Renormalization in the three-body problem with resonant *p*-wave interactions

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Resonant *p*-wave interactions can be described by a minimal zero-range model defined by a truncated effective range expansion so that the only two-body interaction parameters are the inverse scattering volume $1/a_p$ and the *p*-wave effective range r_p. This minimal model can be formulated as a local quantum field theory with a *p*-wave interaction between atom fields and a molecular field. In the two-atom sector, the model is renormalizable, but it has unphysical behavior at high energies because there are negative-probability states with momentum scale r_p. In the sector with three atoms, two of which are identical, renormalization in some parity and angular momentum channels involves an ultraviolet limit cycle, indicating asymptotic discrete scale invariance. The Efimov effect occurs in the unitary limit $a_p^{-1/3}, r_p \to 0$, but this limit is unphysical because there are low-energy states with negative probability. The minimal model can be of physical relevance only at energies small compared to the energy scale set by r_p where the effects of negative-probability states are suppressed.

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I. INTRODUCTION

Systems with an *s*-wave resonance near a threshold arise in many fields of physics, including atomic, condensed-matter, nuclear, and high-energy physics. They are important because they exhibit universal behavior that depends on the energy of the resonance but is otherwise insensitive to the nature of the constituents and the form of the interactions between them as long as they have a short range. The universal aspects of the few-body problem have been studied in detail [1]. The universal properties depend on the *s*-wave scattering length *a*. If the Efimov effect occurs in the three-body problem, the universal properties also depend on a three-body scale parameter upon which observables can only depend log-periodically. In the many-body problem with identical fermions with two spin states, the universal properties have also been studied thoroughly, theoretically and also experimentally using ultracold atoms [2–5]. If the Efimov effect occurs in the three-body problem, the many-body problem is more complicated and may not even be well defined.

Systems with a *p*-wave resonance near a threshold also arise in many fields of physics. A prominent example in nuclear physics is the low-energy *p*-wave resonance in $n - {}^4\mathrm{He}$ scattering, which corresponds to the unstable ${}^5\mathrm{He}$ nucleus [6]. The universal aspects of such systems are not as clear-cut as in the *s*-wave case. They involve at least two parameters: the *p*-wave scattering volume a_p and the *p*-wave effective range r_p. The many-body physics of fermions with two spin states that interact through a *p*-wave resonance has been studied using mean-field approximations [7–11]. If the *p*-wave interactions are strongly resonant, the many-body physics may reveal qualitatively new features that are not captured by mean-field approximations [12].

Ultracold atoms provide a promising laboratory for the experimental study of both few-body and many-body systems of particles with a *p*-wave resonance near threshold. The inverse scattering length a_p can be controlled experimentally and can be made arbitrarily large by tuning the magnetic field to a *p*-wave Feshbach resonance. The first such studies were

for a *p*-wave Feshbach resonance between identical fermions using ${}^{40}\mathrm{K}$ atoms [13]. Fermionic ${}^6\mathrm{Li}$ atoms have been studied near a Feshbach resonance between the lowest two hyperfine spin states [14,15]. The binding energies of *p*-wave dimers and the atom-dimer and dimer-dimer inelastic collision rates have been measured [16,17]. *p*-wave Feshbach resonances have also been observed in a mixture of ${}^{40}\mathrm{K}$ fermions and ${}^{87}\mathrm{Rb}$ bosons [18]. One obstacle to studying the dependence on a_p is that *p*-wave Feshbach resonances in ultracold atoms are usually very narrow.

In this paper, we study the renormalization in the three-body sector for a minimal field-theoretic model of a system with a *p*-wave threshold resonance. In Sec. II, we explain why the model can only be physically relevant at energies small compared to the energy scale set by r_p. In Sec. III, we write down the integral equation that can be used to calculate the bound-state spectrum of triatomic molecules in the model. In Sec. IV, we identify the angular momentum and parity channels of the three-atom sector in which a three-body parameter is required by renormalization. Finally, in Sec. V, we discuss whether the renormalization properties of the minimal model can be relevant to any real physical systems.

II. TWO-BODY PROBLEM WITH A *p*-WAVE RESONANCE

We begin by discussing *p*-wave scattering at low energy $E = k^2/(2\mu)$, where μ is the reduced mass. If the particles interact through a short-range potential, the *p*-wave phase shift has an effective range expansion in powers of k^2 that has the form

$$k^3 \cot \delta_p(k) = -1/a_p + \tfrac{1}{2} r_p k^2 + \cdots, \qquad (1)$$

where a_p is the scattering volume and r_p is the *p*-wave effective range, which have dimensions (length)^3 and 1/(length), respectively. If the only important interactions are in the *p*-wave channel and if the effective range expansion is truncated after

the second term, the scattering amplitude reduces to

$$f_k(\theta) = \frac{k^2 \cos \theta}{-1/a_p + \frac{1}{2}r_p k^2 - ik^3}. \tag{2}$$

An equivalent expression is obtained if the interactions are dominated by a coupling to a p-wave Feshbach resonance,

$$f_k(\theta) = \frac{k^2 \cos \theta}{(\nu - k^2/2\mu)/g^2 - ik^3}. \tag{3}$$

The parameter $r_p = -1/\mu g^2$, which is negative definite, controls the strength of the coupling to the Feshbach resonance. The detuning of the resonance from the threshold is controlled by the combination $1/a_p r_p = \mu \nu$. We consider p-wave interactions for which the scattering amplitude has the minimal form in Eq. (2) or (3) with the two parameters a_p and r_p.

A point in the parameter space that is of particular interest is $a_p^{-1/3} = r_p = 0$, which is called the *unitary limit* because the unitarity bound for scattering is saturated at this point. Since they provide no length scale, the interactions are scale invariant in the unitary limit. It is easy to see that the unitary limit cannot be realized with interactions through a short-range potential. If the interaction potential vanishes outside the range R, there is an upper bound on r_p called the *Wigner bound* [19]. For $|a_p|^{1/3} \gg R$, the Wigner bound reduces to $r_p \leqslant -2/R$. If we try to take the zero-range limit $R \to 0$ to justify the truncation of the effective range expansion in Eq. (1), r_p is driven to $-\infty$. The constraint $r_p < -2/R$ can also be derived by demanding that the probability for a bound state to be in the region $r > R$ is less than 1 [20,21].

The behavior in the zero-range limit is improved if the potential with range R is supplemented by a van der Waals tail that falls off as $-C_6/r^6$. In this case, the expansion in Eq. (1) also includes a linear term $b_p k$ [22]. In the zero-range limit $R \to 0$, the coefficients b_p and r_p are determined by the scattering volume a_p and the van der Waals length $\beta_6 = (2\mu C_6/\hbar^2)^{1/4}$. If we also take the limit $|a_p|^{1/3} \gg \beta_6$, $b_p \to 0$, and $r_p \to -3.4/\beta_6$. Thus, the low-energy expansion of $f_k(\theta)$ reduces to Eq. (2), but the scale of r_p is set by $1/\beta_6$. The truncation of the expansion is justified only at low energies $|E| \ll 1/\mu\beta_6^2$.

Real atoms interact through a short-range potential with a van der Waals tail, but they can also have couplings to diatomic molecules. The scattering volume a_p can be controlled and can be made arbitrarily large by tuning the magnetic field to a p-wave Feshbach resonance where one of the molecules crosses the two-atom threshold. The conditions for $f_k(\theta)$ to be well approximated by the effective range approximation in Eq. (2) have been studied by Zhang *et al.* [23].

Nishida has pointed out that the assumption that the scattering amplitude has the simple form in Eq. (2) or (3) up to arbitrarily large momentum k necessarily implies the existence of states with negative probability [24]. In the case of a short-range potential, the problem is related to the Wigner bound. If the bound $r_p < -2/R$ is violated, the probability for a bound state to be in the region $r > R$ exceeds 1, and that excess probability must be canceled by negative probability from the region $r < R$ [24].

The problem of negative-probability states can also be seen directly from the expression for the scattering amplitude in Eq. (2). It can be expressed as a function of the energy $E = k^2/2\mu$,

$$f_k(\theta) = \frac{2\mu E \cos \theta}{-1/a_p + \mu r_p E - (-2\mu E - i\varepsilon)^{3/2}}. \tag{4}$$

The poles in E of this function are the energies of bound states. We set $1/a_p = 0$ for simplicity. If $r_p < 0$, as required by the Wigner bound, the denominator then has zeros at $E = 0$ and $E = -r_p^2/8\mu$, which correspond to one bound state at the threshold and another with binding energy $r_p^2/8\mu$. The scattering amplitude can be expressed as a sum of contributions from the two poles and a function of E that is regular at the poles,

$$f_k(\theta) = 2E \cos \theta \left(\frac{1}{r_p E} - \frac{2}{r_p \left(E + r_p^2/8\mu \right)} + (\text{regular}) \right). \tag{5}$$

Note that the residues of the two poles have different signs. The two bound states will contribute to unitarity sums with the opposite sign of the residues in Eq. (5). The probabilities of the bound states with binding energies 0 and $r_p^2/8\mu$ are positive and negative, respectively. If $r_p > 0$, there is only a negative probability pole with energy 0 on the physical sheet. A negative-probability state with energy 0 is fatal for any physical interpretation of the threshold region. If $r_p < 0$, the negative-probability state with binding energy $r_p^2/8\mu$ is not necessarily fatal because its effects are suppressed at sufficiently low energy. Thus, a sensible physical interpretation requires $r_p < 0$ and $|E| \ll r_p^2/8\mu$. If $1/a_p$ is nonzero, the analysis is more complicated. We refer to a pole as *shallow* if it has energy $|E| < r_p^2/8\mu$ and as *deep* if it has energy $|E| \gtrsim r_p^2/8\mu$ The different cases can be classified as:

(1) $1/a_p < 0$: There is only one deep pole on the physical sheet. It has negative probability.

(2) $0 < 1/a_p \leqslant |r_p|^3/54$: There are two poles on the physical sheet, one shallow pole with positive probability and one deep pole with negative probability.

(3) $1/a_p > |r_p|^3/54$: There are two poles with complex energies and complex residues on the physical sheet. This case violates standard analyticity assumptions for the S matrix and should be discarded [25].

The problem of atoms whose pair interactions give the scattering amplitude in Eq. (2) can be formulated as a quantum field theory (for a review of field-theoretical models of atom-atom interactions see, e.g., Ref. [26]) with two scalar fields ψ_1 and ψ_2 (which annihilate atoms of types 1 and 2, respectively) and a vector field $\boldsymbol{d}$ (which annihilates a diatomic molecule). We take the masses of the atoms to be m_1 and m_2. The only interaction is a p-wave contact interaction that allows transitions between the diatomic molecule and a pair of atoms of types 1 and 2. The form of the Lagrangian is constrained by Galilean invariance. It consists of kinetic terms, an energy offset for the diatomic molecule, and a p-wave interaction

term,

$$\mathcal{L} = \sum_{\sigma=1}^{2} \psi_\sigma^\dagger \left(i\frac{\partial}{\partial t} + \frac{1}{2m_\sigma}\nabla^2 \right) \psi_\sigma$$
$$+ \eta \boldsymbol{d}^\dagger \cdot \left(i\frac{\partial}{\partial t} + \frac{1}{2(m_1+m_2)}\nabla^2 \right) \boldsymbol{d} + \Delta_0 \boldsymbol{d}^\dagger \cdot \boldsymbol{d}$$
$$- g_0\mu\{\boldsymbol{d}^\dagger \cdot [\psi_2 i\boldsymbol{\nabla}\psi_1/m_1 - (i\boldsymbol{\nabla}\psi_2)\psi_1/m_2] + (\text{H.c.})\}.$$

$$(6)$$

The parameters Δ_0 and g_0 are bare parameters that depend on the ultraviolet cutoff. The parameter η can be chosen by normalization of the field $\boldsymbol{d}$ to be either $+1$ or -1. If $\eta = +1$, $\boldsymbol{d}$ is an ordinary field, which, in the absence of interactions, would annihilate an ordinary positive-probability molecule. If $\eta = -1$, $\boldsymbol{d}$ is a ghost field, which, in the absence of interactions, would annihilate a negative-probability molecule.

The two-body problem for atoms 1 and 2 can be solved analytically. The minimal model defined by the Lagrangian in Eq. (6) is renormalizable in the two-atom sector [6]. The bare parameters g_0 and Δ_0 can be tuned as functions of the ultraviolet momentum cutoff Λ in such a way that the scattering amplitude reduces to Eq. (2) in the limit $\Lambda \to \infty$. The renormalization conditions can be written

$$\frac{1}{a_p} = \frac{6\pi}{g_0^2\mu}\Delta_0 + \frac{2}{3\pi}\Lambda^3, \qquad (7a)$$

$$\frac{r_p}{2} = -\eta\frac{3\pi}{g_0^2\mu^2} - \frac{2}{\pi}\Lambda. \qquad (7b)$$

In order to have finite limits as $\Lambda \to \infty$, the two terms on the right sides of both Eqs. (7a) and (7b) must have opposite signs. A finite limit for a_p requires that Δ_0 be negative and that Δ_0/g_0^2 scale like Λ^3. A finite limit for r_p requires that g_0^2 scale like Λ^{-1} and that $\eta = -1$. Thus, $\boldsymbol{d}$ must be a ghost field. If there were no interactions, it would annihilate a negative-probability molecule. To see whether there are any negative-probability particles in the presence of interactions, one needs to examine the scattering amplitude in Eq. (2). As we have seen, $r_p < 0$ is necessary to avoid negative-probability particles near the two-atom threshold. The energy restriction $|E| \ll r_p^2/8\mu$ is then necessary to ensure that unphysical effects associated with negative-probability particles are suppressed. The simplest observable that can go wrong is the two-body bound-state spectrum. It contains unphysical low-energy states if the energy restriction is not observed.

Nishida also presented an algebraic argument for the existence of states with negative probability in this model [24]. In the unitary limit $a_p^{-1/3} = r_p = 0$, the model has scaling symmetry as well as Galilean symmetry. It, therefore, also has nonrelativistic conformal symmetry [27]. The asymptotic behavior of the scattering amplitude in Eq. (2) at large k implies that the dimer field $\boldsymbol{d}$ has scaling dimension 1. However, Nishida and Son [27] have used the nonrelativistic conformal algebra together with the assumption that all states have a positive norm to prove that primary operators can only have scaling dimensions greater than or equal to 3/2. The violation of the bound in this model implies that there must be states with a negative norm.

III. *p*-WAVE SKORNYAKOV–TER-MARTIROSIAN EQUATION

We now consider the three-body problem with resonant *p*-wave interactions between pairs of atoms. For three identical bosons, there can be no *p*-wave interactions. The case of three identical fermions has been studied thoroughly by Jona-Lasinio *et al.*. [21]. They considered a two-channel model with atoms and a diatomic molecule that have interactions with a finite range b. They found that r_p has to be negative and that there is a lower bound on $|r_p|$ that is proportional to $1/b$. In the three-fermion sector, they calculated the spectrum of triatomic molecules (trimers), atom-dimer scattering lengths, and three-body recombination rates. For $|a_p| \gg b^3$, the trimer spectrum at energies small compared to $\hbar^2/mb^2$ can consist of either zero or one trimer with positive parity and either zero or one trimer with negative parity. Three-body recombination was also studied previously by Suno *et al.* using the adiabatic hyperspherical representation of the Schrödinger equation for three identical fermions interacting through a short-range potential that is tuned to give a large *p*-wave scattering volume [28].

In the next simplest case of a three-body problem with resonant *p*-wave interactions, there are two identical atoms of type 2 and a third atom of type 1. Atoms of types 1 and 2 interact through a *p*-wave resonance, and we assume that the interaction between the identical atoms can be neglected. We describe this system using the model defined by the Lagrangian in Eq. (6). Transition amplitudes for three atoms can be calculated exactly numerically by solving a single-variable integral equation that is analogous to the Skornyakov-Ter-Martirosian (STM) equation for *s*-wave contact interactions [29]. The equations for different orbital angular momentum and parity quantum numbers J^P can be decoupled. The bound-state problem reduces to solving an eigenvalue equation for the energy $E < 0$ that is a homogeneous integral equation of the form

$$B(p) = \nu\frac{3(-1)^{J+1}}{\pi(2J+1)}\int_0^\Lambda dq\, R^{J^P}(p,q,E)D(q,E)B(q), \quad (8)$$

where the prefactor ν is $+1$ or -1 if the identical atoms are bosons or fermions, respectively. The upper limit Λ of the integral is an ultraviolet momentum cutoff that should be taken to ∞. The dimer propagator $D(q,E)$ is given by

$$D(q,E) = q^2\left[\frac{1}{a_p} + \frac{r_p}{2}b(q,E) + b(q,E)^{3/2}\right]^{-1}, \quad (9a)$$

$$b(q,E) = \frac{r(2+r)}{(1+r)^2}\left[q^2 - \frac{2(1+r)}{2+r}m_2 E\right], \quad (9b)$$

where $r = m_1/m_2$ is the mass ratio. The kernel $R^{J^P}(p,q,E)$ depends on the quantum numbers J^P. For positive parity J^+ with $J > 0$, the bound-state amplitude $B(p)$ is a two-component column vector, and $R^{J^+}(p,q,E)$ is a 2×2 matrix with entries,

$$R_{11}^{J^+} = \left[\frac{J}{1+r} + \frac{1+r}{2J-1}\right]Q_J - J\left[\frac{p}{q} + \frac{q}{p}\right]Q_{J-1}$$
$$+ \frac{(J-1)(2J+1)(1+r)}{2J-1}Q_{J-2}, \quad (10a)$$

$$R_{12}^{J^+} = \sqrt{J(J+1)}\{-[1/(1+r)+1+r]Q_J$$
$$+(q/p)Q_{J-1}+(p/q)Q_{J+1}\}, \quad (10b)$$

$$R_{22}^{J^+} = \left[\frac{J+1}{1+r}+\frac{1+r}{2J+3}\right]Q_J - (J+1)\left[\frac{p}{q}+\frac{q}{p}\right]Q_{J+1}$$
$$+\frac{(J+2)(2J+1)(1+r)}{2J+3}Q_{J+2}, \quad (10c)$$

and $R_{21}^{J^+}(p,q,E) = R_{12}^{J^+}(q,p,E)$. For $J \geqslant 0$, $Q_J(x)$ is a Legendre function of the second kind, and its argument is $x = (1+r)[p/q+q/p]/2 - m_2 r E/(pq)$. For $J < 0$, $Q_J = 0$. For the special case $J^P = 0^+$ and for negative parity J^- with $J > 0$, $B(p)$ has one component, and R^{J^P} is a 1×1 matrix. For 0^+, its entry $R_{22}^{0^+}(p,q,E)$ is given by Eq. (10c). For J^-, its entry is

$$R^{J^-}(p,q,E) = (1+r)[Q_{J+1}(x) - Q_{J-1}(x)]. \quad (11)$$

These equations are valid for $1/a_p \leqslant |r_p|^3/54$.

IV. RENORMALIZATION-GROUP LIMIT CYCLES

In most J^P channels, the solution to the bound-state equation for $B(p)$ decreases rapidly enough at large p that the integral in Eq. (8) converges as $\Lambda \to \infty$. In these channels, the model is renormalizable: The only parameters that are required are a_p and r_p. However, there are also channels in which $B(p)$ approaches a log-periodic function of p at large p, just like in the STM equation. In this case, the integral does not converge but instead, approaches a log-periodic function of Λ as $\Lambda \to \infty$. If the identical particles are bosons, the only channel in which this behavior occurs is 1^+. If the identical particles are fermions, this behavior occurs in the channels 0^+, 1^+, 1^-, and 2^+.

In the field-theory framework, the cutoff dependence implies that the model, defined by the Lagrangian in Eq. (6), is not renormalizable in the three-atom sector. In the case of an s-wave contact interaction, renormalizability can be restored by adding a three-body contact interaction term [30]. In our case with a p-wave interaction, renormalizability can also be restored by adding an appropriate three-body contact interaction for each J^P channel in which there is log-periodic dependence on Λ. The renormalization-group (RG) flow for its coupling constant is governed by an RG limit cycle [1]. An RG limit cycle is characterized by a discrete scaling factor λ_0 such that, every time Λ increases by a factor of λ_0, the coupling constant returns to its original value.

Renormalization can be implemented by eliminating the coupling constant for the three-atom interaction in favor of a renormalization scale Λ_* [30]. The dependence of any renormalized amplitude on Λ_* can only be log-periodic with discrete scaling factor λ_0. The model, therefore, exhibits asymptotic discrete scale invariance at high energies. A discrete scale transformation consists of rescaling all momenta and energies by factors of λ_0^n and λ_0^{2n}, respectively, and rescaling the parameters as $a_p \to \lambda_0^{-3n} a_p$ and $r_p \to \lambda_0^n r_p$, where n is an integer. Under a discrete scale transformation, any amplitude changes simply by an overall power of λ_0^n.

The most dramatic consequence of the discrete scale invariance is the Efimov effect. In 1970, Efimov showed that,

if the two-body s-wave scattering length is tuned to infinity, there can be an infinite sequence of three-body bound states called *Efimov states* that have an accumulation point at the three-body threshold [31]. The binding energies of successive states differ by the square of a discrete scaling factor λ_0 that depends on the mass ratios and symmetries. The cases in which the Efimov effect occurs include three identical bosons, two identical bosons and a third particle, and two identical fermions and a third particle whose mass is smaller than the fermion mass divided by 13.6 (see, e.g., Ref. [1]). In the case of resonant p-wave interactions between one atom and two identical atoms, we have identified all the channels in which the Efimov effect occurs. If the identical particles are bosons, the Efimov effect occurs only in the 1^+ channel. If the identical particles are fermions, the Efimov effect occurs only in the channels 0^+, 1^+, 1^-, and 2^+. The Efimov effect in the 1^- sector was discovered by Macek and Sternberg [32] and Macek [33]. Our results for the 1^- sector are consistent with their results. Unfortunately, the Efimov effect in this model is unphysical. In the unitary limit, there are negative-probability states at the three-atom threshold, which is the accumulation point for the infinitely many Efimov states.

The discrete scaling factor for each J^P channel can be expressed as $\lambda_0 = \exp(\pi/s_0)$, where $s = is_0$ is a pure imaginary solution to the equation,

$$\det\left[\mathbb{1} - \nu\frac{3}{\pi}\frac{(-1)^{J+1}}{2J+1}\left(\frac{(1+r)^2}{r(2+r)}\right)^{3/2}M^{J^P}(s)\right] = 0, \quad (12)$$

and M^{J^P} is essentially the Mellin transform of R^{J^P} in Eq. (8). For positive parity J^+ with $J > 0$, $M^{J^+}(s)$ is a 2×2 matrix with entries,

$$M_{11}^{J^+} = \left[\frac{J}{1+r}+\frac{1+r}{2J-1}\right]f_J - J[\hat{t}^- + \hat{t}^+]f_{J-1}$$
$$+\frac{(J-1)(2J+1)(1+r)}{2J-1}f_{J-2}, \quad (13a)$$

$$M_{12}^{J^+} = \sqrt{J(J+1)}\{-[1/(1+r)+1+r]f_J$$
$$+\hat{t}^+ f_{J-1} + \hat{t}^- f_{J+1}\}, \quad (13b)$$

$$M_{22}^{J^+} = \left[\frac{J+1}{1+r}+\frac{1+r}{2J+3}\right]f_J - (J+1)[\hat{t}^- + \hat{t}^+]f_{J+1}$$
$$+\frac{(J+2)(2J+1)(1+r)}{2J+3}f_{J+2}, \quad (13c)$$

and $M_{21}^{J^+}(s) = M_{12}^{J^+}(-s)$. The functions $f_J(s)$ are

$$f_J(s) = P_J\left(\frac{1+r}{2}[\hat{t}^- + \hat{t}^+]\right)f_0(s), \quad (14)$$

where P_J is a Legendre polynomial if $J \geqslant 0$ and 0 if $J < 0$, $\hat{t}^\pm$ is an operator that shifts the argument of $f_J(s)$ by ± 1, and $f_0(s)$ is

$$f_0(s) = \frac{\pi \sin\{s \arcsin[1/(1+r)]\}}{s \cos(s\pi/2)}. \quad (15)$$

In the special case $J^P = 0^+$ and for negative parity J^- with $J > 0$, $M^{J^P}(s)$ is a 1×1 matrix. For 0^+, its entry $M_{22}^{0^+}(s)$ is
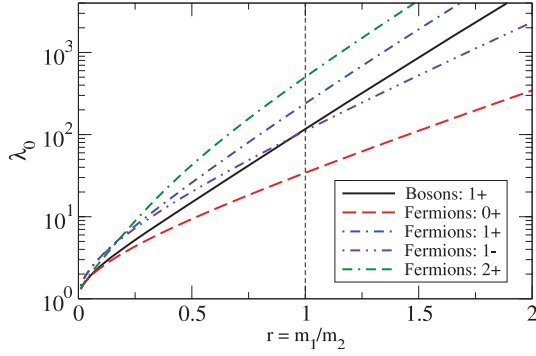
FIG. 1. (Color online) Discrete scaling factor $\lambda_0 = \exp(\pi/s_0)$ as a function of the mass ratio $r = m_1/m_2$. If the identical particles are bosons, the Efimov effect occurs only in the $J^P = 1^+$ sector. If the identical particles are fermions, the Efimov effect occurs in the 0^+, 1^+, 1^-, and 2^+ sectors.

given by Eq. (13c). For J^-, its entry is

$$M^{J^-}(s) = (1+r)[f_{J+1}(s) - f_{J-1}(s)]. \quad (16)$$

If the identical atoms are bosons, Eq. (12) has a pure imaginary solution $s = i s_0$ only for $J^P = 1^+$. For equal masses ($r = 1$), the discrete scaling factor $\lambda_0 = \exp(\pi/s_0)$ is 116.7. If the identical atoms are fermions, Eq. (12) has a pure imaginary solution for 0^+, 1^+, 1^-, and 2^+. For equal masses, the discrete scaling factors are 34.2, 239.3, 111.2, and 501.9, respectively. The dependence of λ_0 on the mass ratio $r = m_1/m_2$ for each of the five cases is shown in Fig. 1. The discrete scaling factors all approach 1 as $r \to 0$.

We proceed to discuss the spectrum of the three-body bound states at $r_p = 0$ predicted by Eq. (8). In the unitary limit defined by $a_p^{-1/3} = r_p = 0$, discrete scale invariance implies the existence of infinitely many Efimov trimers. They have an accumulation point at the three-atom threshold, and the binding energies of successive trimers differ by a factor of λ_0^2. As $|a_p|^{-1/3}$ increases, each of the Efimov trimers disappears through the three-atom threshold at a negative a_p and through the atom-dimer threshold at a positive a_p. Discrete scale invariance requires that, if an Efimov trimer disappears through a threshold at $(a_p, r_p) = (a_p^*, r_p^*)$, the next deeper Efimov trimer must disappear through the same threshold at $(\lambda_0^{-3} a_p^*, \lambda_0 r_p^*)$.

The bound-state spectrum at $r_p = 0$, predicted by Eq. (8), is illustrated in Fig. 2 for the case with identical fermions,
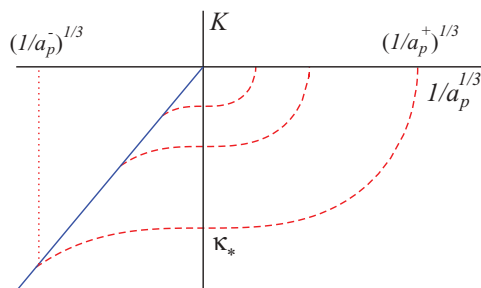
equal masses, and quantum numbers 0^+. The energy variable $K = \text{sgn}(E)(\mu|E|/\hbar^2)^{1/2}$ for several Efimov trimers is shown as a function of $a_p^{-1/3}$ for $r_p = 0$. An Efimov trimer whose binding energy at unitarity is $\hbar^2 \kappa_*^2/m$ disappears through the three-atom threshold at $a_p^- = -0.52/\kappa_*^3$ and through the atom-dimer threshold at $a_p^+ = +0.24/\kappa_*^3$. The behavior in the other channels is similar. The discrete scaling factors for equal masses and the corresponding threshold parameters for all channels J^P are given in Table I.

V. CONCLUSION

Whether there is a universal binding mechanism for shallow p-wave states remains an intriguing and important question. Apart from ultracold atoms, such systems occur frequently in halo nuclei [34]. The application of effective field theory to such a system is called *halo EFT* [6].

In this paper, we have discussed the renormalization of the minimal zero-range model for p-wave interactions defined by Eq. (6). This model is renormalizable in the two-body sector, but it has unphysical negative-norm molecules with momentum scale r_p. It can be used as an effective theory for typical momenta well below the scale r_p where the effects of negative-probability states are suppressed. In the three-body sector, renormalization in some parity and angular momentum channels involves an ultraviolet limit cycle, indicating asymptotic discrete scale invariance for large momenta. In the unitary
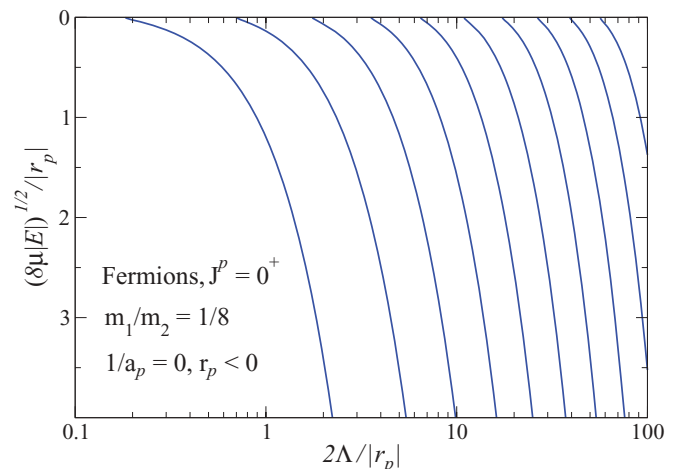
TABLE I. The imaginary solutions $i s_0$ of Eq. (12) and the discrete scaling factors λ_0 for the case of equal masses $m_1 = m_2$. The threshold parameters a_p^- and a_p^+ for $r_p = 0$ are also given.

Identical particles	J^P	s_0	λ_0	$a_p^- \kappa_*^3$	$a_p^+ \kappa_*^3$
Bosons	1^+	0.660	116.7	$-0.134(2)$	0.452(2)
Fermions	0^+	0.889	34.2	$-0.522(2)$	0.244(2)
Fermions	1^+	0.574	239.3	$-0.25(2)$	0.34(2)
Fermions	1^-	0.667	111.2	$-0.576(2)$	0.212(2)
Fermions	2^+	0.505	501.9	$-0.590(1)$	0.19(3)

FIG. 2. (Color online) Efimov plot showing the energy variable K as a function of $(1/a_p)^{1/3}$ with $r_p = 0$ for the case of identical fermions, equal masses, and $J^P = 0^+$. The scaling factor $\lambda_0 \approx 34.2$ was divided by 17 to fit more states in the plot.

FIG. 3. (Color online) The dimensionless binding momentum $\sqrt{8\mu|E|}/|r_p|$ as a function of the dimensionless cutoff $2\Lambda/|r_p|$ for fermions with $J^P = 0^+$, $1/a_p = 0$, $r_p < 0$, and $m_1/m_2 = 1/8$.

limit $a_p^{-1/3}, r_p \to 0$, there is an accumulation of three-body states at the threshold analogous to the Efimov effect. However, this limit is unphysical since the negative-norm states are driven to the threshold and lead to the breakdown of the model.

A promising strategy for dealing with p-wave resonant interactions is to treat the unitarity term ik^3 from Eq. (2) as a perturbation [35]. Such an expansion can be justified rigorously for small momenta $k \ll |r_p|$ and leads to a simplified pole structure of the dimer propagator. As an example, in Fig. 3, we show the dimensionless binding momentum $\sqrt{8\mu|E|}/|r_p|$ as a function of the dimensionless cutoff $2\Lambda/|r_p|$ for fermions in the 0^+ channel, with $1/a_p = 0$, $r_p < 0$, and $m_1/m_2 = 1/8$. There can be zero, one, or possibly even two three-body bound states for dimensionless cutoffs below 1 where the expansion of the dimer propagator can be justified. Increasing the mass ratio m_1/m_2 also increases the critical value of the dimensionless cutoff at which the first three-body bound state appears. For equal masses, this critical value is larger than 1, so there are no such states without introducing a three-body force.

The strategy of treating the unitarity term as a perturbation was recently applied to ^{6}He by Ji *et al.* [36]. This halo nucleus consists of an α particle and two weakly bound neutrons and can be described as an effective three-body system. The α-neutron interaction has a strong p-wave resonance, whereas, the two neutrons interact resonantly in an s-wave channel. Proper renormalization of the halo EFT was achieved by introducing an appropriate three-body force that was tuned to

give the measured binding energy of ^{6}He [36]. Rotureau [37] and Rotureau and van Kolck [38] described ^{6}He in the Gamow shell model using a different strategy. They applied a halo effective theory with separable neutron-α and neutron-neutron interactions and a neutron-neutron-α three-body force. The range of the separable interactions then acts as an ultraviolet cutoff. The unitarity term can be treated nonperturbatively without introducing any negative-probability states. Similar strategies have been used for the pionless EFT in the three-nucleon system.

The successful description of the ^{6}He nucleus within halo EFT [36] suggests that the field theory presented in this paper can describe three-body phenomena in experiments with ultracold atoms, provided that the unitarity term in the p-wave scattering amplitude is treated as a perturbation. Future applications might include the calculation of the scattering-volume dependence of the three-body recombination rate and other few-body reaction rates.

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