Calculation of Ring-Shaped Phase Centers of Feeds for Ring-Focus Paraboloids

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Abstract—Some primary feeds such as the hat feed radiate around the feed waveguide, which also acts as an axial support tube. Such feed antennas have phase centers in the form of rings rather than points. This paper presents a formula to calculate the location of the ring-shaped phase center. The optimum reflector is a ring-focus paraboloid with the ring focus coinciding with the ring-shaped phase center. The phase center formula is applied to two versions of the hat feed and it is shown that the aperture efficiencies can be improved by up to 0.5 dB when optimum ring-focus paraboloids are used instead of point-focus paraboloids.

Index Terms—Hat feed, paraboloids, phase center, reflector antenna, reflector shaping.

I. INTRODUCTION

In most reflector antennas the feed can be treated as a point source, which means that the phase fronts of the feed are spherical with center of curvature in its phase center. The maximum directivity occurs when the phase center coincides with the focal point of the reflector. Several methods have been used to determine the phase center of a feed in a parabolic reflector. The author of [1] considered an idealized feed pattern for which the phase is constant when the phase reference point is at the phase center. In [2] a more general formula based on the maximization of the aperture efficiency of the reflector was presented. An iterative numerical method with the same basis was used in [3]. In [4] the formula of [2] was improved to be valid also when the radiation field of the feed has large phase variations after transformation to the phase center.

For rear-radiating feeds such as the hat feed [5], there may be large gain reductions due to phase errors even when the feed is located with its phase center at the focus of the reflector. The reason is that such feeds radiate from a circumferential aperture rather than from a planar aperture in free-space and, therefore, the radiation field is better characterized in terms of a ring-shaped phase center than a phase center point. For such feeds, the optimum reflector which maximizes the gain is a so-called “ring-focus” paraboloid [6]. In the present paper, we will introduce a formula to determine the location and radius of such a ring-shaped phase center. The formula can also be used to determine the most optimum ring-focus paraboloid when the same reflector is to be used for several feeds operating in different frequency bands.

It has to be pointed out that at a single frequency, it is possible to shape the reflector directly by using the phase variations of the copolar feed pattern in order to obtain a uniform phase for the aperture field and to reduce the phase error losses. In this case, the reflector gets a numerically specified shape which is very close to the analytical ring-focus paraboloid considered in the present work. However, when a reflector is to be used in several frequency bands, the analytical ring-focus paraboloid is easier to use for maximizing the aperture efficiency in all bands. The analytical ring-focus paraboloid is also easier to use for analytical tolerance studies.

The present study is limited to BOR feeds, i.e., bodies of revolution excited with a TE mode in a circular waveguide or similar. The present study is also performed for the case when the reflector is located in the far-field of the feed. This is satisfied for most primary fed reflector. The work is easily extended to near field illumination of a paraboloid or of the hyperboloidal subreflector of a classical Cassegrain, as explained after (2) in Section II.

II. THEORY

Generally, the phase efficiency, which is one of the subaperture efficiencies of a classical rotationally symmetric reflector system, can be expressed as follows [8]:

$$c_\phi = \left[ \int_0^{D/2} E_{\alpha 45}(\rho) \rho \, d\rho \right]^2 \left[ \int_0^{\rho_2} \left| E_{\alpha 45}(\rho) \right| \rho \, d\rho \right]^2$$

where $E_{\alpha 45}(\rho)$ is the copolar field in the 45° plane over the aperture of the reflector. When the feed is located with the origin of its coordinate system (i.e., its phase reference point) in the origin of the coordinate system of the reflector, the aperture field $E_{\alpha 45}(\rho)$, which for convenience is assumed at $z = 0$, can be expressed as

$$E_{\alpha 45}(\rho) = -\frac{1}{\tau(\theta)} G_{\alpha 45}(\theta) e^{j2\kappa z(\theta) \cos \theta}$$

$$= -\frac{1}{\tau(\theta)} G_{\alpha 45}(\theta) e^{-j2\kappa r(\theta) \cos \theta / 2}$$

where $G_{\alpha 45}(\theta) = |G_{\alpha 45}(\theta)| e^{j\varphi(\theta)}$ is the copolar far-field function of the feed, and $\tau(\theta)$ describes the reflector (Fig. 1). Equation (2) means that the reflector is in the far field of the feed. If it is not the case, we can still approximate the copolar aperture field by (2), except that $G_{\alpha 45}(\theta)$ does not represent the far-field function. Actually, $G_{\alpha 45}(\theta)$ will then be different for different focal length of the reflector. This will cause the optimum phase center to be different in reflectors of different...
focal length, even when the subtented angles of the reflectors are the same. We will consider a ring-focus paraboloid defined by

$$\rho = 2F \tan(\theta_F/2) + \rho_0$$

and

$$z = F - F \tan^2(\theta_F/2)$$  \hspace{1cm} (3)

where

- $F$ focal length;
- $\rho_0$ radius of the focus ring;
- $\theta_F$ defined in Fig. 1.

When $\rho_0 \ll F, \theta_F \approx \theta$ and we may use

$$r(\theta) = \sqrt{\rho^2 + z^2} \approx \frac{F}{\cos^2(\theta/2)}$$  \hspace{1cm} (4)

$$r(\theta) = \sqrt{\rho^2 + z^2} \approx \frac{F}{\cos^2(\theta/2)} \left[ 1 + \frac{\rho_0 \sin \theta (1 + \cos \theta)}{2F} \right]$$  \hspace{1cm} (5)

in amplitude factors and phase terms, respectively. Using (2), (4), and (5) we may write the phase efficiency as

$$C_{\phi} = \frac{1}{F_{\phi}} \left[ \int_{0}^{\theta_0} u(\theta) e^{i\phi(\theta)} \cos \theta \sin \theta (1 + \cos \theta) \, d\theta \right]^2$$  \hspace{1cm} (6)

where

- $\theta_0$ subtended half angle of the reflector (see Fig. 1);
- $u(\theta)$ magnitude of $G_{\cos^2}(\theta)$;
- $\phi(\theta)$ phase of $G_{\cos^2}(\theta)$.

We define the ring-shaped phase center of the feed as the location of the focus ring of the optimum ring-focus paraboloid which maximizes the phase efficiency and thereby also the overall aperture efficiency, all in agreement with the definition in [3] for point-focus paraboloids. We will now derive a direct computation formula based on this definition.

The phase function of the copolar pattern of the feed antenna is $\phi(\theta) = \phi(\theta) - k z_0 \cos(\theta)$ when the phase reference point is moved from the origin to a point $z_0$ on the symmetry axis in the coordinate system of the feed. This corresponds to locating the feed in such a way that the point $z = z_0$ coincides with the origin of the coordinate system of the reflector. The phase efficiency for this location of the feed in the ring-focus paraboloid in (3) can be written as follows:

$$C_{\phi} = \frac{1}{F_{\phi}} \left[ \int_{0}^{\theta_0} u(\theta) e^{i\phi(\theta)} \cos \theta \sin \theta (1 + \cos \theta) \, d\theta \right]^2$$  \hspace{1cm} (7)

For convenience, we introduce

$$\phi(\theta) = \phi(\theta) - k z_0 \cos \theta - k \rho_0 \sin \theta (1 + \cos \theta).$$  \hspace{1cm} (8)

We now assume small phase errors, i.e.,

$$\phi(\theta) - \phi(0) \ll \frac{\pi}{2}.$$  \hspace{1cm} (9)

This limits our treatment to feeds which give small phase errors in optimum ring-focus paraboloids, i.e., when $z_0$ and $\rho_0$ are chosen so as to satisfy (9). Then, by Taylor expansion of the phase factor we get [2]

$$C_{\phi} = 1 - \left( \frac{\phi(\theta)}{2} \right)^2$$  \hspace{1cm} (10)
where \( \overline{\phi_\epsilon} \) and \( \overline{\phi^2} \) are the mean and mean-square phase errors, respectively, given by

\[
\overline{\phi_\epsilon} = \frac{1}{I_w} \int_0^{\theta_0} w(\theta)[\phi_\epsilon(\theta) - \phi_0(0)] d\theta
\]
\[
\overline{\phi^2} = \frac{1}{I_w} \int_0^{\theta_0} w(\theta)[\phi_\epsilon(\theta) - \phi_0(0)]^2 d\theta
\]

By substituting (8) into (10), we obtain the following expression:

\[
\epsilon_\phi = a - b(\kappa z_0^2 - c(kq_0)^2)^2 + 2bky_0 + 2c(k\rho_0 - 2f)^2 z_0 \rho_0
\]

where

\[
\begin{align*}
\alpha &= 1 - \frac{I_{\text{ave}2}}{I_w} + \frac{I_{\text{ave}1}^2}{I_{\text{ave}2}^2}; \\
b &= \frac{I_{\text{ave}2}}{I_w} - \frac{I_{\text{ave}1}^2}{I_{\text{ave}2}^2}; \\
c &= \frac{I_{\text{ave}2}}{I_w} - \frac{I_{\text{ave}1}^2}{I_{\text{ave}2}^2}; \\
d &= \frac{I_{\text{ave}1}}{I_w} - \frac{I_{\text{ave}2}}{I_{\text{ave}2}^2}; \\
e &= \frac{I_{\text{ave}1} I_{\text{ave}2}}{I_w} - \frac{I_{\text{ave}1}^2}{I_{\text{ave}2}^2}; \\
f &= \frac{I_{\text{ave}1}}{I_w} - \frac{I_{\text{ave}2}}{I_{\text{ave}2}^2}
\end{align*}
\]

Equation (12) has a simple dependence on both \( z_0 \) and \( \rho_0 \) and, by partial differentiation, \( \epsilon_\phi \) can be easily maximized with respect to \( z_0 \) and \( \rho_0 \). The results \( z_0 = z_{\text{pc}} \) and \( \rho_0 = \rho_{\text{pc}} \) become

\[
kz_{\text{pc}} = \frac{c(b - f^2)}{bc - f^2}; \quad kq_{\text{pc}} = \frac{bc - df}{bc - f^2}
\]

where \( z_{\text{pc}} \) and \( \rho_{\text{pc}} \) represent the axial location and radius of the ring-shaped phase center, respectively. Then \( \rho_0 = \rho_{\text{pc}} \) also defines the optimum ring-focus paraboloid for the feed, which maximizes the aperture efficiency if the feed is located correctly with \( z_0 = z_{\text{pc}} \) in the center of the coordinate system of the reflector when the reflector is defined by (3).

The formula for \( \rho_{\text{pc}} \) is and should be independent of the location of the origin of the coordinate system of the feed, i.e., the initial phase reference point of the radiation pattern of the feed, as long as it is on the symmetry axis. This can be shown as follows. Assume that \( \phi_0(\theta) \) is the phase of the radiation pattern of the feed when the phase reference point is located at \( z_0 \). Then, the phase of the radiation pattern of the feed will be

\[
\phi_\epsilon(\theta) = \phi_0 - k(z_1 - z_0) \cos \theta
\]

when the phase reference point is moved from \( z_0 \) to \( z_1 \). From (13) and (14), the denominator of (15), \( cb - f^2 \), will not be affected by this change of reference point, whereas \( c \) and \( d \) will be affected and from (14) we get

\[
\begin{align*}
I_{\text{ave}1} z_1 &= I_{\text{ave}1} z_0 - k(z_1 - z_0) I_{\text{ave}2} \\
I_{\text{ave}2} z_1 &= I_{\text{ave}2} z_0 - k(z_1 - z_0) I_{\text{ave}1} \\
I_{\text{ave}1} z_1 &= I_{\text{ave}1} z_0 - k(z_1 - z_0) I_{\text{ave}2}
\end{align*}
\]
TABLE I

<table>
<thead>
<tr>
<th>Method</th>
<th>(z_{pc}/\lambda)</th>
<th>(\rho_{pc}/\lambda)</th>
<th>(e_{\phi}(dB))</th>
<th>Method</th>
<th>(z_{pc}/\lambda)</th>
<th>(\rho_{pc}/\lambda)</th>
<th>(e_{\phi}(dB))</th>
</tr>
</thead>
<tbody>
<tr>
<td>Method in [1]</td>
<td>0.84</td>
<td></td>
<td>-0.73</td>
<td>Method in [2]</td>
<td>0.87</td>
<td></td>
<td>-0.63</td>
</tr>
<tr>
<td>Method in [3]</td>
<td>0.89</td>
<td></td>
<td>-0.69</td>
<td>Method in [4]</td>
<td>0.89</td>
<td></td>
<td>-0.45</td>
</tr>
<tr>
<td>Present Method</td>
<td>0.89</td>
<td>0.43</td>
<td>-0.17</td>
<td></td>
<td>0.81</td>
<td>0.33</td>
<td>-0.02</td>
</tr>
</tbody>
</table>

TABLE II

<table>
<thead>
<tr>
<th>(f_0) in different bands</th>
<th>(\Delta \rho/\lambda)</th>
<th>(\Delta z/\lambda)</th>
<th>(e_{\phi}(dB))</th>
<th>(\Delta \rho/\lambda)</th>
<th>(e_{\phi}(dB))</th>
</tr>
</thead>
<tbody>
<tr>
<td>22.4 GHz</td>
<td>-0.08</td>
<td>0.00</td>
<td>-0.25</td>
<td>0.10</td>
<td>-0.10</td>
</tr>
<tr>
<td>25.5 GHz</td>
<td>-0.05</td>
<td>0.00</td>
<td>-0.12</td>
<td>0.06</td>
<td>-0.04</td>
</tr>
<tr>
<td>28.5 GHz</td>
<td>-0.02</td>
<td>0.00</td>
<td>-0.08</td>
<td>0.02</td>
<td>-0.03</td>
</tr>
<tr>
<td>38.5 GHz</td>
<td>0.08</td>
<td>0.00</td>
<td>-0.25</td>
<td>-0.10</td>
<td></td>
</tr>
</tbody>
</table>

By substituting (17) into (13) and (15), we obtain

\[
k\rho_{pc}\big|_{z_1} = \frac{1}{bc} - \frac{f}{b^2} (b\rho_{pc} - d|z_1|f) = \frac{1}{bc} - \frac{f}{b^2} (b\rho_{pc} - d|z_0|f) = k\rho_{pc}\big|_{z_0}.
\]

In the same way, it can be shown that \(z_{pc}\) is also independent of the initial reference point of the radiation pattern of the feed.

Thus, we can conclude that as long as the assumption of (9) is valid, the formula (15) is valid. This corresponds to the requirement that the resulting phase efficiency in the optimum reflector must be better than about \(-1\ dB\).

IV. OPTIMUM AXIAL POSITION WITH GIVEN RING-FOCUS PARABOLOID

We will now consider the optimum axial location of the feed in a given nonoptimum ring-focus paraboloid. Equation (12) is valid also for this case and the optimum location \(z_{opt}\) is found by maximizing \(e_{\phi}\) with respect to \(z_0\) when \(\rho_0\) is constant. This gives

\[
z_{opt} = -\frac{b}{c} f - \frac{f}{b} \rho_0.
\]

We can also introduce \(z_{opt} = z_{pc} + \Delta z\) and \(\rho_0 = \rho_{pc} + \Delta \rho\), for which case the optimum axial location of the feed

\[
\Delta z = -\frac{f}{k} \Delta \rho.
\]

compared to its best location in the optimum ring-focus paraboloid. The corresponding maximum phase efficiency in the given nonoptimum ring-focus paraboloid is

\[
e_{\phi, opt} = e_{\phi, max} - \frac{f^2}{b} (k\Delta \rho)^2.
\]

where \(e_{\phi, max}\) is the phase efficiency in the optimum ring-focus paraboloid. These formulas are very useful when designing reflectors to be used for feeds in multiple frequency bands. It can readily be shown that if a ring-focus paraboloid is to be used with several different feeds between a lowest frequency \(f_{min}\) and a highest frequency \(f_{max}\), which may be several octaves different, the optimum reflector should be designed at the mean frequency \((f_{max} + f_{min})/2\) Then, the phase efficiency has its highest value at the mean frequency, and its lowest value occurs at and is the same for both \(f_{min}\) and \(f_{max}\). Equation (21) is also convenient to find the tolerance requirement on the axial feed location.

V. APPLICATION

The above formulas have been applied to hat fed reflectors. The hat feed is made of a straight circular tube with a corrugated plate (hat) at the end, which is mounted to the tube by means of a small dielectric piece (head) (see Fig. 2). The hat fed reflector can provide extremely low sidelobes and the manufacturing cost is low. Two kinds of hat fed reflectors are used for the present analysis: a Chinese hat feed in a ring-focus paraboloid with a subtended semi-angle of 80° [Fig. 2(a)] [9] and a standard hat feed in a ring-focus paraboloid with a subtended semi-angle of 100° [Fig. 2(b)] [10]. The amplitude and phase functions of the copolar radiation field were calculated by the V2D code [11] based on the finite-difference time-domain (FDTD) method (Fig. 3). In both cases the outer radius of the circular waveguide is \(0.5\lambda\). In comparison, we have used the previous methods, which were mentioned in the introduction, to calculate the phase center of the hat feed and the corresponding phase efficiency. The results are shown in Table I. From Fig. 3 and Table I, one observes that, by using a ring-focus paraboloid, the phase efficiencies are considerably better than those for a standard parab-
oloid. For the standard hat feed, the ring-focus paraboloid has a considerably high phase efficiency of $-0.02$ dB. An interesting phenomenon is that the radius of the ring focus is not equal to the outer radius of the circular waveguide. One can also observe that for the two cases in Table I, the axial locations of the phase centers are equal to those determined by using the point phase center formulas in [3] and [4]. This is because the iterative numerical methods in [3] and [4] are valid for the general cases and the present work gives the analytic solution for a specific case of them.

In order to reduce manufacture cost, the same reflector may be used for different hat feeds in several frequency bands. Then, by using (21), it is very easy to find the optimum axial position of the feed and the corresponding phase efficiency. Table II presents the results for the geometry considered in this work. One can see that the phase efficiency can be improved by 0.15 dB at both the highest and the lowest frequency simply by axially relocating the feeds 0.14 away from their phase center in the optimum reflector.

VI. CONCLUSION

We have presented a method of determining the optimum ring-focus paraboloid for a feed with a ring-shaped phase center. The method provides both the location of the phase center of the feed and the optimum reflector shape. For the examples studied, the phase efficiency can be improved by 0.5 dB by using the ring-focus phase center formula. The method also provides a very efficient and easy way to maximize the aperture efficiency in all frequency bands when the same ring-focus reflector is used in several frequency bands, simply by relocating the feeds in this given nonoptimum ring-focus paraboloid reflector. The improvement of the aperture efficiency can in this case be 0.15 dB. It is also easy to use this method for studying the tolerances of the reflector and feed location.

ACKNOWLEDGMENT

The authors would like to thank J. Flodin for his inspiration and his remark from which this work was initiated; he showed the authors that the optimum reflector, which was calculated from the phase pattern of the feed and presented as a list of coordinates has nearly the same shape as a ring-focus paraboloid.

REFERENCES


Jian Yang was born in Nanjing, China, on August 2, 1960. He received the B.S. degree from the East China Institute of Technology, Nanjing, China, in 1982, and the M.S. degree from Nanjing Research Centre of Electronic Engineering in 1985, Nanjing, China, both in electrical engineering, and the Swedish Licentiate Eng. degree from Chalmers University of Technology, Gothenburg, Sweden, in 1998. He is currently working toward the Ph.D. degree from Chalmers University of Technology.

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Per-Simon Kildal (M’82–SM’84–F’95) was born in Norway on July 4, 1951. He received his Ph.D. degree from Chalmers University of Technology, Gothenburg, Sweden, in 1984, he has been a Consultant for Cornell University, Ithaca, NY, in connection with the upgrading of the radio telescope in Arecibo, Puerto Rico. He is also consulting for others via his own company, Kildal Antenna Consulting AB. This company also develops and markets commercial software for synthesis and analysis of reflector antennas. Kildal is an efficient and accurate programmer in both FORTRAN and Mathcad. He has been largely involved in the electrical design of some large antennas and is the inventor of three granted patents, and he has in addition applications that are not yet granted. He is also involved in new company in Gothenburg, which has been formed for developing and marketing microwave sensors for different application. The sensors are based on a recent patent application. He is the principal author of more than 40 articles in international journals and more than 40 papers at international conferences. All papers are within the area of antenna theory and design. His specialty is reflector antennas.

Dr. Kildal served from 1991 to 1994 as a Distinguished Lecturer of the IEEE Antennas and Propagation Society. He offered two lectures on the concept of artificially soft and hard surfaces for electromagnetic waves and on the techniques for synthesis and analysis of reflector antennas that was developed in connection with the design of the dual-reflector feed for the radio telescope in Arecibo. He was an elected member of IEEE AP-S AdCom (Administration Committee of IEEE Antennas and Propagation Society) for the period 1995–1997, and served as an Associate Editor of the IEEE TRANSACTIONS ON ANTENNAS AND PROPAGATION from June 1995 until July 1998. ELAB award for his project, in connection with an industrial project where the results of his reflector antenna research were applied. He received the R.W. King Award for the Best Paper by a Young Author in IEEE TRANSACTIONS ON ANTENNAS AND PROPAGATION in 1984 (about the EISCAT cylindrical reflector antenna). Later he received the S.A. Schellkunoff Transactions Prize Paper Award for the Best Paper in 1990 in the same IEEE TRANSACTIONS (about the synthesis of the dual-reflector feed for the Arecibo radio telescope). He has given invited lectures in plenary sessions at four conferences (Antem 92 in Winnipeg, 23rd EuMC 93 in Madrid, MIKON 94 in Poland, and JINA 94 in Nice).