle has to be stabilized in a constant radius curve on a slippery surface where instability might occur due to the lateral tire force saturation at the rear axle. We start from a bicycle model [28] where the tire forces are computed through nonlinear static characteristics and piecewise linearize them as in [10]. We present simulation results showing that a vehicle operating in an unstable region of the state and input space can be stabilized through the proposed approach.

Before presenting the reference governor technique we will give a short overview on multiparametric programming and on invariant sets.

II. Definitions and Basic Results

In this section we introduce a few definitions and then recall some basic results on multi-parametric programming and invariant set theory.

Definition 1: A polyhedron is a set that equals the intersection of a finite number of closed halfspaces. An open set $\mathcal{R}$ whose closure $\bar{\mathcal{R}}$ is a polyhedron is called open polyhedron. A "neither open nor closed polyhedron" is a neither open nor closed set $\mathcal{R}$ whose closure $\bar{\mathcal{R}}$ is a polyhedron. A non-Euclidean polyhedron is a set whose closure equals the union of a finite number of polyhedra.

Definition 2: A collection of sets $\mathcal{R}_1, \ldots, \mathcal{R}_N$ is a *partition* of a set Θ if *(i)* $\bigcup_{i=1}^{N} \mathcal{R}_i = \Theta$, *(ii)* $\mathcal{R}_i \cap \mathcal{R}_j = \emptyset$, $\forall i \neq j$. Moreover $\mathcal{R}_1, \ldots, \mathcal{R}_N$ is a *polyhedral partition* of a polyhedral set Θ if $\mathcal{R}_1, \ldots, \mathcal{R}_N$ is a partition of Θ and the $\bar{\mathcal{R}}_i$'s are polyhedral sets, where $\bar{\mathcal{R}}_i$ denotes the closure of the set $\mathcal{R}_i$.

Definition 3: A function $h : \Theta \to \mathbb{R}^k$, where $\Theta \subseteq \mathbb{R}^s$, is *piecewise affine (PWA)* if there exists a partition $\mathcal{R}_1,\ldots,\mathcal{R}_N$ of Θ and $h(\theta) = H^i\theta + k^i$, $\forall\theta \in \mathcal{R}_i$, $i = 1, \ldots, N$.

Definition 4: A function $h : \Theta \to \mathbb{R}^k$, where $\Theta \subseteq \mathbb{R}^s$, is *PWA on polyhedra (PPWA)* if there exists a polyhedral partition $\mathcal{R}_1,\ldots,\mathcal{R}_N$ of Θ and $h(\theta) = H^i\theta + k^i$, $\forall\theta \in \mathcal{R}_i$, $i = 1, \ldots, N$.

Piecewise quadratic functions (PWQ) and piecewise quadratic functions on polyhedra (PPWQ) are defined analogously.

A. Background on Multiparametric programming

Consider the nonlinear mathematical program dependent on a parameter vector x appearing in the cost function and in the constraints

$$\begin{array}{rl} J^*(x) = \inf\limits_{z} & f(z, x) \\ \text{subj. to} & g(z, x) \leq 0 \\ & z \in M, \end{array} \tag{1}$$

where $z \in \mathbb{R}^s$ is the optimization vector, $x \in \mathbb{R}^n$ is the parameter vector, $f : \mathbb{R}^s \times \mathbb{R}^n \to \mathbb{R}$ is the cost function, $g : \mathbb{R}^s \times \mathbb{R}^n \to \mathbb{R}^{n_g}$ are the constraints and $M \subseteq \mathbb{R}^s$. A small perturbation of the parameter x in (1) can cause a variety of outcomes, i.e., depending on the properties of the functions f and g the solution $z^*(x)$ may vary smoothly or change abruptly as a function of x. We denote by K^* the set of feasible parameters, i.e.,

$$K^* = \{x \in \mathbb{R}^n \mid \exists z \in M, g(z, x) \leq 0\}, \tag{2}$$

by $R : \mathbb{R}^n \to 2^{\mathbb{R}^s}$, where $2^{\mathbb{R}^s}$ denotes the set of all subsets of $\mathbb{R}^s$, the point-to-set map that assigns the set of feasible z

$$R(x) = \{z \in M \mid g(z, x) \leq 0\} \tag{3}$$

to a parameter x, by $J^* : K^* \to \mathbb{R} \cup \{-\infty\}$ the real-valued function which expresses the dependence on x of the minimum value of the objective function over K^*, i.e.

$$J^*(x) = \inf_{z}\{f(z, x) \mid x \in K^*, z \in R(x)\}, \tag{4}$$

and by $Z^* : K^* \to 2^{\mathbb{R}^s}$ the point-to-set map which expresses the dependence on x of the set of optimizers, i.e, $Z^*(\bar{x}) = \{z \in R(\bar{x}) \mid f(z, \bar{x}) = J^*(\bar{x})\}$ with $\bar{x} \in K^*$.

$J^*(x)$ will be referred to as the optimal value function or simply *value function*, $Z^*(x)$ will be referred to as the *optimal set*. We will denote by $z^* : \mathbb{R}^n \to \mathbb{R}^s$ one of the possible single valued functions that can be extracted from Z^*, z^* will be called the *optimizer function*. If $Z^*(x)$ is a singleton for all x, then $z^*(x)$ is the only element of $Z^*(x)$.

Optimal control problems for nonlinear systems can be reformulated as the mathematical program (1) where z is the input sequence to be optimized and x the initial state of the system. Therefore, the study of the properties of J^* and Z^* is fundamental for the study of properties of state-feedback optimal controllers. Fiacco ([15, Chapter 2]) provides conditions under which the solution of nonlinear multiparametric programs (1) is locally well behaved and establishes properties of the solution as a function of the parameters. In this note we restrict our attention to the following special class of multiparametric programming:

$$J^*(x) = \frac{1}{2}x'Yx + \begin{array}{ll} \min\limits_{z} & \frac{1}{2}z'Hz + z'Fx \\ \text{subj. to} & Cz \leq c + Sx \end{array} \tag{5}$$

where $z \in \mathbb{R}^{n_z}$ is the optimization vector, $x \in \mathbb{R}^n$ is the vector of parameters, and $C \in \mathbb{R}^{q \times n_z}$, $c \in \mathbb{R}^q$, $S \in \mathbb{R}^{q \times n}$ are constant matrices. We refer to the problem of computing $z^*(x)$ and $J^*(x)$ in (5) as (right-hand-side) *multi-parametric quadratic program* (mp-QP).

Theorem 1 ([4]): Consider the mp-QP (5). Assume $H \succ 0$ and $\left[\begin{smallmatrix} Y & F' \\ F & H \end{smallmatrix}\right] \succeq 0$. The set K^* is a polyhedral set, the value function $J^* : K^* \to \mathbb{R}$ is PPWQ, convex and continuous and the optimizer $z^* : K^* \to \mathbb{R}^{n_z}$ is PPWA and continuous.

B. Background on Invariant Sets

This section adopts the notation used in [19], [31], [21] and provides the basic definitions for invariant sets for constrained systems. A comprehensive survey of papers on set invariance theory can be found in [12].

Denote by f_a the state update function of an autonomous systems

$$x(k+1) = f_a(x(k)) \tag{6}$$

subject to the constraints

$$x \in \mathcal{X} \tag{7}$$

Let $\phi(k; x(0); f_a)$ denote the solution of $x(k+1) = f_a(x(k), w(k))$ at time k if the initial state is $x(0)$. For the autonomous system (6)-(7), we will denote the k-step reachable set for initial states x contained in the set $\mathcal{S}$ as

$$\text{Reach}(k; f_a; \mathcal{S}) \triangleq \{\phi(k; x(0); f_a) \in \mathcal{X} \mid x(0) \in \mathcal{S}\} \tag{8}$$

while

$$\text{Pre}_{f_a}(\mathcal{S}) \triangleq \{x \in \mathcal{X} \mid f_a(x) \in \mathcal{S}\} \tag{9}$$

denotes the set of states that evolves to $\mathcal{S}$ in one step.

Equivalently, for the system with inputs

$$x(k+1) = f(x(k), u(k)), \tag{10}$$

subject to the constraints

$$x \in \mathcal{X},\ u \in \mathcal{U}, \tag{11}$$

the k-step reachable set for initial states x contained in the set $\mathcal{S}$ is defined as

$$\text{Reach}(k; f; \mathcal{S}) \triangleq \{\phi_f(k; x(0); U_k) \in \mathcal{X} \mid x(0) \in \mathcal{S},\ U_k \in \mathcal{U}^k\} \tag{12}$$

while

$$\text{Pre}_f(\mathcal{S}) \triangleq \{x \in \mathcal{X} \mid \exists u \in \mathcal{U} \text{ s.t. } f(x,u) \in \mathcal{S}\} \tag{13}$$

is the set of states which can be driven into the target set $\mathcal{S}$ in one time step.

Two different types of sets are considered in this note: *invariant sets* and *control invariant sets*. We will first discuss invariant sets. The invariant sets are computed for autonomous systems and can be used to "find, for a *given* feedback controller $u = k(x)$, the set of states whose trajectory will never violate the system constraints". The following definitions are derived from [12], [9], [8], [22], [16].

Definition 5 (Positive Invariant Set): A set $\mathcal{O}$ is said to be a positive invariant set for the autonomous system in (6) if $\text{Reach}(1; \mathcal{O}) \subseteq \mathcal{O}$.

Definition 6 (Maximal Positive Invariant Set $\mathcal{O}^\infty$): The set $\mathcal{O}^\infty$ is the maximal invariant set of the autonomous system (6) if $0 \in \mathcal{O}^\infty$, $\mathcal{O}^\infty$ is invariant and $\mathcal{O}^\infty$ contains all invariant sets that contain the origin.

Control invariant sets are defined for systems subject to external inputs and can be used to "find the set of states for which *there exists* a controller such that the system constraints are never violated". The following definitions are derived from [12], [9], [8], [22].

Definition 7 (Control Invariant Set): A set $\mathcal{C} \subseteq \mathbb{X}$ is said to be a control invariant set for the system in (10) if for every $x(k) \in \mathcal{C}$ there exists a $u(k) \in \mathbb{U}$ such that $f(x(k), u(k)) \subseteq \mathcal{C}$.

Definition 8 (Maximal Control Invariant Set $\mathcal{C}_\infty$): The set $\mathcal{C}_\infty$ is said to be the maximal control invariant set for the PWA system in (10) if it is control invariant and contains all control invariant sets contained in $\mathbb{X}$.

For all states contained in the maximal control invariant set $\mathcal{C}_\infty$ there exists a control law, such that the system constraints are never violated. This does not imply that there exists a control law which can drive the state into a user-specified target set. This issue is addressed in the following by introducing the concept of stabilizable sets.

Definition 9 (N-Step Stabilizable Set $\mathcal{K}_N(f_a; \mathcal{O})$): For a given invariant target set $\mathcal{O} \subseteq \mathcal{X}$, the N-step stabilizable set $\mathcal{K}_N(f_a; \mathcal{O})$ of the system (6) subject to the constraints (7) is defined as:

$$\mathcal{K}_N(f_a; \mathcal{O}) \triangleq \text{Pre}_{f_a}(\mathcal{K}_{N-1}(f_a; \mathcal{O})),\ N \in \mathbb{N}^+$$
$$\mathcal{K}_0(f_a; \mathcal{O}) = \mathcal{O}.$$

From Definition 9, all states x_0 belonging to the N-Step Stabilizable Set $\mathcal{K}_N(f_a; \mathcal{O})$ can be driven, through a time-varying control law, to the target set $\mathcal{O}$ in N steps and stay in in $\mathcal{O}$ for all $t \geq N$ while satisfying input and state constraints.

III. Problem Formulation

Consider the PWA system

$$\begin{array}{rcl} x(t+1) & = & A^i x(t) + B^i u(t) + f^i \\ & & \text{if } \begin{bmatrix} x(t) \\ u(t) \end{bmatrix} \in \mathcal{P}^i,\ i = \{1, \ldots, s\}, \end{array} \tag{14}$$

where $x \in \mathbb{R}^n$, $u \in \mathbb{R}^m$, $\{\mathcal{P}^i\}_{i=1}^s$ is a polyhedral partition of the set of the state and input space $\mathcal{P} \subset \mathbb{R}^{n+m}$. The current index i will be called the *system mode*, i.e., the PWA system (14) is *in mode* i at time t if $\left[\begin{smallmatrix} x(t) \\ u(t) \end{smallmatrix}\right] \in \mathcal{P}^i$.

System (14) is subject to hard input and state constraints

$$Ex(t) + Lu(t) \leq M_c \tag{15}$$

for $t \geq 0$, and we denote by *Constrained PWA system* (CPWA) the restriction of the PWA system (14) over the set of states and inputs defined by (15),

$$x(t+1) = A^i x(t) + B^i u(t) + f^i \quad \text{if } \left[\begin{smallmatrix} x(t) \\ u(t) \end{smallmatrix}\right] \in \tilde{\mathcal{P}}^i, \tag{16}$$

where $\{\tilde{\mathcal{P}}^i\}_{i=1}^s$ is the new polyhedral partition of the sets of state and input space $\mathbb{R}^{n+m}$ obtained by intersecting the sets $\mathcal{P}^i$ in (14) with the polyhedron described by (15). We assume the following.

Assumption 1: For a given reference state x_{ref} there is a unique input $u_{ref} = u_{ref}(x_{ref})$ such that $x_{ref} = A^i x_{ref} + B^i u_{ref} + f^i$ if $\left[\begin{smallmatrix} x_{ref} \\ u_{ref} \end{smallmatrix}\right] \in \tilde{\mathcal{P}}^i$.

The function $u_{ref}(x_{ref})$ is unique either from the properties of system (16) (there is one mode and one u_{ref} for each x_{ref}) or by construction (i.e., for the given x_{ref} the user specifies the desired mode and the corresponding u_{ref}).

Assumption 2: In system (16), the constrained set $\tilde{\mathcal{P}}^i$ is the cross product of a set in the input space and a set in the state space, i.e., $\tilde{\mathcal{P}}^i = \tilde{\mathcal{P}}^i_u \otimes \tilde{\mathcal{P}}^i_x$.

Assumptions 1 and 2 are introduced for the sake of simplicity and are not restrictive. They could be easily removed at the cost of a more complex notation.

Our objective is to design a state feedback control law $u(x, x_{ref})$ such that the closed loop system

$$\begin{array}{rcl} x(t+1) & = & A^i x(t) + B^i u(x(t), x_{ref}) + f^i \\ & & \text{if } \left[\begin{smallmatrix} x(t) \\ u(x(t), x_{ref}) \end{smallmatrix}\right] \in \tilde{\mathcal{P}}^i, \end{array} \tag{17}$$

converges to x_{ref} and satisfies state and input constraints.

A systematic approach to design constrained reference tracking controllers is to use a receding horizon control policy. We define the following cost function

$$\begin{array}{rl} J_N(U_N, x(0), x_{ref}) & \triangleq \|x_N - x_{ref}\|_P^2 \\ & + \sum_{k=0}^{N-1} \left[\|x_k - x_{ref}\|_Q^2 + \|u_k - u_{ref}(x_{ref})\|_R^2\right] \end{array} \tag{18}$$

with $Q = Q' \succeq 0$, $R = R' \succ 0$, $P \succeq 0$ and consider the constrained finite-time optimal control (CFTOC) problem

$$J_0^*(x(0), x_{ref}) \triangleq \min_{U_N} J(U_N, x(0), x_{ref}) \tag{19}$$

$$\text{subj. to} \left\{ \begin{array}{l} x_{k+1} = A^i x_k + B^i u_k + f^i \\ \quad \text{if } \left[\begin{smallmatrix} x_k \\ u_k \end{smallmatrix}\right] \in \tilde{\mathcal{P}}^i,\ i = 1, \ldots, s \\ x_{ref,k+1} = x_{ref,k} \\ [x_N, x_{ref}] \in \tilde{\mathcal{X}}_f \\ x_0 = x(0),\ x_{ref,0} = x_{ref} \end{array} \right. \tag{20}$$

where the column vector $U_N \triangleq [u_0', \ldots, u_{N-1}']' \in \mathbb{R}^{mN}$, is the optimization vector, N is the optimal control horizon. $\mathcal{X}_f$ is a polyhedral terminal region in the (x, x_{ref})-space. In (18) $\|x\|_Q^2$ denotes $x'Qx$. Note that we distinguish between the input $u(t)$ and the state $x(t)$ of plant (16) at time t and the variables u_k and x_k of the optimization problem (20).

We will also denote by $\tilde{\mathcal{X}}_k \subseteq \mathbb{R}^{2n}$ the set of states x_k and references x_{ref} that are feasible for (18)-(20):

$$\begin{array}{l} \tilde{\mathcal{X}}_k = \left\{ \begin{array}{l|l} x \in \mathbb{R}^n, & \exists u \in \mathbb{R}^m, \\ x_{ref} \in \mathbb{R}^n & \exists i \in \{1, \ldots, s\} \\ & \left[\begin{smallmatrix} x \\ u \end{smallmatrix}\right] \in \tilde{\mathcal{P}}^i \text{ and} \\ & \left[A^i x + B^i u + f^i, x_{ref}\right] \in \tilde{\mathcal{X}}_{k+1} \end{array} \right\}, \\ \quad k = 0, \ldots, N-1, \\ \tilde{\mathcal{X}}_N = \tilde{\mathcal{X}}_f. \end{array} \tag{21}$$

Note that the optimizer function U_N^* may not be uniquely defined if the optimal set of problem (18)-(20) is not a singleton for some $x(0)$. The next theorem shows the properties of the optimal control solution.

Theorem 2: Consider the optimal control problem (18)-(20). Then, there exists a solution in the form of a PWA state-feedback control law

$$\begin{array}{rcl} u_k^*(x(k), x_{ref}) & = & F_k^{x,i} x(k) + F_k^{u,i} u_{ref} + F_k^{r,i} x_{ref} + G_k^i \\ & & \text{if } [x(k), x_{ref}] \in \mathcal{R}_k^i, \end{array} \tag{22}$$

where $\mathcal{R}_k^i$, $i = 1, \ldots, N_k$ is a partition of the set $\tilde{\mathcal{X}}_k$ of feasible states $x(k)$ and reference x_{ref}. The boundaries of the sets $\mathcal{R}_k^i$ are linear and quadratic inequalities in $x(k)$ and x_{ref}.

Proof. Contained in [14] $\square$

An infinite horizon controller can be obtained by implementing in a receding horizon fashion a finite-time optimal control law. In this case the control law is simply obtained by repeatedly evaluating at each time t the PWA controller (22) for $k = 0$:

$$u(t) = u_0^*(x(t), x_{ref}) \text{ for } \left[\begin{smallmatrix} x(t) \\ x_{ref} \end{smallmatrix}\right] \in \tilde{\mathcal{X}}_0. \tag{23}$$

If $\tilde{\mathcal{X}}_f$ is a control invariant set and the terminal cost P is a control lyapunov function, then for all $[x(0), x_{ref}] \in \tilde{\mathcal{X}}_0$ the systems state $x(k)$ will converge to the a desired constant reference x_{ref} while satisfying input and state constraints [26]. Note that the sets $\tilde{\mathcal{X}}_k$ are defined in the state and reference space.

The number of regions in the solution to (23) might prohibit the real-time implementation for systems with limited computational and storage resources. In the next section we propose an alternative approach based on the results presented in [5], [6], [7], [2], [1] and show how to design a low-complexity controller which guarantees constraint satisfaction by using the idea of reference governor.

IV. Reference Governor

Consider the constrained PWA system (16). The proposed control design approach is based on the three main steps described next.

A. Local Control design

For each region $\tilde{\mathcal{P}}^i$, the following reference tracking controller is considered

$$u = \mathbf{k}^i(x, x_{ref}) \tag{24}$$

where $\mathbf{k}^i(x, x_{ref})$ is a linear control law or a PWA control law. For each region $\tilde{\mathcal{P}}^i$ we compute a positive invariant set $\mathcal{O}^i$ for the closed loop system:

$$x_{k+1} = A^i x_k + B^i \mathbf{k}^i(x_k, x_{ref,k}) + f^i, \tag{25a}$$

$$x_{ref,k+1} = x_{ref,k}, \tag{25b}$$

subject to the constraints

$$\begin{bmatrix} x_k \\ \mathbf{k}^i(x_k, x_{ref,k}) \end{bmatrix} \in \tilde{\mathcal{P}}^i, \quad \begin{bmatrix} x_{ref,k} \\ u_{ref}(x_{ref,k}) \end{bmatrix} \in \tilde{\mathcal{P}}^i. \tag{26}$$

We remark that $\mathcal{O}^i$ is a set in the (x, x_{ref}) space. We assume that $\mathbf{k}^i$ guarantees the convergence of $x(k)$ to a constant reference x_{ref} for system (25).

In addition to standard linear control design techniques, the controller $\mathbf{k}^i(x, x_{ref})$ can be designed as a receding horizon controller. Consider the following optimal control problem in mode i (denoted as "Problem i").

$$J_0^{*,i}(x(0), x_{ref}) \triangleq \min_{U_N} J(U_{N_i}, x(0), x_{ref})$$

$$\text{subj. to} \quad \begin{cases} x_{k+1} = A^i x_k + B^i u_k + f^i \\ \left[\begin{smallmatrix} x_k \\ u_k \end{smallmatrix}\right] \in \tilde{\mathcal{P}}^i, \ \left[\begin{smallmatrix} x_{ref} \\ u_{ref} \end{smallmatrix}\right] \in \tilde{\mathcal{P}}^i \\ [x_{N_i}, x_{ref}] \in \mathcal{X}_f^i \\ x_0 = x(0). \end{cases} \tag{27}$$

Denote by $\mathcal{X}_0^i$ is the feasible set of initial condition for Problem i (27) and the associated PWA RHC control law

$$\mathbf{k}^i(x, x_{ref}) = u_0^*(x, x_{ref}) \text{ for } x \in \mathcal{X}_0^i. \tag{28}$$

If persistent feasibility and convergence are ensured, then $\mathcal{X}_0^i$ is a positive invariant set for system (25)-(26) and $x(k) \to x_{ref}$. We set $\mathcal{O}^i = \mathcal{X}_0^i$.

Remark 1: Note that in problem (18)-(20) the terminal set $\tilde{\mathcal{X}}_f$ is an invariant set for the PWA system (11). In problem (27) $\mathcal{X}_f^i$ is a "local" invariant set, i.e., an invariant in mode i. $\mathcal{X}_f^i$ is empty if $\begin{bmatrix} x_{ref} \\ u_{ref}(x_{ref}) \end{bmatrix} \notin \tilde{\mathcal{P}}^i$.

Assume $\left[\begin{smallmatrix} x_{ref} \\ u_{ref} \end{smallmatrix}\right] \in \tilde{\mathcal{P}}^i$. If $\left[\begin{smallmatrix} x_0 \\ x_{ref} \end{smallmatrix}\right] \in \mathcal{O}^i$ then the controller $\mathbf{k}^i(x, x_{ref})$ will (i) guarantee constraint satisfaction at all time instants, (ii) keep the system in mode i and (iii) guarantee convergence to $\left[\begin{smallmatrix} x_{ref} \\ u_{ref} \end{smallmatrix}\right]$ (step 3 of the Online Algorithm). If $\left[\begin{smallmatrix} x_0 \\ x_{ref} \end{smallmatrix}\right] \notin \mathcal{O}^i$ then the local controller $\mathbf{k}^i$ will not guarantee feasibility and will not drive x_0 towards x_{ref}. However, a $\bar{x}_{ref}$ might exist such that $\left[\begin{smallmatrix} x_0 \\ \bar{x}_{ref} \end{smallmatrix}\right] \in \mathcal{O}^l$ (with $l \neq i$) or a $\bar{u}$ such that $\left[\begin{smallmatrix} x_0 \\ \bar{u} \end{smallmatrix}\right] \in \tilde{\mathcal{P}}^l$ and a "transition controller" $\mathbf{k}^{l,i}(x, \bar{x}_{ref})$ that steers the system from mode l to mode i through a modified $\bar{x}_{ref,k}$. The design of such transition controller is described next.

B. Transition Control Design

For each (i,j), $i \neq j$, select an horizon $N^{i,j}$. For a given linear or PWA transition controller $\mathbf{k}^{i,j}(x, x_{ref})$, denote by $f_a^{i,j}$ the closed loop PWA system in region i, i.e., $\left[\begin{smallmatrix} x_{k+1} \\ x_{ref,k+1} \end{smallmatrix}\right] = f_a^{i,j}(x_k, x_{ref,k}) \triangleq \left[\begin{smallmatrix} A^i x_k + B^i \mathbf{k}^{i,j}(x_k, x_{ref,k}) + f^i \\ x_{ref,k} \end{smallmatrix}\right]$ and by $\mathcal{X}^{i,j}$ the set of states which are steered from mode i to the set $\mathcal{O}^j$ in mode j in at most $N^{i,j}$ steps, i.e., $\mathcal{X}^{i,j} \triangleq \mathcal{K}_{N^{i,j}}(f_a^{i,j}, \mathcal{O}^j)$. Note that in general $\bar{x}_{ref}$ is not necessary equal to the original reference x_{ref}. In fact if $\left[\begin{smallmatrix} x_0 \\ x_{ref} \end{smallmatrix}\right] \notin \mathcal{X}^{i,j}$, then the reference x_{ref} can be modified to $\bar{x}_{ref}$ in order to have $\left[\begin{smallmatrix} x_0 \\ \bar{x}_{ref} \end{smallmatrix}\right] \in \mathcal{X}^{i,j}$ and steer the system to mode j by using the controller $\mathbf{k}^{i,j}$. Clearly $\mathcal{X}^{i,j}$ might be empty.

In addition to standard linear control design techniques, $\mathbf{k}^{i,j}$ can be designed as a constrained minimum time controller as in [19], [20], [11], [25]. Consider the minimum time control problem

$$J_{0,T}^{*,i,j}(x(0), x_{ref}) \triangleq \min_{U_T} J(U_T, x(0), x_{ref}) \tag{29}$$

$$\text{subj. to} \quad \begin{cases} x_{k+1} = A^i x_k + B^i u_k + f^i \\ \left[\begin{smallmatrix} x_k \\ u_k \end{smallmatrix}\right] \in \tilde{\mathcal{P}}^i, \ k = 0, \ldots, T-1 \\ [x_T, x_{ref}] \in \mathcal{O}^j \\ x_0 = x(0) \end{cases} \tag{30}$$

and solve it for $T = 0, \ldots, N^{i,j} - 1$. This can be done by using a sequence of multi-parametric programs of prediction horizon 1 as proposed in [19]:

$$J_p^{*,i,j}(x(p), x_{ref}) \triangleq \min_{u(p)} J(u(p), x(p), x_{ref}) + J_{p+1}^{*,i,j}(x(p+1), x_{ref})$$

$$\text{subj. to} \quad \begin{cases} x_{p+1} = A^i x_p + B^i u_p + f^i \\ \left[\begin{smallmatrix} x_p \\ u_p \end{smallmatrix}\right] \in \tilde{\mathcal{P}}^i, \ k = 0, \ldots, p-1 \\ [x_p, x_{ref}] \in \mathcal{X}_{p+1}^{i,j} \end{cases} \tag{31}$$

for $p = N^{i,j} - 1, N^{i,j} - 2 \ldots, 0$ with

$$\mathcal{X}_{N^{i,j}}^{i,j} = \mathcal{O}^j$$

and $\mathcal{X}_p^{i,j}$ being the set of feasible states $x(p)$ and references x_{ref} for which (31) is feasible at time p. Therefore $N^{i,j}$ multi-parametric programs are solved yielding $u_p^{*,i,j}$ and $\mathcal{X}_p^{i,j}$ for $p = N^{i,j} - 1, N^{i,j} - 2 \ldots, 0$. Therefore

$$\mathcal{X}^{i,j} = \bigcup_{p=1}^{N^{i,j}} \mathcal{X}_p^{i,j} \tag{32}$$

Note that since $N^{i,j}$ multi-parametric programs are solved, several controller regions in $\mathcal{X}_p^{i,j}$ may overlap. In order to guarantee minimum-time convergence and feasibility, the feedback law $u_c^{*,i,j}$ associated with the region computed at the smallest horizon c is selected for any given state x. More details can be found in [19].

Next we denote by $\mathcal{X}_x^{i,j}$ the projection of $\mathcal{X}^{i,j}$ on the x space, i.e. $\mathcal{X}_x^{i,j} = Proj_x(\mathcal{X}^{i,j})$. If x belongs to $\mathcal{X}_x^{i,j}$ then there exist a reference which will bring the PWA system from mode i to the invariant set $\mathcal{O}^j$ in mode j.

C. The Weighted Graph

For each mode we have designed a local controller $\mathbf{k}^i$ and computed a corresponding invariant $\mathcal{O}^i$. For each pair of modes we have designed a transition controller $\mathbf{k}^{i,j}$ and computed a set $\mathcal{X}^{i,j}$ of states and references in mode i which reach $\mathcal{O}^j$ in mode j in at most $N^{i,j}$ steps.

Clearly, if the current state is in mode i_1 and the reference in mode i_n, the system could still be controlled to the reference even if $\mathcal{X}^{i_1,i_n}$ is empty. Therefore, the last step is to compute the optimal transition sequence $i_1, i_2, \ldots, i_n$

between any two modes i_1 and i_n. We propose to use the properties of the sets $\mathcal{O}^i$ and $\mathcal{X}^{i,j}$ in order to avoid the inherent exponential complexity of the problem at the price of smaller regions of attraction.

In particular we can move from i_1 to i_2 through the set $\mathcal{X}^{i_1,i_2}$ by using $\mathbf{k}^{i_1,i_2}$. Then, when the system is in mode i_2, we can move through the set $\mathcal{O}^{i_2}$ by using $\mathbf{k}^{i_2}$ and reach $\mathcal{X}^{i_2,i_3}$ and so on. In this way input and state constraints are always satisfied. The feasibility property of this approach is described in following proposition.

Proposition 1: Let $x(k)$ be the current system state and $x_{ref,k}$ the current reference. Assume the current reference $x_{ref,k}$ is in mode j. Assume there exists $\bar{x}_{ref,k}$ such that $[x(k), \bar{x}_{ref,k}] \in \mathcal{O}^i$, i.e., $x(k) \in Proj_x\mathcal{O}^i$. Define the set $\bar{\mathcal{X}}^{i,j}$ as

$$\bar{\mathcal{X}}^{i,j} \triangleq Proj_x\mathcal{O}^i \bigcap Proj_x \left\{[x,\ x_{ref}] \in \mathcal{X}^{i,j} |\ x = x_{ref}\right\} \tag{33}$$

If $\bar{\mathcal{X}}^{i,j}$ is not empty, then there exists a time varying reference and a feasible state feedback control law such that the system (16) with initial state $x(k)$ in mode i can be steered to the set $\mathcal{O}^j$ in mode j.

Proof.

Consider $\bar{\mathcal{X}}^{i,j}$ in (33). If $x(k) \in Proj_x(\mathcal{O}^i)$ and $x(k) \in Proj_x(\mathcal{X}^{i,j})$ then compute $\bar{x}_{ref,k}$ such that $[x(k), \bar{x}_{ref,k}] \in \mathcal{X}^{i,j}$ and apply $\mathbf{k}^{i,j}(x(k), \bar{x}_{ref,k})$. By construction of $\mathcal{X}^{i,j}$, there exists a sequence of references such that the state will reach $\mathcal{O}^j$ in at most $N^{i,j}$ steps.

If $x(k) \in Proj_x(\mathcal{O}^i)$ and $x(k) \notin Proj_x(\mathcal{X}^{i,j})$, then pick $\begin{bmatrix} \hat{x}_{ref} \\ \hat{x}_{ref} \end{bmatrix} \in \mathcal{X}^{i,j}$ and solve the following problem

$$\tilde{x}_{ref} \triangleq \quad \text{arg}\ \min_{\bar{x}_{ref}} (\|\bar{x}_{ref} - \hat{x}_{ref}\|) \tag{34a}$$

$$\text{subj. to} \begin{bmatrix} x(k) \\ \bar{x}_{ref} \end{bmatrix} \in \mathcal{O}^i \tag{34b}$$

Problem (34) is feasible by assumptions. Apply $\mathbf{k}^i(x, \tilde{x}_{ref})$. Since $(x(k), \tilde{x}_{ref}) \in \mathcal{O}^i$ then $x(k) \to \tilde{x}_{ref}$ and since $\mathcal{O}^i$ is a connected set $\tilde{x}_{ref} \to \hat{x}_{ref}$ with $\begin{bmatrix} \hat{x}_{ref} \\ \hat{x}_{ref} \end{bmatrix} \in \mathcal{X}^{i,j}$. □

Proposition 1 shows that we can transition from mode i to mode j from $\mathcal{X}^{i,j}$ or from $\mathcal{O}^i$ if $\bar{\mathcal{X}}^{i,j}$ is not empty. Therefore one can steer the state from mode i_1 to mode i_n by applying the sequence of controllers $\mathbf{k}^{i_1,i_2}$, $\mathbf{k}^{i_2}$,$\mathbf{k}^{i_2,i_3}$, $\mathbf{k}^{i_3}$,…,$\mathbf{k}^{i_n}$.

The concept of weighted graph will be used to compute the "best" transition sequence from $\mathcal{O}^i$ to $\mathcal{O}^j$ for any two modes i, j. A weighted graph $\mathcal{G}$ is defined as

$$\mathcal{G} = (\mathcal{V}, \mathcal{A}) \tag{35}$$

where $\mathcal{V}$ is the set of nodes (or vertices) $\mathcal{V} = \{1, \ldots, N\}$ and $\mathcal{A} \subseteq \mathcal{V} \times \mathcal{V}$ the set of edges (i, j) with $i \in \mathcal{V},\ j \in \mathcal{V}$. Let $\mathbf{A}_{i,j} \in \mathbb{R}$ be the i, j element of the weighted adjacency matrix $\mathbf{A}$ of the graph $\mathcal{G}$. Then, $\mathbf{A}_{i,j} = 0$ if there is no edge connecting the vertex i with the vertex j, i.e., $(i, j) \notin \mathcal{A}$. The elements of $\mathbf{A}$ are computed as follows:

$$\mathbf{a}_{i,j} = \alpha \frac{1}{\text{vol}(\bar{\mathcal{X}}^{i,j})} + \beta N^{i,j}$$

where vol$(\mathcal{P})$ is the volume of the polyhedron $\mathcal{P}$. The positive real numbers α and β are tuning parameters. Given the weighted graph $\mathcal{G}$, $\mathbf{u}$ = SPath$(\mathcal{G}, i_1, i_n)$ is the vector which describes the shortest path $\mathbf{u} = \{i_1,\ i_2, \ldots, i_n\}$ between node i_1 and node i_n and SPathCost$(\mathcal{G}, i_1, i_n)$ is the corresponding optimal cost.

Remark 2: Note that for the existence of a feasible path $\mathbf{v} = \{i, i_1,\ i_2, \ldots, i_n\}$, between mode i and mode i_n, the set $\mathcal{X}^{i,i_1}$ must be non empty and the sets $\bar{\mathcal{X}}^{i_1,i_2}, \ldots, \bar{\mathcal{X}}^{i_{n-1},i_n}$ must be full dimensional.

D. On-line Reference Governor Algorithm

Once all the elements have been computed off-line, the following algorithm is implemented on-line.

Algorithm 4.1:

Input: Current state $x(t)$ and reference $x_{ref} = x_{ref}(t)$

Output: Modified reference $\bar{x}^*_{ref}(t)$ and controller selection

1 **let** $[x_{ref}, x_{ref}] \in \mathcal{O}^f$

2 **if** $x(t) \in Proj_x(\mathcal{O}^f)$ **then** select local controller $\mathbf{k}^f$ and compute the modified reference as follows

$$\bar{x}^*_{ref} = \quad \text{arg}\ \min_{\bar{x}_{ref}} \|\bar{x}_{ref} - x_{ref}\| \tag{36a}$$

$$\text{subj. to} \begin{bmatrix} x(t) \\ \bar{x}_{ref} \end{bmatrix} \in \mathcal{O}^f \tag{36b}$$

3 **else**

4 **let** $v = \{v_1, \ldots, v_n\}$ the set of modes such that $x(t) \in Proj_x(\mathcal{X}^{l,v_i})$ and **let** $u = \{u_1, \ldots, u_m\}$ the set of modes such that $x(t) \in Proj_x(\mathcal{O}^{u_i})$. (note that x(t) can be in multiple modes because the system partition depends on the input as well).

5 Compute $v^* \in v \cup u$ with the associated shortest path $\{v^*, i_1, \ldots, i_n, f\}$ and cost s^* = SPathCost$(\mathcal{G}, v^*, f)$.

6 **if** $s^* = \infty$ **then** "Infeasible Reference", EXIT

7 **if** $v^* \in v$ **then** select transition controller $\mathbf{k}^{l,v^*}$ and compute the modified reference as follows

$$\bar{x}^*_{ref} = \quad \text{arg}\ \min_{\bar{x}_{ref}} \|\bar{x}_{ref} - x_{ref}\| \tag{37}$$

$$\text{subj. to} \begin{bmatrix} x(t) \\ \bar{x}_{ref} \end{bmatrix} \in \mathcal{X}^{l,v^*} \tag{38}$$

8 **else** select local controller $\mathbf{k}^{v^*}$ and compute the modified reference as follows

$$\bar{x}^*_{ref} \triangleq \quad \text{arg}\ \min_{\bar{x}_{ref}, \hat{x}_{ref}} (\|\bar{x}_{ref} - \hat{x}_{ref}\|) \tag{39}$$

$$\text{subj. to} \begin{bmatrix} x(t) \\ \bar{x}_{ref} \end{bmatrix} \in \mathcal{O}^{v^*} \tag{40}$$

$$\begin{bmatrix} \hat{x}_{ref} \\ \hat{x}_{ref} \end{bmatrix} \in \mathcal{X}^{v^*,i_1} \tag{41}$$

9 Goto Step 1

Remark 3: Note that the sets $\mathcal{X}^{i,j}$ might be described as the union of polyhedra $\mathcal{X}^{i,j}_k$, for $k = 1, \ldots, N^{i,j}$ where $\mathcal{X}^{i,j}_k$ represents the $k - steps$ reachable set. In this case Step 7 in Algorithm 4.1 can be modified as follows:

$$\bar{x}^*_{ref} = \quad \text{arg}\ \min_{\bar{x}_{ref}, k} \|\bar{x}_{ref} - x_{ref}\| \tag{42a}$$

$$\text{subj. to} \begin{bmatrix} x(t) \\ \bar{x}_{ref} \end{bmatrix} \in \mathcal{X}^{l,v^*}_k \tag{42b}$$

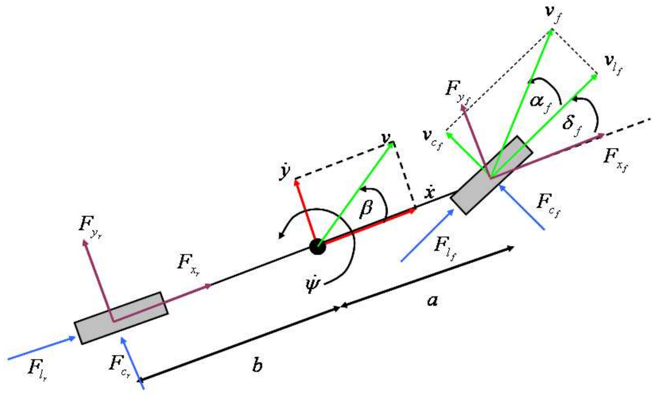


Fig. 1. The bicycle model.

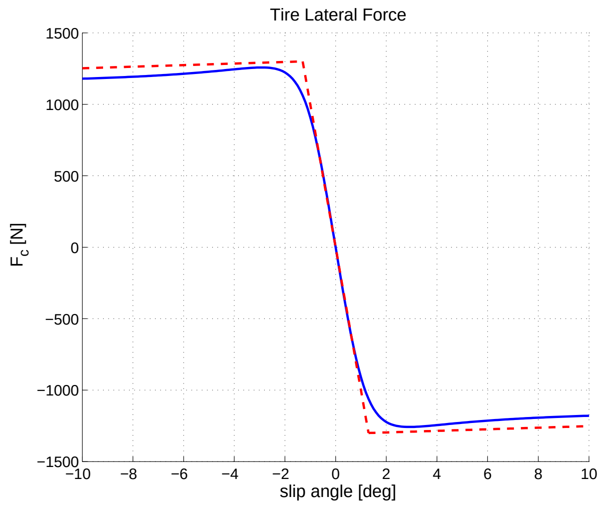


Fig. 2. Piece-wise linear approximation of the lateral tire force on a snow surface. The solid line is the nonlinear lateral tire force characteristic and the dashed line is its piece-wise linear approximation.

The same modification can be applied to Step 8 in Algorithm 4.1.

Remark 4: Note that the QP problem defined in Step 2 in Algorithm 4.1 can be solved explicitly as has been shown in [30], [27].

V. Example

Next we present a vehicle dynamics example for the algorithm presented in Section IV. We start from the bicycle model sketched in Figure 1 and refer to the nomenclature therein.

The lateral and yaw dynamics are described by the following system of nonlinear differential equations:

$$m\ddot{y} = -m\dot{x}\dot{\psi} + 2\left[F_{y_f} + F_{y_r}\right], \tag{43a}$$

$$I\ddot{\psi} = 2\left[aF_{y_f} - bF_{y_r}\right]. \tag{43b}$$

We assume *(i)* small steering angle [29] and *(ii)* constant longitudinal speed, i.e., $\dot{x} = \text{const}$. Note that, according to the assumption *(ii)*, pure cornering maneuvers are considered where the lateral forces can be modeled as nonlinear functions of the tire slip angles α, as shown in Figure 2. By the assumption *(i)*, the front and rear tire slip angles can be approximated as follows [29]:

$$\alpha_f = \delta_f - \frac{\dot{y} + a\dot{\psi}}{\dot{x}}, \tag{44a}$$

$$\alpha_r = \frac{b\dot{\psi} - \dot{y}}{\dot{x}}. \tag{44b}$$

In order to derive a piece-wise approximation of the model (43), the front and rear lateral forces F_{y_f} and F_{y_r} are approximated as piece-wise linear functions of the tire slip angles as shown in Figure 2. In particular the nonlinear tire characteristic (solid line) has been approximated with the following piece-wise linear function (dashed line):

$$\textbf{??}F_y(\alpha) = \begin{cases} -C_c^{lin}\alpha^* + C_c^{sat}(\alpha + \alpha^*), & \text{for } \alpha < -\alpha^*, \\ C_c^{lin}\alpha, & \text{for } -\alpha^* \le \alpha \le \alpha^*, \\ C_c^{lin}\alpha^* + C_c^{sat}(\alpha - \alpha^*), & \text{for } \alpha > \alpha^*, \end{cases} \tag{45}$$

where C_c^{lin} and C_c^{sat} are the slopes of the lateral tire force characteristic in the intervals $[-\alpha^*, \alpha^*]$ and $[-\infty, -\alpha^*] \bigcup [\alpha^*, \infty]$, respectively.

For the sake of simplicity we consider only two operating modes of the system. In particular we consider the mode, next referred to as *mode 1*, where at both front and rear axles $\alpha \in [-\alpha^*, \alpha^*]$ and the mode, next referred to as *mode 2*, where $\alpha_f \in [-\alpha^*, \alpha^*]$ and $\alpha_r > \alpha^*$.

We focus only on the following transition: steer the state of the system (43) from mode 2 to mode 1.

In the final version of the paper we will show the simulation results of the considered vehicle dynamics control problem, if accepted. In Figure 3, for illustrative purposes, we report the simulation results of a numerical example similar to the vehicle dynamics control problem considered here, i.e., second order system with one input, two modes and initial and final states belonging to modes 2 and 1, respectively.

References

[1] A. Bemporad. Reference governor for constrained nonlinear systems. *Automatic Control, IEEE Transactions on*, 43(3):415–419, Mar 1998.

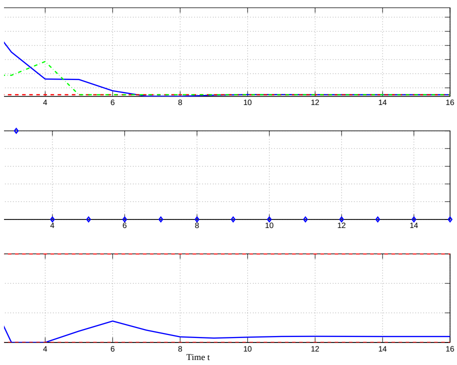
[2] A. Bemporad, A. Casavola, and E. Mosca. Nonlinear control of constrained linear systems via predictive reference management. *Automatic Control, IEEE Transactions on*, 42(3):340–349, Mar 1997.

[3] A. Bemporad and M. Morari. Control of systems integrating logic, dynamics, and constraints. *Automatica*, 35(3):407–427, March 1999.

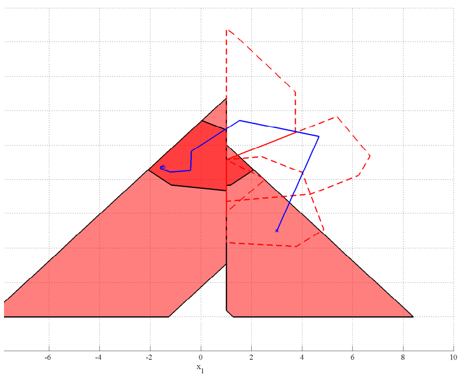
[4] A. Bemporad, M. Morari, V. Dua, and E.N. Pistikopoulos. The explicit linear quadratic regulator for constrained systems. *Automatica*, 38(1):3–20, 2002.

[5] A. Bemporad and E. Mosca. Constraint fulfilment in feedback control via predictive reference management. *Control Applications, 1994., Proceedings of the Third IEEE Conference on*, pages 1909–1914 vol.3, Aug 1994.

[6] A. Bemporad and E. Mosca. Constraint fulfilment in feedback control via predictive reference management. *Control Applications, 1994., Proceedings of the Third IEEE Conference on*, pages 1909–1914 vol.3, Aug 1994.



(a) **Upper plot.** Output trajectory (solid line), reference trajectory x_{ref} (dashed line), updated reference trajectory (dash-dotted line) for the first state. **Middle plot.** Modes. **Lower plot.** Control input (solid line), input bounds (dashed lines).



(b) The light colored regions are the polyhedra $\tilde{\mathcal{P}}_x^i$, while the dark regions represent the projection of the invariant sets $\mathcal{O}_{\infty,x}^i$. The solid line is the state trajectory. The regions delimited by dashed lines are the projections $\mathcal{X}_x^{2,1}$.The initial and the final states are marked with 'o' and '*', respectively.

Fig. 3. Simulation results.

[7] A. Bemporad and E. Mosca. Nonlinear predictive reference governor for constrained control systems. *Decision and Control, 1995., Proceedings of the 34th IEEE Conference on*, 2:1205–1210 vol.2, Dec 1995.

[8] D. P. Bertsekas. *Control of Uncertain Systems with a set–membership description of the uncertainty*. PhD thesis, MIT, 1971.

[9] D. P. Bertsekas and I. B. Rhodes. On the minimax reachability of target sets and target tubes. *Automatica*, 7:233–247, 1971.

[10] Th. Besselmann, Ph. Rostalski, and M. Morari. Hybrid Parameter-Varying Model Predictive Control for Lateral Vehicle Stabilization. In *European Control Conference*, July 2007.

[11] F. Blanchini. Minimum-time control for uncertain discrete-time linear systems. In *Proc. 31rd IEEE Conf. on Decision and Control*, pages 2629–2634, Tucson, Arizona, USA, December 1992.

[12] F. Blanchini. Set invariance in control — a survey. *Automatica*, 35(11):1747–1768, November 1999.

[13] F. Borrelli. *Constrained Optimal Control of Linear & Hybrid Systems*, volume 290. Springer Verlag, 2003.

[14] F. Borrelli, M. Baotic, A. Bemporad, and M. Morari. Dynamic programming for constrained optimal control of discrete-time hybrid systems. *Automatica*, 41:1709–1721, January 2005.

[15] A. V. Fiacco. *Introduction to sensitivity and stability analysis in nonlinear programming*. Academic Press, London, U.K., 1983.

[16] E. G. Gilbert and K. T. Tan. Linear systems with state and control constraints: The theory and application of maximal output admissible sets. *IEEE Trans. Automatic Control*, 36(9):1008–1020, September 1991.

[17] E.G. Gilbert and I. Kolmanovsky. A generalized reference governor for nonlinear systems. *Decision and Control, 2001. Proceedings of the 40th IEEE Conference on*, 5:4222–4227 vol.5, 2001.

[18] E.G. Gilbert, I. Kolmanovsky, and Kok Tin Tan. Nonlinear control of discrete-time linear systems with state and control constraints: a reference governor with global convergence properties. *Decision and Control, 1994., Proceedings of the 33rd IEEE Conference on*, 1:144–149 vol.1, Dec 1994.

[19] P. Grieder. *Efficient Computation of Feedback Controllers for Constrained Systems*. PhD thesis, ETH-Zurich, Automatic Control Laboratory, 2004.

[20] S.S. Keerthi and E.G. Gilbert. Computation of minimum-time feedback control laws for discrete-time systems with state-control constraints. *IEEE Trans. Automatic Control*, AC-32:432–435, May 1987.

[21] E. C. Kerrigan. *Robust Constraints Satisfaction: Invariant Sets and Predictive Control*. PhD thesis, Department of Engineering, University of Cambridge, Cambridge, England, 2000.

[22] I. Kolmanovsky and E. G. Gilbert. Theory and computation of disturbance invariant sets for discrete-time linear systems. *Mathematical Problems in Egineering*, 4:317–367, 1998.

[23] I. Kolmanovsky and Jing Sun. Approaches to computationally efficient implementation of gain governors for nonlinear systems with pointwise-in-time constraints. *Decision and Control, 2005 and 2005 European Control Conference. CDC-ECC '05. 44th IEEE Conference on*, pages 7564–7569, Dec. 2005.

[24] J. Lygeros, C. Tomlin, and S. Sastry. Controllers for reachability specifications for hybrid systems. *Automatica*, 35(3):349–370, 1999.

[25] D. Q. Mayne and W. R. Schroeder. Robust time–optimal control of constrained linear systems. *Automatica*, 33(12):2103–2118, 1997.

[26] D.Q. Mayne, J.B. Rawlings, C.V. Rao, and P.O.M. Scokaert. Constrained model predictive control: Stability and optimality. *Automatica*, 36(6):789–814, June 2000.

[27] S. Olaru and D. Dumur. Compact explicit mpc with guarantee of feasibility for tracking. In *Proc. 44th IEEE Conf. on Decision and Control*, pages 969–974, 2005.

[28] Hans B. Pacejka. *Tire and Vehicle Dynamics*. Society of Automotive Engineers (SAE), second edition, 2006.

[29] Hans B. Pacejka. *Tire and Vehicle Dynamics*, chapter 1, pages 23–24. Society of Automotive Engineers (SAE), second edition, 2006.

[30] J. Pekar and V. Havlena. Model predictive control with invariant sets: Set-point tracking. In *Proc. 15th International Conference of Process Control 05*, 2005.

[31] Sasa V. Rakovic. *Robust Control of Constrained Discrete Time Systems: Characterization and Implementation*. PhD thesis, Imperial College London, London, United Kingdom, July 2005.

[32] Paolo Santesso, João Pedro Hespanha, and Greg E. Stewart. Reset map design for switching between stabilizing controllers. In *Proc. 46th IEEE Conf. on Decision and Control*, December 2007.

[33] E.D. Sontag. Nonlinear regulation: The piecewise linear approach. *IEEE Trans. Automatic Control*, 26(2):346–358, April 1981.

[34] A. Vahidi, I. Kolmanovsky, and A. Stefanopoulou. Constraint handling in a fuel cell system: A fast reference governor approach. *Control Systems Technology, IEEE Transactions on*, 15(1):86–98, Jan. 2007.