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STATIONARY RANDOM GRAPHS ON $\mathbb{Z}$ WITH PRESCRIBED IID DEGREES AND FINITE MEAN CONNECTIONS

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Abstract
Let F be a probability distribution with support on the non-negative integers. A model is proposed for generating stationary simple graphs on $\mathbb{Z}$ with degree distribution F and it is shown for this model that the expected total length of all edges at a given vertex is finite if F has finite second moment. It is not hard to see that any stationary model for generating simple graphs on $\mathbb{Z}$ will give infinite mean for the total edge length per vertex if F does not have finite second moment. Hence, finite second moment of F is a necessary and sufficient condition for the existence of a model with finite mean total edge length.

1 Introduction

In the simplest random graph model, given a set of n vertices, an edge is drawn independently between each pair of vertices with some probability p. This model goes back to Erdős and Rényi (1959) and dominated the field of random graphs for decades after its introduction. However, during the last few years there has been a growing interest in the use of random graphs as models for various types of complex network structures, see e.g. Newman (2003) and the references therein. In this context it has become clear that the Erdős-Rényi graph fails to reflect a number of important features of real-life networks. For instance, an important quantity in a random graph is the degree distribution, and, in an n-vertex Erdős-Renyi graph, if the edge probability is scaled by $1/n$, the vertex degree is asymptotically Poisson distributed. In many real-life networks however, the degree sequence has been observed to follow a power law, that is, the number of vertices with degree k is proportional to $k^{-\tau}$ for some exponent $\tau > 1$. Hence, the Erdős-Renyi graph provides a bad model of reality at this point.

The fact that the Erdős-Renyi model cannot give other degree distributions than Poisson has inspired a number of new graph models that take as input a probability distribution F with support on the non-negative integers and give as output a graph with degree distribution F. The most well-known model in this context is the so called configuration model, formulated in Wormald (1978) and later studied by Molloy and Reed (1995,1998) and van der Hofstad et. al. (2005) among many others. It works so that each vertex is assigned a random number of stubs according to the desired degree distribution F and these stubs are then paired at random to form edges. A drawback with the configuration model is that it can give self-loops and multiple edges between vertices, something that in most cases is not desirable in the applications. See however Britton et. al. (2005) for modifications of the model that give simple graphs – that is, graphs without self-loops and multiple edges – as final result. A different model for generating graphs with given degrees is described in Chung and Lu (2002:1,2).

A natural generalization of the problem of generating a random graph with a prescribed degree distribution is to consider spatial versions of the same problem, that is, to ask for a way to generate edges between vertices arranged on some spatial structure so that the vertex degrees have a certain specified distribution. This problem was introduced in Deijfen and Meester (2006), where also a model is formulated for generating stationary graphs on $\mathbb{Z}$ with a given degree distribution F. Roughly the model works so that a random number of "stubs" with distribution F is attached to each vertex. Each stub is then randomly and indepedently of other stubs assigned a direction, left or right, and the graph is obtained by stepwise pairing stubs that point to each other. Under the assumption that F has finite mean, this is shown to lead to well-defined configurations, but the expected length of the edges is infinite. It is conjectured that, in fact, all stationary procedures for pairing stubs with random, independent directions give connections with infinite mean. This conjecture has been proved for stub distributions F with bounded support.

The purpose of the present paper is to formulate a model for generating stationary simple graphs on $\mathbb{Z}$ with prescribed degree distribution and finite expected edge length. Just as in Deijfen and Meester (2006) we will begin by attaching stubs to the vertices according to the desired degree distribution and then we will look for a stationary way to pair these stubs to create edges. The aim is to do this in such a way that multiple edges are avoided. Furthermore, in view of the conjecture from Deijfen and Meester, if we want to achieve finite mean for the edge length, the pairing step cannot involve giving independent directions to the stubs, but the stubs have to be connected in a more "effective" way. For distributions with bounded support it turns out that there is a quite simple way of doing this, while the case with unbounded support requires a bit more work. We mention that related matching problems have been studied for instance by Holroyd and Peres (2003,2005).

Let $D \sim F$ be the degree of the origin in a stationary simple graph on $\mathbb{Z}$ and write T for the total length of all edges at the origin. We then have that $T \geq 2\sum_{k=1}^{\lfloor D/2 \rfloor} k$, where the lower bound is attained when there is one edge to each nearest neighbor, one edge to each second nearest neighbor, and so on. Since $2\sum_{k=1}^{\lfloor D/2 \rfloor} k \geq (D^2-1)/4$, it follows that finite second moment of the degree distribution is a necessary condition for the possibility of generating a stationary simple graph with finite mean total edge length per vertex. In this paper we propose a model that indeed gives finite mean for the total edge length when $\mathbb{E}[D^2] < \infty$. This establishes the following theorem.

Theorem 1.1 *Let F be a probability distribution with support on the non-negative integers. It is possible to generate a simple stationary graph on $\mathbb{Z}$ with degree distribution F and $\mathbb{E}[T] < \infty$ if and only if F has finite second moment.*

The rest of the paper is organized as follows. In Section 2, a model is described that works for degree distributions with bounded support, that is, for distributions with bounded support the expected total length of all edges at a given vertex is finite. In Section 3, this model is refined in that vertices with high degree are treated separately. Finally in Section 4, the refined model is shown to give finite mean for the total edge length per vertex if the degree distribution has finite second moment.

2 A basic model

Let F be a probability distribution with support on the non-negative integers. In this section we formulate a basic model for generating a stationary simple graph on $\mathbb{Z}$ with degree distribution F. We also show that the expected length of the edges is finite if F has bounded support.

The basis for the model proposed in this section – and also for the refined model in the following section – is a stub configuration on $\mathbb{Z}$. This configuration is obtained by associating independently to each vertex $i \in \mathbb{Z}$ a random degree D_i with distribution F and then attach D_i stubs to vertex i. The question is how the stubs should be connected to create edges. As mentioned, an important restriction is that the pairing procedure is required to be stationary. Also, the resulting graph is not allowed to contain multiple edges between vertices.

Our first suggestion of how to join the stubs is as follows. Let Γ_j be the set of all vertices with degree at least j, that is, $\Gamma_j = \{i \in \mathbb{Z} : D_i \geq j\}$. Furthermore, for each vertex i, label the stubs $\{s_{i,j}\}_{j=1}^{D_i}$ and define $\Lambda_j = \bigcup_{i \in \Gamma_j} s_{i,j}$ so that each vertex $i \in \Gamma_j$ has exactly one stub $s_{i,j}$ in Λ_j connected to it. Stubs in Λ_j will be referred to as belonging to *level* j. The stubs in the sets $\{\Lambda_j\}$ are now connected to each other within the sets, starting with Λ_1, as follows:

1. Imagine that a fair coin is tossed at the first vertex $i_1 \geq 0$ with degree at least 1. If the coin comes up heads, the level 1 stub $s_{i_1,1}$ at i_1 is turned to the right and if the coin comes up tails, the stub $s_{i_1,1}$ is turned to the left. The other stubs in Λ_1 are directed so that, at every second vertex in Γ_1, the level 1 stub points to the right and, at every second vertex, it points to the left. Edges are then created by connecting stubs that are pointing at each other. More precisely, a level 1 stub at vertex i pointing to the right (left) is joined to the level 1 stub at the next vertex $j > i$ ($j < i$) in Γ_1 – by the construction, this stub is pointing to the left (right).

2. The stubs in Λ_2 are assigned directions analogously by letting every second stub in Λ_2 point to the right and every second stub to the right, the direction of the stub at the first vertex $i_2 \geq 0$ with degree at least 2 being determined by a coin toss. To avoid multiple edges between vertices, the stubs are then connected so that a right (left) stub $s_{i,2}$ at a vertex i is joined with the *second* left (right) stub encountered to the right (left) of i. Since every second stub in Λ_2 is pointing to the right (left), this means that $s_{i,2}$ is linked to the level 2 stub at the third vertex in Γ_2 to the right of i. This vertex cannot have an edge to i from step 1, since the level 1 stub at i was connected to the first vertex in Γ_1 either to the left or to the right, and $\Gamma_2 \subset \Gamma_1$.

$\vdots$

n. In general, in step n, the stubs in Λ_n are connected by first randomly choosing one of the two possible configurations where every second stub is pointing to the right and every second stub is pointing to the left, and then link a given stub to the nth stub pointing in the opposite direction encountered in the direction of the stub. A level n stub at vertex i pointing to the right (left) is hence connected to vertex number $2n-1$ to the right (left) of i in Γ_n. Since $\Gamma_n \subset \Gamma_{n-1}$, this does not give rise to multiple edges.

$$\vdots$$

This procedure is clearly stationary and will be referred to as the *Coin Toss* (CT) *model*. Our first result is a formula for the expected total edge length per vertex in the CT-model. To formulate it, write p_j for the probability of the outcome j in the degree distribution F and note that, by stationarity, the distribution of the edge length is the same at all vertices. Hence it suffices to consider the total length T of all edges at the origin.

Proposition 2.1 *In the CT-model, assume that F has bounded support and let $u = \max\{j : p_j > 0\}$. Then*

$$\mathbb{E}[T] = u^2.$$

Proof: In what follows we will drop the vertex index for the degree at the origin and write $D_0 = D$. For $j = 1, \ldots, u$, let K_j denote the length of the jth edge at the origin, that is, K_j is the length of the edge created by the stub $s_{0,j}$ (if $D < j$, we set $K_j = 0$). Also, define $p_j^+ = P(D \geq j) = 1 - F(j-1)$. Then

$$\begin{aligned} \mathbb{E}[T] &= \mathbb{E}\Big[\sum_{j=1}^{u} K_j\Big] \\ &= \sum_{j=1}^{u} p_j^+ \mathbb{E}[K_j | D \geq j]. \end{aligned}$$

The jth edge at the origin is equally likely to point to the right as to the left, and, by symmetry, the expected length of the edge is the same in both cases. If the edge points to the right (left), its other endpoint is vertex number $2j-1$ to the right (left) of the origin with degree at least j. The distance to this vertex has a negative binomial distribution with mean $(2j-1)/p_j^+$, and it follows that

$$\mathbb{E}[T] = \sum_{n=1}^{u} \frac{2j-1}{p_j^+} p_j^+ = 2\sum_{j=1}^{u} j - u = u^2,$$

as desired. □

It follows from the calculations in the proof that T has infinite mean when the support of F is unbounded and hence we have the following corollary:

Corollary 2.1 *In the CT-model $\mathbb{E}[T] < \infty$ iff F has bounded support.*

Roughly, the reason the CT-model gives infinite mean for T when the degrees are not bounded is that vertices with high degree are connected to other vertices with high degree. In Section 3, a model is formulated where high degree vertices are connected in a more effective way in order to get edges with finite mean length.

According to Corollary 2.1, the mean total edge length at the origin in the CT-model with i.i.d. degrees is finite for degree distributions with bounded support. The following proposition – which will be needed in the proof of Theorem 4.1 – asserts that the same result holds also for stationary degrees.

Proposition 2.2 *Let $\{D'_i\}$ be a stationary stub configuration on $\mathbb{Z}$ with $D'_i \sim G$ and connect the stubs as in the CT-model. Then, if G has bounded support, we have $\mathbb{E}[T] < \infty$.*

The proof of the proposition is based on the following lemma.

Lemma 2.1 *Let $\{X_i\}_{i\in\mathbb{Z}}$ be a $\{0,1\}$-valued stationary process with an almost surely infinite number of 1's. Define $\tau_i = \inf\{k > i : X_k = 1\}$ and write $P(X_i = 1) = p$. Then*

$$\mathbb{E}[\tau_i | X_i = 1] = \frac{1}{p}.$$

Proof of Lemma 2.1: By stationarity, it suffices to consider $i = 0$. We have

$$\mathbb{E}[\tau_0 | X_0 = 1] = \frac{\mathbb{E}[\tau_0 1_{\{X_0=1\}}]}{p},$$

where the numerator can be written as

$$\begin{aligned}
\mathbb{E}[\tau_0 1_{\{X_0=1\}}] &= \sum_{k=0}^{\infty} P(\tau_0 1_{\{X_0=1\}} > k) \\
&= \sum_{k=0}^{\infty} P(X_0 = 1, X_1 = 0, \ldots, X_k = 0) \\
&= \sum_{k=0}^{\infty} P(X_{-k} = 1, X_{-(k-1)} = 0, \ldots, X_0 = 0) \\
&= \sum_{k=0}^{\infty} P(\min\{j \geq 0 : X_{-j} = 1\} = k) \\
&= 1.
\end{aligned}$$

□

Proof of Proposition 2.2: Write $D'_0 = D'$, let $u = \max\{j : p_j > 0\}$, and remember from the proof of Proposition 2.1 that K_j denotes the length of the jth edge at the origin for $j = 1, \ldots, u$ (if $D' < j$, we set $K_j = 0$). Clearly we are done if we can show that $\mathrm{E}[K_j] < \infty$ for all j. We have $\mathrm{E}[K_j] = \mathbb{E}[K_j | D' \geq j] p_j^+$, and to see that $\mathbb{E}[K_j | D' \geq j]$ is finite, define

$$X_i^j = \begin{cases} 1 & \text{if } D' \geq j; \\ 0 & \text{otherwise.} \end{cases}$$

The jth edge at the origin is equally likely to point to the right as to the left, and, by symmetry, the expected length of the edge is the same in both cases. If the edge points to the right (left), its other endpoint is vertex number $2j - 1$ to the right (left) of the origin with degree at least j. Hence $\mathbb{E}[K_j | D' \geq j]$ is the expected distance to the right of the origin until $2j - 1$ 1's have been encountered in the process $\{X_i^j\}$, given that $X_0 = 1$. It follows from Lemma 2.1 that the expected distance between two successive 1's in the process is $1/p_j^+$, which gives the desired result exactly as in the proof of Proposition 2.1. □

3 A refined model

In this section we describe a model designed to give finite mean for the total edge length per vertex for degree distribution with finite second moment. The idea is to truncate the degrees at some high level d and connect stubs at level $j \geq d+1$ separately. The remaining stubs at level $1, \ldots, d$ after this has been done are then connected according to the CT-model. The model is slightly easier to define when the degrees are almost surely non-zero, and hence, in what follows, we will assume that $P(D_i \geq 1) = 1$. This means no loss of generality, since removing vertices with degree 0 only shrinks expected edge lengths by a factor $1 - P(D_i = 0)$.

First we introduce some notation and terminology. For a fixed $d \in \mathbb{N}$, let D_i^d be the "tail" above level d at vertex i, that is, $D_i^d = \max\{D_i - d, 0\}$. Stubs at level $j \geq d+1$ will be referred to as *bad* and D_i^d thus indicates the number of bad stubs at vertex i. A vertex with bad stubs on it – that is, a vertex with degree strictly larger than d – will be called *high*. Of course, stubs that are not bad will be called *good* and vertices that are not high will be called *low*. The model for connecting the stubs is based on the concept of *claimed* vertices:

Definition 3.1 *A vertex $i \in \mathbb{Z}$ is said to be claimed on level d iff $\sum_{k=i-m}^{i+m} D_k^d \geq m$ for some $m \geq 1$.*

In words, a vertex is claimed if, either there are bad stubs on the vertex itself (this means that a high vertex is by definition claimed), or there is a symmetric interval of width $2m+1$ for some positive m around the vertex in which the total number of bad stubs is at least m. The set of claimed vertices at level d will be denoted by $\mathcal{C}_d$. Finally, by a *cluster* of claimed vertices we mean a set of consecutive vertices $i, \ldots, i+n$ with $\{i, \ldots, i+n\} \subset \mathcal{C}_d$ but $i-1 \notin \mathcal{C}_d$ and $i+n+1 \notin \mathcal{C}_d$.

The idea with these definitions is that a claimed vertex that is not high is in some sense close to a high vertex and might therefore be used for the bad stubs at the high vertices to connect to. Indeed, in the model that we will soon propose for connecting the stubs, bad stubs are always connected within the claimed cluster that their vertex belongs to. To make sure that this can be done without creating multiple edges we need to see that the total number of bad stubs in a claimed cluster is strictly smaller than the number of vertices in the cluster. Hence, for a given subset A of $\mathbb{Z}$, let $b(A)$ be the number of bad stubs at vertices in A, that is,

$$b(A) = \sum_{i \in A} D_i^d.$$

Then the following holds:

Lemma 3.1 *For each cluster C in $\mathcal{C}_d$, we have $b(C) \leq |C| - 1$.*

Proof: Consider a given claimed cluster $C = [i+1, i+n]$ and assume that $b(C) \geq |C|$, that is, $b(C) \geq n$. We then have for the vertex i next to the left endpoint of the cluster that

$$\sum_{k=i-n}^{k=i+n} D_i^d \geq b(C) \geq n$$

so that hence i is claimed as well, which is a contradiction. □

We are now ready to describe the refined model. The model requires that the claimed clusters are almost surely finite. This will indeed be the case if d is large, as follows from Proposition

4.1 in the next section (which stipulates that the expected cluster size is finite for large d). For now we take this for granted. Hence, fix d large enough to ensure that the clusters are finite and, to make step 2 below easier, assume without loss of generality that d is even.

1. Bad stubs are connected within the claimed clusters. In a cluster with only one single high vertex i, this is done by choosing a random subset of D_i^d low vertices in the cluster, pick one stub from each of these vertices and connect to a bad stub of i. This is indeed possible, since, by Lemma 3.1, there are at least D_i^d low vertices in the cluster (in fact, at least $2D_i^d$) and, by assumption, there is at least one stub at each vertex. If there is more than one high vertex in a cluster C, the bad stubs are connected as follows.

   (i) Write $h = h(C)$ for the number of high vertices in C and let these high vertices be denoted by $i_1, \dots, i_h$ (ordered from the left to the right). First we use some of the bad stubs to create edges between high vertices: If h is even, consider the pairs $(i_1, i_2), \dots, (i_{h-1}, i_h)$ and connect the two vertices in each pair by using one bad stub from each vertex. If h is odd, with probability 1/2, leave the last high vertex i_h out and connect the pairs $(i_1, i_2), \dots (i_{h-2}, i_{h-1})$, and, with probability 1/2, leave the first high vertex i_1 out and connect the pairs $(i_2, i_3), \dots (i_{h-1}, i_h)$. This means that one bad stub from each high vertex (except i_h or i_1) is connected if h is even (odd). In any case, at least $h - 1$ bad stubs are used.

   (ii) Let $b_r(C)$ denote the number of remaining unconnected bad stubs in C after step (i). By Lemma 3.1, the total number of bad stubs in C is at most $|C| - 1$, implying that $b_r(C) \leq |C| - h$, that is, the number of unconnected bad stubs in C does not exceed the number $|C| - h$ of low vertices. Hence, to connect the remaining bad stubs, chose randomly $b_r(C)$ low vertices, take one stub from each of these vertices and pair these stubs randomly with the bad stubs.

2. In this step, the good stubs at the high vertices $\{i_j\}$ are connected to each other. Each high vertex has d good stubs attached to it and, for a given high vertex i_j, these stubs are linked to good stubs at the vertices $i_{j+2}, \dots, i_{j+d/2+1}$ and $i_{j-2}, \dots, i_{j-d/2-1}$, that is, half of the good stubs are pointed to the left and half of them to the right. Consecutive high vertices might already have an edge between them from step 1(i) and therefore, to avoid multiple edges, we do not connect i_j to i_{j-1} or i_{j+1}.

3. The remaining stubs at low vertices are connected according to the CT-model. There are no edges between low vertices from the previous steps and hence multiple edges will not arise here.

This model will be referred to as the *cluster model*. The next task is to show that the mean total edge length per vertex is finite provided the degree distribution has finite second moment.

4 The mean total edge length per vertex

The truncation level d is of course important for the properties of the cluster model. Write T^d for the total length of all edges at the origin for a given value of d. The aim in this section is to prove the following theorem.

Theorem 4.1 *If F has finite second moment, then, for large d, we have $\mathbb{E}[T^d] < \infty$ in the cluster model.*

A large part of the work in proving this theorem lies in showing that the expected size of a claimed clusters is finite if d is large. Since the bad stubs are connected within the claimed clusters, this ensures that the expected length of the edges created by the bad stubs is finite. For technical reasons, we will need a slightly more general result. To formulate it, first generalize the definition of a claimed vertex to incorporate a parameter α.

Definition 4.1 *A vertex is α-claimed on level d iff $\sum_{k=i-m}^{i+m} D_k^d \geq \alpha m$ for some $m \geq 1$.*

Given a stub configuration $\{D_i\}_{i\in\mathbb{Z}}$, we can now talk about clusters of α-claimed vertices. Let $C^{d,\alpha}$ be the α-claimed cluster of the origin. Also, let $\mu_d = \mathbb{E}[D_i^d]$, that is, μ_d is the expected number of stubs above level d at a given vertex. Clearly $\mu_d \to 0$ as $d \to \infty$ and hence, by picking d large, we can make μ_d arbitrarily small. The result concerning the expected cluster size now runs as follows.

Proposition 4.1 *Fix $\alpha > 0$. If d is large enough to ensure that $\mu_d < \alpha/18$, then $\mathbb{E}\left[|C^{d,\alpha}|\right] < \infty$.*

Proof: We will show that

$$\left\{|C^{d,\alpha}| \geq n\right\} \subset \left\{\exists m \geq n : \sum_{k=-m}^{m} D_k^d \geq \frac{\alpha m}{6}\right\}. \tag{1}$$

With $\widetilde{D}_k^d = D_k^d - \mu_d$, this implies that

$$\begin{aligned}\left\{|C^{d,\alpha}| \geq n\right\} &\subset \left\{\exists m \geq n : \sum_{k=-m}^{m} \widetilde{D}_k^d \geq \frac{\alpha m}{6} - (2m+1)\mu_d\right\} \\ &\subset \left\{\exists m \geq n : \sum_{k=-m}^{m} \widetilde{D}_k^d \geq c_{d,\alpha} m\right\},\end{aligned}$$

where $c_{d,\alpha} := \alpha/6 - 3\mu_d > 0$ (to get the last inclusion, we have used that $2m+1 \leq 3m$). Hence

$$\begin{aligned}P\big(|C^{d,\alpha}| \geq n\big) &\leq P\left(\exists m \geq n : \sum_{k=-m}^{m} \widetilde{D}_k^d \geq c_{d,\alpha} m\right) \\ &= P\left(\sup_{m\geq n} \frac{1}{m}\sum_{k=-m}^{m} \widetilde{D}_k^d \geq c_{d,\alpha}\right),\end{aligned}$$

and consequently

$$\mathbb{E}\big[|C^{d,\alpha}|\big] \leq \sum_{n=1}^{\infty} P\left(\sup_{m\geq n}\left(\frac{1}{m}\sum_{k=-m}^{m} \widetilde{D}_k^d\right) \geq c_{d,\alpha}\right).$$

By a standard result on convergence rate in the law of large numbers from Baum and Katz (1965; Theorem 3 with $t = r = 2$), the sum on the right hand side is convergent iff the $\widetilde{D}_k^d$'s have finite second moment. This proves the proposition.

It remains to show (1). To this end, for $i \in C^{d,\alpha}$, let $\mathcal{I}_i$ be the shortest interval around i where the condition for i to be α-claimed is satisfied. More precisely, if

$$m_i = \inf\Big\{m : \sum_{k=i-m}^{i+m} D_k^d \geq \alpha m\Big\},$$

we have $\mathcal{I}_i = [i - m_i, i + m_i]$. We now claim that we can pick a subset $\{\mathcal{I}_{i_j}\}_{i_j \in C^{d,\alpha}}$ of these intervals that completely covers the cluster ($C^{d,\alpha} \subset \cup_j \mathcal{I}_{i_j}$) and where only consecutive intervals intersect ($\mathcal{I}_{i_j} \cap \mathcal{I}_{i_k} = \emptyset$ if $|j-k| \geq 2$). To construct this subsequence, let $\mathcal{I}_{i_1}$ be the interval in $\{\mathcal{I}_i\}_{i\in C^{d,\alpha}}$ that reaches furthest to the left, that is, $\mathcal{I}_{i_1}$ is the interval with $l = \inf\{k : k \in \cup_{i\in C^{d,\alpha}} \mathcal{I}_i\}$ as its left endpoint. If there is more than one interval in $\{\mathcal{I}_i\}_{i\in C^{d,\alpha}}$ with l as its left endpoint, we take $\mathcal{I}_{i_1}$ to be the largest one. Next, consider the set S_1 of intervals $\mathcal{I}_k$ with $k \in C^{d,\alpha}$ that intersect $\mathcal{I}_{i_1}$ and define $\mathcal{I}_{i_2}$ to be the interval in S_1 that reaches furthest to the right. If there is more than one interval in S_1 that ends at the same maximal right endpoint, we let $\mathcal{I}_{i_2}$ be the largest of those intervals. Let S_2 be the set of intervals that intersect $\mathcal{I}_{i_2}$. The interval $\mathcal{I}_{i_3}$ is set to be the member in S_2 that reaches furthest to the right and, as before, if there is more than one candidate, we pick the largest one. This interval cannot intersect $\mathcal{I}_{i_1}$, since then it would have been chosen already in the previous step when $\mathcal{I}_{i_2}$ was defined. In general, given $\mathcal{I}_{i_1}, \ldots, \mathcal{I}_{i_j}$ the interval $\mathcal{I}_{i_{j+1}}$ is defined as follows.

(i) Let $S_j = \{\mathcal{I}_i : i \in C^{d,\alpha} \text{ and } \mathcal{I}_i \cap \mathcal{I}_{i_j} \neq \emptyset\}$ and write $r_j = \sup\{k : k \in S_j\}$.

(ii) Take $\mathcal{I}_{i_{j+1}}$ to be the largest interval in S_j with its right endpoint at the vertex r_j.

We repeat this procedure until an interval $\mathcal{I}_{i_s}$ whose right endpoint is outside the cluster $C^{d,\alpha}$ is picked. The entire cluster is then covered by $\cup_{j=1}^{s} \mathcal{I}_{i_j}$ and, by construction, non-consecutive intervals do not intersect, as desired.

Now let $A = \cup_{k\geq 1} \mathcal{I}_{i_{2k-1}}$ and $B = \cup_{k\geq 1} \mathcal{I}_{i_{2k}}$, that is, every second interval in $\{\mathcal{I}_{i_j}\}$ is placed in A and every second interval is placed in B. Then A and B are both unions of mutually disjoint intervals. Remember that $b(\cdot)$ denotes the number of bad stubs in a given set and note that, by the definition of badness, for a given interval $\mathcal{I}_{i_j}$, we have $b(\mathcal{I}_{i_j}) \geq \alpha|\mathcal{I}_{i_j}|/3$. Since the intervals in A are disjoint, it follows that $b(A) \geq \alpha|A|/3$, and similarly, $b(B) \geq \alpha|B|/3$. Hence

$$\begin{aligned} b(A \cup B) &\geq \max\{b(A), b(B)\} \\ &\geq \frac{\alpha}{3}\max\{|A|, |B|\} \\ &\geq \frac{\alpha}{6}|A \cup B|. \end{aligned}$$

With $m = |A \cup B|$, we have

$$\begin{aligned} b([-m, m]) &\geq b(A \cup B) \\ &\geq \frac{\alpha}{6}|A \cup B| \\ &= \frac{\alpha m}{6}. \end{aligned}$$

This establishes (1). □

We are now ready to prove Theorem 4.1. The proof is based on Proposition 4.1, which ensures that the expected length of the edges created by the bad stubs in step 1 in the description of the cluster model is finite, and 2.2, which guarantees that the edges created in step 2 and 3 have finite mean.

Proof of Theorem 4.1: Write T_1^d, T_2^d and T_3^d for the total length of the edges created at the origin in step 1, 2 and 3 respectively in the description of the cluster model.

First we attack T_1^d. To this end, given a stub configuration $\{D_i\}_{i\in\mathbb{Z}}$, write $C_i^{d,\alpha}$ for the α-claimed cluster of the vertex i. Now, T_1^d is the total length of all edges created by bad stubs at the origin. The number of such edges is clearly smaller than the total number D of edges at the origin and they are all connected within the claimed cluster of the origin. Hence $T_1^d \le D|C_0^{d,1}|$ (the cluster model is based on $\alpha = 1$). If D and $|C_0^{d,1}|$ were independent it would follow immediately from Proposition 4.1 that $\mathrm{E}[T_1^d] < \infty$ for large d. However, D and $|C_0^{d,1}|$ are of course not independent. To get around this, introduce a coupled degree configuration $\{\widehat{D}_i\}_{i\in\mathbb{Z}}$ where $\widehat{D}$ is generated independently, while $\widehat{D}_i = D_i$ for $i \ne 0$. Quantities based on $\{\widehat{D}_i\}_{i\in\mathbb{Z}}$ will be equipped with a hat-symbol. We will show that

$$\big|C_0^{d,1}\big| \le 4D + \big|\widehat{C}_{-2D}^{d,1/2}\big| + \big|\widehat{C}_{2D}^{d,1/2}\big|. \tag{2}$$

Since $\widehat{C}_i^{d,1/2}$ is clearly independent of D for all i, this implies that

$$\begin{aligned}\mathbb{E}[T_1^d] &\le \mathbb{E}\left[D\left(4D + \big|\widehat{C}_{-2D}^{d,1/2}\big| + \big|\widehat{C}_{2D}^{d,1/2}\big|\right)\right]\\ &= 4\mathbb{E}\left[D^2\right] + 2\mathbb{E}\left[D\right]\cdot\mathbb{E}\left[\big|\widehat{C}_{2D}^{d,1/2}\big|\right].\end{aligned}$$

If F has finite second moment and d is large so that $\mu_d \le 1/36$, then, by Proposition 4.1, we have $\mathrm{E}\big[\big|\widehat{C}_{2D}^{d,1/2}\big|\big] < \infty$. It follows that $\mathrm{E}[T_1^d]$ is finite under the same conditions.

To establish (2), it suffices to observe that each vertex $i \notin [-2D-1, 2D+1]$ that is claimed for $\alpha = 1$ in the original configuration $\{D_i\}$ is still claimed for $\alpha = 1/2$ in the coupled configuration $\{\widehat{D}_i\}$. Hence, pick a vertex i with $|i| \ge 2D$ that is claimed for $\alpha = 1$ in the original configuration. Write $\mathcal{I}_i^m = [i-m, i+m]$ for the smallest interval such that $b(\mathcal{I}_i^m) \ge m$ and assume that $m \ge 2D$ so that hence $0 \in \mathcal{I}_m$ (if this is not the case, i is obviously claimed for $\alpha = 1$ also in the coupled configuration $\{\widehat{D}_i\}$, since $\widehat{D}_i = D_i$ for all $i \ne 0$). For such i, we have

$$\begin{aligned}\widehat{b}(\mathcal{I}_i^m) &\ge b(\mathcal{I}_i^m) - D\\ &\ge m - D\\ &\ge m/2,\end{aligned}$$

meaning that i is claimed for $\alpha = 1/2$ in $\{\widehat{D}_i\}$, as desired.

Next, consider the total length T_2^d of the edges created at the origin in step 2, where good stubs at high vertices are connected. If the origin is not high, that is, if $D \le d$, then clearly $T_2^d = 0$. Hence assume that $D \ge d+1$. Then d edges will be created at the origin in step 2 – half of them will point to the right and half of them to the left. The longest edge to the right (left) runs to vertex number $2d+1$ to the right (left) of the origin with degree larger than or equal to $d + 1$. The distance to this vertex has a negative binomial distribution with finite mean, and it follows that T_2^d has finite mean.

All that remains is to see that the total length T_3^d of the edges created in step 3 – when the CT-model is applied to connect remaining stubs after steps 1 and 2 – has finite expectation. This however is an immediate consequence of Proposition 2.2, since, if D_i' denotes the number of stubs at vertex i that are not connected after steps 1 and 2, then $D_i' \leq d$ and $\{D_i'\}$ is clearly a stationary sequence.

To sum up, we have shown that $\mathbb{E}[T] = \mathbb{E}[T_1^d + T_2^d + T_3^d] < \infty$, as desired. $\square$

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