MARTIN BOUNDARY OF A FRACTAL DOMAIN

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Dedicated to Professors Masayuki Itoh and Hisako Watanabe
on the occasion of their 60th birthdays

Abstract. A uniformly John domain is a domain intermediate between a John domain and a uniform domain. We determine the Martin boundary of a uniformly John domain D as an application of a boundary Harnack principle. We show that a certain self-similar fractal has its complement as a uniformly John domain. In particular, the complement of the 3-dimensional Sierpiński gasket is a uniform domain and its Martin boundary is homeomorphic to the Sierpiński gasket itself.

1. Introduction

In the previous paper [1] the first author proved a uniform boundary Harnack principle for a bounded uniform domain. As a result, it is shown that the Martin boundary of a bounded uniform domain is homeomorphic to the Euclidean boundary. In this paper, we shall study more general domains, mainly uniformly John domains introduced by Balogh and Volberg [5, 6]. A uniformly John domain is a domain intermediate between a John domain and a uniform domain. In the first part we shall establish a certain uniform boundary Harnack principle for a uniformly John domain. Its Martin boundary will be determined as a corollary to the boundary Harnack principle. The Martin boundary is no longer expected to be homeomorphic to the Euclidean boundary. Instead, it will turn out to be homeomorphic to the ideal boundary with respect to the internal metric (See below). The second part of the paper deals with more concrete examples of John domains and uniformly John domains. We shall provide two axioms for a self-similar fractal which

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ensure that the complement of the fractal is a John domain, and two more axioms for a uniformly John domain. Among the axioms we have a certain *nesting axiom* which is similar to Lindstrøm's ramified condition in [18].

Let us begin with the definitions of a John domain, a uniform domain and a uniformly John domain. Throughout the paper, let D be a proper subdomain in $\mathbb{R}^d$, $d \geq 2$, and let $\delta_D(x) = \mathrm{dist}(x, \partial D)$. We say that D is a John domain if there are $x_0 \in D$ (John center) and $A_1 \geq 1$ (John constant) such that each $x \in D$ can be connected to x_0 by a rectifiable curve $\gamma \subset D$ with

$$\ell(\gamma(x,z)) \leq A_1 \delta_D(z) \quad \text{for all } z \in \gamma, \tag{1.1}$$

where $\gamma(x,z)$ is the subarc of γ from x to z and $\ell(\gamma(x,z))$ is the length of $\gamma(x,z)$. We say that D is a uniform domain if there exists $A_2 \geq 1$ (uniform constant) such that each pair of points $x, y \in D$ can be connected by a rectifiable curve $\gamma \subset D$ for which

$$\min\{\ell(\gamma(x,z)), \ell(\gamma(z,y))\} \leq A_2 \delta_D(z) \quad \text{for all } z \in \gamma, \tag{1.2}$$

$$\ell(\gamma) \leq A_2 |x - y|. \tag{1.3}$$

We note that (1.3) is regarded as the bounded turning condition of γ (cf. [21]). Apparently, a uniform domain is a John domain.

In connection with conformal dynamics, Balogh and Volberg [5, 6] introduced a uniformly John domain. It is a domain intermediate between a John domain and a uniform domain. Let us give the definition. First we define the internal metric $\rho_D(x,y)$ by

$$\rho_D(x,y) = \inf\{\mathrm{diam}(\gamma) : \ \gamma \text{ is a curve connecting } x \text{ and } y \text{ in } D\}$$

for $x, y \in D$. Here $\mathrm{diam}(\gamma)$ denotes the diameter of γ. Obviously $|x - y| \leq \rho_D(x,y)$. We say that D is a uniformly John domain if there exists a constant $A_3 \geq 1$ (uniform John constant) such that each pair of points $x, y \in D$ can be connected by a curve $\gamma \subset D$ for which

$$\min\{\ell(\gamma(x,z)), \ell(\gamma(z,y))\} \leq A_3 \delta_D(z) \quad \text{for all } z \in \gamma, \tag{1.4}$$

$$\ell(\gamma) \leq A_3 \rho_D(x,y). \tag{1.5}$$

By definition

$$\text{uniform} \subsetneqq \text{uniformly John} \subsetneqq \text{John}.$$

The difference between a uniform domain and a uniformly John domain arises from the difference between the right hand sides of (1.3) and (1.5). One may say that a uniform domain is a uniformly John domain with internal metric satisfying $\rho_D(x,y) \le A|x-y|$ for $x, y \in D$ with positive constant A.

Let us illustrate the above definitions by a Denjoy domain, the complement of a closed set in a hyperplane. By $B(x,r)$ we denote the open ball with center at x and radius r. We identify the hyperplane $\{(x_1, \dots, x_d) \in \mathbb{R}^d : x_d = 0\}$ with $\mathbb{R}^{d-1}$. By $B'(x,r)$ we denote the $(d-1)$-dimensional ball with center at x and radius r, i.e., $B'(x,r) = \mathbb{R}^{d-1} \cap B(x,r)$, for $x \in \mathbb{R}^{d-1}$. Let E be a closed set in $\mathbb{R}^{d-1}$ such that $D = B(0,1) \setminus E$ is connected, i.e. $B'(0,1) \setminus E \neq \emptyset$. We call D is a (bounded) Denjoy domain. We have the following criteria for D.

Proposition 1.1. *Let E and D be as above. Then we have the following:*

(i) *D is a John domain.*

(ii) *D is a uniformly John domain if and only if there are $\alpha > 0$ and $r_0 > 0$ such that*

$$\sup_{z \in B'(x,r) \cap B'(0,1)} \delta_D(z) \ge \alpha r \quad \text{for } 0 < r < r_0 \tag{1.6}$$

whenever $x \in B'(0,1) \setminus E$.

(iii) *D is a uniform domain if and only if there are $\alpha > 0$ and $r_0 > 0$ such that* (1.6) *holds whenever $x \in B'(0,1)$.*

It is well-known that a bounded Lipschitz domain, and more generally a bounded NTA domain, have the Martin compactification homeomorphic to the Euclidean closure (Hunt and Wheeden [16], Jerison and Kenig [17]). In the previous paper [1], the first author showed that the Martin compactification of a bounded uniform domain is homeomorphic to the Euclidean closure. This gives an alternative proof of the results of Hunt-Wheeden and Jerison-Kenig, since a Lipschitz domain and an NTA domain are uniform domains.

The Martin compactification of a uniformly John domain is more complicated. We shall show that it is homeomorphic to the completion D^* with respect to the internal metric. That is, D^* is the equivalence class of all ρ_D-Cauchy sequences with equivalence relation "$\sim$", where we say $\{x_j\} \sim \{y_j\}$ if $\{x_j\} \cup \{y_j\}$ is a ρ_D-Cauchy sequence. Let $\partial^* D = D^* \setminus D$, the boundary with respect to ρ_D. Take $\xi^* \in D^*$. Suppose ξ^* is represented by a ρ_D-Cauchy sequence $\{x_j\}$. Since $\{x_j\}$ is also a usual Cauchy sequence, it follows that x_j converges to some point $\xi \in \overline{D}$. The point ξ is independent of the representative

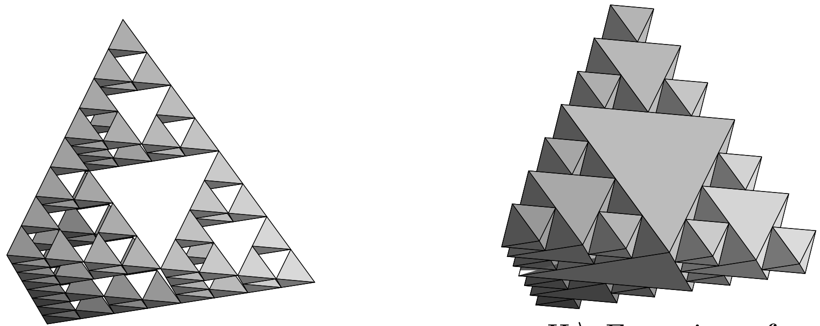
$\{x_j\}$ and uniquely determined by ξ^*. We say that ξ^* lies over $\xi \in \overline{D}$. If $\xi \in D$, then ξ and ξ^* coincide. We say that $\xi \in \partial D$ is a simple boundary point if there is exactly one boundary point of $\partial^* D$ over ξ. In other words, ξ is a simple boundary point if and only if every sequence $\{x_j\} \subset D$ converging to ξ also converges to the same boundary point with respect to the internal metric ρ_D. Define the projection $\pi : D^* \to \overline{D}$ by $\pi(\xi^*) = \xi$. It is easy to see that π is a continuous contraction mapping, i.e. $|\pi(\xi_1^*) - \pi(\xi_2^*)| \le \rho_D(\xi_1^*, \xi_2^*)$. If ξ is a simple point, we identify ξ and the point over ξ in $\partial^* D$ and write $\pi(\xi) = \xi$.

One of the main results of this paper is the following theorem.

Theorem 1.2. *Let D be a bounded uniformly John domain with uniform John constant A_4. Then the Martin compactification of D is homeomorphic to D^* and each boundary point $\xi^* \in \partial^* D$ is minimal. Moreover, for every boundary point $\xi \in \partial D$, the number of Martin boundary points over ξ is bounded by a constant depending only on A_4.*

The above theorem will be proved as a corollary to a uniform boundary Harnack principle for a uniformly John domain. Balogh and Volberg [6] proved a uniform boundary Harnack principle for a planar uniformly John domain with uniformly perfect boundary. Having a uniform perfect boundary is an additional assumption. In the present paper we assume neither the uniform perfectness of the boundary nor any other exterior conditions. Balogh and Volberg also demonstrated, in their setting, that the harmonic measure satisfies the doubling condition with respect to the internal metric [6, Theorem 3.1]. In the present setting, the harmonic measure needs not satisfy the doubling condition, because of the lack of exterior condition. This is a significant difference between [6] and the present paper. Moreover, we should remark that our domain may admit an irregular boundary point. Hence, we always consider a generalized Dirichlet problem, i.e. boundary values have meaning outside a polar set. For simplicity, we shall say that a property holds q.e. (quasi everywhere) if it holds outside a polar set.

Our second purpose is to give some axioms for a self-similar fractal such that the complement of the fractal becomes a John domain, or a uniformly John domain. See Theorems 4.16 and 5.3 below. One of our conditions is a *nesting axiom* which is similar to Lindstrøm's [18]. A typical example of self-similar fractals satisfying our axioms is the 3-dimensional Sierpiński gasket. Consider a tetrahedron H and four similarities each of which is a composition of a translation and a dilation with fixed point at a vertex of H. The 3-dimensional Sierpiński gasket F is given as the fixed set of the above four similarities. We see that $\mathrm{int}(F)=\emptyset$ and that $H\setminus F$ consists of octahedra. See Figure 1.1. Let B be an open ball containing H. We shall show that $B\setminus F$ is a uniform domain



Third Step.

$H\setminus F$ consists of octahedra.

FIGURE 1.1. The 3-dimensional Sierpiński gasket F. $D=B\setminus F$ is a uniform domain.

and hence its Martin boundary coincides with $F\cup\partial B$ (Corollary 6.9). The connectivity among octahedra will play an important role. For details see Sections 4, 5 and 6 below.

Once we have obtained a uniformly John domain, then we can easily modify it to have another uniformly John domain. The following offers one of such modifications.

Proposition 1.3. *Let D be a bounded uniformly John domain. Then a domain $\widetilde{D}$ between D and $\mathrm{int}(\overline{D})$ such that $\widetilde{D}\setminus D$ consists of simple boundary points is a uniformly John domain. In particular, if D is a bounded uniform domain, then every domain $\widetilde{D}$ between D and $\mathrm{int}(\overline{D})$ is a uniform domain.*

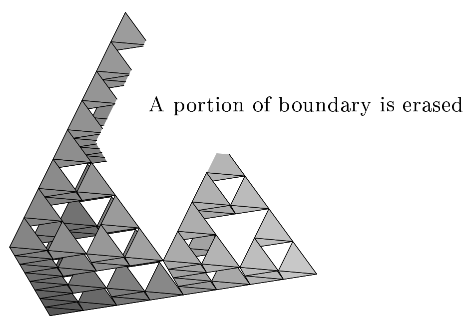


FIGURE 1.2. $\widetilde{D}=B\setminus F'$ is a uniform domain for any $F'\subset F$.

See Figure 1.2 for an example of the above 3-dimensional Sierpiński gasket F. In general, if a domain is given as the complement of a self-similar fractal, then its boundary enjoys a nice uniform condition because of the homogeneity of the fractal. By the above proposition we can add some part of the boundary to obtain a uniformly John domain without uniform exterior or boundary condition. Nevertheless, its Martin compactification is homeomorphic to D^* with the aid of Theorem 1.2.

The plan of the paper is as follows. In the next section we shall give several geometrical notions and properties of a uniformly John domain. In particular, Propositions 1.1 and 1.3 will be proved. In Section 3 we shall state the boundary Harnack principle (Theorem 3.1) and prove it along a line similar to [1]. Then Theorem 1.2 will be proved as its corollary. In Section 4 we shall state several notions and terminologies for self-similar fractals and their complements. Then Theorem 4.16 will give sufficient conditions for the complement of a self-similar fractal to be a John domain. It is much more difficult to show that a domain is a uniformly John domain than a John domain. Theorem 5.3 in Section 5 will give sufficient conditions for the complement of a self-similar fractal to be a uniformly John domain. As a corollary we shall observe in Section 6 that the complement of the 3-dimensional Sierpiński gasket is a uniform domain (Corollary 6.9).

We shall use the following notation. By the symbol A we denote an absolute positive constant whose value is unimportant and may change even in the same line. If necessary, we use $A_0, A_1, \dots,$ to specify them. We shall say that two positive functions f_1 and f_2 are comparable, written $f_1 \approx f_2$, if and only if there exists a constant $A \geq 1$ such that $A^{-1} f_1 \leq f_2 \leq A f_1$. The constant A will be called the constant of comparison. By $B(x,r)$, $C(x,r)$ and $S(x,r)$ we denote the open ball, the closed ball and the sphere with center at x and radius r, respectively.

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2. Geometric properties of a uniformly John domain

In view of [19, Lemma 2.7] and [21, Theorem 2.18], we observe that (1.4) and (1.5) are equivalent to

$$\min\{|x-z|,|z-y|\} \le A_4\delta_D(z) \quad \text{for all } z\in\gamma, \tag{2.1}$$

$$\operatorname{diam}(\gamma) \le A_4\rho_D(x,y) \tag{2.2}$$

with another positive constant A_4 depending only on A_3. For simplicity we call a curve satisfying (2.1) a *cigar curve* or more precisely *distance-cigar curve*. This terminology comes from the fact that the union

$$\bigcup_{z\in\gamma} B(z, A_4^{-1}\min\{|x-z|,|z-y|\})$$

of cigar like shape is included in D. On the other hand, a curve satisfying (1.1) is said to be a *carrot curve*. If a curve satisfies (1.2), then it is said to be a *length-cigar curve*.

Let us begin with the proof of Proposition 1.1. The proof is straightforward and may help the reader's understanding of the different classes of domains studied.

Proof of Proposition 1.1. We can easily show (i). Let us prove (ii). We assume (1.6) for $x\in B'(0,1)\setminus E$ and we are going to show that D is a uniformly John domain. Take arbitrary points $x=(x_1,\dots,x_d)$ and $y=(y_1,\dots,y_d)$ in D. If both x_d and y_d have the same sign, then we can easily construct a cigar curve $\widetilde{xy}$ connecting x and y in D with $\operatorname{diam}(\widetilde{xy}) \le A|x-y| = A\rho_D(x,y)$. Hence, we may assume that x_d and y_d have the different signs. Consider an arbitrary curve γ connecting x and y in D and let $r=\operatorname{diam}(\gamma)$. Then $0<r<2$ and γ must intersect $B'(0,1)\setminus E$ at some point $z\in B'(0,1)\setminus E$. If necessary taking $\alpha>0$ smaller, we may assume that $r_0>2$. By assumption we find a point $z^*\in B'(z,r)$ such that $\delta_D(z^*)\ge\alpha r$. We can easily construct cigar curves $\widetilde{xz^*}$ and $\widetilde{z^*y}$ connecting x to z^* and z^* to y such that $\operatorname{diam}(\widetilde{xz^*})\le Ar$ and $\operatorname{diam}(\widetilde{z^*y})\le Ar$, respectively. Let $\widetilde{\gamma}=\widetilde{xz^*}\cup\widetilde{z^*y}$. Then $\operatorname{diam}(\widetilde{\gamma})\le 2Ar$ and $\widetilde{\gamma}$ is a cigar curve in D by $\delta_D(z^*)\ge\alpha r$. Since γ is an arbitrary curve connecting x and y in D, it follows that D is a uniformly John domain.

Conversely, we suppose D is a uniformly John domain satisfying (2.1) and (2.2) and we are going to show that (1.6) holds for $r_0=1$ and $\alpha=\sqrt{3}/(8A_4^2)$ whenever $x\in B'(0,1)\setminus E$, where A_4 is the constant in (2.1) and (2.2). Fix $x\in B'(0,1)\setminus E$ and $0<r<1$. By an elementary geometrical observation we find a point $y \in B(0,1)$ such that

$$\frac{r}{4A_4} = |y-x| \le \frac{2}{\sqrt{3}} \operatorname{dist}(y, \mathbb{R}^{d-1}).$$

Let $\overline{y}$ be the reflection of y with respect to $\mathbb{R}^{d-1}$. Then y and $\overline{y}$ are connected by the union of the line segments from y to x and from x to $\overline{y}$, whose diameter is not greater than $r/(2A_4)$. Hence $\rho_D(y,\overline{y}) \le r/(2A_4)$. In view of (2.1) and (2.2), we find a cigar curve $\gamma \subset D$ connecting y and $\overline{y}$ such that $\operatorname{diam}(\gamma) \le r/2$ and

$$\min\{|y-z|, |z-\overline{y}|\} \le A_4 \delta_D(z) \quad \text{for all } z \in \gamma.$$

This curve γ must intersect $B'(0,1)$ at some point z_0, so that

$$\delta_D(z_0) \ge \frac{1}{A_4} \min\{|y-z_0|, |z_0-\overline{y}|\} \ge \frac{1}{A_4} \operatorname{dist}(y, \mathbb{R}^{d-1}) \ge \frac{\sqrt{3}r}{8A_4^2}.$$

Since $z_0 \in B'(x, |x-y| + \operatorname{diam}(\gamma)) \subset B'(x,r)$, we obtain (1.6). Thus the necessity part of (ii) is proved.

Finally we prove (iii). The proof of the sufficiency part is similar to that of (ii). In fact, take two points x and y in D with different signs of x_d and y_d. Instead of the curve connecting x and y in D, we simply consider the line segment $\overline{xy}$ and let z be the intersection of this line segment with $\mathbb{R}^{d-1}$. Since we assume that (1.6) holds for every point in $B'(0,1)$, it applies to this z and the same argument as for (ii) yields a required cigar curve $\widetilde{\gamma}$ connecting x and y in D. For the necessity part we suppose D is a uniform domain. Then D is a uniformly John domain in particular, and hence by (ii) there are $\alpha > 0$ and $r_0 > 0$ such that (1.6) holds for every point in $B'(0,1) \setminus E$. Since the internal metric and the Euclidean metric are comparable, E cannot include a relatively open set in $\mathbb{R}^{d-1}$. Hence $B'(0,1)$ is included in the closure of $B'(0,1) \setminus E$, so that (1.6) actually holds for every point in $B'(0,1)$. The proof is complete. $\square$

Balogh and Volberg [5] proved a very deep property of a planar uniformly John domain; a geometric localization. In the course of the proof of Theorem 1.2 we shall not use their result. Instead, we shall need some elementary properties of a uniformly John domain. The purpose of this section is to show these properties with purely geometrical proofs. No potential theory will be involved in this section. Let us first show that the completion D^* is a compact space. This property holds even for a bounded John domain.

Proposition 2.1. *Let D be a bounded John domain. Then D^* is a compact space and each boundary point $\xi^* \in \partial^* D$ is accessible from D, i.e., there is an arc $\gamma \subset D$ converging to ξ^*. Moreover, for every boundary point $\xi \in \partial D$, the number of points in $\partial^* D$ over ξ is bounded by a constant depending only on the John constant A_1.*

Proof. Take a sequence $\{x_m^*\}$ in D^*. We need to show that there exists a subsequence of $\{x_m^*\}$ converging to some point in D^* with respect to ρ_D. Suppose that each x_m^* is represented by a ρ_D-Cauchy sequence $\{x_m^j\}_j \subset D$. Since $\{x_m^j\}_j$ is also a usual Cauchy sequence, it must converge to $x_m = \pi(x_m^*) \in \overline{D}$ with respect to the usual metric. Taking a subsequence, if necessary, we may assume that $\{x_m\}_m$ is a Cauchy sequence converging to some $\xi \in \overline{D}$ with respect to the usual metric. If $\xi \in D$, then it is easy to show that x_m^* converges to ξ with respect to ρ_D. So, we may assume that $\xi \in \partial D$.

Let $r > 0$ be so small that the John center x_0 lies outside $B(\xi, r)$. Observe that $D \cap B(\xi, r)$ consists of countably many open connected components $B_i(r)$. Obviously

$$\rho_D(x, y) \le 2r \quad \text{for } x, y \in B_i(r). \tag{2.3}$$

Let us count the number $\nu(r)$ of components $B_i(r)$ having a point x_m with $|x_m - \xi| < r/2$. We claim that

$$\nu(r) \le N, \tag{2.4}$$

where the number N depends only on the John constant A_1. By definition x_m is connected to x_0 by a carrot curve γ satisfying (1.1). Hence it follows from the definition of a John domain that the Lebesgue measure of $B_i(r)$ is comparable to r^d with constant of comparison depending only on the John constant A_1. Therefore, (2.4) holds.

Now let $r_k = 2^{-k} \downarrow 0$. Then we infer from (2.4) that there exists a decreasing sequence of components $B_{i_k}(r_k)$ each of which contains infinitely many x_m. We find $\xi^* \in \partial^* D$ such that

$$B_{i_1}(r_1) \supset B_{i_2}(r_2) \supset \cdots \to \xi^* \in \partial^* D,$$

and a subsequence of $\{x_m^*\}$ converges along $\{B_{i_k}(r_k)\}$ to ξ^* with respect to ρ_D by (2.3). Obviously $\pi(\xi^*) = \xi$. This shows that D^* is compact and ξ^* is accessible from D. Moreover, the second assertion follows, since every point on $\partial^* D$ has a ρ_D-Cauchy sequence converging to it.

Finally let $\xi \in \partial D$ and suppose k distinct points $\xi_1^*, \dots \xi_k^* \in \partial^* D$ lie over ξ. Then there is an $\varepsilon > 0$ such that $\rho_D(\xi_i^*, \xi_j^*) > 2\varepsilon$ for $i \ne j$. By V_i we denote the component of

$D \cap B(\xi, \varepsilon)$ from which ξ_i^* is accessible. Then $V_1, \dots, V_k$ are distinct. In fact, if $V_i = V_j$ for some $i \neq j$, then ξ_i^* and ξ_j^* would be accessible from the same component. That is, there would be an arc γ in $V_i = V_j$ connecting ξ_i^* and ξ_j^*. By definition, $\rho_D(\xi_i^*, \xi_j^*) \le \operatorname{diam}(\gamma) \le 2\varepsilon$; a contradiction would arise. Thus $V_1, \dots, V_k$ are distinct and hence disjoint by definition. We may assume that the John center x_0 lies outside $B(\xi, \varepsilon)$. Then each ξ_i^* can be connected to x_0 by a carrot curve, say γ_i, in D with (1.1). Let $x_i \in \gamma_i \cap V_i \cap S(\xi, \varepsilon/2)$. Then $B(x_i, \varepsilon/(2A_1)) \subset V_i$ by (1.1), so that the Lebesgue measure of V_i is comparable to ε^d. Since $V_1, \dots, V_k$ are disjoint subsets of $B(\xi, \varepsilon)$, it follows that the number k is bounded by a constant depending only on the John constant A_1. The proof is complete. $\square$

Remark 2.2. In general, a minimal boundary point of the Martin boundary is accessible from the domain (e.g. [12, Satz 13.3]). Hence, if we have shown Theorem 1.2, the above proposition follows automatically. The above argument proves the accessibility without potential theoretic consideration. We also note that there is a bounded John domain having non minimal Martin boundary point. Such a domain can be easily constructed as a Denjoy domain. See Ancona [3, 4], Benedicks [8], Chevallier [11], Segawa [20] and references therein.

Hereafter we let D be a bounded uniformly John domain with uniform John constant A_4. We extend $\rho_D(x, y)$ for $x, y \in D^*$ by $\rho_D(x, y) = \lim \rho_D(x_j, y_j)$ if x and y are represented by ρ_D-Cauchy sequences $\{x_j\}$ and $\{y_j\}$ in D. It is easy to see that $\rho_D(x, y)$ is independent of the choice of the ρ_D-Cauchy sequences $\{x_j\}$ and $\{y_j\}$. The connectivity given by (2.1) and (2.2) also extends to points in D^*.

Lemma 2.3. *Every pair of points $x, y \in D^*$ can be connected by a curve γ for which $\gamma \setminus \{x, y\} \subset D$ and*

$$\min\{|\pi(x) - z|, |z - \pi(y)|\} \le A\delta_D(z) \quad \text{for all } z \in \gamma, \tag{2.5}$$

$$\operatorname{diam}(\gamma) \le A\rho_D(x, y), \tag{2.6}$$

where A is a constant depending only on the uniform John constant A_4 for D.

Proof. If both x and y are points in D, then there is nothing to prove. Let us assume that $x \in D$ and $y \in \partial^* D$. In view of Proposition 2.1 we find a sequence $\{y_j\} \subset D$ converging to y with respect to ρ_D. Each point y_j can be connected to the John center x_0 by a carrot curve, on which we find points y_j' such that $\rho_D(y_j', y) \to 0$ and $|y_j' - \pi(y)| \le A\delta_D(y_j')$.

Hence, we may assume, from the beginning, that

$$\delta_D(y_j) \le |y_j - \pi(y)| \le A\delta_D(y_j), \tag{2.7}$$

where $A > 1$ is a constant depending only on A_4. Moreover, taking a subsequence, if necessary, we may assume that

$$\rho_D(x,y) \ge 2\rho_D(y,y_1) \ge 2^2\rho_D(y,y_2) \ge \cdots \tag{2.8}$$

By definition we find cigar curves $\widetilde{xy_1}$ such that

$$\begin{aligned} &\min\{|x-z|, |z-y_1|\} \le A_4\delta_D(z) \quad \text{for all } z \in \widetilde{xy_1}, \\ &\operatorname{diam}(\widetilde{xy_1}) \le A_4\rho_D(x,y_1) \end{aligned} \tag{2.9}$$

and $\widetilde{y_jy_{j+1}}$ such that

$$\begin{aligned} &\min\{|y_j-z|, |z-y_{j+1}|\} \le A_4\delta_D(z) \quad \text{for all } z \in \widetilde{y_jy_{j+1}}, \\ &\operatorname{diam}(\widetilde{y_jy_{j+1}}) \le A_4\rho_D(y_j,y_{j+1}) \end{aligned} \tag{2.10}$$

for $j = 1, 2, \dots$. We claim that

$$\gamma = \widetilde{xy_1} \cup \widetilde{y_1y_2} \cup \cdots \cup \widetilde{y_jy_{j+1}} \cup \cdots$$

is a required curve connecting x and y. We have from (2.8), (2.9) and (2.10)

$$\begin{aligned} \operatorname{diam}(\gamma) &\le A_4(\rho_D(x,y_1) + \sum_{j=1}^{\infty}\rho_D(y_j,y_{j+1})) \\ &\le A_4\left(\rho_D(x,y) + \rho_D(y,y_1) + \sum_{j=1}^{\infty}(\rho_D(y,y_j) + \rho_D(y,y_{j+1}))\right) \\ &\le 3A_4\rho_D(x,y). \end{aligned}$$

Thus (2.6) holds.

Let us prove (2.5). First examine (2.5) for $z \in \widetilde{xy_1}$. If $|z-y_1| \le \frac{1}{2}\delta_D(y_1)$, then $\delta_D(z) \ge \frac{1}{2}\delta_D(y_1)$ and

$$|z-\pi(y)| \le |z-y_1| + |y_1-\pi(y)| \le \frac{1}{2}\delta_D(y_1) + A\delta_D(y_1) \le (1+2A)\delta_D(z)$$

by (2.7), so that (2.5) holds in this case. If $|z-y_1| \ge \frac{1}{2}\delta_D(y_1)$, then

$$|z-\pi(y)| \le |z-y_1| + |y_1-\pi(y)| \le (1+2A)|z-y_1|$$

by (2.7), so that (2.9) yields

$$A_4\delta_D(z) \ge \frac{1}{1+2A}\min\{|x-z|, |z-\pi(y)|\}.$$

Thus (2.5) holds for all $z \in \widetilde{xy_1}$. Now, we examine (2.5) for $z \in \widetilde{y_j y_{j+1}}$. If $|z - y_j| \le \frac{1}{2}\delta_D(y_j)$, then $\delta_D(z) \ge \frac{1}{2}\delta_D(y_j)$ and $|z - \pi(y)| \le (1 + 2A)\delta_D(z)$, so that (2.5) holds in the same way as above. Similarly, (2.5) holds if $|z-y_{j+1}| \le \frac{1}{2}\delta_D(y_{j+1})$. If $|z-y_j| \ge \frac{1}{2}\delta_D(y_j)$ and $|z-y_{j+1}| \ge \frac{1}{2}\delta_D(y_{j+1})$, then $|z-\pi(y)| \le (1+2A)|z-y_j|$ and $|z-\pi(y)| \le (1+2A)|z-y_{j+1}|$ by (2.7), so that (2.10) yields (2.5). Thus (2.5) holds for all $z \in \gamma$.

Finally, in the case when $x, y \in \partial^* D$, we take a sequence $\{x_j\}$ converging to x. Then the same argument as above to x_j yields a required curve. The proof is complete. □

We shall define 'balls' with respect to the internal metric. For this purpose it is convenient to modify the internal metric slightly. For $x \in D$ and $\gamma \subset D$ we let

$$\widehat{r}(x, \gamma) = \sup_{z \in \gamma} |z - x|,$$

i.e., the the infimum of radii r for which $\gamma \subset B(x, r)$. Observe that $\widehat{r}(x, \gamma) \le \operatorname{diam}(\gamma) \le 2\widehat{r}(x, \gamma)$ for $x \in \gamma$. Let

$$\widehat{\rho}_D(x, y) = \inf\{\widehat{r}(x, \gamma) : \ \gamma \text{ is a curve connecting } x \text{ and } y \text{ in } D\}$$

for $x, y \in D$. The quantity $\widehat{\rho}_D$ is not symmetric. It is related to the internal metric ρ_D as follows:

$$\widehat{\rho}_D(x, y) \le \rho_D(x, y) \le 2\widehat{\rho}_D(x, y).$$

Therefore the convergence with respect to ρ_D is equivalent to the convergence with respect to $\widehat{\rho}_D$. We can also show the following inequalities

$$\widehat{\rho}_D(x, z) \le \widehat{\rho}_D(x, y) + \widehat{\rho}_D(y, z),$$
$$\widehat{\rho}_D(x, z) \le \widehat{\rho}_D(x, y) + 2\widehat{\rho}_D(z, y)$$

for $x, y, z \in D$. We extend $\rho_D(x, y)$ and $\widehat{\rho}_D(x, y)$ for $x, y \in D^*$ by $\rho_D(x, y) = \lim \rho_D(x_j, y_j)$ and $\widehat{\rho}_D(x, y) = \lim \widehat{\rho}_D(x_j, y_j)$ if x and y are represented by ρ_D-Cauchy sequences $\{x_j\}$ and $\{y_j\}$ in D. It is easy to see that the quantities $\rho_D(x, y)$ and $\widehat{\rho}_D(x, y)$ are independent of the choice of the ρ_D-Cauchy sequences $\{x_j\}$ and $\{y_j\}$. Let $\xi^* \in \partial^* D$ and put

$$B_\rho(\xi^*, r) = \{x \in D : \widehat{\rho}_D(\xi^*, x) < r\}.$$

Moreover, let $S_\rho(\xi^*, r) = D \cap \partial B_\rho(\xi^*, r)$ and $C_\rho(\xi^*, r) = D \cap \overline{B_\rho(\xi^*, r)}$. Here, '$\partial$' and '$\overline{}$' mean the boundary and the closure in the Euclidean space, respectively. These sets correspond to $D \cap B(x, r)$, $D \cap C(x, r)$ and $D \cap S(x, r)$. The following observation enables us to use many arguments in [1].

Lemma 2.4. *The set $B_\rho(\xi^*, r)$ is the open connected component of $D \cap B(\pi(\xi^*), r)$ which can be connected to ξ^* in itself, i.e. there is an arc $\gamma \subset B_\rho(\xi^*, r)$ converging to ξ^*.*

Proof. It is sufficient to show the following (i)–(iv).

(i) $B_\rho(\xi^*, r) \subset D \cap B(\pi(\xi^*), r)$.

(ii) $B_\rho(\xi^*, r)$ is open.

(iii) Every point $x \in B_\rho(\xi^*, r)$ is connected to ξ^* by an arc in $B_\rho(\xi^*, r)$.

(iv) $B_\rho(\xi^*, r)$ is the maximal set with the above properties (i)–(iii).

Let ξ^* be represented by a ρ_D-Cauchy sequence $\{x_j\}$. First, we prove (i), (ii) and (iii). Suppose $x \in B_\rho(\xi^*, r)$. Then $\varepsilon = r - \widehat{\rho}_D(\xi^*, x) > 0$. Since $\widehat{\rho}_D(\xi^*, x) = \lim_{j\to\infty} \widehat{\rho}_D(x_j, x) < r - \varepsilon$, there exists a positive integer j_0 such that $\widehat{\rho}_D(x_j, x) < r - \varepsilon/2$ for $j \ge j_0$. By the definition of $\widehat{\rho}_D$ we find a curve $\widetilde{x_j x} \subset D$ connecting x_j and x with

$$|x_j - x| \le \widehat{r}(x_j, \widetilde{x_j x}) < r - \varepsilon/2 \tag{2.11}$$

for $j \ge j_0$. Hence

$$|\pi(\xi^*) - x| = \lim_{j\to\infty} |x_j - x| \le r - \varepsilon/2 < r.$$

Therefore, $x \in D \cap B(\pi(\xi^*), r)$ and (i) follows. Now x lies in the open set $D \cap B(\pi(\xi^*), r)$. We find r_0, $0 < r_0 < \varepsilon/2$, such that $B(x, r_0) \subset D \cap B(\pi(\xi^*), r)$. For (ii) it suffices to show that $B(x, r_0) \subset B_\rho(\xi^*, r)$. In fact, every $y \in B(x, r_0)$ can be connected to x_j by $\widetilde{x_j x} \cup \overline{xy}$ for $j \ge j_0$, where $\overline{xy}$ denotes the line segment between x and y. Hence, (2.11) yields

$$\widehat{\rho}_D(\xi^*, y) = \lim_{j\to\infty} \widehat{\rho}_D(x_j, y) \le \limsup_{j\to\infty} \widehat{r}(x_j, \widetilde{x_j x} \cup \overline{xy}) \le r - \frac{\varepsilon}{2} + r_0 < r,$$

so that $B(x, r_0) \subset B_\rho(\xi^*, r)$ and (ii) follows. In order to prove (iii) we may assume that

$$\rho_D(x_j, x_{j+1}) < 2^{-j}\varepsilon, \tag{2.12}$$

by taking a subsequence of $\{x_j\}$. Then each pair of points x_j and x_{j+1} can be connected by a curve $\widetilde{x_j x_{j+1}} \subset D$ with $\operatorname{diam}(\widetilde{x_j x_{j+1}}) < 2^{-j}\varepsilon$. Let

$$\gamma = \widetilde{x x_{j_0}} \cup \left(\bigcup_{j=j_0}^{\infty} \widetilde{x_j x_{j+1}} \right).$$

Then, by (2.11) and (2.12), γ is an arc in D connecting x and ξ^* such that

$$\widehat{r}(\xi^*, \gamma) \le \widehat{r}(x_{j_0}, \widetilde{x x_{j_0}}) + \sum_{j=j_0}^{\infty} \operatorname{diam}(\widetilde{x_j x_{j+1}}) < r - \frac{\varepsilon}{2} + \sum_{j=j_0}^{\infty} 2^{-j}\varepsilon.$$

Without loss of generality, we may assume that $j_0 \geq 2$, so that $\widehat{r}(\xi^*, \gamma) < r$. Hence $\gamma \subset B_\rho(\xi^*, r)$ and (iii) follows. We remark that (iii) implies that $B_\rho(\xi^*, r)$ is connected.

Finally we prove (iv). Suppose that D_1 is a subset of $D \cap B(\pi(\xi^*), r)$ such that every $x \in D_1$ is connected to ξ^* by an arc in D_1. We have to show that $\widehat{\rho}_D(\xi^*, x) < r$ for $x \in D_1$. Suppose $x \in D_1$. Then there is an arc $\gamma \subset D_1$ connecting ξ^* and x. By the compactness of γ we see that $\gamma \subset B(\pi(\xi^*), r - \eta)$ for some $\eta > 0$. By the definition of $\widehat{r}$

$$\widehat{\rho}_D(\xi^*, x) = \lim_{\substack{y \to \xi^* \\ y \in \gamma}} \widehat{\rho}_D(y, x) \leq \limsup_{\substack{y \to \xi^* \\ y \in \gamma}} \widehat{r}(y, \gamma) \leq \limsup_{\substack{y \to \xi^* \\ y \in \gamma}} |y - \pi(\xi^*)| + r - \eta = r - \eta < r.$$

Hence (iv) follows. □

As a corollary to Lemma 2.4 we have the following.

Lemma 2.5. *Let V be a connected open subset of $D \cap B(\pi(\xi^*), r)$. If $V \cap B_\rho(\xi^*, r) \neq \emptyset$, then $V \subset B_\rho(\xi^*, r)$. In particular, if $\xi_1^* \in \partial^* D$ is accessible from $B_\rho(\xi^*, r)$ and $r_1 + |\pi(\xi^*) - \pi(\xi_1^*)| < r$, then $B_\rho(\xi_1^*, r_1) \subset B_\rho(\xi^*, r)$.*

Now let us prove Proposition 1.3. The following lemma says that the internal metric is invariant by adding simple boundary points.

Lemma 2.6. *Let $\widetilde{D}$ be a domain between D and $\operatorname{int}(\overline{D})$ such that $\widetilde{D} \setminus D$ consists of simple boundary points. Then $\rho_D(x, y) = \rho_{\widetilde{D}}(x, y)$ for $x, y \in D$.*

Proof. Let $x, y \in D$. By definition $\rho_{\widetilde{D}}(x, y) \leq \rho_D(x, y)$. Let us prove the opposite inequality. It is sufficient to show that if $\widetilde{\gamma}$ is a curve in $\widetilde{D}$ connecting x and y, then for each $\varepsilon > 0$ there is a curve $\gamma \subset D$ connecting x and y with

$$\operatorname{diam}(\gamma) \leq \operatorname{diam}(\widetilde{\gamma}) + \varepsilon. \tag{2.13}$$

Observe from Lemma 2.4 that if $\xi \in \partial D$ is a simple boundary point, then ξ is accessible from only one connected component $V(\xi)$ of $D \cap B(\xi, \varepsilon/2)$. This means that there is $\eta(\xi) > 0$ such that $D \cap B(\xi, 2\eta(\xi)) \subset V(\xi)$. If $\xi \in D$, then we define $\eta(\xi) = \frac{1}{4} \min\{\delta_D(\xi), \varepsilon\}$ and $V(\xi) = B(\xi, 2\eta(\xi))$. Since $\widetilde{\gamma}$ consists of points of D and simple boundary points, we can find finitely many points $\xi_j \in \widetilde{\gamma}$ and $\eta_j = \eta(\xi_j) > 0$ such that

$$\begin{aligned} &\widetilde{\gamma} \subset \bigcup_j B(\xi_j, \eta_j), \\ &D \cap B(\xi_j, 2\eta_j) \subset V(\xi_j) \end{aligned} \tag{2.14}$$

by the compactness of $\widetilde{\gamma}$. Changing the number j, we may assume that $x \in B(\xi_1, \eta_1)$. Let x_1 be the last point of the curve $\widetilde{\gamma}$ in $C(\xi_1, \eta_1)$. If $x_1 = y$, then we stop. Otherwise, x_1 lies in some $B(\xi_j, \eta_j)$, say $B(\xi_2, \eta_2)$ by (2.14). Let x_2 be the last point of $\widetilde{\gamma}$ in $C(\xi_2, \eta_2)$ and continue in the same fashion. Then we obtain a finite sequence of points $x_1, \dots, x_n = y$ such that each x_j is the last point of $\widetilde{\gamma}$ in $C(\xi_j, \eta_j)$ and $x_j \in B(\xi_{j+1}, \eta_{j+1})$ for $j = 1, \dots, n-1$. Observe that either $x_j \in D$ or x_j is accessible from $V(\xi_j)$ by (2.14) and Lemma 2.4. Hence we find $x'_j \in D \cap B(\xi_j, 2\eta_j) \cap B(\xi_{j+1}, 2\eta_{j+1})$ for $j = 1, \dots, n-1$. Let $x'_0 = x$ and $x'_n = y$ for convention. Then $x'_{j-1}, x'_j \in V(\xi_j)$ by (2.14) and we find a curve $\widetilde{x'_{j-1}x'_j} \subset V(\xi_j) \subset D \cap B(\xi_j, \varepsilon/2)$ connecting x'_{j-1} and x'_j for $j = 1, \dots, n$. Then x and y are connected by the curve

$$\gamma = \widetilde{x'_0x'_1} \cup \widetilde{x'_1x'_2} \cup \cdots \cup \widetilde{x'_{n-1}x'_n} \subset D \cap \left(\bigcup_{j=1}^{n} B(\xi_j, \varepsilon/2) \right).$$

Since each $\xi_j \in \widetilde{\gamma}$, we have (2.13). The proof is complete. $\square$

Now we can prove Proposition 1.3.

Proof of Proposition 1.3. By Lemma 2.6 we have $\rho_D(x,y) = \rho_{\widetilde{D}}(x,y)$ for $x, y \in D$, and hence for $x, y \in \widetilde{D}$ by extending ρ_D. By definition $\delta_D(z) \le \delta_{\widetilde{D}}(z)$ for $z \in D$. Now let $x, y \in \widetilde{D}$. Note that $\pi(x) = x$ and $\pi(y) = y$ since x and y are points of D or simple boundary points. By Lemma 2.3 we find a curve $\gamma \subset \widetilde{D}$ connecting x and y with

$$\min\{|x-z|, |z-y|\} \le A\delta_D(z) \le A\delta_{\widetilde{D}}(z) \quad \text{for all } z \in \gamma,$$
$$\operatorname{diam}(\gamma) \le A\rho_D(x,y) = A\rho_{\widetilde{D}}(x,y),$$

where A depends only on A_4. Thus $\widetilde{D}$ is a uniformly John domain. $\square$

For a moment let D be a general proper subdomain of $\mathbb{R}^d$. We define the quasi-hyperbolic metric $k_D(x,y)$ by

$$k_D(x,y) = \inf_\gamma \int_\gamma \frac{ds(z)}{\delta_D(z)},$$

where the infimum is taken over all rectifiable curves γ connecting x to y in D. Observe that $k_D(x,y)$ is monotone decreasing with respect to D, i.e., if $x, y \in D_1 \subset D$, then $k_{D_1}(x,y) \ge k_D(x,y)$. The converse estimate will be needed in the sequel. Observe that if $z \in D$, then

$$(2.15) \qquad k_D(x,y) \le k_{D\setminus\{z\}}(x,y) \le k_D(x,y) + A \quad \text{for } x, y \in D \setminus B(z, 2^{-1}\delta_D(z)).$$

This observation will be useful to estimate the Green function with pole at z.

Now let D be a bounded uniformly John domain. Then the following uniform quasi hyperbolic boundary condition holds.

Lemma 2.7. *Let D be a bounded uniformly John domain. Then*

$$k_D(x,y) \le A \log \frac{\rho_D(x,y)}{\min\{\delta_D(x), \delta_D(y)\}} + A',$$

where A and A' depend only on the uniform John constant A_4.

Proof. If $y \in B(x, \delta_D(x)/2)$ or $x \in B(y, \delta_D(y)/2)$, then the lemma is obvious. Hence, suppose $|x-y| \ge \frac{1}{2}\max\{\delta_D(x), \delta_D(y)\}$. Let γ be a curve connecting x to y with (2.1) and (2.2). Then

$$\begin{aligned}\int_\gamma \frac{ds(z)}{\delta_D(z)} &\le \int_0^{\delta_D(x)/2} \frac{ds}{\delta_D(x)/2} + \int_{\delta_D(x)/2}^{\ell(\gamma)/2} \frac{A_4 ds}{s} + \int_{\ell(\gamma)/2}^{\ell(\gamma)-\delta_D(y)/2} \frac{A_4 ds}{s} + \int_0^{\delta_D(y)/2} \frac{ds}{\delta_D(y)/2} \\ &\le 2 + 2A_4 \log \frac{A_4 \rho_D(x,y)}{\min\{\delta_D(x), \delta_D(y)\}}.\end{aligned}$$

Thus the lemma follows. □

Let $x_0 \in D$ be fixed. Then every point $x \in D$ can be connected to x_0 by γ along which the distance to the boundary increases as in (2.2). Hence, there is A_5, $0 < A_5 < 1$ such that

$$A_5 R \le \sup_{x \in S_\rho(\xi^*, R)} \delta_D(x) \le R$$

for sufficiently small R, say $0 < R < \delta_D(x_0)/2$. Let us take $\xi_R \in S_\rho(\xi^*, 4R)$ with $4A_5 R \le \delta_D(\xi_R) \le 4R$. Then, we have the following.

Lemma 2.8. *Let D be a bounded uniformly John domain. Then there exists a constant $A_6 > 9$ depending only on D such that*

$$k_{B_\rho(\xi^*, A_6 R)}(x,y) \le A \log \frac{\rho_D(x,y)}{\min\{\delta_D(x), \delta_D(y)\}} \quad \text{for } x, y \in B_\rho(\xi^*, 9R). \tag{2.16}$$

where $\xi^* \in \partial^* D$, $R > 0$ is sufficiently small and A depends only on D. In particular,

$$k_{B_\rho(\xi^*, A_6 R)}(x, \xi_R) \le A \log \frac{18R}{\delta_D(x)} \quad \text{for } x \in B_\rho(\xi^*, 9R), \tag{2.17}$$

where A is independent of the choice of ξ_R. In the sequel, estimates will be independent of the choice of ξ_R.

Proof. Let $x, y \in B_\rho(\xi^*, 9R)$. Suppose γ is a curve connecting x to y with (2.1) and (2.2). Then

$$\widehat{\rho}_D(\xi^*, z) \le \widehat{\rho}_D(\xi^*, x) + \widehat{\rho}_D(x, z) < 9R + \operatorname{diam}(\gamma) \le AR \quad \text{for } z \in \gamma.$$

Let A_6 be the twice of the above A. Then $\gamma \subset B_\rho(\xi^*, \frac{1}{2}A_6 R)$ and $\delta_{B_\rho(\xi^*, A_6 R)}(z) = \delta_D(z)$ for $z \in \gamma$. Hence the proof of the preceding lemma yields (2.16). Since $\rho_D(x, \xi_R) < 18R$ and $\delta_D(\xi_R) \ge 4A_5 R$, we have (2.17) from (2.16). $\square$

3. Boundary Harnack Principle

The main aim of this section is to show the following boundary Harnack principle.

Theorem 3.1. *Let D be a bounded uniformly John domain. Then there exists a constant $A_7 > 1$ depending only on D with the following property: Let $\xi^* \in \partial^* D$ and let $R > 0$ be sufficiently small. Suppose u and v are positive bounded harmonic functions on $B_\rho(\xi^*, A_7 R)$ vanishing q.e. on $\partial D \cap \overline{B_\rho(\xi^*, A_7 R)}$. Then*

$$\frac{u(x)}{v(x)} \approx \frac{u(x')}{v(x')} \qquad \text{uniformly for } x, x' \in B_\rho(\xi^*, R),$$

where the constant of comparison depends on D.

Theorem 3.1 can be proved in a way similar to that of [1, Theorem 1] with the aid of Lemma 2.4. However, we must be careful about the fact that D^* is the completion of D with respect to the internal metric. It is, in general, different from the Euclidean closure. The proof is inspired by the probabilistic work of Bass and Burdzy [7]. See Ferrari [14] for an analytic proof. It should be noted that Bass-Burdzy and Ferrari gave a non-uniform boundary Harnack principle. To determine the Martin boundary, we need a uniform or scale invariant boundary Harnack principle. Our boundary Harnack principle is uniform with respect to the internal metric.

We say that $x, y \in D$ is connected by a Harnack chain $\{B(x_j, \frac{1}{2}\delta_D(x_j))\}_{j=1}^k$ if $x \in B(x_1, \frac{1}{2}\delta_D(x_1))$, $y \in B(y_k, \frac{1}{2}\delta_D(y_k))$, and $B(x_j, \frac{1}{2}\delta_D(x_j)) \cap B(x_{j+1}, \frac{1}{2}\delta_D(x_{j+1})) \ne \emptyset$ for $j = 1, \dots, k-1$. The number k is called the length of the Harnack chain. We observe that the shortest length of the Harnack chain connecting x and y is comparable to $k_D(x, y)$. Therefore, the Harnack inequality yields that there is a positive constant A depending only on d such that

$$\exp(-Ak_D(x, y)) \le \frac{h(x)}{h(y)} \le \exp(Ak_D(x, y))$$

for every positive harmonic function h on D.

Our proof of Theorem 3.1 will be based on a certain estimate of harmonic measure. By $\omega(x, E, U)$ we denote the harmonic measure of E for an open set U evaluated at x. For $r > 0$ let $U(r) = \{x \in D : \delta_D(x) < r\}$. Since every point $x \in U(r)$ can be connected to x_0 by a curve γ along which the distance to the boundary increases as in (2.2), it follows that if $r > 0$ is sufficiently small, then there is a point $z \in D \cap S(x, A_8 r)$ with $\delta_D(z) > 2r$, where $A_8 > 1$ is a constant depending only on D. Hence there is a ball $B(z, r) \subset B(x, A_8 r) \setminus U(r)$. This implies that

$$\omega(x, \overline{U(r)} \cap S(x, A_8 r), U(r) \cap B(x, A_8 r)) \le 1 - \varepsilon_0 \quad \text{for } x \in U(r)$$

with $0 < \varepsilon_0 < 1$ depending only on A_8 and the dimension. Let $R \ge r$ and repeat this argument with the maximum principle. Then there exist positive constants A_9 and A_{10} such that

$$\omega(x, \overline{U(r)} \cap S(x, R), U(r) \cap B(x, R)) \le \exp(A_9 - A_{10} R/r). \tag{3.1}$$

See [1, Lemma 1] for details.

Let us compare the Green function and the harmonic measure. For simplicity we let $D_R = B_\rho(\xi^*, (A_6 + 7)R)$ and $D'_R = B_\rho(\xi^*, A_6 R)$ with A_6 as in Lemma 2.8. By G_R and G'_R we denote the Green functions for D_R and D'_R, respectively.

Lemma 3.2. *If $R > 0$ is sufficiently small, then*

$$\omega(\cdot, S_\rho(\xi^*, 2R), B_\rho(\xi^*, 2R)) \le A R^{d-2} G'_R(\cdot, \xi_R) \le A R^{d-2} G_R(\cdot, \xi_R) \quad \text{on } B_\rho(\xi^*, R),$$

where A depends only on D.

Proof. It is sufficient to show the first inequality. We follow the idea of [7] and [1]. We find $A_{11} > 0$ depending only on D such that $A_{11} R^{d-2} G'_R(\cdot, \xi_R) < 1/e$ on $B_\rho(\xi^*, 2R)$. Then

$$B_\rho(\xi^*, 2R) = \bigcup_{j \ge 0} D_j \cap B_\rho(\xi^*, 2R), \tag{3.2}$$

where

$$D_j = \{x \in D : \exp(-2^{j+1}) \le A_{11} R^{d-2} G'_R(x, \xi_R) < \exp(-2^j)\}.$$

Let $U_j = (\cup_{k \ge j} D_k) \cap B_\rho(\xi^*, 2R) = \{x \in B_\rho(\xi^*, 2R) : A_{11} R^{d-2} G'_R(x, \xi_R) < \exp(-2^j)\}$. First we observe

$$U_j \subset \{x \in D : \delta_D(x) < AR \exp(-2^j/\lambda)\} \tag{3.3}$$

with some $\lambda > 0$ depending only on D. For a moment fix $z \in S(\xi_R, \frac{1}{2}\delta_D(\xi_R))$. Then $G'_R(z, \xi_R) \approx R^{2-d}$ and

$$k_{D'_R \setminus \{\xi_R\}}(x, z) \le k_{D'_R}(x, \xi_R) + A \le A \log \frac{18R}{\delta_D(x)}$$

for $x \in B_\rho(\xi^*, 9R) \setminus B(\xi_R, \frac{1}{2}\delta_D(\xi_R))$ by (2.15) and (2.17). We see from the Harnack inequality that there is $\lambda > 0$ such that

$$\begin{aligned} \exp(-2^j) > A_{11} R^{d-2} G'_R(x, \xi_R) &\ge A R^{d-2} G'_R(z, \xi_R) \exp(-A k_{D'_R \setminus \{\xi_R\}}(x, z)) \\ &\ge A \exp\left(-\lambda \log \frac{18R}{\delta_D(x)}\right) = A \left(\frac{\delta_D(x)}{18R}\right)^\lambda \end{aligned}$$

for $x \in U_j$. Thus (3.3) follows.

Let $r_j = AR \exp(-2^j/\lambda)$ with A in (3.3). We take a slowly decreasing sequence $\{R_j\}$ converging to R such that

$$\sum_{j=1}^{\infty} \exp\left(2^{j+1} - \frac{A_{10}(R_{j-1} - R_j)}{r_j}\right) < \infty, \tag{3.4}$$

where the value of the summation is independent of R. In fact, if we let $R_0 = 2R$ and $R_j = \left(2 - \frac{6}{\pi^2} \sum_{k \le j} \frac{1}{k^2}\right) R$ for $j \ge 1$, then (3.4) holds. For simplicity we let $\omega_0 = \omega(\cdot, S_\rho(\xi^*, 2R), B_\rho(\xi^*, 2R))$ and

$$d_j = \begin{cases} \displaystyle\sup_{x \in D_j \cap B_\rho(\xi^*, R_j)} \frac{\omega_0(x)}{R^{d-2} G'_R(x, \xi_R)} & \text{if } D_j \cap B_\rho(\xi^*, R_j) \ne \emptyset, \\ 0 & \text{if } D_j \cap B_\rho(\xi^*, R_j) = \emptyset. \end{cases}$$

In view of (3.2) it is sufficient to show that

$$\sup_{j \ge 0} d_j \le A < \infty, \tag{3.5}$$

where A is independent of R.

Let $j > 0$. Let us apply the maximum principle over $U_j \cap B_\rho(\xi^*, R_{j-1})$. Observe that $D \cap \partial(U_j \cap B_\rho(\xi^*, R_{j-1}))$ is included in the union of $\overline{U_j} \cap S_\rho(\xi^*, R_{j-1})$ and $\{x \in B_\rho(\xi^*, R_{j-1}) : A_{11} R^{d-2} G'_R(x, \xi_R) = \exp(-2^j)\}$. By definition the last set is included in $D_{j-1} \cap B_\rho(\xi^*, R_{j-1})$, on which $\omega_0 \le d_{j-1} R^{d-2} G'_R(\cdot, \xi_R)$ holds. Hence the maximum principle yields that

$$\omega_0(x) \le \omega(x, \overline{U_j} \cap S_\rho(\xi^*, R_{j-1}), U_j \cap B_\rho(\xi^*, R_{j-1})) + d_{j-1} R^{d-2} G'_R(x, \xi_R). \tag{3.6}$$

for $x \in U_j \cap B_\rho(\xi^*, R_{j-1})$.

Now let $x \in U_j \cap B_\rho(\xi^*, R_j)$. We apply the maximum principle over the connected component V_x of $U_j \cap B(x, R_{j-1} - R_j)$ containing x. In view of Lemma 2.4 we have $|x - \pi(\xi^*)| < R_j$, so that $V_x \subset B(\pi(\xi^*), R_{j-1})$. Hence Lemma 2.5 yields that $V_x \subset B_\rho(\xi^*, R_{j-1})$. Moreover, we have

$$D \cap \partial V_x \subset (D \cap \overline{V_x} \cap S(x, R_{j-1} - R_j)) \cup (B_\rho(\xi^*, R_{j-1}) \cap \partial U_j). \tag{3.7}$$

In fact, suppose $y \in D \cap \partial V_x$ and $|y - x| < R_{j-1} - R_j$. Then there is $\varepsilon > 0$ such that $B(y, \varepsilon) \subset D \cap B(\pi(\xi^*), R_{j-1})$. By definition $V_x \cap B(y, \varepsilon) \neq \emptyset$, and hence $y \in B(y, \varepsilon) \subset B_\rho(\xi^*, R_{j-1})$ by Lemma 2.5. It is easy to see that $y \in \partial U_j$, so that (3.7) follows.

Since $\omega(\cdot, \overline{U_j} \cap S_\rho(\xi^*, R_{j-1}), U_j \cap B_\rho(\xi^*, R_{j-1}))$ vanishes q.e. on $\partial D \cup (B_\rho(\xi^*, R_{j-1}) \cap \partial U_j)$, it is less than or equal to

$$\omega(x, \overline{V_x} \cap S(x, R_{j-1} - R_j), V_x) \le \omega(x, \overline{U_j} \cap S(x, R_{j-1} - R_j), U_j \cap B(x, R_{j-1} - R_j))$$

by the maximum principle and (3.7). The last harmonic measure is less than or equal to $\exp(A_9 - A_{10}(R_{j-1} - R_j)/r_j)$ by (3.1) and (3.3). Since $A_{11} R^{d-2} G'_R(x, \xi_R) \ge \exp(-2^{j+1})$ for $x \in D_j$ by definition, (3.6) now becomes

$$\omega_0(x) \le \left\{ A_{11} \exp\left(2^{j+1} + A_9 - \frac{A_{10}(R_{j-1} - R_j)}{r_j}\right) + d_{j-1} \right\} R^{d-2} G'_R(x, \xi_R)$$

for $x \in D_j \cap B_\rho(\xi^*, R_j)$. Dividing both sides by $R^{d-2} G'_R(x, \xi_R)$ and taking the supremum over $x \in D_j \cap B_\rho(\xi^*, R_j)$, we obtain

$$d_j \le A_{11} \exp\left(2^{j+1} + A_9 - \frac{A_{10}(R_{j-1} - R_j)}{r_j}\right) + d_{j-1}.$$

Hence (3.5) follows from (3.4). □

Lemma 3.3. *If $R > 0$ is sufficiently small, then*

$$\frac{G_R(x, y)}{G_R(x', y)} \approx \frac{G_R(x, y')}{G_R(x', y')} \quad \text{for } x, x' \in B_\rho(\xi^*, R) \text{ and } y, y' \in S_\rho(\xi^*, 6R)$$

with constant comparison depending only on D.

Proof. Let us take $x_R \in S_\rho(\xi^*, R)$ and $y_R \in S_\rho(\xi^*, 6R)$ such that $A_5 R \le \delta_D(x_R) \le R$ and $6A_5 R \le \delta_D(y_R) \le 6R$. It is sufficient to show

$$G_R(x, y) \approx \frac{G_R(x_R, y)}{G_R(x_R, y_R)} G_R(x, y_R) \tag{3.8}$$

for $x \in B_\rho(\xi^*, R)$ and $y \in S_\rho(\xi^*, 6R)$. For simplicity we fix $y \in S_\rho(\xi^*, 6R)$ and let $u(x)$ (resp. $v(x)$) be the left (resp. right) hand side of (3.8).

First we show that $u \ge Av$ on $B_\rho(\xi^*, R)$ with A independent of y. Observe that

(i) u is a positive harmonic function on $D_R \setminus \{y\}$ with vanishing q.e. on ∂D_R;

(ii) v is a positive harmonic function on $D_R \setminus \{y_R\}$ with vanishing q.e. on ∂D_R.

Since u is superharmonic on D_R and $B_\rho(\xi^*, R) \subset D_R \setminus B(y_R, A_5 R)$, it is sufficient to show that $u \ge Av$ on $S(y_R, A_5 R)$ by the maximum principle. Take $z \in S(y_R, A_5 R)$. Then $k_{D_R\setminus\{y_R\}}(z, x_R) \le A$ by (2.15), and hence

$$v(z) \approx \frac{G_R(x_R, y)}{G_R(x_R, y_R)} G_R(x_R, y_R) = G_R(x_R, y) \le AR^{2-d}. \tag{3.9}$$

If $y \in B(y_R, 2A_5 R)$, then $u(z) = G_R(z, y) \ge AR^{2-d}$, so that $u(z) \ge Av(z)$. If $y \in D \setminus B(y_R, 2A_5 R)$, then (2.15) and Lemma 2.8 yield

$$k_{D_R\setminus\{y\}}(z, x_R) \le k_{D_R}(z, x_R) + A \le A,$$

so that $v(z) \approx G_R(x_R, y) \approx G_R(z, y) = u(z)$ by (3.9). Hence we have $u \ge Av$ on $S(y_R, A_5 R)$ in any case.

In order to show that $u(x) \le Av(x)$, we make use of Lemma 3.2. It is clear that $G_R(x, z) \le AR^{2-d} \approx G_R(x_R, y_R)$ for $x \in C_\rho(\xi^*, 2R)$ and $z \in B_\rho(\xi^*, 9R) \setminus B(\xi, 3R)$, where $\xi = \pi(\xi^*)$. Since $S_\rho(\xi^*, 2R) \subset C_\rho(\xi^*, 2R)$, it follows from the maximum principle that

$$G_R(\cdot, z) \le AG_R(x_R, y_R)\omega(\cdot, S_\rho(\xi^*, 2R), B_\rho(\xi^*, 2R)) \quad \text{on } B_\rho(\xi^*, 2R).$$

Since $G_R(x_R, y_R) \approx R^{2-d}$ and $G_R(x, \xi_R) \approx G_R(x, y_R)$, it follows from Lemma 3.2 and the Harnack inequality that

$$G_R(x, z) \le AG_R(x_R, y_R)R^{d-2}G_R(x, \xi_R) \le AG_R(x, y_R) \tag{3.10}$$

for $x \in B_\rho(\xi^*, R)$ and $z \in B_\rho(\xi^*, 9R) \setminus B(\xi, 3R)$.

Now fix $x \in B_\rho(\xi^*, R)$ and $y \in S_\rho(\xi^*, 6R)$. If $\delta_D(y) \ge 2^{-1}A_5 R$, then $k_{D_R}(y, y_R) \le A$ by Lemma 2.8, so that $G_R(x, y) \approx G_R(x, y_R)$ and $G_R(x_R, y) \approx G_R(x_R, y_R)$ by the Harnack inequality. Hence (3.8) follows. Therefore, we may assume that $\delta_D(y) < 2^{-1}A_5 R$. Then there is $\xi_1 \in \partial D$ such that $|y - \xi_1| = \delta_D(y) < 2^{-1}A_5 R$. In view of Lemma 2.4, we find $\xi_1^* \in \partial^* D$ such that $\pi(\xi_1^*) = \xi_1$ and $y \in B_\rho(\xi_1^*, 2^{-1}A_5 R)$ since $B(y, \delta_D(y)) \subset D$. Since $5R < 6R - 2^{-1}A_5 R \le |\xi - \xi_1| \le 6R + 2^{-1}A_5 R < 7R$, it follows from Lemmas 2.4 and 2.5 that $B_\rho(\xi_1^*, 2R) \subset B_\rho(\xi^*, 9R) \setminus B(\xi, 3R)$, and hence from (3.10) that $G_R(x, z) \le AG_R(x, y_R)$ for $z \in B_\rho(\xi_1^*, 2R)$. Hence the maximum principle yields that

$$G_R(x, y) \le AG_R(x, y_R)\omega(y, S_\rho(\xi_1^*, 2R), B_\rho(\xi_1^*, 2R)). \tag{3.11}$$

Using Lemma 3.2 with replacing ξ^* by ξ_1^*, we obtain

$$\omega(y, S_\rho(\xi_1^*, 2R), B_\rho(\xi_1^*, 2R)) \le AR^{d-2} G_{B_\rho(\xi_1^*, A_6 R)}(y, \xi_R')$$

with $\xi_R' \in S_\rho(\xi_1^*, 4R)$ such that $4A_5 R \le \delta_D(\xi_R') \le 4R$. Since $|\xi - \xi_1| < 7R$, it follows from Lemma 2.5 that $B_\rho(\xi_1^*, A_6 R) \subset B_\rho(\xi^*, (A_6 + 7)R) = D_R$, so that

$$\omega(y, S_\rho(\xi_1^*, 2R), B_\rho(\xi_1^*, 2R)) \le AR^{d-2} G_R(y, \xi_R') = AR^{d-2} G_R(\xi_R', y).$$

Hence (3.11) becomes

$$G_R(x, y) \le A G_R(x, y_R) R^{d-2} G_R(\xi_R', y) \le A G_R(x, y_R) R^{d-2} G_R(x_R, y)$$

by the Harnack inequality. Since $G_R(x_R, y_R) \approx R^{2-d}$, we have $u(x) \le Av(x)$. Thus (3.8) is proved. The proof is complete. □

Proof of Theorem 3.1. We prove the theorem with $A_7 = A_6 + 7$. Since u is a positive harmonic function on D_R, we can consider the regularized reduced function $\widehat{R}_u^{S_\rho(\xi^*, 6R)}$ of u to $S_\rho(\xi^*, 6R)$ with respect to D_R. This regularized reduced function is a superharmonic function on D_R such that $\widehat{R}_u^{S_\rho(\xi^*, 6R)} = u$ q.e. on $S_\rho(\xi^*, 6R)$ and harmonic on $D_R \setminus S_\rho(\xi^*, 6R)$. Moreover, $\widehat{R}_u^{S_\rho(\xi^*, 6R)} = 0$ q.e. on ∂D_R by assumption. Since u is bounded on D_R, it follows from the maximum principle that $u = \widehat{R}_u^{S_\rho(\xi^*, 6R)}$ on $B_\rho(\xi^*, 6R)$. It is easy to see that $\widehat{R}_u^{S_\rho(\xi^*, 6R)}$ is a Green potential of a measure μ supported on $S_\rho(\xi^*, 6R)$, i.e.

$$u(x) = \int_{S_\rho(\xi^*, 6R)} G_R(x, y) d\mu(y) \quad \text{for } \in B_\rho(\xi^*, 6R).$$

Let $x, x' \in B_\rho(\xi^*, R)$ and $y, y' \in S_\rho(\xi^*, 6R)$. Then

$$G_R(x, y) \approx \frac{G_R(x, y')}{G_R(x', y')} G_R(x', y)$$

by Lemma 3.3. Hence

$$u(x) \approx \frac{G_R(x, y')}{G_R(x', y')} \int_{S_\rho(\xi^*, 6R)} G_R(x', y) d\mu(y) = \frac{G_R(x, y')}{G_R(x', y')} u(x').$$

Therefore,

$$\frac{u(x)}{u(x')} \approx \frac{G_R(x, y')}{G_R(x', y')} \quad \text{uniformly for } y' \in S_\rho(\xi^*, 6R).$$

Similarly,

$$\frac{v(x)}{v(x')} \approx \frac{G_R(x, y')}{G_R(x', y')}.$$

Hence the theorem follows. □

Remark 3.4. In view of the above proof, the assertion of Theorem 3.1 holds for an unbounded uniformly John domain if ξ^* lies over a finite boundary point ξ of D.

Let $\mathscr{H}_{\xi^*}$ be the family of all positive harmonic functions h on D vanishing q.e. on ∂D, bounded on $D \setminus B_\rho(\xi^*, r)$ for each $r > 0$ and taking value $h(x_0) = 1$. A function h in $\mathscr{H}_{\xi^*}$ is called a *kernel function* at ξ normalized at x_0.

Lemma 3.5. *There is a constant $A \geq 1$ depending only on D such that*

$$A^{-1} \leq \frac{u}{v} \leq A \quad \textit{for } u, v \in \mathscr{H}_{\xi^*}.$$

Proof. Let $u, v \in \mathscr{H}_{\xi^*}$ and let $r > 0$. Then u and v be bounded on $B_\rho(\xi_1^*, 2^{-1}r)$ for $\xi_1^* \in \partial D \cap \overline{S_\rho(\xi^*, r)}$. Hence Theorem 3.1 yields

$$\frac{u(x)}{v(x)} \approx \frac{u(x')}{v(x')} \qquad \text{for } x, x' \in B_\rho(\xi_1^*, 2^{-1}r/A_7),$$

where A_7 is as in Theorem 3.1. This, together with the Harnack inequality, shows that

$$\frac{u(x)}{v(x)} \approx \frac{u(x')}{v(x')} \qquad \text{for } x, x' \in S_\rho(\xi^*, r),$$

where the constant of comparison is independent of r. Then the same comparison holds for $x, x' \in D \setminus B_\rho(\xi^*, r)$ by the maximum principle. Since $u(x_0) = v(x_0) = 1$, it follows that

$$\frac{u(x)}{v(x)} \approx 1 \qquad \text{for } x \in D \setminus B_\rho(\xi^*, r).$$

Since $r > 0$ is arbitrary small and the constant of comparison is independent of r, the lemma follows. $\square$

Proof of Theorem 1.2. Lemma 3.5 actually shows that $\mathscr{H}_{\xi^*}$ is a singleton and that the function $u \in \mathscr{H}_{\xi^*}$ is minimal. This is proved by Ancona [2, Lemma 6.2]. For a short proof see [1, Theorem 3]. Let $G(x, y)$ be the Green function for D. Put $K(x, y) = G(x, y)/G(x_0, y)$ for $x \in D$ and $y \in D \setminus \{x_0\}$. The Martin kernel is given as the limit of $K(x, y)$ when y tends to a ideal boundary point. If $y \to \xi^* \in \partial^* D$, then some subsequence of $\{K(\cdot, y)\}$ converges to a positive harmonic function in $\mathscr{H}_{\xi^*}$. However, since $\mathscr{H}_{\xi^*}$ is a singleton, it follows that all sequences $\{K(\cdot, y)\}$ must converge to the same positive harmonic function, the Martin kernel $K(\cdot, \xi^*)$ at ξ^*. Therefore $K(x, \cdot)$ extends continuously to $D^* \setminus \{x_0\}$. The kernel function $K(\cdot, \xi^*)$ should be minimal. It is easy to see that distinct ideal boundary points on $\partial^* D$ have different kernel functions. Hence the Martin compactification of D is homeomorphic to D^*. The last assertion now follows from Proposition 2.1. The theorem is proved. $\square$

Using Theorem 3.1, we can show the following theorems in the same way as in [1, Section 4]. We omit the details.

Theorem 3.6. *Let D be a uniformly John domain and let V be an open set and K a compact subset of V intersecting ∂D. Then there are $A > 0$ and $\varepsilon > 0$ depending on D, V and K such that*

$$\left| \frac{u(x)/v(x)}{u(y)/v(y)} - 1 \right| \le A\rho_D(x,y)^\varepsilon \quad \text{for } x, y \in D \cap K,$$

whenever u and v are positive harmonic functions on D, bounded on $D \cap V$ and vanishing q.e. on $\partial D \cap V$. Moreover, the ratio u/v extends to $D^* \cap \pi^{-1}(K)$ as a Hölder continuous function with respect to ρ_D.

This theorem is deduced from the following local version.

Theorem 3.7. *Let D be a uniformly John domain. Then there exist positive constants A and ε depending only on D with the following property: Let $\xi^* \in \partial^* D$ and $R > 0$ be sufficiently small. Suppose u and v are positive bounded harmonic functions on $B_\rho(\xi^*, A_7 R)$ vanishing q.e. on $\partial D \cap \overline{B_\rho(\xi^*, A_7 R)}$. Then*

$$\operatorname*{osc}_{B_\rho(\xi^*,r)} \frac{u}{v} \le A' \left(\frac{r}{R}\right)^\varepsilon \operatorname*{osc}_{B_\rho(\xi^*,R)} \frac{u}{v} \quad \text{for } 0 < r \le R.$$

Similarly, the Martin kernel $K(x,\xi^*)$ for D is Hölder continuous function with respect to ρ_D.

Theorem 3.8. *Let D be a bounded uniformly John domain. If $\xi_1^*, \xi_2^* \in \partial^* D$ and $R \ge 4\rho_D(\xi_1^*, \xi_2^*)$, then*

$$\operatorname*{osc}_{D \setminus B_\rho(\xi^*,R)} \frac{K(\cdot,\xi_1^*)}{K(\cdot,\xi_2^*)} \le A \left(\frac{\rho_D(\xi_1^*,\xi_2^*)}{R}\right)^\varepsilon.$$

Moreover, if $x \in D \setminus B_\rho(\xi_1^*, R)$, then

$$\left| \frac{K(x,\xi_1^*)}{K(x,\xi_2^*)} - 1 \right| \le A \left(\frac{\rho_D(\xi_1^*,\xi_2^*)}{R}\right)^\varepsilon.$$

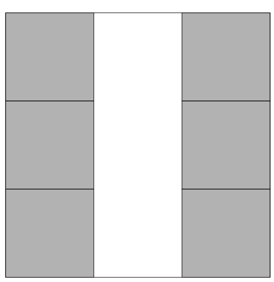
4. Fractal John domain

The main aim of this section is to show that the complement of a certain self-similar fractal is a John domain. Let $\Psi = \{\psi_1, \dots, \psi_\nu\}$ be a finite union of contractive similarities ψ_i, i.e., $|\psi_i(x) - \psi_i(y)| = \lambda_i |x - y|$ for any $x, y \in \mathbb{R}^d$ with $0 < \lambda_i < 1$. We note that each ψ_i is homeomorphism from $\mathbb{R}^d$ to itself, so that set operations and topological operations, such as taking boundary, closure and interior, commute ψ_i. We let $\Psi(E) = \cup_{i=1}^{\nu} \psi_i(E)$. It is known that there is a unique compact set F invariant under Ψ, i.e.,

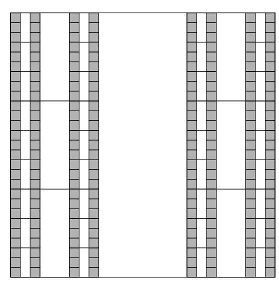
$$F = \Psi(F) = \bigcup_{i=1}^{\nu} \psi_i(F).$$

Moreover, $\Psi^n(K)$ converges to F in the Hausdorff metric for any nonempty compact set K. The set F is the self-similar fractal constructed from $\Psi = \{\psi_1, \dots, \psi_\nu\}$. Let B be a sufficiently large open ball containing F. We are interested in the conditions for $D = B \setminus F$ to be a John domain.

One might think that $B \setminus F$ is a John domain whenever it is connected. This is not the case. The following *filled Cantor set* has a connected complement and yet it is not a John domain. Let $d = 2$ and S a unit square. We divide S into 9 small squares with side $1/3$. We remove 3 small squares in the middle column and repeat the same procedure to the remaining 6 squares. This is equivalent to consider 6 similarities with similitude $1/3$; 4 of them have a vertex of S as a fixed point; the other two shrink and translate S to the midst small squares in the left and right columns. Then $D = B \setminus F$ has arbitrary narrow vertical corridor with length 1, so that it can not be a John domain. See Figure 4.1.



First Step.



Third Step.

FIGURE 4.1. The complement of the filled Cantor set is not a John domain.

Hence, it is worthwhile to find conditions which guarantee that $D = B \setminus F$ is a John domain. In what follows we assume that $\mathrm{int}(F) = \emptyset$ to exclude the trivial case. It is convenient to start with a compact set H whose image under Ψ is included in itself. Then the iteration of Ψ gives a decreasing sequence of compact sets converging to F, i.e.,

$$H \supset \Psi(H) \supset \cdots \supset \Psi^n(H) \supset \cdots \to \bigcap_{n=0}^{\infty} \Psi^n(H) = F.$$

To make it precise, let us start with a compact convex polyhedron H with $\operatorname{int}(H) \neq \emptyset$ and $\Psi(H) \subset H$. Here a set is called a closed convex polyhedron if it is given by a finite intersection of closed half spaces. For fundamental geometrical notions of convex polyhedra we refer to Berger [9, 10] and Grünbaum [15]. If $\operatorname{int}(H) \neq \emptyset$, then there is a unique minimal family of closed half spaces Π_j^+ whose intersection is H. The boundary ∂H consists of $(d-1)$-dimensional compact convex polyhedra L_j whose $(d-1)$-dimensional interiors $\operatorname{int}_{d-1}(L_j)$ are nonempty. Each compact convex polyhedron L_j is given as the intersection of H and Π_j, the boundary of the half space Π_j^+. Thus $\partial H = \cup_j L_j$ and $\operatorname{int}_{d-1}(L_j) \neq \emptyset$. We call L_j and $\operatorname{int}_{d-1}(L_j)$ a closed face and an open face of H, respectively. A subset M of L_j is said to be a subface of L_j. If $\operatorname{int}_{d-1}(M) = M$, then M is said to be an open subface. Observe that open faces of one convex polyhedron are mutually disjoint. We say that Π is a *supporting hyperplane* to H at $x \in \partial H$ if $x \in \Pi$ and Π is the boundary of the closed half space Π^+ including H. We say that $x \in \partial H$ has order α if the intersection of all supporting hyperplanes to H at x is an affine subspace of dimension α (Berger [9, Chapter 11]). We observe that $x \in \operatorname{int}_{d-1}(L_j)$ has order $d-1$. This is equivalent to say that there is a small $\varepsilon > 0$ such that $B(x, \varepsilon) \cap H$ is a half ball. We have the same supporting hyperplane Π at every point of the open face $\operatorname{int}_{d-1}(L_j)$. For simplicity we call Π the supporting hyperplane of the open face $\operatorname{int}_{d-1}(L_j)$. We also say that Π is the supporting hyperplane of the face L_j. Moreover, if M is a nonempty open subface of L_j, then we say that Π is the supporting hyperplane of the open subface M.

We need an assumption which ensures iterative arguments. By I, J, K and so on we denote the multiindices like $(i_1, \ldots, i_n)$ taken from $\{1, \ldots, \nu\}$. By $I|_k$ we denote the truncated index $(i_1, \ldots, i_{\min\{k,n\}})$ and by $I \circ J$ the composition $(i_1, \ldots, i_n, j_1, \ldots, j_m)$ with $J = (j_1, \ldots, j_m)$. Moreover, $I \circ j$ stands for $(i_1, \ldots, i_n, j)$. Let $|I| = n$ be the length of I and write $\psi_I = \psi_{i_1} \circ \cdots \circ \psi_{i_n}$. By $I \subset J$ we mean that $|I| \leq |J|$ and the truncated $J|_{|I|}$ coincides with I. By definition $I = J$ if and only if $I \subset J$ and $I \supset J$. Hereafter, we assume the following nesting axiom which rules out the above filled Cantor set.

Axiom 1. (Nesting Axiom) If $i \neq j$, then

$$\psi_i(H) \cap \psi_j(H) = \psi_i(F) \cap \psi_j(F).$$

In fact, this axiom is equivalent to the following stronger statement..

Lemma 4.1. (Indefinite Nesting) *If $|I| = |J|$ and $I \neq J$, then*

$$\psi_I(H) \cap \psi_J(H) = \psi_I(F) \cap \psi_J(F),$$

and in particular $\psi_I(H) \cap \psi_J(H) \subset F$.

Proof. First, we claim

$$\psi_i(H) \cap F = \psi_i(F). \tag{4.1}$$

It is easy to see that $\psi_i(F) \subset \psi_i(H) \cap F$. Let us prove the opposite inclusion. We have $F = \Psi(F) = \cup_j \psi_j(F)$, so that

$$\psi_i(H) \cap F = \bigcup_j \psi_i(H) \cap \psi_j(F).$$

If $i = j$, then $\psi_i(H) \cap \psi_j(F) = \psi_i(F)$. If $i \neq j$, then $\psi_i(H) \cap \psi_j(F) \subset \psi_i(F) \cap \psi_j(F) \subset \psi_i(F)$ by Axiom 1. Hence (4.1) holds.

Second, we show that (4.1) has a generalization

$$\psi_I(H) \cap F = \psi_I(F). \tag{4.2}$$

Let us prove (4.2) by induction on $n = |I|$. If $n = 1$, then (4.2) is nothing but (4.1). Let $n > 1$ and write $I = (i_1, \ldots, i_n) = i_1 \circ I'$ with $I' = (i_2, \ldots, i_n)$. Then

$$\psi_I(H) \cap F = \psi_{i_1}(\psi_{I'}(H)) \cap F \subset \psi_{i_1}(H) \cap F = \psi_{i_1}(F)$$

by (4.1), so that $\psi_{I'}(H) \cap \psi_{i_1}^{-1}(F) \subset F$. By the induction assumption $\psi_{I'}(H) \cap F = \psi_{I'}(F)$, which, together with the previous inclusion, implies that

$$\psi_{I'}(H) \cap \psi_{i_1}^{-1}(F) = F \cap \psi_{I'}(H) \cap \psi_{i_1}^{-1}(F) = \psi_{I'}(F) \cap \psi_{i_1}^{-1}(F).$$

Hence $\psi_I(H) \cap F = \psi_I(F) \cap F = \psi_I(F)$. Thus (4.2) follows.

Finally, we prove the assertion of the lemma by induction on $n = |I| = |J|$. If $n = 1$, then it is nothing but Axiom 1. Let $n > 1$ and write $I = i_1 \circ I'$ and $J = j_1 \circ J'$ in the same way as in the preceding paragraph. If $i_1 = j_1$, then $I' \neq J'$, so that the induction assumption yields

$$\psi_I(H) \cap \psi_J(H) = \psi_{i_1}(\psi_{I'}(H) \cap \psi_{J'}(H)) = \psi_{i_1}(\psi_{I'}(F) \cap \psi_{J'}(F)) = \psi_I(F) \cap \psi_J(F).$$

If $i_1 \neq j_1$, then

$$\psi_I(H) \cap \psi_J(H) \subset \psi_{i_1}(H) \cap \psi_{j_1}(H) = \psi_{i_1}(F) \cap \psi_{j_1}(F) \subset F$$

by Axiom 1, so that

$$\psi_I(H) \cap \psi_J(H) = \psi_I(H) \cap \psi_J(H) \cap F = \psi_I(F) \cap \psi_J(F)$$

by (4.2). The proof is complete. $\square$

Remark 4.2. We have from Axiom 1

$$\psi_i(\mathrm{int}(H)) \cap \psi_j(\mathrm{int}(H)) = \mathrm{int}(\psi_i(H) \cap \psi_j(H)) \subset \mathrm{int}(F) = \emptyset \quad \text{for } i \neq j.$$

Thus the open set condition follows from our nesting axiom.

Remark 4.3. Lindstrøm [18] defined a similar nesting axiom. Namely, if $|I| = |J|$ and $I \neq J$, then he assumes that

$$\psi_I(F) \cap \psi_J(F) = \psi_I(F_0) \cap \psi_J(F_0),$$

where F_0 is the set of the essential fixed points of Ψ. Thus, $\psi_I(F) \cap \psi_J(F)$ is a finite set in his setting. (Note that he used the letter F for the set of the essential fixed points and the letter E for the fractal.) On the other hand our nesting axiom allows for the intersection to be an infinite set. The usual 3-dimensional Sierpiński gasket (depicted in Figure 1.1) fulfills our Axiom 1 and the above Lindstrøm's axiom. There are fractals which satisfy Axiom 1 and fail to satisfy Lindstrøm's axiom. A typical example is a base-covered 3-dimensional Sierpiński gasket. See Figure 4.2. The bottom three tetrahedra in the fist step intersects each other with a line segment. For the precise definition see the explanation before Proposition 6.5.

We observe that the family of $\psi_I(H)$ has an inclusion property similar to Whitney cubes.

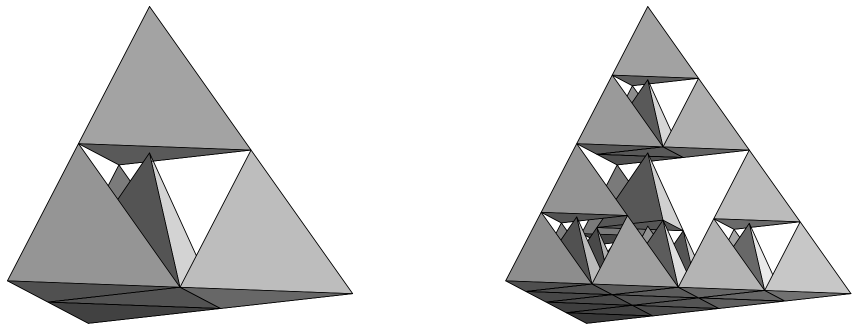
Lemma 4.4. *Let $\psi_I(H) \cap \psi_J(H) \setminus F \neq \emptyset$. Then one of the following holds:*

(i) *$I = J$ and $\psi_I(H) = \psi_J(H)$.*

(ii) *$I \subsetneqq J$ and $\psi_I(H) \supsetneqq \psi_J(H)$.*

(iii) *$I \supsetneqq J$ and $\psi_I(H) \subsetneqq \psi_J(H)$.*

Proof. We assume that $I \neq J$ and show either (ii) or (iii) holds. If $|I| = |J|$, then $\psi_I(H) \cap \psi_J(H) \subset F$ by Lemma 4.1. This contradicts the assumption. Hence we have



First Step. Second Step.

FIGURE 4.2. Base-covered 3-dimensional Sierpiński gasket.

only to consider the case when $|I| \neq |J|$. Without loss of generality, we may assume $|I| < |J|$. Let $J' = J|_{|I|}$. Suppose $J' \neq I$. Then $\psi_I(H) \cap \psi_J(H) \subset \psi_I(H) \cap \psi_{J'}(H) \subset F$ by Lemma 4.1 again. This is a contradiction. Hence $J|_{|I|} = I$. This means $I \subsetneqq J$ and $\psi_I(H) \supsetneqq \psi_J(H)$. The lemma is proved. $\square$

As a result we have the following.

Corollary 4.5. *Let I and J be multiindices. Then*

(i) $I = J \iff \psi_I(H) = \psi_J(H)$.
(ii) $I \subsetneqq J \iff \psi_I(H) \supsetneqq \psi_J(H)$.
(iii) $I \supsetneqq J \iff \psi_I(H) \subsetneqq \psi_J(H)$.

Proof. For every statement " $\Longrightarrow$ " is trivial. For the opposite implication we observe that the condition in the right hand side for each statement implies $\psi_I(H) \cap \psi_J(H) \setminus F \neq \emptyset$, since $\mathrm{int}(F) = \emptyset$ and $\mathrm{int}(H) \neq \emptyset$. Hence the above lemma yields " $\Longleftarrow$ ". $\square$

Let

$$H \setminus \Psi(H) = P^1 \cup \cdots \cup P^\mu,$$

where P^i is a connected component of $H \setminus \Psi(H)$. We call P^i a *pocket* (of generation 0). The following lemma says that the complement of the fractal is decomposed into the union of images of $P^1, \ldots, P^\mu$ under combinations of $\{\psi_1, \ldots, \psi_\nu\}$.

Lemma 4.6. *Let $n \geq 0$. Then*

$$\Psi^n(H) \setminus \Psi^{n+1}(H) = \Psi^n(H \setminus \Psi(H)).$$

Moreover, $\psi_I(P^i)$ is a connected component of $\Psi^n(H) \setminus \Psi^{n+1}(H)$, i.e. if $|I| = |J| = n$ and $(I, i) \neq (J, j)$, then $\psi_I(P^i)$ and $\psi_J(P^j)$ are disconnected. The domain $D = B \setminus F$ has the following decomposition

$$D = O \cup \bigcup_{|I| \geq 0} \bigcup_{i=1}^{\mu} \psi_I(P^i) \quad \textit{disjoint union,} \tag{4.3}$$

where we recall $O = B \setminus H$ and $\psi_I(P^i) = P^i$ if $|I| = 0$.

Proof. Observe

$$\Psi^n(H) \setminus \Psi^{n+1}(H) = \bigcup_{|I|=n} \psi_I(H) \setminus \bigcup_{|J|=n} \psi_J(\Psi(H)) \subset \bigcup_{|I|=n} \psi_I(H \setminus \Psi(H)) = \Psi^n(H \setminus \Psi(H)).$$

For the opposite we need the nesting axiom. Suppose to the contrary, there is a point x in

$$\Psi^n(H \setminus \Psi(H)) \setminus (\Psi^n(H) \setminus \Psi^{n+1}(H)) = (\bigcup_{|I|=n} \psi_I(H \setminus \Psi(H))) \cap (\bigcup_{|J|=n} \psi_J(\Psi(H)).$$

Then there are I, J with $|I| = |J| = n$ such that $x \in \psi_I(H \setminus \Psi(H)) \cap \psi_J(\Psi(H))$. If $I = J$, then $\psi_I^{-1}(x) \in (H \setminus \Psi(H)) \cap \Psi(H) = \emptyset$, a contradiction. If $I \neq J$, then Lemma 4.1 implies that

$$x \in \psi_I(H \setminus \Psi(H)) \cap \psi_J(\Psi(H)) \subset \psi_I(H) \cap \psi_J(H) = \psi_I(F) \cap \psi_J(F) \subset \psi_I(\Psi(H)),$$

a contradiction.

We claim that $\psi_I(P^i)$ and $\psi_J(P^j)$ are disconnected if $(I, i) \neq (J, j)$. If $I = J$, then $i \neq j$. By definition P^i and P^j are disconnected, so that $\psi_I(P^i)$ and $\psi_J(P^j) = \psi_I(P^j)$ are disconnected and the claim follows in this case. Suppose $I \neq J$. Then Lemma 4.1 implies

$$\overline{\psi_I(P^i)} \cap \overline{\psi_J(P^j)} \subset \psi_I(H) \cap \psi_J(H) \subset F \subset \Psi^{n+1}(H). \tag{4.4}$$

On the other hand both $\psi_I(P^i)$ and $\psi_J(P^j)$ are subsets of $\Psi^n(H) \setminus \Psi^{n+1}(H)$, so that $\psi_I(P^i) \cap \psi_J(P^j) = \emptyset$ by (4.4). Thus they are disjoint. Moreover, we have $\psi_I(P^i) \cap \overline{\psi_J(P^j)} = \overline{\psi_I(P^i)} \cap \psi_J(P^j) = \emptyset$. Thus $\psi_I(P^i)$ and $\psi_J(P^j)$ are distinct connected components of $\Psi^n(H) \setminus \Psi^{n+1}(H)$. In particular, $\psi_I(P^i)$ and $\psi_J(P^j)$ are disconnected and the claim follows in this case too. Since $\Psi^n(H) \downarrow F$ as $n \uparrow \infty$, the decomposition of D holds. □

Let $\mathscr{F} = \{O\} \cup \{\psi_I(P^i) : |I| \geq 0, 1 \leq i \leq \mu\}$. Then

$$D = \bigcup_{Q \in \mathscr{F}} Q. \tag{4.5}$$

We call $Q = \psi_I(P^i)$ a *pocket* (of generation $|I|$). Each pocket Q has a unique expression $\psi_I(P^i)$. We let $g(Q) = |I|$, the *generation* of Q. We put $g(O) = -1$ and call O a pocket of generation -1 for convention. By an elementary geometrical observation we see that the interior $\mathrm{int}(P^i)$ is a uniformly John domain. Since each pocket $Q \neq O$ is one of the images of $P^1, \ldots, P^\mu$ under similarities, we have the following.

Lemma 4.7. *For each pocket Q the interior $\mathrm{int}(Q)$ is a uniformly John domain with universal uniformly John constant.*

In view of (4.5) and the above lemma we can conclude D is a John domain if pockets are well connected. To describe the connection among pockets we divide their boundaries into two parts:

Definition 4.8. Let Q be a pocket of generation $g(Q) = n$. We let

$$e(Q) = \begin{cases} \partial Q \setminus \Psi^{n+1}(H) & \text{if } Q \neq O, \\ \emptyset & \text{if } Q = O, \end{cases} \qquad i(Q) = \begin{cases} \partial Q \cap \Psi^{n+1}(H) & \text{if } Q \neq O, \\ \partial H & \text{if } Q = O. \end{cases}$$

We say that $e(Q)$ (resp. $i(Q)$) is the *exterior* (resp. *interior*) part of the boundary of Q.

We assume the following.

Axiom 2. (Pocket Axiom) For each pocket P^i of generation 0 we assume:

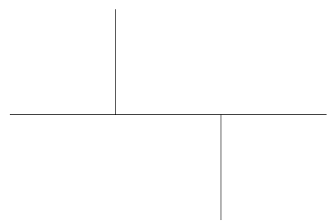
(i) $e(P^i) \neq \emptyset$ and it consists of finitely many *open subfaces* of H.

(ii) $i(P^i)$ consists of finitely many *faces* of some polyhedra appearing in $\Psi(H)$.

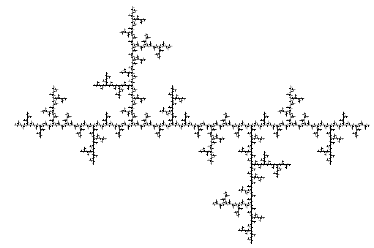
(iii) $i(P^i) \cap \partial H \subset F$.

As an example we give a picture for Example 9.7 of Falconer [13]. See Figure 4.3. This is a fractal constructed from a *generator* of five line segments. We start with the convex hull H of these five line segments. We have five similarities corresponding to five line segments of the generator. The difference $H \setminus \Psi(H)$ consists of four pockets P^1, P^2, P^3 and P^4 of generation 0. The pockets P^1 and P^3 are congruent; the pockets P^2 and P^4 are congruent. Each pocket P^i has $e(P^i)$ of one open line segment. The pocket P^1 has $i(P^1)$ of five line segments and the pocket P^2 has $i(P^2)$ of three line segments. We observe that Axioms 1 and 2 hold.

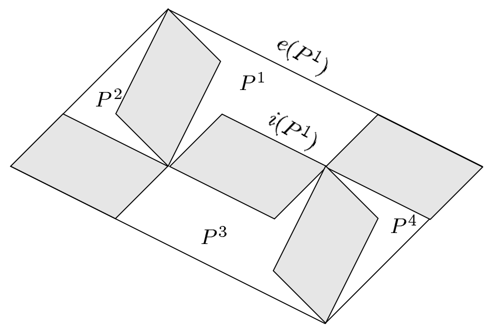
Remark 4.9. It is easy to see that the 3-dimensional Sierpiński gasket and the base-covered gasket both fulfill Axiom 2. See Section 6 for details on these examples.



(a) Generator of fractal.



(b) Fifth step.



(c) $H \setminus \Psi(H) = P^1 \cup P^2 \cup P^3 \cup P^4$.

FIGURE 4.3. Example 9.7 of Falconer [13]. See also Figure 4.4 below for a part of the next generation.

Remark 4.10. Observe from Lemma 4.6 that $e(Q) = \psi_I(e(P^i))$ and $i(Q) = \psi_I(i(P^i))$ for $Q = \psi_I(P^i)$, and that the above properties are inherited:

(i) $e(Q) \neq \emptyset$ consists of finitely many *open subfaces* L_Q of $\psi_I(H)$.
(ii) $i(Q)$ consists of finitely many *faces* M_Q of some polyhedra appearing in $\psi_I(\Psi(H))$.
(iii) $i(Q) \cap \partial H \subset i(Q) \cap \partial \psi_I(H) \subset F$.

Here the first inclusion of (iii) follows from $i(Q) \subset \psi_I(H) \subset H$. We call L_Q and M_Q a face of $e(Q)$ and a face of $i(Q)$, respectively. Since P^i is a connected component of $H \setminus \Psi(H)$, it follows that $e(P^i) \subset P^i$, so that $e(Q) \subset Q$ by Lemma 4.6. On the other hand, $i(P^i) \cap P^i = \emptyset$, and hence $i(Q) \cap Q = \emptyset$.

The following lemma gives fundamental relationship among $e(Q)$ and $i(Q)$ for pockets Q.

Lemma 4.11. *Let Q and R be distinct pockets. Then $e(Q) \cap e(R) = \emptyset$ and $i(Q) \cap i(R) \subset F$. Moreover,*

$$\overline{Q} \cap \overline{R} \setminus F = (e(Q) \cap i(R)) \cup (i(Q) \cap e(R))$$

and either the set $e(Q) \cap i(R)$ or the set $i(Q) \cap e(R)$ is empty.

Proof. Since $e(Q) \subset Q$ by Remark 4.10, it follows from Lemma 4.6 that $e(Q) \cap e(R) = \emptyset$. Let us prove the second assertion. We claim

$$i(P^i) \cap i(P^j) \subset F \quad \text{if } i \neq j. \tag{4.6}$$

In view of Axiom 2 (iii) we have $i(P^i) \cap i(P^j) \cap \partial H \subset F$. Hence, it is sufficient to show

$$\partial P^i \cap \partial P^j \cap \mathrm{int}(H) \subset F. \tag{4.7}$$

Let x be a point of the set in the left hand side and take $\varepsilon > 0$ such that $B(x, \varepsilon) \subset \mathrm{int}(H)$. Since x is a limit point of distinct connected components P^i and P^j of $H \setminus \Psi(H)$, it follows from the connectedness of $B(x, \varepsilon) \setminus \psi_\alpha(H)$ and Axiom 1 that there exist distinct α and β such that

$$x \in \partial\psi_\alpha(H) \cap \partial\psi_\beta(H) \subset \psi_\alpha(H) \cap \psi_\beta(H) \subset F.$$

This implies (4.7) and hence (4.6).

Now we prove $i(Q) \cap i(R) \subset F$ for the general case. If one pocket, say R, is O, then

$$i(Q) \cap i(R) = i(Q) \cap \partial H \subset F$$

by Remark 4.10 (iii). Let $Q = \psi_I(P^i)$ and $R = \psi_J(P^j)$. Suppose $|I| = |J|$. If $I \neq J$, then Lemma 4.1 implies

$$i(Q) \cap i(R) \subset \psi_I(H) \cap \psi_J(H) \subset F.$$

If $I = J$, then $i \neq j$, so that

$$i(Q) \cap i(R) = \psi_I(i(P^i) \cap i(P^j)) \subset \psi_I(F) \subset F$$

by (4.6). To complete the proof we let $|I| \neq |J|$. We may assume that $|I| > |J|$ and hence $\Psi^{|I|}(H) \subset \Psi^{|J|+1}(H)$. We have from Remark 4.10 (iii)

$$\begin{aligned} i(Q) \cap i(R) &= (\partial\psi_I(H) \cap i(Q) \cap i(R)) \cup (\mathrm{int}(\psi_I(H)) \cap i(Q) \cap i(R)) \\ &\subset F \cup (\mathrm{int}(\Psi_{|I|}(H)) \cap \partial\Psi^{|J|+1}(H)) = F. \end{aligned}$$

Moreover, Lemma 4.6 yields

$$\begin{aligned}\overline{Q} \cap \overline{R} \setminus F &= (\mathrm{int}(Q) \cup e(Q) \cup i(Q)) \cap (\mathrm{int}(R) \cup e(R) \cup i(R)) \setminus F \\ &= (e(Q) \cap i(R)) \cup (i(Q) \cap e(R)).\end{aligned}$$

Finally, the last assertion follows from Lemma 4.4. The lemma is proved. □

Now we introduce a relationship among pockets Q.

Definition 4.12. Let Q and R be pockets. If $e(Q) \cap i(R) \neq \emptyset$, then we write $Q \precsim R$ and say that Q is a *child* of R and that R is a *mother* of Q. If either $Q \precsim R$ or $Q \succsim R$ holds, then we write $Q \sim R$ and say that Q and R are linked. (Note that $Q \not\precsim Q$.) Moreover, we put

$$[Q, R] = \begin{cases} e(Q) \cap i(R) & \text{if } Q \precsim R, \\ i(Q) \cap e(R) & \text{if } Q \succsim R \end{cases}$$

and call $[Q, R]$ the *door* between Q and R. If there is a chain $Q_1 \precsim Q_2 \precsim \cdots \precsim Q_k$, then we write $Q_1 \prec Q_k$.

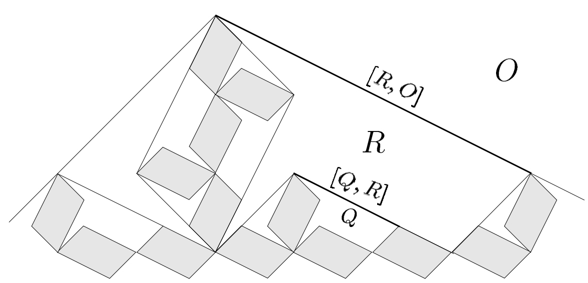


FIGURE 4.4. Example 9.7 of Falconer [13]: $Q \precsim R \precsim O$, the doors $[Q, R]$ and $[R, O]$.

We readily have the following lemma from Lemmas 4.4 and 4.11.

Lemma 4.13. *Let Q and R be distinct pockets. Then the following statements holds:*

(i) *$Q \sim R$ if and only if $\overline{Q} \cap \overline{R} \setminus F \neq \emptyset$.*
(ii) *If $Q \sim R$ and a curve $\gamma \subset Q \cup R$ connects a point in Q to a point in R, then γ goes through the door $[Q, R]$, i.e., $\gamma \cap [Q, R] \neq \emptyset$.*
(iii) *If $Q = \psi_I(P^i) \precsim R = \psi_J(P^j)$, then $I \supsetneq J$ and $\psi_I(H) \subsetneq \psi_J(H)$.*
(iv) *If $g(Q) \geq 0$, then $Q \prec O$, i.e., there is a chain $Q = Q_1 \precsim \cdots \precsim Q_k = O$.*

(v) *If* $Q_1 \precsim \cdots \precsim Q_k$, *then* $\operatorname{diam}(Q_1) \le A\lambda^k \operatorname{diam}(Q_k)$, *where* $\lambda = \max\{\lambda_1, \dots, \lambda_\nu\} < 1$ *with* λ_j *being the similitude for* ψ_i. *In particular, for every pocket* Q

$$\operatorname{diam}(\bigcup_{Q' \prec Q} Q') \le A \operatorname{diam}(Q),$$

where $A > 1$ *is independent of* Q.

In Remark 4.10 we have observed that $e(Q)$ consists of open subfaces of $\psi_I(H)$, where $Q = \psi_I(P^i)$. Now we use Axiom 2 (i) and (ii) to show that if $Q \precsim R$, then $e(Q)$ consists of open subfaces disjoint from $i(R)$ and open subfaces included in some face of $i(R)$. More precisely, we have the following lemma.

Lemma 4.14. *Let* $Q \precsim R$. *Let* L_Q *be an open face of* $e(Q)$ *such that* $L_Q \cap i(R) \neq \emptyset$. *Then there exists a face* M_R *of* $i(R)$ *such that* $L_Q \subset M_R$.

Proof. By definition there is a face M_R of $i(R)$ such that $L_Q \cap M_R \neq \emptyset$. We, in fact, show that $L_Q \subset M_R$. Let $Q = \psi_I(P^i)$ and $R = \psi_J(P^j)$. In view of Remark 4.10 we see that M_R is a face of $\psi_{J\circ\alpha}(H)$ for some $\alpha \in \{1, \dots, \nu\}$. Since

$$\emptyset \neq L_Q \cap M_R \subset \psi_I(H) \cap \psi_{J\circ\alpha}(H) \setminus F,$$

it follows from Lemma 4.4 that either $I = J \circ \alpha$ or $I \supsetneqq J \circ \alpha$. Suppose first $I = J \circ \alpha$. Then L_Q and M_R are an open subface and a face of the same convex polyhedron $\psi_I(H)$. Hence, $L_Q \cap M_R \neq \emptyset$ implies $L_Q \subset M_R$. Suppose next $I \supsetneqq J \circ \alpha$. Since L_Q intersects the face M_R of $\psi_{J\circ\alpha}(H)$, it follows that $L_Q \cap \partial\psi_{J\circ\alpha}(H) \neq \emptyset$. Let Π be the supporting hyperplane of L_Q. If $\Pi \cap \operatorname{int}(\psi_{J\circ\alpha}(H)) \neq \emptyset$, then $\Pi \cap \partial(\psi_{J\circ\alpha}(H))$ would be the $(d-2)$-dimensional boundary of the $(d-1)$-dimensional convex polyhedron $\Pi \cap \psi_{J\circ\alpha}(H)$. Since $L_Q \subset \psi_I(H) \subset \psi_{J\circ\alpha}(H)$, a point of $L_Q \cap \partial\psi_{J\circ\alpha}(H)$ could not be a $(d-1)$-dimensional interior point of L_Q. This would contradict the fact that L_Q is an open subface. Hence, $\Pi \cap \operatorname{int}(\psi_{J\circ\alpha}(H)) = \emptyset$ and so $L_Q \subset \partial\psi_{J\circ\alpha}(H)$. Now, L_Q and M_R are an open subface and a face of the same convex polyhedron $\psi_{J\circ\alpha}(H)$. Hence, $L_Q \cap M_R \neq \emptyset$ implies $L_Q \subset M_R$. □

Let L^i be a face of either $e(P^i)$ or $i(P^i)$ for a pocket P^i of generation 0. It is easy to see that

$$\operatorname{dist}(x, \partial P^i \setminus L^i) \ge \frac{1}{A} \operatorname{dist}(x, \partial_{d-1}(L^i)) \quad \text{for } x \in L^i_j,$$

where $\partial_{d-1}(L^i) = \overline{L^i} \setminus L^i$ stands for the the boundary of the face L^i in the supporting hyperplane of L^i. We note that the constant A can be taken independent of P^i and L^i

since there are only finitely many polyhedra and faces. In view of Lemma 4.1, the above properties are inherited by each pocket Q: If L_Q is a face of either $e(Q)$ or $i(Q)$, then

$$\operatorname{dist}(x, \partial Q \setminus L_Q) \geq \frac{1}{A} \operatorname{dist}(x, \partial_{d-1}(L_Q)) \quad \text{for } x \in L_Q, \tag{4.8}$$

where A is independent of Q and L_Q. Moreover, $\operatorname{diam}(L_Q)$ is comparable to $\operatorname{diam}(Q)$. This observation, together with Lemma 4.14, yields the following lemma.

Lemma 4.15. *Let $Q \precsim R$ and let L_Q be a face of $e(Q)$ included in the door $[Q, R] = e(Q) \cap i(R)$. Then there is a point $\xi \in L_Q$ such that*

$$\delta_{Q \cup R}(\xi) \geq \frac{1}{A} \operatorname{diam}(Q).$$

Moreover, the door $[Q, R]$ consists of such L_Q and $[Q, R] \subset \operatorname{int}(Q \cup R)$.

Proof. We infer from Lemma 4.14 that $L_Q \subset \operatorname{int}(Q \cup R)$ and

$$\partial(Q \cup R) \subset (\partial Q \setminus L_Q) \cup (\partial R \setminus L_Q).$$

With the aid of (4.8) we find a point $\xi \in L_Q$ such that

$$\operatorname{dist}(\xi, \partial Q \setminus L_Q) \geq \frac{1}{A} \operatorname{dist}(\xi, \partial_{d-1}(L_Q)) \geq \frac{1}{A} \operatorname{diam}(Q).$$

By Lemma 4.14 there is a face M_R of $i(R)$ such that $L_Q \subset M_R$. Then

$$\begin{aligned}
\operatorname{dist}(\xi, \partial R \setminus L_Q) &\geq \min\{\operatorname{dist}(\xi, \partial R \setminus M_R), \operatorname{dist}(\xi, M_R \setminus L_Q)\} \\
&\geq \frac{1}{A} \min\{\operatorname{dist}(\xi, \partial_{d-1}(M_R)), \operatorname{dist}(\xi, \partial_{d-1}(L_Q))\} \\
&\geq \frac{1}{A} \operatorname{diam}(Q)
\end{aligned}$$

by (4.8). The last assertion follows from Lemma 4.14. The proof is complete. □

Now we are in a position to prove the Johnness under Axioms 1 and 2.

Theorem 4.16. *Assume Axioms 1 and 2. Then D is a John domain.*

Proof. Let $x_0 \in O$ be fixed. It is sufficient to show that each point $x \in D$ can be connected to x_0 by a cigar curve. In view of (4.5) it is sufficient to show that an arbitrary point x in an arbitrary pocket Q can be connected to x_0 by a cigar curve. If $Q = O$, then this is trivial. Hence we assume $g(Q) \geq 0$. By Lemma 4.13 we obtain a chain

$$Q = Q_1 \precsim \cdots \precsim Q_k = O.$$

By Lemma 4.15 we find points $\xi_i \in [Q_i, Q_{i+1}]$ such that

$$\delta_D(\xi_i) \geq \delta_{Q_i \cup Q_{i+1}}(\xi_i) \geq \frac{1}{A} \operatorname{diam}(Q_i) \tag{4.9}$$

for $1 \leq i \leq k-1$. Let $\xi_0 = x$ and $\xi_k = x_0$ as a convention. Since each $\operatorname{int}(Q_i)$ is a uniformly John domain with universal John constant by Lemma 4.7, we find cigar curves $\widetilde{\xi_{i-1}\xi_i}$ connecting ξ_{i-1} and ξ_i in Q_i for $1 \leq i \leq k$. We claim that

$$\gamma = \widetilde{\xi_0\xi_1} \cup \cdots \cup \widetilde{\xi_{k-1}\xi_k}$$

is a distance-carrot curve connecting $x = \xi_0$ and $x_0 = \xi_k$, i.e.,

$$\delta_D(z) \geq \frac{1}{A}|x - z| \tag{4.10}$$

for all $z \in \gamma$. Then the equivalence among the length-cigar-condition, the diameter-cigar-condition and the distance-cigar-condition ([19, Lemma 2.7] and [21, Theorem 2.18]) proves that D is a John domain.

Now let us prove (4.10). Since $\widetilde{\xi_0\xi_1} \cup \cdots \cup \widetilde{\xi_{i-1}\xi_i}$ is covered by the chain $Q_1 \precsim \cdots \precsim Q_i$, it follows from Lemma 4.13 (v) and (4.9) that

$$|x - \xi_i| \leq \operatorname{diam}(\widetilde{\xi_0\xi_1} \cup \cdots \cup \widetilde{\xi_{i-1}\xi_i}) \leq A \operatorname{diam}(Q_i) \leq A\delta_D(\xi_i). \tag{4.11}$$

This means that (4.10) holds at $z = \xi_i$ for $i = 0, \ldots, k$. Let us consider other $z \in \gamma$. If $z \in B(\xi_i, \frac{1}{2}\delta_D(\xi_i))$, then $\delta_D(z) \geq \frac{1}{2}\delta_D(\xi_i)$ and by (4.11),

$$|x - z| \leq |x - \xi_i| + |\xi_i - z| < |x - \xi_i| + \frac{1}{2}\delta_D(\xi_i) \leq (A + \frac{1}{2})\delta_D(\xi_i).$$

Hence (4.10) holds for $z \in \gamma \cap B(\xi_i, \frac{1}{2}\delta_D(\xi_i))$ and hence for $z \in \gamma \cap (\cup_{i=0}^k B(\xi_i, \frac{1}{2}\delta_D(\xi_i)))$. On the other hand, if $z \notin B(\xi_i, \frac{1}{2}\delta_D(\xi_i))$, then

$$|x - z| \leq |x - \xi_i| + |\xi_i - z| \leq A\delta_D(\xi_i) + |\xi_i - z| \leq (2A + 1)|\xi_i - z|$$

by (4.11). Since $\widetilde{\xi_{i-1}\xi_i}$ is a cigar curve in Q_i, it follows that

$$\delta_D(z) \geq \delta_{Q_i}(z) \geq \frac{1}{A}\min\{|\xi_{i-1} - z|, |z - \xi_i|\} \geq \frac{1}{A}|x - z|$$

for $z \in \widetilde{\xi_{i-1}\xi_i} \setminus (B(\xi_i, \frac{1}{2}\delta_D(\xi_i)) \cup B(\xi_{i-1}, \frac{1}{2}\delta_D(\xi_{i-1})))$. Hence (4.10) holds for $z \in \gamma \setminus (\cup_{i=0}^k B(\xi_i, \frac{1}{2}\delta_D(\xi_i)))$. Thus (4.10) holds for all $z \in \gamma$. The proof is complete. $\square$

5. Fractal uniformly John domain

It is much more difficult to show that D is a uniformly John domain than a John domain, because we have to treat arbitrary two points in D and to connect them by a cigar curve with diameter bounded by the internal metric between the points up to a multiplicative constant. To this end we shall, from now on, assume further two axioms, viz. Axioms 3 and 4. These axioms look rather technical. We do not know whether they are sharp or not. One of them is the following.

Axiom 3. (Linkage Axiom) Suppose distinct pockets R and S have a common child Q, i.e., $Q \precsim R$ and $Q \precsim S$. Then R and S are linked, $R \sim S$, i.e., either $R \precsim S$ or $R \succsim S$ holds.

Remark 5.1. We can view the structure of pockets as a graph where the pockets are nodes, and the connections are given by the linkage, $\sim$. The linkage axiom above guarantees that this graph is a *chordal graph*. If each pocket has a unique mother, then we have a tree (with infinite degree).

Recall the definition of the internal metric in the introduction. We use the same definition for a general arcwise connected set E, i.e.,

$$\rho_E(x,y) = \inf\{\mathrm{diam}(\gamma) : \ \gamma \text{ is a curve connecting } x \text{ and } y \text{ in } E\}$$

for $x, y \in E$. As before Lemma 2.3, we extend $\rho_E(x,y)$ up to the closure of E with respect to ρ_E. By definition ρ_E is decreasing with respect to E, i.e., if $E' \subset E$, then

$$\rho_E(x,y) \le \rho_{E'}(x,y) \quad \text{for } x, y \in E'.$$

We assume the following axiom, which gives a reverse inequality in some sense.

Axiom 4. (Stability of the Internal Metric Axiom) We assume that

$$\rho_Q(x,y) \le A\rho_D(x,y) \quad \text{for } x, y \in Q. \tag{5.1}$$

Moreover, we assume that if $Q \precsim R$, then

$$\rho_{Q\cup R}(x,y) \le A\rho_D(x,y) \quad \text{for } x, y \in Q \cup R. \tag{5.2}$$

Here A is a universal constant independent of Q and R.

Remark 5.2. We note that Axiom 4 is equivalent to the following: If $x, y \in Q$ (resp. $x, y \in Q \cup R$) are connected by a curve $\gamma \subset D$, then they are connected by a curve $\widetilde{\gamma} \subset Q$ (resp. $\widetilde{\gamma} \subset Q \cup R$) with $\operatorname{diam}(\widetilde{\gamma}) \le A \operatorname{diam}(\gamma)$. In view of Lemma 4.13, if $x \in Q$ and $y \in R$, then $\widetilde{\gamma}$ goes through the door $[Q, R]$, i.e., $\widetilde{\gamma} \cap [Q, R] \neq \emptyset$.

With the aid of Lemma 4.6, it is sufficient to verify (5.1) only for the pockets $P^1, \dots, P^\mu$ of generation 0. On the other hand, (5.2) is not so obvious, since there are infinitely many essentially different possibilities of a pair $Q \precsim R$. However, it can be verified for particular examples, including the 3-dimensional Sierpiński gasket, the base-covered 3-dimensional Sierpiński gasket and the 2-dimensional Sierpiński gasket with gap. See Section 6.

Theorem 5.3. *Assume Axioms 1, 2, 3 and 4. Then D is a uniformly John domain.*

We prepare the proof of Theorem 5.3 with the following two lemmas.

Lemma 5.4. *There exists an integer $N \ge 3$ such that every chain $Q_1 \precsim \cdots \precsim Q_N$ of pockets of length N has a pocket Q_j, $3 \le j \le N$, with*

$$\operatorname{dist}(Q_1, Q_j) \ge \operatorname{diam}(Q_1).$$

Proof. Suppose $Q_1 \precsim \cdots \precsim Q_N$. Let $U = \{x \in \mathbb{R}^d : \operatorname{dist}(x, Q_1) \le \operatorname{diam}(Q_1)\}$ and $V = \{x \in \mathbb{R}^d : \operatorname{dist}(x, Q_1) \le 2 \operatorname{diam}(Q_1)\}$. Then $|V| \le A \operatorname{diam}(Q_1)^d$. On the other hand Lemmas 4.7 and 4.13 imply

$$|Q_j| \ge \frac{1}{A} \operatorname{diam}(Q_j)^d \ge \frac{1}{A} \operatorname{diam}(Q_1)^d.$$

Hence, the number of j such that $Q_j \subset V$ is bounded. Suppose $Q_j \setminus V \neq \emptyset$ and $Q_j \cap U \neq \emptyset$. Then the uniform Johnness implies that there is a ball lying in $Q_j \cap (V \setminus U)$ with radius comparable to $\operatorname{diam}(Q_1)$. Hence, the number of such j is bounded. Thus there is a pocket Q_j, $3 \le j \le N$, with $Q_j \cap U = \emptyset$, and hence $\operatorname{dist}(Q_1, Q_j) \ge \operatorname{diam}(Q_1)$, provided N is sufficiently large. $\square$

The following lemma asserts that every curve γ connecting two points x and y in D can be modified so as to be covered by a chain with a certain property. Axioms 3 and 4 are used only in this lemma.

Lemma 5.5. *Suppose two points $x, y \in D$ are connected by a curve $\gamma \subset D$. Then there are a curve $\widetilde{\gamma} \subset D$ connecting x and y with $\operatorname{diam}(\widetilde{\gamma}) \le A \operatorname{diam}(\gamma)$, and a chain $Q_1 \sim \cdots \sim Q_k$ such that $\widetilde{\gamma} \subset Q_1 \cup \cdots \cup Q_k$, $\widetilde{\gamma} \cap [Q_i, Q_{i+1}] \neq \emptyset$ for $1 \le i \le k-1$ and*

$Q_1 \precsim \cdots \precsim Q_m \succsim \cdots \succsim Q_k$ *with* $1 \le m \le k$*. Here the constant* A *is independent of* x*,* y *and* γ*.*

Proof. Since $B \setminus \Psi^n(H)$ is an increasing sequence of open sets converging to $D = B \setminus F$, it follows from the compactness that $\gamma \subset B \setminus \Psi^n(H)$ for some n. Hence, we find finitely many mutually disjoint pockets $\{Q, \dots, R\}$ whose union covers γ. Without loss of generality, we may assume that each pocket intersects γ. By induction on the number of $\{Q, \dots, R\}$ we claim that there exist $Q_1, \dots, Q_k \in \{Q, \dots, R\}$ such that $Q_1 \sim \cdots \sim Q_k$, $x \in Q_1$ and $y \in Q_k$. If both x and y belong to the same pocket in $\{Q, \dots, R\}$, then the claim trivially holds. Now we assume that $\{Q, \dots, R\}$ has at least two pockets and x and y belong to different pockets. ¿From $\{Q, \dots, R\}$ we find a pocket, say Q_1, such that $x \in Q_1$. Then $y \notin Q_1$. Let $t_1 = \sup\{t : z(t) \in Q_1\}$, where $z = z(t)$, $0 \le t \le 1$, is a parameterization of γ such that $z(0) = x$ and $z(1) = y$. Then $0 \le t_1 \le 1$ and $x_1 = z(t_1) \in \partial Q_1 = e(Q_1) \cup i(Q_1)$. Hence, from $\{Q, \dots, R\}$ we find a pocket, say Q_2, such that $Q_1 \sim Q_2$ and $x_1 \in \gamma \cap [Q_1, Q_2]$. If $t_1 = 1$, then $x_1 = y$ and $Q_1 \sim Q_2$ is the required chain. Suppose $t_1 < 1$. Since $[Q_1, Q_2] \subset \operatorname{int}(Q_1 \cup Q_2)$ by Lemma 4.15, we find t_2 such that $t_1 < t_2 < 1$ and $x_2 = z(t_2) \in Q_2$. Then the subcurve $z = z(t)$, $t_2 \le t \le 1$, is covered by $\{Q, \dots, R\} \setminus \{Q_1\}$. By induction we can extract a chain $Q_2 \sim \cdots \sim Q_k$ from $\{Q, \dots, R\} \setminus \{Q_1\}$ such that $x_2 \in Q_2$ and $y \in Q_k$. Now $Q_1 \sim Q_2 \sim \cdots \sim Q_k$ is a required chain. Thus the claim is proved by induction. Note that $\gamma \cap Q_i \neq \emptyset$ for $i = 1, \dots, k$ and yet the union $Q_1 \cup \cdots \cup Q_k$ may no longer cover γ.

Next, we remove *small* pockets from the chain $Q_1 \sim \cdots \sim Q_k$. We say that Q_i is removable if $2 \le i \le k-1$ and $Q_{i-1} \succsim Q_i \precsim Q_{i+1}$. If there is a removable Q_i, then we remove it from the chain $Q_1 \sim \cdots \sim Q_k$. By Axiom 3 we have either $Q_1 \sim \cdots \sim Q_{i-1} \sim Q_{i+1} \sim \cdots \sim Q_k$ or $Q_1 \sim \cdots \sim Q_{i-1} = Q_{i+1} \sim \cdots \sim Q_k$. Hence we may assume that there is no removable Q_i in the chain $Q_1 \sim \cdots \sim Q_k$; in other words, $Q_1 \precsim \cdots \precsim Q_m \succsim \cdots \succsim Q_k$. Note that $\gamma \cap Q_i \neq \emptyset$ for $i = 1, \dots, k$.

Finally we construct a modified curve $\widetilde{\gamma} \subset Q_1 \cup \cdots \cup Q_k$ with the required properties. At this stage Axiom 4 plays an important role. Let $\xi_i \in \gamma \cap Q_i$ for $i = 1, \dots, k$. In particular, we may let $\xi_1 = x$ and $\xi_k = y$. By Axiom 4 we find a curve $\widetilde{\xi_i \xi_{i+1}} \subset Q_i \cup Q_{i+1}$ such that

$$\operatorname{diam}(\widetilde{\xi_i \xi_{i+1}}) < A \rho_D(\xi_i, \xi_{i+1}) \le A \operatorname{diam}(\gamma)$$

for $i = 1, \ldots, k-1$. Since $\xi_i \in \gamma$, it follows that $\widetilde{\xi_i \xi_{i+1}} \subset B(x, (1+A)\operatorname{diam}(\gamma))$, so that

$$\widetilde{\gamma} = \widetilde{\xi_1 \xi_2} \cup \cdots \cup \widetilde{\xi_{k-1}\xi_k} \subset B(x, (1+A)\operatorname{diam}(\gamma)).$$

Hence $\operatorname{diam}(\widetilde{\gamma}) \le 2(1+A)\operatorname{diam}(\gamma)$. Of course the curve $\widetilde{\gamma}$ connects x to y and $\widetilde{\gamma} \subset Q_1 \cup \cdots \cup Q_k$. In view or Lemma 4.13 we see that the curve $\widetilde{\xi_i \xi_{i+1}}$ goes through the door $[Q_i, Q_{i+1}]$ and so does $\widetilde{\gamma}$, i.e., $\widetilde{\gamma} \cap [Q_i, Q_{i+1}] \neq \emptyset$. The lemma is proved. $\square$

Now we are in a position to prove the main theorem.

Proof of Theorem 5.3. Let $x, y \in D$ and suppose γ connects x and y in D. It is sufficient to show that there is a cigar curve $\widehat{\gamma}$ connecting x and y with $\operatorname{diam}(\widehat{\gamma}) \le A \operatorname{diam}(\gamma)$, where A is independent of x, y and γ. In view of Lemma 5.5, we may assume that γ is covered by a chain $Q_1 \sim \cdots \sim Q_k$ with $x \in Q_1$, $y \in Q_k$, $Q_1 \precsim \cdots \precsim Q_m \succsim \cdots \succsim Q_k$ and $\gamma \cap [Q_i, Q_{i+1}] \neq \emptyset$ for $1 \le i \le k-1$. Take $x_i \in \gamma \cap [Q_i, Q_{i+1}]$. The point x_i may be close to the boundary. In order to construct a cigar curve, we shall choose another point $x_i^* \in [Q_i, Q_{i+1}]$ which is far from the boundary and yet close to x_i. To this end let

$$\ell = \max_{1 \le i \le k} \operatorname{diam}(\gamma \cap \overline{Q_i}).$$

We claim

$$\ell \le \operatorname{diam}(\gamma) \le A\ell. \tag{5.3}$$

The first inequality is obvious. Let $N \ge 3$ be as in Lemma 5.4. We have

$$\operatorname{diam}\left(\gamma \cap \Big(\bigcup_{i=m-N+2}^{m+N-2} \overline{Q_i}\Big)\right) \le \sum_{i=m-N+2}^{m+N-2} \operatorname{diam}(\gamma \cap \overline{Q_i}) \le (2N-3)\ell,$$

where $Q_i = \emptyset$ for $i < 1$ and for $i > k$ as a convention.

Now let us estimate $\operatorname{diam}(\gamma \cap (\cup_{i=1}^{m-N+1} \overline{Q_i}))$ in case $m \ge N$. Since m may be arbitrarily large, the above summation estimate does not work. Instead, by Lemmas 4.13 and 5.4 we have

$$\operatorname{diam}\left(\gamma \cap \Big(\bigcup_{i=1}^{m-N+1} \overline{Q_i}\Big)\right) \le A \operatorname{diam}(Q_{m-N+1}) \le A \operatorname{dist}(Q_{m-N+1}, Q_j)$$

for some j, $m-N+3 \le j \le m$. Recall we have $x_i \in \gamma \cap [Q_i, Q_{i+1}]$. Since both x_{i-1} and x_i belong to $\gamma \cap \overline{Q_i}$, it follows that $|x_{i-1} - x_i| \le \ell$, so that

$$\operatorname{dist}(Q_{m-N+1}, Q_j) \le |x_{m-N+1} - x_j| \le |x_{m-N+1} - x_{m-N+2}| + \cdots + |x_{j-1} - x_j| \le (N-1)\ell.$$

Hence $\operatorname{diam}(\gamma \cap (\cup_{i=1}^{m-N+1}\overline{Q_i})) \le A\ell$. If $k \ge m+N-1$, then similarly

$$\operatorname{diam}\left(\gamma \cap (\bigcup_{i=m+N-1}^{k} \overline{Q_i})\right) \le A\ell.$$

Collecting the above inequalities, we obtain the second inequality of (5.3). Now we construct a cigar curve $\widehat{\gamma}$ in $\cup_{z\in\gamma} B(z, A\ell)$ by modifying γ, where $A \ge 1$ is independent of x, y and γ. In view of Lemma 4.15 we find points x_i^* in the door $[Q_i, Q_{i+1}]$ such that

$$\begin{aligned} &\delta_D(x_i^*) \ge \delta_{Q_i \cup Q_{i+1}}(x_i^*) \ge \frac{1}{A}\min\{\ell, \operatorname{diam}(Q_i), \operatorname{diam}(Q_{i+1})\}, \\ &\rho_D(x_i^*, x_i) \le \rho_{Q_i \cup Q_{i+1}}(x_i^*, x_i) \le A\ell \end{aligned} \tag{5.4}$$

for $1 \le i \le k-1$. See Figure 5.1. As a convention we let $x_0^* = x$ and $x_k^* = y$. Observe

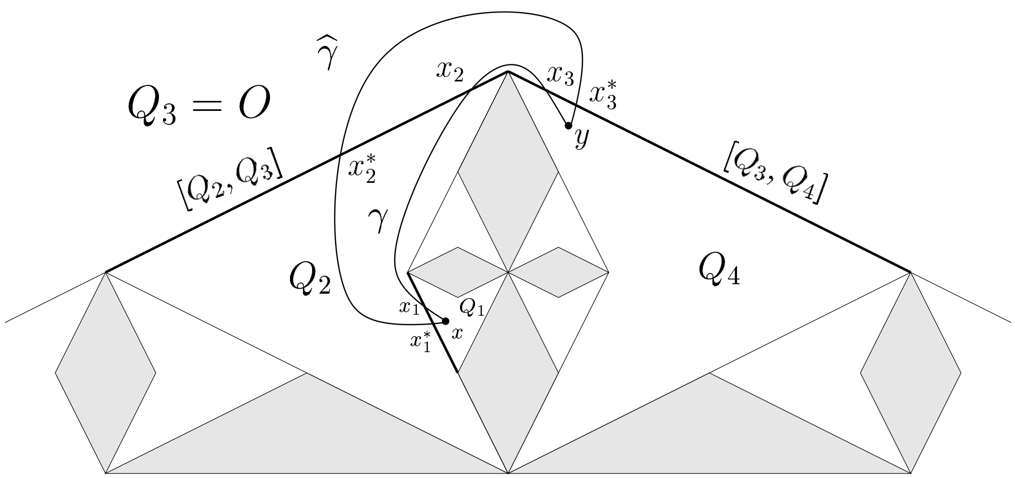


FIGURE 5.1. An illustration of the proof of Theorem 5.3 for the example in Figure 6.5 in Section 6. The case γ is covered by $Q_1 \precsim Q_2 \underset{\sim}{\precsim} Q_3 \succsim Q_4$. Note $\operatorname{dist}(Q_1, Q_3) \ge \operatorname{diam}(Q_1)$. The maximum ℓ of $\operatorname{diam}(\gamma \cap \overline{Q_i})$ is taken by Q_2 not by the biggest pocket Q_3.

from Axiom 4 that

$$\rho_{Q_i}(x_{i-1}^*, x_i^*) \le A\rho_D(x_{i-1}^*, x_i^*) \le A(\rho_D(x_{i-1}^*, x_{i-1}) + \rho_D(x_{i-1}, x_i) + \rho_D(x_i, x_i^*)) \le A\ell$$

for $i = 1, \ldots, k$. Since each $\mathrm{int}(Q_i)$ is a uniformly John domain by Lemma 4.7, we can find, by Lemma 2.3, a cigar curve $\widetilde{x_{i-1}^* x_i^*}$ such that

$$
\begin{aligned}
&\delta_D(z) \geq \delta_{Q_i}(z) \geq \frac{1}{A} \min\{|x_{i-1}^* - z|, |z - x_i^*|\} \quad \text{for all } z \in \widetilde{x_{i-1}^* x_i^*}, \\
&\mathrm{diam}(\widetilde{x_{i-1}^* x_i^*}) \leq A\ell.
\end{aligned}
\tag{5.5}
$$

Finally, we show that $\widehat{\gamma} = \widetilde{x_0^* x_1^*} \cup \cdots \cup \widetilde{x_{k-1}^* x_k^*}$ is a required cigar curve connecting x and y. The second assertions of (5.4) and (5.5) show that $\widehat{\gamma} \subset \cup_{z \in \gamma} B(z, A\ell)$, so that by (5.3),

$$
\mathrm{diam}(\widehat{\gamma}) \leq A\ell. \tag{5.6}
$$

We claim

$$
\delta_D(x_i^*) \geq \begin{cases} \frac{1}{A}|x - x_i^*| & \text{if } 0 \leq i \leq m-1, \\ \\ \frac{1}{A}|y - x_i^*| & \text{if } m \leq i \leq k. \end{cases} \tag{5.7}
$$

Let us prove (5.7) for $0 \leq i \leq m-1$. Since $x_0^* = x$, (5.7) is obvious for $i = 0$. Suppose $1 \leq i \leq m-1$. Since $\widetilde{x_0^* x_1^*} \cup \cdots \cup \widetilde{x_{i-1}^* x_i^*}$ is covered by the chain $Q_1 \precsim \cdots \precsim Q_i$, it follows from Lemma 4.13 that $|x - x_i^*| = |x_0^* - x_i^*| \leq A \,\mathrm{diam}(Q_i) \leq A \,\mathrm{diam}(Q_{i+1})$. It also follows from (5.6) that $|x - x_i^*| \leq \mathrm{diam}(\widehat{\gamma}) \leq A\ell$. Hence the first assertion of (5.4) yields (5.7) for $1 \leq i \leq m-1$. Similarly, we can prove (5.7) for $m \leq i \leq k$. Now in the same way as in the proof of Theorem 4.16 we can prove

$$
\delta_D(z) \geq \frac{1}{A} \min\{|x - z|, |z - y|\} \quad \text{for all } z \in \widehat{\gamma}.
$$

This, together with (5.6), shows that $\widehat{\gamma}$ is a required cigar curve connecting x and y. The proof is complete. □

6. Examples of fractal uniformly John domains

In this section we verify the axioms stated in the previous sections for particular examples, including Example 9.7 of Falconer [13], the 3-dimensional Sierpiński gasket, the base-covered 3-dimensional Sierpiński gasket and the 2-dimensional Sierpiński gasket with gap. Main technical difficulty arises for Axioms 3 and 4. We first give sufficient conditions for Axiom 3, which can be verified for particular examples. The following is an obvious one.

Proposition 6.1. *Suppose each pocket Q with $g(Q) \geq 0$ has just one mother. Then Axiom 3 holds.*

Example 9.7 of Falconer [13] satisfies the assumption of this proposition. Unfortunately, the Sierpiński gasket does not satisfies the assumption. To see this and to show Axiom 4, let us illustrate the relationship $Q \precsim R$ for the 3-dimensional Sierpiński gasket. See Figure 6.1. Let H be a regular tetrahedron with vertices v_1, v_2, v_3 and v_4. Let ψ_i be the similarity composed of translation and dilation of factor $1/2$ with fixed point at v_i for $i = 1, \ldots, 4$. We call $i \in \{1, 2, 3, 4\}$ a label. Observe that if $i \neq j$, then two small tetrahedra $\psi_i(H)$ and $\psi_j(H)$ have the common point $\psi_i(v_j) = \psi_j(v_i)$. More generally, we write $\psi_{i_1 \cdots i_n} = \psi_I$ and $(j; i_1, \ldots, i_n) = \psi_{i_1 \cdots i_n}(v_j)$ if $I = (i_1, \ldots, i_n)$. We observe that

$$(j; i_1, \ldots, i_n) = (i_1; j, i_2, \ldots, i_n)$$

and this is the common point of $\psi_{i_1 \cdots i_n}(H)$ and $\psi_{j, i_2, \ldots, i_n}(H)$. This observation determines the combinatorial relationship for the Sierpiński gasket, which has been studied by many authors, particularly in probabilistic context.

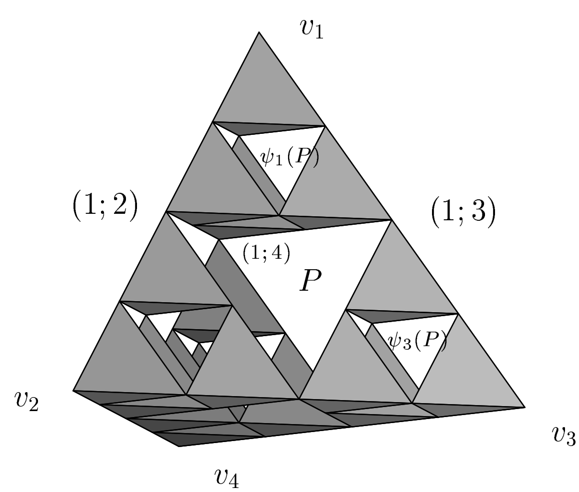


FIGURE 6.1. Relationship for the Sierpiński gasket. Each ψ_i has the fixed point v_i.

In this paper, we are interested in the complement of the Sierpiński gasket. Let $\Psi = \{\psi_1, \psi_2, \psi_3, \psi_4\}$ be the set valued mapping and observe that $H \backslash \Psi(H)$ consists of a regular octahedron, called a pocket P. We see that $i(P)$ consists of four regular triangles with vertices $\{(1;2), (1;3), (1;4)\}, \ldots, \{(4;1), (4;2), (4;3)\}$, respectively. Similarly, $e(P)$ consists of four regular triangles with vertices $\{(1;2), (2;3), (3;1)\}, \ldots, \{(2;3), (3;4), (4;2)\}$, respectively. In particular, $e(P)$ has a subface in each face of H. Our relationship '$\precsim$'

among pockets up to the second generation is as follows:

$$\begin{aligned}&P, \psi_i(P), \psi_{ij}(P) \precsim O = B \setminus H,\\ &\psi_i(P), \psi_{ij}(P) \precsim P \quad \text{for } i \neq j,\\ &\psi_{ii}(P) \not\precsim P.\end{aligned}$$

Relationship for pockets of general order can be obtained by the following consideration. Let $\{i, j, k, \ell\}$ be an enumeration of the labels $\{1, 2, 3, 4\}$. By $\triangle(i, j, k)$ we denote the triangle with vertices v_i, v_j and v_k. Since ψ_i is the composition of translation and dilation, it follows that $\psi_i(\triangle(i, j, k))$ is a triangle lying in $\triangle(i, j, k)$. On the other hand, $\psi_\ell(\triangle(i, j, k))$ is a triangle parallel to $\triangle(i, j, k)$ with

$$\operatorname{dist}(\psi_\ell(\triangle(i, j, k)), \triangle(i, j, k)) = \frac{1}{\sqrt{6}} \operatorname{diam}(H),$$

so that

$$\operatorname{dist}(\psi_\ell(H), \triangle(i, j, k)) = \frac{1}{\sqrt{6}} \operatorname{diam}(H).$$

More generally, we have

$$\begin{aligned}&\psi_I(\triangle(i, j, k)) \subset \triangle(i, j, k) && \text{if } \ell \notin I,\\ &\operatorname{dist}(\psi_I(H), \triangle(i, j, k)) \geq \tfrac{1}{\sqrt{6}} \operatorname{diam}(\psi_I(H)) && \text{if } \ell \in I.\end{aligned} \tag{6.1}$$

Since $i(O)$ (resp. $i(P)$) is the union of four triangles of the form $\triangle(i, j, k)$ (resp. $\psi_\ell(\triangle(i, j, k))$), we obtain the following proposition from the above observation.

Proposition 6.2. *The relationship '$\precsim$' for the 3-dimensional Sierpiński gasket is characterized as follows:*

(i) $\psi_I(P) \precsim O$ *if and only if there is a label not appearing in* I.

(ii) $\psi_I(P) \precsim \psi_J(P)$ *with* $I = (i_1, \dots, i_n)$ *and* $J = (j_1, \dots, j_m)$ *if and only if* $n > m$ *and* i_{m+1} *does not appear in* $\{i_{m+2}, \dots, i_n\}$.

Now we observe that the Sierpiński gasket satisfies the assumption of the following lemma.

Lemma 6.3. *Assume that*

$$e(Q) \subset \widetilde{e}(R) \cup \operatorname{int}(\psi_J(H)) \tag{6.2}$$

for every pair of pockets $Q = \psi_I(P^i)$ *and* $R = \psi_J(P^j)$ *with* $Q \precsim R$, *where* $\widetilde{e}(R)$ *is the union of all open faces of* $\partial\psi_J(H)$ *intersecting* $e(R)$. *Then Axiom 3 holds.*

Remark 6.4. Suppose $e(R)$ intersects every open face of $\partial\psi_J(H)$. Then $\widetilde{e}(R) = \partial\psi_J(H)$ and (6.2) holds for every $Q \precsim R$. A typical example is the 3-dimensional Sierpiński gasket.

Proof. Suppose distinct pockets $R = \psi_J(P^j)$ and $S = \psi_K(P^k)$ satisfy $Q \precsim R$ and $Q \precsim S$ with $Q = \psi_I(P^i)$. By Lemma 4.4 we have $J \subsetneqq I$, $K \subsetneqq I$ and $Q \subset \psi_I(H) \subset \psi_J(H) \cap \psi_K(H)$. Moreover, either $J \subsetneqq K$ or $J \supsetneqq K$ holds. Without loss of generality, we may assume that $J \supsetneqq K$. Then

$$i(S) \cap \mathrm{int}(\psi_J(H)) = \emptyset,$$

since $i(S) \subset \partial\Psi_{|K|+1}(H)$ and $\mathrm{int}(\psi_J(H)) \subset \mathrm{int}(\Psi_{|J|}(H)) \subset \mathrm{int}(\Psi_{|K|+1}(H))$. By definition $e(Q) \cap i(S) \neq \emptyset$, so that $\widetilde{e}(R) \cap i(S) \neq \emptyset$ by (6.2). Observe that $\widetilde{e}(R)$ consists of open faces $\widetilde{L_R}$ of $\partial\psi_J(H)$ and that $i(S)$ consists of faces M_S of $\partial\Psi_{|K|+1}(H)$. Since $\widetilde{e}(R) \cap i(S) \neq \emptyset$, we find $\widetilde{L_R}$ and M_S such that $\widetilde{L_R} \cap M_S \neq \emptyset$, which automatically implies that $\widetilde{L_R} \subset M_S$. By definition there is an open face $L_R \subset e(R)$ which is included in $\widetilde{L_R}$. Hence $L_R \subset M_S$, which means $R \precsim S$. The lemma is proved. $\square$

The base-covered 3-dimensional Sierpiński gasket provides an example satisfying (6.2) and yet $\widetilde{e}(R) \neq \partial\psi_J(H)$. See Figures 4.2 and 6.2. Let us give the precise definition. We use the same notation as before Proposition 6.2. Besides the similarities $\psi_1, \ldots, \psi_4$, we consider one more similarity ψ_5 which maps H to the small tetrahedron with base $\triangle((2;3)(3;4)(4;2))$. Observe that the base $M = \triangle(234)$ of H is covered by the bases of $\psi_2(H), \ldots, \psi_5(H)$. We note that ψ_5 involves a rotation. We assume that the rotation–axis goes thorough v_1 and is perpendicular to M. The set of labels for the base-covered 3-dimensional Sierpiński gasket is $\{1, \ldots, 5\}$ and $\Psi = \{\psi_1, \ldots, \psi_5\}$. We see that $H \setminus \Psi(H) = P$ with P the octahedron minus $\psi_5(H)$. Observe that M and its image $\psi_I(M)$ lie in the fractal F. Hence $e(P)$ consists of three regular triangles $\triangle((1;2)(2;3)(3;1))$, $\triangle((1;2)(2;4)(4;1))$ and $\triangle((1;3)(3;4)(4;1))$. Moreover, $\widetilde{e}(P) = \partial H \setminus M$ consists of three regular triangles $\triangle(123)$, $\triangle(124)$ and $\triangle(134)$. Observe that $\psi_5(P) \subset \psi_5(H) \subset \mathrm{int}(H) \cup M$, so that $\psi_5(P) \not\precsim O$. From this observation as well as Lemma 4.6 and Proposition 6.2, we obtain the following proposition.

Proposition 6.5. *The relationship '$\precsim$' for the base-covered 3-dimensional Sierpiński gasket is characterized as follows:*

(i) $\psi_I(P) \precsim O$ *if and only if the label 5 does not appear in I and one of the labels in $\{2, 3, 4\}$ does not appear in I.*

(ii) $\psi_I(P) \precsim \psi_J(P)$ *with* $I = (i_1, \dots, i_n)$ *and* $J = (j_1, \dots, j_m)$ *if and only if* $n > m$, $i_{m+1} \neq 1$ *and the label 5 and* i_{m+1} *do not appear in* $\{i_{m+2}, \dots, i_n\}$.

In particular, (6.2) *and hence Axiom 3 hold.*

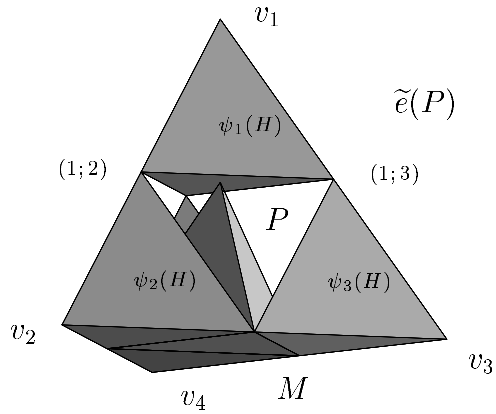


FIGURE 6.2. Base-covered Sierpiński gasket.

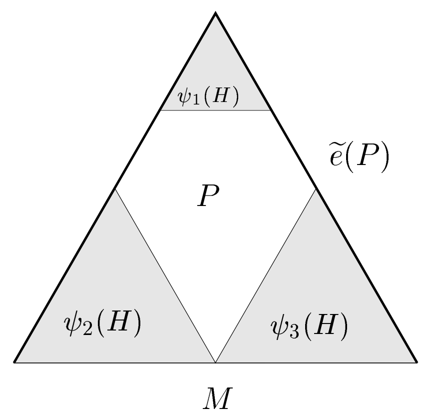


FIGURE 6.3. Sierpiński gasket with gap.

Remark 6.6. A similar situation occurs for a 2-dimensional Sierpiński gasket with gap. We start with the regular triangle H and similarities ψ_1, ψ_2 and ψ_3 composed of translation and dilation each of which has a fixed point at the corresponding vertex of H. Let us suppose ψ_1 corresponds to the top of H and its dilation factor is less than 1/2. Both ψ_2 and ψ_3 have dilation factor 1/2. Then the bottom line segment M of H is covered by $\psi_2(H)$ and $\psi_3(H)$. We have one pocket P of generation 0 with $\widetilde{e}(P) = \partial H \setminus M$. It is again easy to show (6.2) and Axiom 3 for this example. See Figure 6.3.

Now let us consider Axiom 4. First we prepare the following lemma.

Lemma 6.7. *Suppose every pocket* P^i *of generation 0 is convex. Then* (5.2) *holds for* $Q \precsim R$ *with* $g(R) \geq 0$.

Proof. By assumption and Lemma 4.6 every pocket of nonnegative generation is convex. Suppose $Q \precsim R$ with $g(R) \geq 0$. Then there is a face L of $e(Q)$ lying in $i(R)$ by Lemma 4.14. Let Π be the supporting hyperplane of L. Since Q and R are convex, it follows that Π separates them. Take $x \in Q$ and $y \in R$. Observe that

$$\operatorname{dist}(x, L) \leq A \operatorname{dist}(x, \Pi) \leq A|x - y|.$$

Hence we find $z \in L$ with $|x-z| \le A|x-y|$. By convexity the line segments $\overline{xz}$ and $\overline{zy}$ lie in Q and R, respectively. This implies

$$\rho_{Q\cup R}(x,y) \le |x-z| + |z-y| \le 2|x-z| + |x-y| \le (2A+1)|x-y|.$$

Since $|x-y| \le \rho_D(x,y)$, this proves (5.2) for $g(R) \ge 0$. $\square$

Proposition 6.8. *The 3-dimensional Sierpiński gasket satisfies Axiom 4.*

Proof. Since the unique pocket P of generation 0 is an octahedron, a convex polyhedron, it follows that $\rho_P(x,y) = |x-y|$, so that (5.1) holds. Lemma 6.7 shows (5.2) for $Q \precsim R$ with $g(R) \ge 0$. Hence, it is sufficient to show (5.2) for $Q \precsim O$. Let $x \in Q$ and $y \in O$. Since $y \notin H$, there is a face M of H whose supporting hyperplane Π separates H and y. If $e(Q)$ has a face lying in Π, then the same argument as in Lemma 6.7 shows that

$$\rho_{Q\cup O}(x,y) \le A|x-y| \le A\rho_D(x,y),$$

so that (5.2) follows in this case. Suppose $e(Q)$ has no face lying in Π. Then, it follows from (6.1) that

$$|x-y| \ge \operatorname{dist}(Q,y) \ge \frac{1}{\sqrt{6}} \operatorname{diam}(\psi_I(H)) = \frac{1}{\sqrt{3}} \operatorname{diam}(Q),$$

where $Q = \psi_I(H)$. Now we take $\widetilde{x} \in e(Q) \subset i(O)$. Then

$$\begin{aligned}\rho_{Q\cup O}(x,y) &\le \rho_Q(x,\widetilde{x}) + \rho_O(\widetilde{x},y) \le |x-\widetilde{x}| + A|\widetilde{x}-y| \le (1+A)|x-\widetilde{x}| + A|x-y| \\ &\le (1+A)\operatorname{diam}(Q) + A|x-y| \le A'|x-y| \le A'\rho_D(x,y)\end{aligned}$$

with $A' = \sqrt{3}(1+A) + A$. Thus (5.2) holds in this case too. $\square$

Corollary 6.9. *Let F be the 3-dimensional Sierpiński gasket. Then $D = B \setminus F$ is a uniform domain.*

Proof. ¿From Remarks 4.3 and 4.9 we have Axioms 1 and 2. Axioms 3 and 4 follow from Remark 6.4 and Proposition 6.8. Hence Theorem 5.3 implies that D is a uniformly John domain. It is easy to see that the internal metric $\rho_D(x,y)$ and the Euclidean metric are comparable, so that D is a uniform domain. $\square$

Proposition 6.10. *The 3-dimensional base-covered Sierpiński gasket satisfies Axiom 4.*

Proof. It is easy to show (5.1). Let us prove (5.2). We use the same notation as in Proposition 6.5. First we prove (5.2) for $\psi_I(P) = Q \precsim O$. In view of Proposition 6.5, the label 5 does not appear in I. Let $x \in Q$ and $y \in O$. Let Π_M be the supporting hyperplane of the base $M = \triangle(234)$ and let Π_M^+ be the half space bounded by Π_M and including H. Suppose $y \in \Pi_M^+ \setminus H$. Then there is a supporting hyperplane $\Pi \neq \Pi_M$ of a face of H separating x and y. Since the label 5 does not appear in I, the same argument as in Proposition 6.8 yields (5.2) in this case. Suppose $y \notin \Pi_M^+$. Then every curve $\widetilde{xy}$ connecting x and y in D must intersect Π_M. Hence we find $\widetilde{y} \in \Pi_M \cap O$ such that

$$\rho_D(x,\widetilde{y}) \le \rho_D(x,y) + \varepsilon \quad \text{and} \quad \rho_D(\widetilde{y},y) \le \rho_D(x,y) + \varepsilon$$

for $\varepsilon > 0$. We also observe that $|\widetilde{y} - y| = \rho_D(\widetilde{y},y) = \rho_{Q\cup O}(\widetilde{y},y)$. We have

$$\rho_{Q\cup O}(x,y) \le \rho_{Q\cup O}(x,\widetilde{y}) + \rho_{Q\cup O}(\widetilde{y},y) \le A\rho_D(x,\widetilde{y}) + |\widetilde{y} - y| \le 2A(\rho_D(x,y) + \varepsilon),$$

where the second inequality follows from the first case applied to x and $\widetilde{y}$. Since $\varepsilon > 0$ is arbitrary, we have (5.2) in this case, too. Second we prove (5.2) for $Q = \psi_I(P) \precsim R = \psi_J(P)$. If the label 5 does not appear in $I \setminus J$, the same argument as in Proposition 6.8 shows (5.2). If the label 5 appears in $I \setminus J$, then it must appear at the first place and the same argument as above can be made by the pull back ψ_J^{-1}. Hence (5.2) holds in any case. □

Corollary 6.11. *Let F be the 3-dimensional base-covered Sierpiński gasket. Then $D = B \setminus F$ is a uniformly John domain.*

Proof. ¿From Remarks 4.3 and 4.9 we have Axioms 1 and 2. Propositions 6.5 and 6.10 prove Axioms 3 and 4. Hence Theorem 5.3 completes the proof. □

Remark 6.12. In contrast the usual the 3-dimensional Sierpiński gasket, the base-covered Sierpiński gasket is *not* a uniform domain since $\rho_D(x,y)$ and $|x - y|$ are not comparable.

If the domain D is *simply connected* in the following sense, then Axiom 4 can be verified rather easily.

Proposition 6.13. *Suppose each pocket Q with $g(Q) \ge 0$ has just one direct predecessor. If* (5.1) *holds, then Axiom 4 holds.*

Proof. Let $Q \precsim R$ and let $x \in Q$ and $y \in R$. Let $\gamma \subset D$ be a curve connecting x to y with parameterization: $z = z(t)$, $0 \le t \le 1$, $z(0) = x$ and $z(1) = y$. Put $\widetilde{t} = \sup\{t : z(t) \in Q\}$. Then $\widetilde{x} = z(\widetilde{t}) \in e(Q) \cap i(R)$ by assumption. Observe that

$$\rho_D(x,\widetilde{x}) \le \operatorname{diam}(\gamma) \quad \text{and} \quad \rho_D(\widetilde{x},y) \le \operatorname{diam}(\gamma),$$

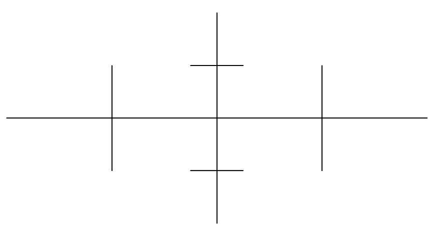
so that by (5.1)

$$\rho_{Q\cup R}(x,y) \le \rho_Q(x,\widetilde{x}) + \rho_R(\widetilde{x},y) \le A(\rho_D(x,\widetilde{x}) + \rho_D(\widetilde{x},y)) \le 2A\,\mathrm{diam}(\gamma).$$

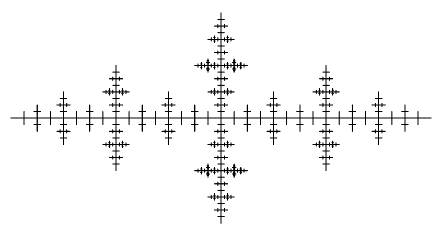
Taking the infimum with respect to γ, we obtain (5.2). $\square$

The hypotheses of Proposition 6.13 hold for many *simply connected* fractal domains. For example the complement of the fractal of Example 9.7 of Falconer [13] (see Figure 4.3) satisfies the hypotheses.

Another example of a simply connected fractal is given as the closure of the union of horizontal and vertical line segments as follows: We identify $\mathbb{R}^2$ and $\mathbb{C}$ and write $z = x+iy$ for a generic point. By $[z,w]$ we denote the closed line segment connecting z and w. We start with four line segments $[0,1]$, $[0,\frac{i}{2}]$, $[0,-1]$ and $[0,-\frac{i}{2}]$. At the second stage we add four line segments $[\frac{1}{2}-\frac{i}{4},\frac{1}{2}+\frac{i}{4}]$, $[-\frac{1}{8}+\frac{i}{4},\frac{1}{8}+\frac{i}{4}]$, $[-\frac{1}{2}-\frac{i}{4},-\frac{1}{2}+\frac{i}{4}]$ and $[-\frac{1}{8}-\frac{i}{4},\frac{1}{8}-\frac{i}{4}]$, each of which perpendicularly bisects the first line segment in this order. We repeat the same procedure and take the closure of the union of the resulting line segments. See Figure 6.4.



(a) Second step.



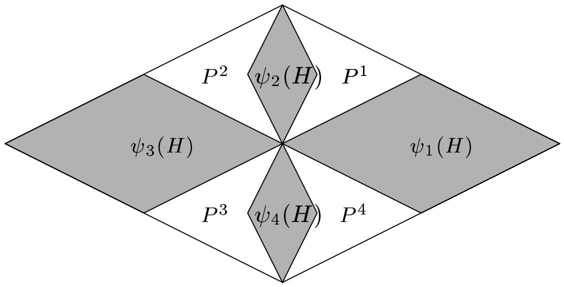
(b) Fifth step.

FIGURE 6.4. Fractal given as the closure of line segments.

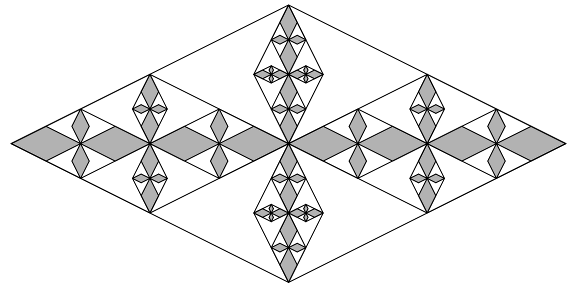
This fractal is actually given as the self-similar fractal of $\Psi = \{\psi_1,\psi_2,\psi_3,\psi_4\}$, where $\psi_1(z) = \frac{1}{2}z+\frac{1}{2}$, $\psi_2(z) = \frac{i}{4}z+\frac{i}{4}$, $\psi_3(z) = \frac{1}{2}z-\frac{1}{2}$ and $\psi_4(z) = \frac{i}{4}z-\frac{i}{4}$. The system Ψ has two different scaling factors $\frac{1}{2}$ and $\frac{1}{4}$. We observe that the rhombus with vertices at 1, $\frac{i}{2}$, -1 and $-\frac{i}{2}$ satisfies $\Psi(H) \subset H$ and the difference $H \setminus \Psi(H)$ consists of four pockets P^1, P^2, P^3 and P^4 of generation 0. All of them are congruent to each other. Each pocket P^i has $e(P^i)$ of one line segment. Every pocket Q of nonnegative generation has just one mother. See Figure 6.5.

Corollary 6.14. *Assume Axioms 1 and 2. Suppose each pocket Q with $g(Q) \ge 0$ has just one direct predecessor and*

$$\rho_Q(x,y) \le A|x-y| \quad \textit{for } x,y \in Q$$



(a) $H \setminus \Psi(H) = P^1 \cup P^2 \cup P^3 \cup P^4$.



(b) Third step.

FIGURE 6.5. D is given as the union of pockets with unique mother.

with A independent of a pocket Q. Then $D = B \setminus F$ is a uniformly John domain. In particular, if F is the fractal of Example 9.7 of Falconer [13] *or the fractal explained above, then $D = B \setminus F$ is a uniformly John domain.*

Proof. Propositions 6.1 and 6.13 prove Axioms 3 and 4. Hence Theorem 5.3 completes the proof. □

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