MARTIN BOUNDARY POINTS OF JOHN DOMAINS AND UNIONS OF CONVEX SETS

HIROAKI AIKAWA, KENTARO HIRATA, AND TORBJÖRN LUNDH

Dedicated to Professor Matts Essén on the occasion of his 70th birthday

Abstract. We show that a John domain has finitely many minimal Martin boundary points at each Euclidean boundary point. The number of minimal Martin boundary points is estimated by the John constant. In particular, if the John constant is bigger than $\sqrt{3}/2$, then there are one or two minimal Martin boundary points at each Euclidean boundary point. For a special John domain represented as the union of convex sets we give a sufficient condition for the Martin boundary and the Euclidean boundary to coincide.

1. Introduction

Let D be a domain in $\mathbb{R}^n$ with $n \geq 2$. Let $\delta_D(x) = \operatorname{dist}(x, \partial D)$ and $x_0 \in D$. We say that D is a John domain with John constant $c_J > 0$ and John center at x_0 if each $x \in D$ can be joined to x_0 by a rectifiable curve γ such that

$$\delta_D(y) \geq c_J \ell(\gamma(x,y)) \quad \text{for all } y \in \gamma, \tag{1}$$

where $\gamma(x,y)$ is the subarc of γ from x to y and $\ell(\gamma(x,y))$ is the length of $\gamma(x,y)$. Since we are interested in the boundary behavior, we may replace x_0 by a compact subset K_0 of D. We call such a domain a *general John domain with center K_0 and John constant c_J*. A general John domain with John constant c_J is a John domain with John constant $c'_J \leq c_J$. Several interesting domains studied in connection with the Martin boundary fall into this category:

- (A) Let F be a compact set on a hyperplane and let B be an open ball containing F. Then $B \setminus F$ (a Denjoy domain) is a general John domain with John constant 1. (See Benedicks [**?**]).
- (B) Let Σ be a Lipschitz surface and F a compact set on Σ. Let B be an open ball containing F. Then $B \setminus F$ (a Lipschitz Denjoy domain) is a general John domain. Moreover, if Σ is given as the graph of a Lipschitz function with Lipschitz constant k, then the John constant of $B \setminus F$ is $1/\sqrt{k^2+1}$. (See Ancona [**?**, **?**] and Chevallier [**?**]).
- (C) A planar domain with boundary lying on the union of finitely many rays leaving the origin is called a sectorial domain (Cranston-Salisbury [**?**]). A sectorial domain

---

2000 *Mathematics Subject Classification.* 31B05, 31B25, 31C35.

Key words and phrases. John domain, convex set, Martin boundary, quasihyperbolic metric, Carleson estimate, Domar's theorem, tract, weak boundary Harnack principle.

This work was supported in part by Grant-in-Aid for Scientific Research (A) (No. 11304008), (B) (No. 12440040) and Exploratory Research (No. 13874023) Japan Society for the Promotion of Science.

is a general John domain with John constant $c_J = \sin(\theta/2)$, where θ is the smallest angle between two rays. A higher dimensional analogue of a sectorial domain is called a quasi-sectorial domain (Lömker [?]). A quasi-sectorial domain is a general John domain.

(D) The union of a family of open balls with the same radius is a general John domain with John constant 1, provided it is connected (Ancona [?]).

(E) The complement of a certain self similar fractal is a general John domain (Aikawa-Lundh-Mizutani [?, Section 4]).

In [?] the first author showed that the Martin boundary of a bounded uniform domain consists of minimal boundary points and it is homeomorphic to the Euclidean boundary; in [?] the first and the third authors and Mizutani showed that the Martin boundary of a uniformly John domain consists of minimal boundary points and it is homeomorphic to the ideal boundary with respect to the internal metric. No exterior conditions are assumed both in [?, ?]. Bonk, Heinonen and Koskela [?] called a uniformly John domain an inner uniform domain. In a very general framework of Gromov hyperbolicity, but under the additional assumption of the existence of a strong barrier, they identified the Martin boundary of an inner uniform domain. The existence of a strong barrier is an exterior condition. The usage of strong barriers to the Martin boundary was first introduced by Ancona [?, ?]. See [?] for the relationship between a strong barrier and other exterior conditions, such as the capacity density condition.

The Martin boundary of a John domain is much more complicated; it may admit a non minimal boundary point. Our first purpose of this paper is to show that a general John domain has finitely many minimal Martin boundary points at each Euclidean boundary point. Moreover, the number of minimal Martin boundary points is estimated in terms of the John constant.

Theorem 1. *Let D be a general John domain with John constant c_J and generalized John center K_0. Let $\xi \in \partial D$.*

(i) *The number of minimal Martin boundary points at ξ is bounded by a constant depending only on the John constant c_J.*

(ii) *If $c_J > \sqrt{3}/2$, then there are one or two minimal Martin boundary points at ξ .*

Remark 1. Let D be a sectorial domain whose boundary near the origin lies on three equally distributed rays leaving the origin. Then D is a general John domain with John constant $\sin(\pi/3) = \sqrt{3}/2$. There may be three different minimal Martin boundary points over the origin. This simple example shows that the bound $c_J > \sqrt{3}/2$ in Theorem 1 is sharp. Note that the same bound $c_J > \sqrt{3}/2$ also applies to the higher dimensional case.

Remark 2. Theorem 1 generalizes some parts of [?], [?, ?], [?], [?] and [?]. One of the main interests of these papers was to give a criterion for the number of minimal Martin boundary points at a fixed Euclidean boundary point (via Kelvin transform for [?]). Such a criterion seems to be very difficult for a general John domain, since the boundary may disperse at every point (See e.g. [?, Figure 4.1 (b)]).

Our second purpose is to find a certain class of John domains whose boundary points have one minimal Martin boundary point. In view of Benedicks' work on a Denjoy domain ([?]), we observe that the John constant c_J is not sufficient to give a condition for a boundary point to have one minimal Martin boundary point. We need some other information. Ancona [**?**, Théorème] gave a condition for the union of a family of open balls with the same radius to have one minimal Martin boundary point at each Euclidean boundary point. By $B(x,r)$ we denote the open ball with center at x and radius r. Let x and y be distinct points in $\mathbb{R}^n$ and $\theta > 0$. We denote by $\Gamma_\theta(x,y)$ the open circular cone $\{z \in \mathbb{R}^n : \angle zxy < \theta\}$ with vertex at x, axis $\overline{xy}$ and aperture θ. Ancona says that a domain D is *admissible* if

(A1) D is the union of a family of open balls with the same radius ρ_0.
(A2) Let $\xi \in \partial D$. If D includes two open balls B_1 and B_2 with radius ρ_0 tangential to each other at ξ, then D includes a truncated circular cone $\Gamma_\theta(\xi,y) \cap B(\xi,r)$ for some $\theta > 0$, $r > 0$ and y in the hyperplane tangent to B_i at ξ.

Theorem A (Ancona)**.** *Let D be a bounded admissible domain. Then every Euclidean boundary point of D has one Martin boundary point and it is minimal. Moreover, the Martin boundary of D is homeomorphic to the Euclidean boundary.*

Let us generalize both (A1) and (A2). As observed previously, (A1) implies that D is a general John domain with John constant 1. We would like to consider general convex sets rather than balls with the same radius. They need not to be congruent. Observe that Ancona's condition (A2) implies that two balls B_1 and B_2 are *connected* by a truncated cone $\Gamma_\theta(\xi,y)\cap B(\xi,r)$. As a result, the union of truncated cones $\Gamma_{\theta'}(\xi,y)\cap B(\xi,r')$ included in D is connected for $0 < \theta' \le \theta$, i.e.,

$$\bigcup_{\substack{y\in D,\\ \Gamma_{\theta'}(\xi,y)\cap B(\xi,r')\subset D}} \Gamma_{\theta'}(\xi,y) \cap B(\xi,r') \text{ is connected,}$$

provided $r' > 0$ is sufficiently small. In view of this observation, we generalize (A1) and (A2) as follows. Let $A_0 \ge 1$ and $\rho_0 > 0$. We consider a bounded domain D such that

(I) D is the union of a family of open convex sets $\{C_\lambda\}_{\lambda\in\Lambda}$ such that $B(z_\lambda,\rho_0) \subset C_\lambda \subset B(z_\lambda, A_0\rho_0)$.
(II) Let $\xi \in \partial D$. Then there are positive constants $\theta_1 \le \sin^{-1}(1/A_0)$ and $\rho_1 \le \rho_0\cos\theta_1$ such that the union of truncated cones $\Gamma_{\theta_1}(\xi,y) \cap B(\xi,2\rho_1)$ included in D is connected, i.e.,

$$\bigcup_{\substack{y\in D,\\ \Gamma_{\theta_1}(\xi,y)\cap B(\xi,2\rho_1)\subset D}} \Gamma_{\theta_1}(\xi,y) \cap B(\xi,2\rho_1) \text{ is connected.}$$

Theorem 2. *Let D be a bounded domain satisfying (I) and (II). Then every Euclidean boundary point of D has one Martin boundary point and it is minimal. Moreover the Martin boundary of D is homeomorphic to the Euclidean boundary.*

Remark 3. Ancona's admissible domains satisfy (I) and (II) of Theorem 2. The argument of Ancona depends on the special properties of a ball. His crucial lemma ([**?**, Lemme 1]) relies on the reflection with respect to a hyperplane. His lemma is applied to a ball by the Kelvin transform ([**?**, Corollarie 2]). This approach is not applicable to our domains.

Remark 4. A Denjoy domain can be represented as the union of a family of open balls with the same radius. A Lipschitz Denjoy domain, a sectorial domain and a quasi-sectorial domain can be represented as the union of a family of open convex sets C_λ satisfying (I). However, they are not represented as the union of a family of open balls with the same radius. Thus our Theorem 2 is applicable to these domains, whereas Theorem A is not.

Remark 5. Condition (II) is local in the following sense: Suppose D is the union of a family of open convex sets $\{C_\lambda\}_{\lambda\in\Lambda}$ satisfying (I). If a particular point $\xi \in \partial D$ satisfies (II), then there is one Martin boundary point at ξ and it is minimal.

Remark 6. Note that $0 < \theta_1 < \pi/2$ by $0 < \rho_1 \le \rho_0 \cos\theta_1$. The bounds $\theta_1 \le \sin^{-1}(1/A_0)$ and $\rho_1 \le \rho_0 \cos\theta_1$ are sharp. See Hirata [?]. Under these assumptions, there exists a truncated circular cone $\Gamma_{\theta_1}(\xi, y) \cap B(\xi, 2\rho_1)$ included in D; the union of such cones contains a neighborhood of ξ in some sense. See Lemma 2 below.

Both Theorems 1 and 2 are based on a common geometrical notion, *a system of local reference points*. In Section 2, we shall introduce a quasihyperbolic metric and define a system of local reference points. Then we shall observe that Theorems 1 and 2 are decomposed into three propositions, namely, Propositions 1, 2 and 3. The first two propositions are purely geometric and will be proved in the same section. Proposition 3 involves many potential theoretic arguments. Among them, a Carleson type estimate (Lemma 7 in Section 5) for bounded positive harmonic functions vanishing on a portion of the boundary will be useful. This estimate will be deduced from a Domar's type theorem (Domar [?]) for positive subharmonic functions, as was employed by Benedicks [?] and Chevallier [?].

Because of the intricacy of the boundary of a John domain, we shall give a refinement of Domar's theorem in Section 3 and prepare an integrability of the negative power of the distance function in Section 4. These arguments are necessary to prove a Carleson type estimate since the so-called geometric localization is not available for a general John domain. Even for an NTA domain a geometric localization is difficult. It takes the following form: If D is an NTA domain, then for any $x_0 \in \partial D$ and $r < r_0$ there exists an NTA domain $\Omega \subset D$ such that $B(x_0, r/M) \cap D \subset \Omega \subset B(x_0, Mr) \cap D$. Furthermore, the constant $M > 1$ in the NTA definition for Ω is independent of x_0 and r. The problem is that the intersection $B(x_0, r) \cap D$ is no longer connected; so, complicated modification of the intersection is needed to construct a nice subdomain. See Jones [?] and Jerison and Kenig [?]. For a uniformly John domain see Balogh and Volberg [?]. The approach of this paper is to show potential theoretic estimates directly avoiding a geometric localization. This seems easier than showing a geometric localization for a John domain.

Section 5 will be devoted to the proof of Proposition 3 in the case corresponding to Theorem 1 (i). We shall give a growth estimate of kernel functions at ξ; then we shall apply the tract argument due to Friedland and Hayman [?], as was employed by Benedicks [?]. The tract argument gives a rather coarse estimate of the number of minimal boundary points. In Section 6 we shall show Proposition 3 in the case corresponding to Theorem 1 (ii) and Theorem 2 by establishing a weak boundary Harnack principle (Ancona [?, Théorème 7.3]). The main tool will be the box argument for the estimate of a harmonic measure in terms of the Green function (Bass and Burdzy [?] and Aikawa [?, Lemma 2] for the present form). We shall use a subtle estimate (21) of the Green function, whose proof will be given in Section 7.

By the symbol A we denote an absolute positive constant whose value is unimportant and may change from line to line. If necessary, we use $A_0, A_1, \dots,$ to specify them. We shall say that two positive functions f_1 and f_2 are comparable, written $f_1 \approx f_2$, if and only if there exists a constant $A \ge 1$ such that $A^{-1} f_1 \le f_2 \le A f_1$. The constant A will be called the constant of comparison. We write $B(x, r)$ and $S(x, r)$ for the open ball and the sphere of center at x and radius r, respectively.

2. Local reference points

2.1. **Restatements of Theorems 1 and 2.** We define the quasihyperbolic metric $k_D(x, y)$ by

$$k_D(x, y) = \inf_\gamma \int_\gamma \frac{ds(z)}{\delta_D(z)},$$

where the infimum is taken over all rectifiable curves γ connecting x to y in D. We say that D satisfies a quasihyperbolic boundary condition if

$$k_D(x, x_0) \le A \log \frac{\delta_D(x_0)}{\delta_D(x)} + A' \quad \text{for all } x \in D. \tag{2}$$

A domain satisfying the quasihyperbolic boundary condition is called a Hölder domain by Smith-Stegenga [**?**, **?**]. It is easy to see that a John domain satisfies the quasihyperbolic boundary condition (see [**?**, Lemma 3.11]). We need more precise estimates.

Definition 1. Let N be a positive integer and $0 < \eta < 1$. We say that $\xi \in \partial D$ has a system of local reference points of order N with factor η if there exist $R_\xi > 0$ and $A_\xi > 1$ with the following property: for each positive $R < R_\xi$ there are N points $y_1, \dots, y_N \in D \cap S(\xi, R)$ such that $A_\xi^{-1} R \le \delta_D(y_i) \le R$ for $i = 1, \dots, N$ and

$$\min_{i=1,\dots,N} \{k_{D_R}(x, y_i)\} \le A_\xi \log \frac{R}{\delta_D(x)} + A_\xi \quad \text{for } x \in D \cap \overline{B(\xi, \eta R)},$$

where $D_R = D \cap B(\xi, \eta^{-3} R)$. If η is not so important, we simply say that $\xi \in \partial D$ has a system of local reference points of order N.

Remark 7. The quasihyperbolic metric is a useful tool to study the Martin boundary. See [**?**], [**?**] and [**?**]. Note that no exterior condition is assumed in the first two articles; while Bonk, Heinonen and Koskela [**?**] study a Gromov hyperbolic domain with strong barrier, an exterior condition.

The proofs of Theorems 1 and 2 can be decomposed into the following three propositions. The first and the second are purely geometric; the third is potential theoretic.

Proposition 1. *Let D be a general John domain with John constant c_J. Then every $\xi \in \partial D$ has a system of local reference points of order N with $N \le N(c_J, n) < \infty$. Moreover, if the John constant $c_J > \sqrt{3}/2$, then we can let $N \le 2$ by choosing a suitable factor $0 < \eta < 1$.*

Proposition 2. *Let D be a bounded domain satisfying (I) and (II). Then every $\xi \in \partial D$ has a system of local reference points of order* 1.

Remark 8. In Proposition 1, the constants R_ξ and A_ξ in Definition 1 can be taken uniformly for $\xi \in \partial D$, whereas they may depend on ξ in Proposition 2.

By $\mathscr{H}_\xi$ we denote the family of all kernel functions at ξ normalized at the John center x_0, i.e., the set of all positive harmonic functions h on D such that $h(x_0) = 1$, $h = 0$ q.e. on ∂D and h is bounded on $D \setminus B(\xi, r)$ for each $r > 0$. Here we say that a property holds q.e. (quasi everywhere) if it holds outside a polar set. A Martin kernel at ξ (with reference point x_0) is a limit of the ratio $G(x, y_j)/G(x_0, y_j)$ of Green functions with $y_j \to \xi$. Suppose $y_j \subset D \cap B(\xi, r/2)$. Then the (global) boundary Harnack principle for a John domain (Bass and Burdzy [**?**]) implies that the $G(\cdot, y_j)/G(x_0, y_j)$ is bounded on $D \setminus B(\xi, r)$, and so is a Martin kernel at ξ. Obviously, a Martin kernel at ξ is a positive harmonic function vanishing q.e on ∂D with value 1 at x_0, so that it belongs to $\mathscr{H}_\xi$. Thus Theorems 1 and 2 will follow from Propositions 1, 2 and the following:

Proposition 3. *Let D be a general John domain. Suppose $\xi \in \partial D$ has a system of local reference points of order N.*

(i) *The number of minimal functions in $\mathscr{H}_\xi$ is bounded by a constant depending only on N.*
(ii) *If $N \le 2$, then there are at most N minimal functions in $\mathscr{H}_\xi$. Moreover, if $N = 1$, then $\mathscr{H}_\xi$ itself is a singleton.*

2.2. **Proof of Proposition 1.** For the proof of the second assertion in Proposition 1, we prepare an elementary geometrical observation.

Lemma 1. *Let e_1, e_2 and e_3 be points on the unit sphere $S(0,1)$. Then*

$$\max \min_{i \ne j} |e_i - e_j| = \sqrt{3},$$

where the maximum is taken over all positions of e_1, e_2 and e_3.

Proof. This is a well-known fact (Fejes [**?**]). For the convenience sake of the reader we provide a proof. We can easily prove the lemma for $n = 2$. Let $n \ge 3$. We observe from the compactness of $S(0,1)$ that the maximum d is taken by some points e_1, e_2 and e_3 on $S(0,1)$. There is a unique 2-dimensional plane Π containing e_1, e_2 and e_3, since three distinct points on $S(0,1)$ cannot be collinear by convexity. Observe that $S(0,1) \cap \Pi$ is a circle with radius at most 1. Since e_1, e_2 and e_3 are points on this circle, it follows from the case $n = 2$ that $d \le \sqrt{3}$. The lemma follows. □

Proof of Proposition 1. We prove the proposition with $R_\xi = \delta_D(K_0)$. Let $\xi \in \partial D$ and $0 < R < \delta_D(K_0)$. Let us prove the first assertion with $\eta = 1/2$. Take $x \in D \cap \overline{B(\xi, R/2)}$. By definition there is a rectifiable curve γ starting from x and terminating at K_0 such that (1) holds. Then the first hit $y(x)$ of $S(\xi, R)$ along γ satisfies $2^{-1} c_J R \le \delta_D(y(x)) \le R$ and $k_{D_R}(x, y(x)) \le A \log \dfrac{R}{\delta_D(x)}$. We associate $y(x)$ with x, although it may not be unique.

Consider in general the family of balls $B(y, 4^{-1} c_J R)$ with $y \in S(\xi, R)$. These balls are included in $B(\xi, (4^{-1} c_J + 1)R)$, so that at most $N(c_J, n)$ balls among them can be mutually disjoint. Hence we find N points $x_1, \dots, x_N \in D \cap B(\xi, R/2)$ with $N \le N(c_J, n)$ such that $\{B(y_1, 4^{-1} c_J R), \dots, B(y_N, 4^{-1} c_J R)\}$ is maximal, where $y_j = y(x_j) \in D \cap S(\xi, R)$ is the point associate with x_j as above. This means that if $x \in D \cap B(\xi, R/2)$, then

$B(y(x), 4^{-1}c_J R)$ intersects some of $B(y_1, 4^{-1}c_J R), \dots, B(y_N, 4^{-1}c_J R)$, say $B(y_i, 4^{-1}c_J R)$. Since $B(y(x), 4^{-1}c_J R) \cap B(y_i, 4^{-1}c_J R) \neq \emptyset$ and $B(y(x), 2^{-1}c_J R) \cup B(y_i, 2^{-1}c_J R) \subset D$, it follows that $k_{D_R}(y(x), y_i) \le A'$. Hence

$$k_{D_R}(x, y_i) \le k_{D_R}(x, y(x)) + k_{D_R}(y(x), y_i) \le A \log \frac{R}{\delta_D(x)} + A'.$$

Thus the first assertion follows.

For the proof of the second assertion, let $\sqrt{3}/2 < b' < b < c_J$ and $\eta = 1 - b/c_J > 0$. Let us prove that ξ has a system of local reference points of order at most 2 with factor η. Let $0 < R < \delta_D(K_0)$. Suppose $x \in D \cap \overline{B(\xi, \eta R)}$. In the same way as in the proof of the first assertion, we find $y(x) \in S(\xi, R)$ such that $k_{D_R}(x, y(x)) \le A \log \dfrac{R}{\delta_D(x)}$ and

$$\delta_D(y(x)) \ge c_J(1 - \eta)R = bR > b'R > \frac{\sqrt{3}}{2}R.$$

In view of Lemma 1, we can choose $x_1, x_2 \in D \cap B(\xi, \eta R)$ such that if $x \in D \cap B(\xi, R)$, then $B(y(x), b'R)$ intersects $B(y_i, b'R)$ for some $i = 1, 2$, where $y_i = y(x_i)$. Since $B(y(x), b'R) \cap B(y_i, b'R) \neq \emptyset$ and $B(y(x), bR) \cup B(y_i, bR) \subset D$, it follows that $k_{D_R}(y(x), y_i) \le A$. Hence the proposition follows. □

Remark 9. In case $c_J \le \sqrt{3}/2$, we may have an estimate of N better than the above proof, by considering a lemma similar to Lemma 1. See Proposition 3 and Remark 10.

2.3. **Proof of Proposition 2.** In this subsection, we assume, by translation and dilation, that $\xi = 0$ and $\rho_1 = 1$ for simplicity. The aperture $\theta_1 \le \sin^{-1}(1/A_0)$ is fixed and we write $\Gamma(x, y)$ for $\Gamma_{\theta_1}(x, y)$. Note that $1 = \rho_1 \le \rho_0 \cos\theta_1$, so that $0 < \theta_1 < \pi/2$ and $\rho_0 \ge \sec\theta_1$. Let C_λ be a convex set appearing in (I) and let $B(z_\lambda, \rho_0) \subset C_\lambda \subset B(z_\lambda, A_0\rho_0)$. If $x \in \overline{C_\lambda} \setminus B(z_\lambda, \rho_0)$, then

$$\Gamma(x, z_\lambda) \cap B(x, 2) \subset \mathrm{co}(\{x\} \cup B(z_\lambda, \rho_0)) \subset C_\lambda, \tag{3}$$

where $\mathrm{co}(\{x\} \cup B(z_\lambda, \rho_0))$ is the convex hull of $\{x\} \cup B(z_\lambda, \rho_0)$. Observe that the assumption (II) can be restated as the connectedness of a certain set on the unit sphere $S(0, 1)$. Let $\mathscr{Y} = \{y \in S(0, 1) : \Gamma(0, y) \cap B(0, 2) \subset D\}$. Then (II) holds if and only if

$$= \bigcup_{y \in \mathscr{Y}} B(y, \sin\theta_1) \cap S(0, 1)$$

is a connected domain on $S(0, 1)$. By definition

$$\mathrm{dist}(\mathscr{Y}, S(0, 1) \setminus) \ge \sin\theta_1,$$

and the truncated cone of radius 2 with vertex at 0 subtended by is included in D. Hence, by dilation, if $0 < \eta < 1$ and $0 < R < 2\eta^3$, then

$$k_{D_R}(Ry_1, Ry_2) \le A \quad \text{for } y_1, y_2 \in \mathscr{Y}, \tag{4}$$

where $D_R = D \cap B(0, \eta^{-3}R)$ and A is independent of $y_1, y_2 \in \mathscr{Y}$ and R. Let us show that $\mathscr{Y} \neq \emptyset$ and that the point 0 can be accessible along a ray issuing from the origin toward a point in $\mathscr{Y}$.

Lemma 2. *There is a positive constant* $R_0 < 1$ *such that if* $C_\lambda \cap B(0, R_0) \neq \emptyset$*, then* $C_\lambda \cap \mathscr{Y} \neq \emptyset$*. In particular,* $\mathscr{Y} \neq \emptyset$*.*

Proof. Suppose to the contrary, there is a sequence C_{λ_j} with $\operatorname{dist}(0, C_{\lambda_j}) \to 0$ and $C_{\lambda_j} \cap \mathscr{Y} = \emptyset$. Let z_{λ_j} be such that $B(z_{\lambda_j}, \rho_0) \subset C_{\lambda_j} \subset B(z_{\lambda_j}, A_0\rho_0)$. Taking a subsequence, if necessary, we may assume that z_{λ_j} converges, say to z_0. We claim

$$\Gamma(0, z_0) \cap B(0, 2) \subset \bigcup_j C_{\lambda_j}. \tag{5}$$

We find $x_{\lambda_j} \in \partial C_{\lambda_j}$ with $x_{\lambda_j} \to 0$. Take $x \in \Gamma(0, z_0) \cap B(0, 2)$. Then $\angle x 0 z_0 < \theta_1$ and $|x| < 2$ by definition. If j is sufficiently large, then $\angle x x_{\lambda_j} z_{\lambda_j} < \theta_1$ and $|x - x_{\lambda_j}| < 2$ by continuity, so that

$$x \in \Gamma(x_{\lambda_j}, z_{\lambda_j}) \cap B(x_{\lambda_j}, 2) \subset \operatorname{co}(\{x_{\lambda_j}\} \cup B(z_{\lambda_j}, 1)) \subset C_{\lambda_j},$$

by (3). Thus (5) follows. Now, by definition, $y_0 = z_0/|z_0| \in \mathscr{Y}$ and $y_0 \in \Gamma(0, z_0) \cap B(0, 2) \subset \bigcup_j C_{\lambda_j}$. This contradicts $C_{\lambda_j} \cap \mathscr{Y} = \emptyset$. The lemma follows. □

Observe that if C is a convex set, then the distance function $\delta_C(x) = \operatorname{dist}(x, \partial C)$ is a concave function on $\overline{C}$, i.e.,

$$\delta_C(z) \geq \frac{|z-y|}{|x-y|}\delta_C(x) + \frac{|x-z|}{|x-y|}\delta_C(y) \quad \text{for } z \in \overline{xy}, \tag{6}$$

whenever $x, y \in \overline{C}$. This fact will be used in the following lemma.

Lemma 3. *Let* $0 < R_0 < 1$ *be as in Lemma 2. Suppose* $0 < R < \min\{R_0, 3^{-1}\sin\theta_1\}$*. If* $C_\lambda \cap B(0, R) \neq \emptyset$ *and* $y \in C_\lambda \cap \mathscr{Y}$*, then there exists a point* $w \in C_\lambda \cap \Gamma(0, y) \cap B(0, 3R/\sin\theta_1)$ *such that*

$$\delta_{C_\lambda \cap \Gamma(0,y)}(w) \geq \frac{\sin\theta_1}{4}R.$$

Proof. Take $x \in C_\lambda \cap B(0, R)$. Then $\overline{xy} \subset C_\lambda$. Observe that there is a point $w_1 \in \overline{xy} \cap \overline{\Gamma(0, y)}$ with $|w_1| \leq R/\sin\theta_1$. In fact, if $x \in \overline{\Gamma(0, y)}$, then $w_1 = x$ satisfies the condition. Otherwise, let w_1 be the intersection of $\overline{xy}$ and $\partial\Gamma(0, y)$. By elementary geometry

$$R > \operatorname{dist}(x, \overline{0y}) \geq \operatorname{dist}(w_1, \overline{0y}) = |w_1|\sin\theta_1,$$

so that $|w_1| \leq R/\sin\theta_1$. Since $|w_1 - y| \geq 1 - R/\sin\theta_1$ and $3R/\sin\theta_1 < 1$, we find a point $w_2 \in \overline{w_1 y} \subset \overline{C_\lambda \cap \Gamma(0, y)}$ with $|w_1 - w_2| = R/\sin\theta_1$. By (6) with $C = \Gamma(0, y)$ we obtain

$$\delta_{\Gamma(0,y)}(w_2) \geq \frac{|w_1 - w_2|}{|w_1 - y|}\delta_{\Gamma(0,y)}(y) \geq \frac{R/\sin\theta_1}{R/\sin\theta_1 + 1}\sin\theta_1 > \frac{R}{2}.$$

Moreover $|w_2| \leq 2R/\sin\theta_1$. Since $|w_2 - z_\lambda| \geq \rho_0 - 2R/\sin\theta_1 > R$ by $3R/\sin\theta_1 < 1 \leq \rho_0$, we can take a point $w \in \overline{w_2 z_\lambda} \subset \overline{C_\lambda}$ such that $|w - w_2| = R/4$. Then it follows from (6) with $C = C_\lambda$ that

$$\delta_{C_\lambda}(w) \geq \frac{|w - w_2|}{|z_\lambda - w_2|}\delta_{C_\lambda}(z_\lambda) \geq \frac{R/4}{A_0\rho_0}\rho_0 \geq \frac{\sin\theta_1}{4}R.$$

Hence

$$\delta_{\Gamma(0,y)\cap C_\lambda}(w) \ge \min\left\{\frac{R}{2}-\frac{R}{4}, \frac{\sin\theta_1}{4}R\right\} = \frac{\sin\theta_1}{4}R.$$

Moreover,

$$|w| \le |w-w_2| + |w_2-w_1| + |w_1| \le \frac{R}{4} + \frac{R}{\sin\theta_1} + \frac{R}{\sin\theta_1} < \frac{3R}{\sin\theta_1}.$$

□

Proof of Proposition 2. Let $0 < R_0 < 1$ be as in Lemma 2 and let $0 < \eta^3 < 3^{-1}\sin\theta_1$. Suppose $0 < R < \min\{R_0, 3^{-1}\sin\theta_1\}$. By Lemma 2 we fix $y_0 \in \mathscr{Y}$ and write $y_R = Ry_0$. It is sufficient to show

$$k_{D_R}(x, y_R) \le A\log\frac{R}{\delta_D(x)} + A \quad \text{for } x \in D\cap\overline{B(0,\eta R)}, \tag{7}$$

where A is independent of x and R. Take $x \in D\cap\overline{B(0,\eta R)}$. Then there is a convex set C_λ containing x and there is $y \in C_\lambda\cap\mathscr{Y}$ by Lemma 2. By Lemma 3 we find a point $w \in C_\lambda\cap\Gamma(0,y)\cap B(0,3R/\sin\theta_1)$ such that $\delta_{C_\lambda\cap\Gamma(0,y)}(w) \ge 4^{-1}R\sin\theta_1$. An elementary calculation shows

$$k_{D_R}(x,w) \le \int_{\overline{xw}} \frac{ds(z)}{\delta_D(z)} \le A\log\frac{R}{\delta_D(x)}.$$

Similarly, $k_{D_R}(w, |w|y) \le A$ and $k_{D_R}(|w|y_0, Ry_0) \le A$. Moreover, $k_{D_R}(|w|y, |w|y_0) \le A$ by (4). These altogether imply (7). □

3. Refinement of Domar's theorem

Domar [?, Theorem 2] gave a criterion for the boundedness of a subharmonic function majorized by a positive function. We need its quantitative refinement, i.e., the dependency of the bound is given explicitly.

Lemma 4. *Let u be a nonnegative subharmonic function on a bounded domain Ω. Suppose there is $\varepsilon > 0$ such that*

$$I = \int_\Omega (\log^+ u)^{n-1+\varepsilon} dx < \infty.$$

Then

$$u(x) \le \exp(1 + AI^{1/\varepsilon}\delta_\Omega(x)^{-n/\varepsilon}), \tag{8}$$

where A is a positive constant depending only on the dimension n.

For the proof we prepare the following.

Lemma 5. *Let u be a nonnegative subharmonic function on $B(x,R)$. Suppose $u(x) \ge t > 0$ and*

$$R \ge L_n|\{y\in B(x,R) : e^{-1}t < u(y) \le et\}|^{1/n}, \tag{9}$$

where $L_n = (e^2/v_n)^{1/n}$ and v_n is the volume of the unit ball. Then there exists a point $x' \in B(x,R)$ with $u(x') > et$.

Proof. Observe that (9) is equivalent to

$$\frac{|\{y \in B(x,R) : e^{-1}t < u(y) \le et\}|}{|B(x,R)|} \le \frac{1}{e^2}.$$

Suppose $u \le et$ on $B(x,R)$. Then the mean value property of subharmonic functions yields

$$\begin{aligned} t \le u(x) &\le \frac{1}{|B(x,R)|}\int_{B(x,R)} u(y)dy \\ &= \frac{1}{|B(x,R)|}\left(\int_{B(x,R)\cap\{u\le e^{-1}t\}} udy + \int_{B(x,R)\cap\{u>e^{-1}t\}} udy\right) \\ &\le e^{-1}t + \frac{1}{e^2}et < t. \end{aligned}$$

This is a contradiction. □

Proof of Lemma 4. Since the right hand side of (8) is not less than e, it is sufficient to show that

$$\delta_\Omega(x) \le AI^{1/n}(\log u(x))^{-\varepsilon/n}, \quad \text{whenever } u(x) > e. \tag{10}$$

Fix $x_1 \in \Omega$ with $u(x_1) > e$ and let us prove (10) with $x = x_1$. Let

$$R_j = L_n|\{y \in \Omega : e^{j-2}u(x_1) < u(y) \le e^j u(x_1)\}|^{1/n} \quad \text{for } j \ge 1.$$

We choose a sequence $\{x_j\}$ as follows: If $\delta_\Omega(x_1) < R_1$, then we stop. If $\delta_\Omega(x_1) \ge R_1$, then $B(x_1,R_1) \subset \Omega$, so that there exists $x_2 \in B(x_1,R_1)$ such that $u(x_2) > eu(x_1)$ by Lemma 5. Next we consider $\delta_\Omega(x_2)$. If $\delta_\Omega(x_2) < R_2$, then we stop. If $\delta_\Omega(x_2) \ge R_2$, then $B(x_2,R_2) \subset \Omega$, so that there exists $x_3 \in B(x_2,R_2)$ such that $u(x_3) > e^2u(x_1)$ by Lemma 5. Repeat this procedure to obtain a finite or infinite sequence $\{x_j\}$. We claim

$$\delta_\Omega(x_1) \le 2\sum_{j=1}^{\infty} R_j. \tag{11}$$

Suppose first $\{x_j\}$ is finite. If $\delta_\Omega(x_1) < R_1$, then (11) trivially holds. If $\delta_\Omega(x_1) \ge R_1$, then we have an integer $J \ge 2$ such that

$$\begin{aligned} &\delta_\Omega(x_1) \ge R_1, \dots, \delta_\Omega(x_{J-1}) \ge R_{J-1}, \delta_\Omega(x_J) < R_J, \\ &x_2 \in B(x_1,R_1), x_3 \in B(x_2,R_2), \dots, x_J \in B(x_{J-1},R_{J-1}). \end{aligned}$$

Hence we have

$$\delta_\Omega(x_1) \le |x_1 - x_2| + \cdots + |x_{J-1} - x_J| + \delta_\Omega(x_J) < R_1 + \cdots + R_{J-1} + R_J,$$

so that (11) follows. Suppose next $\{x_j\}$ is infinite. Since $u(x_j) > e^j u(x_1) \to \infty$, it follows from the local boundedness of a subharmonic function that x_j goes to the boundary. Hence, there is an integer $J \ge 2$ such that $\delta_\Omega(x_J) \le \frac{1}{2}\delta_\Omega(x_1)$. Then

$$\delta_\Omega(x_1) \le |x_1 - x_2| + \cdots + |x_{J-1} - x_J| + \delta_\Omega(x_J) \le R_1 + \cdots + R_{J-1} + \frac{1}{2}\delta_\Omega(x_1),$$

so that (11) follows. In view of (11) we observe that (10) follows from

$$\sum_{j=1}^{\infty} R_j \le AI^{1/n}(\log u(x_1))^{-\varepsilon/n}. \tag{12}$$

To show (12), let j_1 be the positive integer such that $e^{j_1} < u(x_1) \le e^{j_1+1}$. Then

$$R_j \le L_n |\{y \in \Omega : e^{j_1+j-2} < u(y) \le e^{j_1+j+1}\}|^{1/n}.$$

Since the family of intervals $\{(e^{j_1+j-2}, e^{j_1+j+1}]\}_j$ overlaps at most 3 times, it follows from Hölder's inequality that

$$\begin{aligned}\sum_{j=1}^{\infty} R_j &\le 3L_n \sum_{j=j_1}^{\infty} |\{y \in \Omega : e^{j-1} < u(y) \le e^j\}|^{1/n} \\ &\le 3L_n \left(\sum_{j=j_1}^{\infty} \frac{1}{j^{(n-1+\varepsilon)/(n-1)}}\right)^{(n-1)/n} \left(\sum_{j=j_1}^{\infty} j^{n-1+\varepsilon}|\{y \in \Omega : e^{j-1} < u(y) \le e^j\}|\right)^{1/n} \\ &\le Aj_1^{-\varepsilon/n}\left(\int_\Omega (\log^+ u)^{n-1+\varepsilon} dy\right)^{1/n} \\ &\le A(\log u(x_1))^{-\varepsilon/n} I^{1/n}.\end{aligned}$$

Thus (12) follows. The lemma is proved. □

4. Integrability of negative power of the distance function

Inspired by Smith and Stegenga [**?**, Theorem 4] we have proved that for a bounded John domain there is a positive constant τ such that

$$\int_D \delta_D(x)^{-\tau} dx < \infty$$

([**?**, Lemma 5]). We need its local version.

Lemma 6. *Let D be a general John domain with John constant c_J and generalized John center K_0. Then there are positive constants τ and A depending on c_J such that*

$$\int_{D\cap B(\xi,R)} \left(\frac{R}{\delta_D(x)}\right)^{\tau} dx \le AR^n$$

for each $\xi \in \partial D$ and $0 < R < \delta_D(K_0)$.

Proof. Let

$$V_j = \{x \in D \cap B(\xi, R + (1 + c_J^{-1})2^{1-j}R) : 2^{-j-1}R \le \delta_D(x) < 2^{-j}R\}$$

for $j \ge 0$. For a moment we fix $x \in \bigcup_{i=j+1}^{\infty} V_i$. By definition there is a rectifiable curve γ connecting x and K_0 with (1). Hence we find $y \in \gamma$ such that $\delta_D(y) = 2^{-j}R \ge c_J|x - y|$. In other words $x \in \overline{B(y, c_J^{-1}2^{-j}R)}$. We observe

$$|B(y, 5c_J^{-1}2^{-j}R)| \le A|V_j \cap B(y, c_J^{-1}2^{-j}R)|. \tag{13}$$

In fact, take $y^* \in \partial D$ such that $|y-y^*| = 2^{-j}R$, and then take and $y' \in \overline{yy^*}$ with $\delta_D(y') = \frac{1}{2}(2^{-j}R + 2^{-j-1}R)$. An elementary geometrical observation gives $B(y', 2^{-j-2}R) \subset V_j \cap B(y, c_J^{-1}2^{-j}R)$, so that (13) follows.

Now the covering lemma yields a sequence $\{y_k\}$ such that

$$\bigcup_{i=j+1}^{\infty} V_i \subset \bigcup_k \overline{B(y_k, 5c_J^{-1}2^{-j}R)}$$

and $\{\overline{B(y_k, c_J^{-1}2^{-j}R)}\}_k$ are disjoint. Hence

$$\sum_{i=j+1}^{\infty} |V_i| \le \left|\bigcup_{i=j+1}^{\infty} V_i\right| \le \sum_k |B(y_k, 5c_J^{-1}2^{-j}R)| \le A_1 \sum_k |V_j \cap B(y_k, c_J^{-1}2^{-j}R)| \le A_1|V_j|$$

by (13). Let $1 < t < 1 + A_1^{-1}$. In the same way as in [?, Lemma 5] we have

$$\sum_{j=0}^{\infty} t^j |V_j| \le \frac{t}{1-(t-1)A_1} \sum_{j=0}^{\infty} |V_j| \le A|B(\xi, R + (1+c_J^{-1})2R)| \le AR^n.$$

Since $t^j < (R/\delta_D(x))^\tau \le t^{j+1}$ on V_j with $\tau = \log t/\log 2 > 0$, it follows that

$$\int_{D\cap B(\xi,R)} \left(\frac{R}{\delta_D(x)}\right)^\tau dx \le \sum_{j=0}^{\infty} t^{j+1}|V_j| \le AR^n.$$

Thus the lemma follows. □

5. Growth of positive harmonic functions

In this section we shall show Proposition 3 (i) by investigating the growth of $h \in \mathscr{H}_\xi$. Throughout this section we let D be a general John domain and let $\xi \in \partial D$ be fixed. We say that $x, y \in D$ are connected by a Harnack chain $\{B(x_j, \frac{1}{2}\delta_D(x_j))\}_{j=1}^k$ if $x \in B(x_1, \frac{1}{2}\delta_D(x_1))$, $y \in B(y_k, \frac{1}{2}\delta_D(y_k))$, and $B(x_j, \frac{1}{2}\delta_D(x_j)) \cap B(x_{j+1}, \frac{1}{2}\delta_D(x_{j+1})) \ne \emptyset$ for $j = 1, \dots, k-1$. The number k is called the length of the Harnack chain. We observe that the shortest length of the Harnack chain connecting x and y is comparable to $k_D(x,y)$. Therefore, the Harnack inequality yields that there is a constant $A_2 > 1$ depending only on n such that

$$\exp(-A_2 k_D(x,y)) \le \frac{h(x)}{h(y)} \le \exp(A_2 k_D(x,y)) \tag{14}$$

for every positive harmonic function h on D. If D is a John domain with John constant c_J and John center x_0, then we have from (2)

$$\frac{h(x)}{h(x_0)} \le A_3 \left(\frac{\delta_D(x_0)}{\delta_D(x)}\right)^\lambda \tag{15}$$

with λ and $A_3 > 0$ depending only on the John constant c_J. If D is a general John domain with John constant c_J and John center K_0, then (15) holds with the same λ and another A_3 depending only on c_J, x_0 and K_0.

Let Ω be an open set intersecting ∂D. Let h be a bounded positive harmonic function in $D \cap \Omega$ vanishing q.e. on $\partial D \cap \Omega$. We extend h to $\Omega \setminus D$ by 0 outside D and denote by h^* its upper regularization. Then we observe that h^* is a nonnegative subharmonic function on Ω ([**?**, Theorem 5.2.1]). We shall apply the refinement of Domar's theorem (Lemma 4) to the subharmonic function h^* to obtain a Carleson type estimate.

Lemma 7. *Let $\xi \in \partial D$ have a system of local reference points $y_1, \dots, y_N \in D \cap S(\xi, R)$ of order N with factor η for $0 < R < R_\xi$. Suppose h is a positive harmonic function in $D \cap B(\xi, \eta^{-3}R)$ vanishing q.e. on $\partial D \cap B(\xi, \eta^{-3}R)$. If h is bounded in $D \cap B(\xi, \eta R) \setminus \overline{B(\xi, \eta^3 R)}$, then*

$$h \le A \sum_{i=1}^{N} h(y_i) \quad \text{on } S(\xi, \eta^2 R), \tag{16}$$

where A is independent of h and R.

Proof. Let $0 < R < R_\xi$. Then we find $y_1, \dots, y_N \in D \cap S(\xi, R)$ with $\delta_D(y_i) \approx R$ such that

$$\min_{i=1,\dots,N} \{k_{D_R}(x, y_i)\} \le A \log \frac{R}{\delta_D(x)} + A \quad \text{for } x \in D \cap \overline{B(\xi, \eta R)}.$$

Hence

$$h(x) \le A \left(\frac{R}{\delta_D(x)} \right)^{\lambda} \sum_{i=1}^{N} h(y_i) \quad \text{for } x \in D \cap \overline{B(\xi, \eta R)}. \tag{17}$$

by (14). Let us apply Lemma 4 to $u = h^* / \sum_{i=1}^{N} h(y_i)$ and $\Omega = B(\xi, \eta R) \setminus \overline{B(\xi, \eta^3 R)}$. Let $\varepsilon > 0$ and $\tau > 0$ be as in Lemma 6. Apply the elementary inequality:

$$(\log t)^{n-1+\varepsilon} \le \left(\frac{n-1+\varepsilon}{\tau} \right)^{n-1+\varepsilon} t^\tau \quad \text{for } t \ge 1$$

to $t = R/\delta_D(x) \ge 1$ for $x \in \Omega$. Then

$$\left[\log^+ \left(\frac{R}{\delta_D(x)} \right) \right]^{n-1+\varepsilon} \le A \left(\frac{R}{\delta_D(x)} \right)^{\tau},$$

so that it follows from (17) and Lemma 6 that

$$I = \int_\Omega (\log^+ u)^{n-1+\varepsilon} dx \le A \int_{B(\xi, \eta R)} \left(\frac{R}{\delta_D(x)} \right)^{\tau} dx \le A R^n.$$

Hence, Lemma 4 yields that $u \le \exp(1 + A I^{1/\varepsilon} R^{-n/\varepsilon}) \le A$ on $S(\xi, \eta^2 R)$, i.e., (16) holds. □

Let us apply Lemma 7 to a kernel function $h \in \mathscr{H}_\xi$ to obtain the following growth estimate.

Lemma 8. *Let $\xi \in \partial D$ have a system of local reference points $y_1, \dots, y_N \in D \cap S(\xi, R)$ of order N with factor η for $0 < R < R_\xi$. Let $h \in \mathscr{H}_\xi$ for $\xi \in \partial D$. Then*

$$h(x) \le A |x - \xi|^{-\lambda} \quad \text{for } x \in D,$$

where $\lambda > 0$ is as in (15) and A is independent of R and h.

Proof. By Lemma 7 we have (16). Since h is bounded apart from a neighborhood of ξ, the maximum principle gives

$$h(x) \le A \sum_{i=1}^{N} h(y_i) \quad \text{for } x \in D \setminus B(\xi, \eta^2 R).$$

Apply (15) to each $y_i \in D \cap S(\xi, R)$ with $\delta_D(y_i) \approx R$. Then obtain $h(y_i) \le AR^{-\lambda}$. This, together with the above estimate, yields $h(x) \le A|x-\xi|^{-\lambda}$ for $x \in D$. The lemma is proved. □

Here we record another application of Lemma 7, as this will be useful later.

Lemma 9. *Let $\xi \in \partial D$ have a system of local reference points $y_1, \dots, y_N \in D \cap S(\xi, R)$ of order N with factor η for $0 < R < R_\xi$. Let h be a bounded positive harmonic function on $D \cap B(\xi, \eta^{-3}R)$ vanishing q.e on $\partial D \cap B(\xi, \eta^{-3}R)$. Then*

$$h(x) \le A \sum_{i=1}^{N} h(y_i) \quad \text{for } x \in D \cap \overline{B(\xi, \eta^2 R)},$$

where A is independent of R and h.

Proof. We have (16). Apply the maximum principle to $D \cap B(\xi, \eta^2 R)$. □

The following lemma is well-known. For the reader's convenience sake, we record it with a proof.

Lemma 10. *Suppose there exist a positive integer M and a positive constant A with the following property: if $h_0, \dots, h_M \in \mathscr{H}_\xi$, then there is j such that*

$$h_j \le A \sum_{i \ne j} h_i \quad \text{on } D.$$

Then $\mathscr{H}_\xi$ has at most M minimal harmonic functions.

Proof. Suppose there are $M+1$ different minimal harmonic functions $h_0, \dots h_M \in \mathscr{H}_\xi$. If necessary relabeling, we may assume that

$$h_0 \le A \sum_{i=1}^{M} h_i \quad \text{on } D.$$

We may also assume that $A \ge 1$. Then

$$h = (A \sum_{i=1}^{M} h_i - h_0)/(AM - 1) \in \mathscr{H}_\xi.$$

Hence

$$\frac{1}{AM} h_0 + (1 - \frac{1}{AM}) h = \frac{1}{M} \sum_{i=1}^{M} h_i.$$

Compare the Martin representation measures for the both sides. The measure for the left hand side has at least $\dfrac{1}{AM}$ mass at h_0, whereas the measure for the right hand side has 0 mass at h_0. This contradicts the uniqueness of the Martin representation. □

Proof of Proposition 3 for $N \geq 3$. Let $h_j \in \mathscr{H}_\xi$ for $j = 0, \dots, M$. Let h_j^* be the upper regularization of the extension of h_j and let H_j be the Kelvin transform of h_j^* with respect to $S(\xi, 1)$, i.e.,

$$H_j(x) = |x - \xi|^{2-n} h_j^*(\xi + |x - \xi|^{-2}(x - \xi)).$$

Observe that H_j is a nonnegative subharmonic function on $\mathbb{R}^n$ which is positive and harmonic on the Kelvin image D^* of D and is equal to 0 q.e. outside D^*. Moreover, Lemma 8 shows

$$H_j(x) \leq A|x - \xi|^{2-n+\lambda}.$$

Thus H_j is of order at most $2 - n + \lambda$. As in Benedicks [**?**, Theorem 2], we let

$$w = \max_{j=0,\dots,M} \{H_j - \sum_{i \neq j} H_i\}$$

and let w^+ be the upper regularization of $\max\{w, 0\}$. Then w^+ is a nonnegative subharmonic function on $\mathbb{R}^n$ of order at most $2 - n + \lambda$. If none of $\{x : H_j(x) > \sum_{i \neq j} H_i(x)\}$ is empty, then w^+ has $M + 1$ tracts. Hence, [**?**, Theorem 3] yields

$$2 - n + \lambda \geq \frac{1}{2} \log \left(\frac{M + 1}{4} \right) + \frac{3}{2} \quad \text{if } M \geq 3.$$

Hence, if $M > 4\exp(1 - 2n + 2\lambda) - 1$, then $\{x : H_j(x) > \sum_{i \neq j} H_i(x)\} = \emptyset$ for some $j = 1, \dots, M$. This means that $H_j \leq \sum_{i \neq j} H_i$ on D^*, so that

$$h_j \leq \sum_{i \neq j} h_i \quad \text{on } D.$$

Hence Lemma 10 implies that $\mathscr{H}_\xi$ has at most M minimal harmonic functions, or equivalently there are at most M minimal Martin boundary points at ξ. Thus the number of minimal Martin boundary points at ξ is bounded by $4\exp(1 - 2n + 2\lambda)$. □

Remark 10. The above proof gives a coarse estimate of the number of minimal harmonic functions of $\mathscr{H}_\xi$ in terms of λ depending on the John constant c_J. For a sharp estimate more delicate argument will be needed.

6. Weak boundary Harnack principle

In this section we shall prove Proposition 3 for $N \leq 2$. Throughout this section we let D be a general John domain and fix $\xi \in \partial D$. By $\omega(x, E, U)$ we denote the harmonic measure of E for an open set U evaluated at x. Let G be the Green function for D. Since many arguments are valid for a general N except for (21), we shall state the results for a general N.

The box argument in [**?**, Lemma 2] (see [**?**] for the original form), gives the following estimate of the harmonic measure.

Lemma 11. *Let $\xi \in \partial D$ have a system of local reference points $y_1, \dots, y_N \in D \cap S(\xi, R)$ of order N with factor η for $0 < R < R_\xi$. If $x \in D \cap \overline{B(\xi, \eta^3 R)}$, then*

$$\omega(x, D \cap S(\xi, \eta^2 R), D \cap B(\xi, \eta^2 R)) \le \begin{cases} A(\log \frac{1}{R})^{-1} \sum_{i=1}^N G(x, y_i) & \text{if } n = 2, \\ AR^{n-2} \sum_{i=1}^N G(x, y_i) & \text{if } n \ge 3, \end{cases} \tag{18}$$

where A depends only on n, c_J, R_ξ and A_ξ.

Remark 11. If $n = 2$, then $(\log 1/R)^{-1}$ appears in (18). This is different from [**?**, Lemma 2]. In [**?**], the harmonic measure is estimated by the Green function for $D \cap B(\xi, AR)$, whereas in (18), it is estimated by the Green function for D itself.

Proof. Let us begin with an estimate of harmonic measure in a John domain. For $0 < r < \delta_D(K_0)$ let $U(r) = \{x \in D : \delta_D(x) < r\}$. Then each point $x \in U(r)$ can be connected to K_0 by a curve with (1). Hence, $B(x, A_4 r) \setminus U(r)$ includes a ball with radius r, provided A_4 is large. This implies that

$$\omega(x, U(r) \cap S(x, A_4 r), U(r) \cap B(x, A_4 r)) \le 1 - \varepsilon_0 \quad \text{for } x \in U(r)$$

with $0 < \varepsilon_0 < 1$ depending only on A_4 and the dimension. Let $R \ge r$ and repeat this argument with the maximum principle. Then there exist positive constants A_5 and A_6 such that

$$\omega(x, U(r) \cap S(x, R), U(r) \cap B(x, R)) \le \exp(A_5 - A_6 R/r). \tag{19}$$

See [**?**, Lemma 1] for details.

Let $0 < R < R_\xi$. For each $x \in D \cap \overline{B(\xi, \eta R)}$ there is a local reference point $y(x) \in \{y_1, \dots, y_N\}$ such that

$$k_{D_R}(x, y(x)) \le A_\xi \log \frac{R}{\delta_D(x)} + A_\xi$$

by definition. Let $y'(x) \in S(y(x), \frac{1}{2}\delta_D(y(x)))$. Then we observe that $k_{D_R \setminus \{y(x)\}}(x, y'(x)) \le A_\xi \log(R/\delta_D(x)) + A_\xi$. It is easy to see that $(\log \frac{1}{R})^{-1} G(y'(x), y(x)) \approx 1$ if $n = 2$, and that $AR^{n-2} G(y'(x), y(x)) \approx 1$ if $n \ge 3$. Hence, letting

$$u(x) = \begin{cases} A(\log \frac{1}{R})^{-1} \sum_{i=1}^N G(x, y_i) & \text{if } n = 2, \\ AR^{n-2} \sum_{i=1}^N G(x, y_i) & \text{if } n \ge 3, \end{cases}$$

we obtain from (14) and (15) that

$$u(x) \ge A \left(\frac{\delta_D(x)}{R} \right)^\lambda \quad \text{for } x \in D \cap \overline{B(\xi, \eta R)}$$

with some $\lambda > 0$ depending only on n, c_J, R_ξ and A_ξ. Let $D_j = \{x \in D : \exp(-2^{j+1}) \le u(x) < \exp(-2^j)\}$ and $U_j = \{x \in D : u(x) < \exp(-2^j)\}$. Then we see

$$U_j \cap B(\xi, \eta R) \subset \left\{ x \in D : \delta_D(x) < AR \exp\left(-\frac{2^j}{\lambda} \right) \right\}.$$

Define a decreasing sequence R_j by $R_0 = \eta^2 R$ and

$$R_j = \left(\eta^2 - \frac{6(\eta^2-\eta^3)}{\pi^2}\sum_{k=1}^{j}\frac{1}{k^2}\right) R \quad \text{for } j \ge 1.$$

Let $\omega_0 = \omega(\cdot, D\cap S(\xi,\eta^2 R), D\cap B(\xi,\eta^2 R))$ and put

$$d_j = \begin{cases} \displaystyle\sup_{x\in D_j\cap B(\xi,R_j)} \frac{\omega_0(x)}{u(x)} & \text{if } D_j\cap B(\xi,R_j)\ne\emptyset, \\ 0 & \text{if } D_j\cap B(\xi,R_j)=\emptyset. \end{cases}$$

It is sufficient to show that d_j is bounded by a constant independent of R, since $R_j > \eta^3 R$ for all $j \ge 0$. Apply the maximum principle to $U_j \cap B(\xi, R_{j-1})$ to obtain

$$\omega_0(x) \le \omega(x, U_j\cap S(\xi,R_{j-1}), U_j\cap B(\xi,R_{j-1})) + d_{j-1}u(x).$$

Divide the both sides by $u(x)$ and take the supremum over $D_j \cap B(\xi, R_j)$. Then (19) yields

$$d_j \le A\exp\left(2^{j+1} + A_5 - A_6\frac{R_{j-1}-R_j}{AR\exp(-2^j/\lambda)}\right) + d_{j-1},$$

provided j is sufficiently large, say $j \ge j_0$, so that

$$\frac{R_{j-1}-R_j}{AR\exp(-2^j/\lambda)} = \frac{6(\eta^2-\eta^3)}{\pi^2}\frac{\exp(2^j/\lambda)}{Aj^2} \ge 1.$$

For $j < j_0$ we have $d_j \le 1 + d_{j-1}$. Since the series

$$\sum_{j=1}^{\infty}\exp\left(2^{j+1} + A_5 - A_6\frac{6(\eta^2-\eta^3)}{\pi^2}\frac{\exp(2^j/\lambda)}{Aj^2}\right)$$

is convergent and independent of R, we obtain $\sup_{j\ge 0} d_j < \infty$. Thus (18) follows. □

Lemma 12. *Let $\xi \in \partial D$ have a system of local reference points $y_1,\dots,y_N \in D\cap S(\xi,R)$ of order N with factor η for $0 < R < R_\xi$. If $x \in D\cap \overline{B(\xi,\eta^3 R)}$ and $y \in D\setminus B(\xi,\eta^{-3}R)$, then*

$$G(x,y) \le \begin{cases} A(\log\frac{1}{R})\sum_{i=1}^N G(x,y_i)\sum_{j=1}^N G(y_j,y) & \text{if } n=2, \\ AR^{n-2}\sum_{i=1}^N G(x,y_i)\sum_{j=1}^N G(y_j,y) & \text{if } n\ge 3, \end{cases} \tag{20}$$

where A depends only on n, c_J, R_ξ and A_ξ.

Proof. For simplicity we give the proof only for $n \ge 3$. In case $n = 2$, we replace R^{n-2} by $(\log 1/R)^{-1}$. Apply Lemma 9 to $h(x) = G(x,y)$ with $y \in D \setminus B(\xi, \eta^{-3}R)$. Then

$$G(x,y) \le A\sum_{j=1}^N h(y_j) \quad \text{for } x\in D\cap S(\xi,\eta^2 R).$$

Hence (18) yields

$$G(x,y) \le AR^{n-2} \sum_{i=1}^{N} G(x,y_i) \sum_{j=1}^{N} h(y_j) \quad \text{for } x \in D \cap \overline{B(\xi,\eta^3 R)}$$

by the maximum principle. The lemma follows. □

For further arguments we need the following improvement of (20): If $x \in D\cap\overline{B(\xi,\eta^9 R)}$ and $y \in D \setminus B(\xi,\eta^{-3}R)$, then

$$G(x,y) \le \begin{cases} A(\log \dfrac{1}{R})^{-1} \sum_{i=1}^{N} G(x,y_i)G(y_i,y) & \text{if } n=2, \\ AR^{n-2} \sum_{i=1}^{N} G(x,y_i)G(y_i,y) & \text{if } n \ge 3, \end{cases} \tag{21}$$

where A depends only on n, c_J, R_ξ and A_ξ. The point is that the cross terms $G(x,y_i)G(y_j,y)$ $(i \ne j)$ disappear from the right hand side of (20).

If $N = 1$, then (21) is nothing but (20). If $N \le 2$, then Ancona's ingenious trick [**?**, Théorème 7.3] gives (21) from (20). However, the proof is rather complicated and we postpone the proof to the next section. The remaining arguments are rather easy and hold for arbitrary $N \ge 1$, provided (21) holds. Let us show the weak boundary Harnack principle defined by Ancona [**?**, Définition 2.3].

Lemma 13. (Weak Boundary Harnack Principle) *Let $\xi \in \partial D$ have a system of local reference points $y_1,\dots,y_N \in D \cap S(\xi,R)$ of order N with factor η for $0 < R < R_\xi$. Moreover, suppose* (21) *holds. Let $h_0,h_1,\dots,h_N \in \mathscr{H}_\xi$. Then*

$$h_0(y) \le A \sum_{i=1}^{N} \frac{h_0(y_i)}{h_i(y_i)} h_i(y) \quad \text{for } y \in D \setminus B(\xi,\eta^{-3}R). \tag{22}$$

where A depends only on n, c_J, R_ξ and A_ξ.

Proof. For simplicity we give the proof only for $n \ge 3$. In case $n = 2$, we replace R^{n-2} by $(\log 1/R)^{-1}$. Let $0 < r < \eta^9 R$. Observe that the regularized reduced function $u_r = \widehat{R}_{h_0}^{D\cap S(\xi,r)}$ is a Green potential of a measure μ_r concentrated on $D \cap S(\xi,r)$. We have from (21)

$$\begin{aligned} u_r(y) &= \int_{D\cap S(\xi,r)} G(x,y)d\mu_r(x) \\ &\le AR^{n-2} \sum_{i=1}^{N} \int_{D\cap S(\xi,r)} G(x,y_i)G(y_i,y)d\mu_r(x) = AR^{n-2} \sum_{i=1}^{N} u_r(y_i)G(y_i,y) \end{aligned}$$

for $y \in D \setminus B(\xi,\eta^{-3}R)$. Letting $r \to 0$, we obtain

$$h_0(y) \le AR^{n-2} \sum_{i=1}^{N} h_0(y_i)G(y_i,y) \quad \text{for } y \in D \setminus B(\xi,\eta^{-3}R).$$

Let $\varepsilon = \min\{\frac{1}{2},\eta^{-3}-1\}$. Then $D \setminus B(\xi,\eta^{-3}R) \subset D \setminus B(y_i,\varepsilon\delta_D(y_i))$. Observe from the Harnack principle that $h_i(y_i)R^{n-2}G(y_i,y) \approx h_i(y)$ for $y \in S(y_i,\varepsilon\delta_D(y_i))$, and so is for $y \in D \setminus B(\xi, \eta^{-3}R) \subset D \setminus B(y_i, \varepsilon\delta_D(y_i))$ by the maximum principle. Hence (22) follows. □

Varying R in Lemma 13, we obtain relationships among kernel functions in $\mathscr{H}_\xi$, which yield Proposition 3.

Proof of Proposition 3 (ii) for $N = 1$. Obviously, (21) holds, and hence (22) holds for $N = 1$ by Lemma 13. Let $h_0, h_1 \in \mathscr{H}_\xi$. Let $R_j \to 0$ and take a local reference point $y_1^j \in D \cap S(\xi, R_j)$. Then one of the inequalities $h_0(y_1^j) \le h_1(y_1^j)$ and $h_1(y_1^j) \le h_0(y_1^j)$ holds for infinitely many j. Hence $h_0 \le Ah_1$ or $h_1 \le Ah_0$ holds on D by (22) with $N = 1$. Moreover suppose that h_0 and h_1 are minimal. Then $h_0 \equiv h_1$ in any case. This implies that $\mathscr{H}_\xi$ has just one minimal kernel function. Take $h \in \mathscr{H}_\xi$. By the Martin representation theorem h is given as the integral of Martin kernel by a measure μ over the minimal Martin boundary. Since h vanishes q.e. on ∂D and bounded apart from a neighborhood of ξ, it follows that μ is a point measure at ξ, so that h must coincide with a unique minimal harmonic function in $\mathscr{H}_\xi$. Thus, $\mathscr{H}_\xi$ is a singleton. □

Proof of Proposition 3 (ii) for $N = 2$. As we shall show in the next section (21) holds for $N = 2$, and hence (22) holds for $N = 2$ by Lemma 13. We follow the proof of Ancona [**?**, Théoremè 2.5]. We slightly generalize the proof of Proposition 3 for $N = 1$. Let $h_0, h_1, h_2 \in \mathscr{H}_\xi$. Take a decreasing sequence $R_j \to 0$. For each R_j sufficiently small we find reference points $y_i^j \in D \cap S(\xi, R_j)$ with $i = 1, 2$. For a moment fix j and consider $\max_{0 \le k \le 2} h_k(y_1^j)$. Then we find $k(j)$ such that $h_{k(j)} = \max_{0 \le k \le 2} h_k(y_1^j)$. This holds for infinitely many j, so that we find $k_1 \in \{0, 1, 2\}$ such that

$$h_{k_1}(y_1^j) = \max_{0 \le k \le 2} h_k(y_1^j) \tag{23}$$

for infinitely many j. Then consider j satisfying (23) and find $k_2 \in \{0, 1, 2\}$ such that

$$h_{k_2}(y_2^j) = \max_{0 \le k \le 2} h_k(y_2^j)$$

for infinitely many j. Thus

$$h_k(y_i^j) \le h_{k_i}(y_i^j) \quad \text{for all } i, k \in \{0, 1, 2\}$$

holds for infinitely many j. If necessary relabeling h_0, h_1, h_2, we may assume that $k_1 \ne 0$ and $k_2 \ne 0$. Then (22) yields

$$h_0(y) \le A \sum_{i=1}^{2} \frac{h_0(y_i^j)}{h_{k_i}(y_i^j)} h_{k_i}(y) \le A \sum_{k=1}^{2} h_k(y) \quad \text{for } y \in D \setminus B(\xi, \eta^{-3}R_j).$$

This holds for infinitely many j. Letting $j \to \infty$, we obtain

$$h_0 \le A \sum_{k=1}^{2} h_k \quad \text{on } D.$$

This, together with Lemma 10, completes the proof. □

Remark 12. We do not know whether the weak boundary Harnack principle holds for $N \ge 3$. In special cases, such as a sectorial domain whose boundary lies on N rays leaving ξ, we can apply the weak boundary Harnack principle repeatedly to subdomains containing just one ray and conclude the weak boundary Harnack principle for the sectorial domain itself (cf. Cranston and Salisbury [**?**, (2.2) Lemma]).

7. Proof of (21)

In this section we shall prove the following:

Lemma 14. *Let $\xi \in \partial D$ have a system of local reference points $y_1, y_2 \in D \cap S(\xi, R)$ of order 2 with factor η for $0 < R < R_\xi$. If $x \in D \cap \overline{B(\xi, \eta^9 R)}$ and $y \in D \setminus B(\xi, \eta^{-3} R)$, then (21) holds, i.e.,*

$$G(x,y) \le \begin{cases} A(\log \dfrac{1}{R})^{-1} \sum_{i=1}^{2} G(x,y_i)G(y_i,y) & \text{if } n = 2, \\ \\ AR^{n-2} \sum_{i=1}^{2} G(x,y_i)G(y_i,y) & \text{if } n \ge 3, \end{cases}$$

where A depends only on n, c_J, R_ξ and A_ξ.

We employ Ancona's ingenious trick [**?**, Théorème 7.3]. Since our setting is slightly different from Ancona's, we provide a proof for the sake of the reader's convenience.

Proof. For simplicity we give the proof only for $n \ge 3$. In case $n = 2$, we replace R^{n-2} by $(\log 1/R)^{-1}$. Besides the local reference points $y_1, y_2 \in D \cap S(\xi, R)$, we take local reference points $y_1^*, y_2^* \in D \cap S(\xi, \eta^6 R)$ with

$$\min_{i=1,2}\{k_{D\cap B(\xi,\eta^3 R)}(x, y_i^*)\} \le A_\xi \log \frac{\eta^6 R}{\delta_D(x)} + A_\xi \quad \text{for } x \in D \cap \overline{B(\xi, \eta^7 R)}.$$

Then

$$\min_{j=1,2}\{k_{D_R}(y_i^*, y_j)\} \le A_\xi \log \frac{R}{\delta_D(y_i^*)} + A_\xi \le A_\xi.$$

So, we may assume either

$$k_{D_R}(y_1^*, y_1) \le A \text{ and } k_{D_R}(y_2^*, y_1) \le A, \tag{24}$$

or

$$k_{D_R}(y_1^*, y_1) \le A \text{ and } k_{D_R}(y_2^*, y_2) \le A, \tag{25}$$

by replacing the roles of y_1 and y_2, if necessary.

First consider the case when (24) holds. Suppose $x \in D \cap \overline{B(\xi, \eta^9 R)}$ and $y \in D \setminus B(\xi, \eta^3 R)$. Then (14) and (20) for y_1^*, y_2^* yield

$$G(x,y) \le AR^{n-2} \sum_{i,j} G(x, y_i^*)G(y_j^*, y) \le AR^{n-2} G(x, y_1)G(y_1, y).$$

Hence the lemma follows in this case.

Next consider the case when (25) holds. Let $\Phi = \{z \in D : G(z, y_1) \ge G(z, y_2)\}$. If either $x, y \in \Phi$ or $x, y \in D \setminus \Phi$, then (21) follows from (20). Let us consider the remaining cases. If necessary, exchanging the roles of y_1 and y_2, we may assume that $x \in \Phi \cap \overline{B(\xi, \eta^9 R)}$ and $y \in (D \setminus \Phi) \setminus B(\xi, \eta^{-3} R)$. Let $E = \Phi \setminus B(\xi, \eta^3 R)$ and consider the regularized reduced function

$$\widehat{R}^E_{G(\cdot,y)} = \int_E G(\cdot, z)\, d\mu(z),$$

which is represented as the Green potential of a measure μ concentrated on ∂E. Observe that (20) for y_1^*, y_2^* and (25) imply

$$G(x,z) \le AR^{n-2} \sum_{i,j} G(x,y_i^*)G(y_j^*,z) \le AR^{n-2} G(x,y_1)G(y_1,z) \quad \text{for } z \in E.$$

Hence

$$\begin{aligned} \widehat{R}^E_{G(\cdot,y)}(x) &\le AR^{n-2}G(x,y_1)\int_E G(y_1,z)d\mu(z) \\ &= AR^{n-2}G(x,y_1)\widehat{R}^E_{G(\cdot,y)}(y_1) \le AR^{n-2}G(x,y_1)G(y_1,y). \end{aligned} \tag{26}$$

Let $v_y = G(\cdot,y) - \widehat{R}^E_{G(\cdot,y)}$. Then

$$v_y = 0 \quad \text{q.e. on } E = \Phi \setminus B(\xi, \eta^3 R). \tag{27}$$

By (20) we have

$$v_y(z) \le G(z,y) \le AR^{n-2}G(z,y_2)G(y_2,y) \quad \text{for } z \in (D \setminus \Phi) \cap \overline{B(\xi,\eta^3 R)}. \tag{28}$$

Observe that

$$D \cap \partial(\Phi \cap B(\xi,\eta^3 R)) \subset (\Phi \setminus B(\xi,\eta^3 R)) \cup (D \cap \overline{B(\xi,\eta^3 R)} \cap \partial\Phi).$$

Hence (27), (28) and the maximum principle yield

$$v_y \le AR^{n-2}G(\cdot,y_2)G(y_2,y) \quad \text{on } \Phi \cap \overline{B(\xi,\eta^3 R)}.$$

This, together with (26), implies

$$G(x,y) \le AR^{n-2}(G(x,y_1)G(y_1,y) + G(x,y_2)G(y_2,y)).$$

The proof is complete. □

Department of Mathematics, Shimane University, Matsue 690-8504, Japan
E-mail address: haikawa@math.shimane-u.ac.jp

Department of Mathematics, Shimane University, Matsue 690-8504, Japan
E-mail address: hirata@math.shimane-u.ac.jp

Department of Mathematics, Chalmers University of Technology, SE-412 96 Göteborg, Sweden
E-mail address: torbjrn@math.chalmers.se