Aspects of MMOSPA Estimation

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Abstract—We expand upon existing literature regarding using Minimum Mean Optimal Sub-Pattern Assignment (MMOSPA) estimates in multitarget tracking, noting its advantages in comparison to Maximum Likelihood (ML) and Minimum Mean Squared Error (MMSE) estimation, and look at the practical computation of MMOSPA estimates. We demonstrate the use of MMOSPA estimation in a two-target tracking scenario as well as outside of tracking in a radar angular superresolution scenario.

I. INTRODUCTION

It has been noted that certain “optimal” tracking algorithms can be “beaten” by approximations [4]. The problem is illustrated in Figure 1 which represents the evolution of particles in a particle filter tracking two targets. Initially, the hypotheses for each of the targets are well separated, but once the targets are closely-spaced for a while, the particles for each target mix, which leads to the MMSE estimates of the target locations being in between both targets. This type of coalescence problem has been extensively studied in the target identities. Figure 1 shows how identity un-

The problem stems from the fact that using the minimum mean squared error estimate for track display can result in track coalescence when there is a high degree of uncertainty in the target identities. Figure 1 shows how identity uncertainty can lead to coalescence. To estimate the states of targets, we shall look at the MOSPA estimate, first considered in [12] and the underlying Optimal Sub-Pattern Assignment Metric (OSPA), introduced in [16], which approximates joint track hypotheses at each step by a single Gaussian and will be present in any tracker that uses an MMSE estimate for display.

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Portions of this work focussing on tracking, with a more detailed tracking simulation, have been submitted for journal publication [7].

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In Section II we review aspects of MOSPA optimization as applied to full joint target PDFs as well as to particle filter and Gaussian mixture (e.g. the MHT [13] and [2]) approximations of PDFs. We then expand upon this work considering the problem of practical MMOSPA estimation for tracking and provide an optimal solution in the case of a particle filter tracker, demonstrating that the Set JPDAF [18] can be thought of as an approximate MMOSPA tracker. In Section IV we use MMOSPA estimation in a tracking scenario as well as in an angular superresolution problem for a linear array, showing that MMOSPA estimation is useful in areas outside of tracking. We conclude in Section V.

II. THE MOSPA METRIC

The general notion behind the OSPA metric has existed for many years. For example, a similar measure is defined in [11]. However, a general definition with a rigorous proof that the quantity being discussed was indeed a metric was first given in [16]. This metric is used to evaluate the performance of tracking algorithms without taking into account target labeling.

Suppose that a tracking algorithm outputs $T$ tracks, whereas there is a total of $N_T$ true targets. Let $\hat{x}$ be the stacked set of $D$-dimensional state estimates for the tracks and $\bar{x}$ be the stacked state vectors of the targets at a particular time. Also let $\hat{x}_t$ be the state estimate for the $t$th track and $x_t$ be the true state of the $t$th target at time $k$. The OSPA error statistic is given by Equation (1).

The permutation vector $a$ determines the order of the states; its elements consist of a particular permutation of the integers from 1 to $N_T$. Thus, $a$ and $\hat{a}_a$ can be written as follows:

$$a = [a(1), a(2), ..., a(N_T)]$$

$$\hat{a}_a = [\hat{x}_{a(1)}, \hat{x}_{a(2)}, ..., \hat{x}_{a(N_T)}]$$

$$\overset{1}{\text{Though we are discussing this in terms of tracks and truth, the metric may be used with any two sets of objects whose ordering may be changed.}}$$
The permutation change can also be denoted using a permutation matrix, \( \chi \), instead of a subscript. The permutation matrix times the vector puts the vector elements in the desired order. This is the notation used in Appendix I.

The variable \( p \) is an arbitrary number larger than one. The distance metric \( d^{(c)} \) is defined to be

\[
d^{(c)}(\hat{x}_{a(t)}, x_t) = \min \left[ c, d(\hat{x}_{a(t)}, x_t) \right]
\]

(4)

where \( d \) is an arbitrary distance metric. We shall use the distance metric that has been most commonly used in previous work, namely, \( d(\hat{x}_{a(t)}, x_t) = \| \hat{x}_{a(t)} - x_t \|^2 \). The value \( c \) is a cutoff for the maximum allowable error added by a single track. Basically, this metric tries to find the best assignment of targets to tracks, capping the maximum allowed error and penalizing mismatches in the number of targets to tracks with the maximum possible error for each difference. The expected value of this metric is the MOSPA error.

For the purposes of discussing optimization in this paper, we shall use a specific version of this metric for optimization, the same version that is used in [12]. For our purposes, \( T = N_T, p = 1, d \) will be the \( l_2 \) norm squared, as previously mentioned, and \( c \to \infty \) (since \( T - N_T = 0 \), the \( c^p \) term is not present in the limit). This gives us a metric of

\[
d(\hat{x}, x) = \frac{1}{N_T} \min_{a} \sum_{t=1}^{N_T} \| \hat{x}_t - x_{a(t)} \|^2 = \frac{1}{N_T} \min_{a} \| x_a - \hat{x} \|^2
\]

(5)

The MOSPA error of an estimate whereby the underlying PDF is \( p(x) \) is simply the expected value of the OSPA error in (5):

\[
d(\hat{x}) \triangleq \frac{1}{N_T} \mathbb{E} \left[ \min_{a} \| x_a - \hat{x} \|^2 \right]
\]

(6)

\[
d(\hat{x}) = \frac{1}{N_T} \int_{x} \min (d_1, d_2 \ldots d_{N_T}) p(x) \, dx
\]

(7)

where

\[
d_i \triangleq \| x_{a_i} - \hat{x} \|^2 = \| x - \hat{x}_{a_i} \|^2
\]

(8)

The MMOSPA estimate is by definition

\[
\hat{x}_M \triangleq \arg \min_{\hat{x}} \mathbb{E} \left[ \min_{a} \| x_a - \hat{x} \|^2 \right]
\]

(9)

In general, no explicit formulation for the MMOSPA error estimate exists. However, in [8], an explicit formulation was found for the case where \( N_T = 2 \) and the target states are scalar. In Appendix I we show that this can be generalized to an arbitrary number of scalar targets.

It has been shown that the MMOSPA error estimate in (9) is equivalent to finding the expected value of \( x \) over a “folded” version of \( p(x) \) [12]. As a result, the problem of “symmetric” hypotheses causing track coalescence is largely avoided.

### III. Practical MMOSPA Optimization

We shall prove that the algorithm used to generate the merged state in the Set JPDAF [18] is an approximation to the MMOSPA estimate when used to generate estimates of the target locations for display in the MHT. We will also prove that the expression is exact in the particle filter case, and highlight the analogy between the MMOSPA and the MMSE estimates.

Let us assume that the PDF of the target states at a particular time can be decomposed across \( N_H \) global hypotheses; the \( i \)th hypothesis shall be designated by \( u_i \) and have probability \( w_i \). In the MHT, the \( u_i \)'s would represent the individual Gaussian PDFs in a Gaussian mixture, whereas in a particle filter, these would be delta functions representing the individual particles. To derive an approximation to the MMOSPA estimate, we shall use the following definition of an unordered joint covariance matrix for a particular estimate \( \hat{x}_M \),

\[
P \triangleq \mathbb{E} \left[ \min_{a} (x_a - \hat{x}_M) (x_a - \hat{x}_M)' \right]
\]

(10)

\[
= \sum_{i=1}^{N_H} w_i \int_{x} \min \{ (x_a - \hat{x}_M) (x_a - \hat{x}_M)' \} p(x|u_i) \, dx
\]

(11)

\[
\approx \sum_{i=1}^{N_H} w_i \min_{a_i} \mathbb{E} \left[ (x_a - \hat{x}_M) (x_a - \hat{x}_M)' | u_i \right]
\]

(12)

Note that the minimization operator in (10) is a slight abuse of notation in that we are actually minimizing the trace of the argument rather than the argument itself, which is a matrix. The approximation in (12) decouples the \( x \) between conditional expected values and sets a single ordering for each hypothesis rather than for every single point in the integral. In other words, whereas the changes in ordering

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2For example, for two targets, the two possible permutation matrices are

\[
\chi_1 = \begin{bmatrix} I_D & 0_D \\ 0_D & I_D \end{bmatrix} \quad \chi_2 = \begin{bmatrix} 0_D & I_D \\ I_D & 0_D \end{bmatrix}
\]

where \( I_D \) is the \( D \times D \) identity matrix and \( 0_D \) is a \( D \times D \) matrix of zeros.

3This cutoff, in effect, keeps a single lost track from masking the performance of the tracker on all other tracks. It is also the penalty added for having the wrong number of targets and/or tracks.

4The covariance in (15) is just the usual equation for the covariance of a mixture, except some of the orderings are switched. Thus, this is the covariance of the PDF with the switched orderings. It is always less than or equal to the covariance of the original mixture.
in (11) move individual points, the changes in ordering in (12) move individual PDFs. If the PDFs are “peaky”, then this is a good approximation.

As used in the optimization on the right-hand side of (9), the MOSPA error is defined to be

\[
d(\hat{\mathbf{x}}_M) \triangleq \frac{1}{N_T} \mathbb{E} \left[ \min_a \| \mathbf{x}_a - \hat{\mathbf{x}}_M \|^2 \right] \\
= \frac{1}{N_T} \int \min_a \left\{ \sum_{i=1}^{N_H} w_i \| \mathbf{x}_a - \hat{\mathbf{x}}_M \|^2 p(\mathbf{x}|u_i) \right\} d\mathbf{x}
\]

(13)

The MOSPA error is equivalent to 1/N_T times the trace of the unordered joint covariance matrix in (10). The optimization for the MMOSPA error is, in general, a difficult problem that must be solved iteratively, utilizing numerical integration [12]. Thus, the SJPDAF utilizes the approximation in (12), moving the minimization from being on every point in every integral to simply being fixed over every integral, to lower the complexity of the filter.

Due to its relationship to the expected value, if we know the correct ordering of all possible \( x \), the is we know the a term in (14) for all \( x \), then we can write the MMOSPA estimate as

\[
\hat{\mathbf{x}}_M \triangleq \sum_{i=1}^{N_H} w_i \hat{\mathbf{x}}_{i,a_i}
\]

(15)

(the SJPDAF mean), whereby \( \hat{\mathbf{x}}_{i,a_i} \) is the vector of the means of the states of all of the tracks according to the \( i \)th hypothesis, and in which the ordering of the tracks in the vector is given by \( a_i \). We shall also define \( \hat{\mathbf{x}}_i \) to be the same with the original ordering. The covariance matrix of this estimate is \( \mathbf{P}_{i,a_i} \), the covariance of the \( i \)th component of the mixture such that the ordering of the targets has been rearranged according to \( a_i \). By adding and subtracting \( \hat{\mathbf{x}}_{i,a_i}, \hat{\mathbf{x}}'_{i,a_i} \) to each term, we can continue the simplification from (12), as shown below

\[
\mathbf{P} = \sum_{i=1}^{N_H} w_i \min_{\mathbf{a}_i} \left[ \mathbf{P}_{i,a_i} + (\hat{\mathbf{x}}_{i,a_i} - \hat{\mathbf{x}}_M)(\hat{\mathbf{x}}_{i,a_i} - \hat{\mathbf{x}}_M)' \right]
\]

(16)

All together, the SJPDAF uses the state estimates from (15) with \( \{a_1, \ldots, a_{N_H}\} \) chosen to minimize the trace of (16), that is

\[
\{a_1, \ldots, a_{N_H}\} = \arg_{a_1 \ldots a_{N_H}} \text{tr} \left[ \mathbf{P} \right]
\]

\[
= \arg \min_{a_1 \ldots a_{N_H}} \sum_{i=1}^{N_H} \left( w_i \| \mathbf{x}_{i,a_i} - \hat{\mathbf{x}}_M \|^2 \right) + C
\]

\[
= \arg \max_{a_1 \ldots a_{N_H}} \hat{\mathbf{x}}_M \hat{\mathbf{x}}_M
\]

(17)

(18)

(19)

where \( C \) is the weighted sum of the traces of \( \mathbf{P}_{i} \) and does not depend upon the ordering. Equation (19) follows by expanding the norm in (18), substituting (15), and noting

To put it another way, if \( \chi_i \hat{\mathbf{x}}_i = \hat{\mathbf{x}}_{i,a_i} \), in other words \( \chi_i \) is a permutation matrix corresponding to the permutation given in \( a_i \), then \( \mathbf{P}_{i,a_i} = \chi_i \mathbf{P} \chi_i' \).

that the quantity \( \sum_{i=1}^{N_H} w_i \hat{\mathbf{x}}'_{i,a_i} \hat{\mathbf{x}}_{i,a_i} \) does not depend upon the target ordering. In Appendix III, we prove that this optimization is equivalent to a multiframe assignment problem (also known as an S-D assignment problem), as well as a quadratic semi-assignment problem. Additionally, we shall note that the optimization can be formulated explicitly as a quadratic programming problem with \( N_T \times N_H \) terms as described in [17]. In Subsection III-B we present a simple, greedy solution to the problem.

In summary, we can approximate the MMOSPA estimate for the targets by determining the ordering \( a_1 \ldots a_{N_H} \) that minimizes (19) where our final estimate is given by (15).

Each state estimate has an unordered covariance matrix given by one of the diagonal blocks from (16). This covariance matrix worked well in the SJPDAF [18], [6].

Other methods for estimating the target locations from a posterior PDF without causing coalescence, i.e., the problem in Figure 1 have been considered in past work. For example, a method of decomposing an arbitrary two-target PDF into strictly permutation variant and permutation invariant components was introduced in [3] and generalized to an arbitrary number of targets is [8]. The mean of the strictly permutation variant component has many of the properties of the MMOSPA estimate. Similarly, in [9] an approximation to the MMOSPA estimate for an arbitrary PDF was derived for the case wherein the OSPA metric used was \( d(\hat{x}_{a(t)}, x_t) = \|\hat{x}_{a(t)} - x_t\|^2 \) (the \( l_2 \)-norm to the fourth, rather than squared).

Note that though the MMOSPA estimate says where targets are, it says nothing about the identities of the targets.

A. Optimality for the Particle Filter

We shall show that the formulation in (12) is exact in the particle filter case, and thus, the optimization of the simplified cost function in (19) gives us the exact MMOSPA estimate. In this instance, the “hypotheses” consist of \( N_H \) particles, meaning that \( p(\mathbf{x}|u_i) \) is a delta function and here \( u_i \) is conditioning on which particle is “true”. Let us designate the location of the delta function corresponding to \( u_i \) as \( \hat{\mathbf{x}}_i \). We can thus rewrite the MOSPA error in (13) as

\[
d(\hat{\mathbf{x}}_M) = \frac{1}{N_T} \int \min_a \left\{ \| \mathbf{x}_a - \hat{\mathbf{x}}_M \|^2 \right\} \sum_{i=1}^{N_H} w_i \delta [\hat{\mathbf{x}}_i - \mathbf{x}] \ d\mathbf{x}
\]

(20)

\[
= \frac{1}{N_T} \sum_{i=1}^{N_H} w_i \min_{\mathbf{a}_i} \left\{ \| \mathbf{x}_{i,a_i} - \hat{\mathbf{x}}_M \|^2 \right\}
\]

(21)

The simplification from (20) to (21) comes naturally, assuming that no two particles are identical within a permutation of the targets. If that is not the case, it can be shown that there is no advantage to assigning such particles different orderings (i.e. it makes the MOSPA error larger), so (21) still holds. It can be seen that the trace of (12) is equal to (21), so the solution is exact. The same conclusion can also be inferred from the work in [12].
B. A Greedy Solution

The optimization in [19] is quite complex. A sequential method of performing the optimization was presented in [18]. We shall give a simple, alternate approach.

Suppose that we have determined the optimal ordering of hypotheses 1 through \( N_H - 1 \). Let \( \hat{x}_{\text{sum}} \) be the partial sum of terms \( i = 1 \) through \( i = N_H - 1 \) in \([15]\), having weight \( w_{\text{sum}} = \sum_{i=1}^{N_H-1} w_i \). Let \( \hat{x}_{N_H, i} \) be the last term in the sum in \([15]\), having weight \( w_{N_H} \). The optimal ordering of the last term according to \([19]\) after discarding constant terms is

\[
\alpha_{\text{opt}} = \arg\max_a \{ \hat{x}_{\text{sum}} \hat{x}_{N_H, a} \}
\]

This is a two-dimensional assignment problem. We are assigning each state in \( \hat{x}_{N_H} \) to one in \( \hat{x}_{\text{sum}} \). The cost of assigning state vector \( i \) in \( \hat{x}_{\text{sum}} \) to state vector \( j \) in \( \hat{x}_{N_H} \) is \( \hat{x}_{\text{sum}}(i)^\dagger \hat{x}_{N_H}(j) \). For example, for three targets, we have a cost matrix of the form

\[
A = \begin{bmatrix}
\hat{x}_{\text{sum}}(1)^\dagger \hat{x}_{N_H}(1) & \hat{x}_{\text{sum}}(1)^\dagger \hat{x}_{N_H}(2) & \hat{x}_{\text{sum}}(1)^\dagger \hat{x}_{N_H}(3) \\
\hat{x}_{\text{sum}}(2)^\dagger \hat{x}_{N_H}(1) & \hat{x}_{\text{sum}}(2)^\dagger \hat{x}_{N_H}(2) & \hat{x}_{\text{sum}}(2)^\dagger \hat{x}_{N_H}(3) \\
\hat{x}_{\text{sum}}(3)^\dagger \hat{x}_{N_H}(1) & \hat{x}_{\text{sum}}(3)^\dagger \hat{x}_{N_H}(2) & \hat{x}_{\text{sum}}(3)^\dagger \hat{x}_{N_H}(3)
\end{bmatrix}
\]

We have to choose exactly one element in each row and column. This may be solved using the auction or JVC algorithms [15], or simply by trying all \( N_T! \) possible assignments. Thus, we can sequentially determine the optimal orderings in \([19]\) as follows:

1) Fix the ordering of the first hypothesis. Set \( i = 2 \).
2) Determine the ordering for the \( i \)th hypothesis, assuming the orderings for hypotheses less than \( i \) are correct, by solving the optimization problem in \([22]\) as if \( N_H = i \).
3) Increment \( i \). If \( i > N_H \), stop; otherwise, go to step 2.

IV. SIMULATION EXAMPLES

A. A Tracking Example

To consider the improvement offered from using M莫斯SPA estimates in tracking, we ran an MHT using the approximate MMOSPA estimate for Gaussian mixtures, as described in Section II, the MMSE estimate and the ML estimate to determine the position of two targets at each step.

Two targets moved on trajectories from left to right that approached within 40 m, traveled parallel to each other, and separated, similar to the scenario in Figure 1. The angle with respect to the horizontal axis at which the targets approached was \( \pm 0.6 \) rad. There were 100 m/s. The targets moved toward each other, and after the turns was 800 m. The targets were together for a period of 2 km. The targets moved at a constant speed of 100 m/s. The targets changed direction according to a coordinated turn model at an angular velocity of \( \pm 0.3 \) rad/s. Observations were made every \( \tau = 1 \) s. Track initialization was done for each track by feeding two correctly associated measurements to an Information Filter [1]. One thousand Monte Carlo runs were performed.

Two-dimensional measurements were taken in polar coordinates with \( \sigma_r = 8 \) m and \( \sigma_u = 0.52 \) mrad and converted to Cartesian coordinates. The unbiased conversion method of [1] was used. A single sensor was placed at \((x, y)\) coordinates of \((1 \text{ km}, -40 \text{ km})\). The number of clutter points at each step was determined according to a Poisson process with mean \( \lambda = 2.179 \times 10^{-6} \text{ V} \) where \( \lambda \) is the area of the surveillance region. Clutter points were placed uniformly over a surveillance region spanning \(-1049 \text{ m} < x < 30489 \text{ m} \) and \(-730 \text{ m} < y < 730 \text{ m}\). The average number of clutter measurements per scan was 13. Both targets had a detection probability of 80%.

The discretized continuous white-noise acceleration model (DCWNA) [1] was used in the trackers. We used a process noise power spectral density of \( q_0 = 400 \text{ m}^2/\text{s}^3 \).

The performance of the methods was evaluated by looking at the jitter of the estimate.

<table>
<thead>
<tr>
<th>Metric</th>
<th>ML</th>
<th>MMSE</th>
<th>MOSPA</th>
</tr>
</thead>
<tbody>
<tr>
<td>Jitter</td>
<td>683</td>
<td>502</td>
<td>493</td>
</tr>
</tbody>
</table>

We can see that MMOSPA estimation provides the lowest level of jitter.

B. Radar Superresolution

Consider a linear array receiving a reflected signal from two far-field targets (a discussion on angular superresolution is given in [19]). The complex signal received by the \( i \)th antenna is

\[
z_i = \sum_{m=1}^{2} b_m e^{-j \omega x_i u_m + w_i}
\]

where \( z_i \) is the location of the \( i \)th antenna along the linear array, \( w_i \) is noise, \( b_m \) is the complex amplitude and \( u_m \) is a value between \(-1 \) and \( 1 \) representing the cosine of the angle of arrival of the return from the \( m \)th target. We shall let \( z \) be the vector of returns from all of the antennas. \( a_m \) shall be a steering vector such that the \( i \)th element is

\[
a_i(u_m) = e^{-j \omega x_i u_m}
\]

Assuming that the noise is Gaussian, the PDF of \( z \) is given in [27], where \( Q \) is the noise covariance, which in this simulation we shall take to be \( 0.5 \text{I} \). We would like to estimate the directions of arrival of the two waves. In this example, we will assume that \( b_1 = b_1 = 1 \) and are known. We will place the targets at \( u_1 = -0.3 \) and \( u_2 = 0.1 \), assume that \( \omega = 2 \pi \) and place the array elements at \( x = \{-1,-0.5,0,0.5,1\} \). The prior probability of the angles shall be uniformly distributed. Using Bayes’ rule we can write

\[
f(u_1, u_2|x) = \frac{p(z|u_1, u_2)p(u_1, u_2)}{p(z)}
\]

\[
\text{Jitter} = \frac{1}{N_{\text{num}}} \sum_{m=1}^{N} \sum_{k=1}^{N} ||H \hat{x}^m(k|k) - H \hat{x}^m(k|k-1)||^2
\]

\( ^{6} \text{Two definitions of track jitter are given in [10]. We shall use the second definition of the metric, since it is also commonly used to assess the accuracy of trackers when truth data is not available.}

\( ^{7} \text{To get a unique solution, we assume that the target is in front of the linear array.}

\( ^{8} \text{Thus } p(u_1, u_2) = 1/4 \text{ if } -1 \leq \{u_1 \text{ and } u_2\} \leq 1. \)
\[ p(z | u_1, u_2) = |\pi Q|^{-1} \exp \left\{ - [z - a(u_1)b_1 - a(u_2)b_2]^H Q^{-1} [z - a(u_1)b_1 - a(u_2)b_2] \right\} \tag{27} \]

where

\[ p(z) = \frac{1}{4} \int_{-1}^{1} \int_{-1}^{1} p(z | u_1, u_2) du_1 du_2 \tag{29} \]

Using the results of Appendix I, an explicit solution for the MMOSPA estimate of the angles of arrival of the two targets is given by

\[ \hat{u}_M = 2 \frac{\int_{-1}^{1} \int_{-1}^{1} p(z | u_1, u_2) du_1 du_2}{\int_{-1}^{1} \int_{-1}^{1} p(z | u_1, u_2) du_1 du_2} \tag{30} \]

The 2 comes from the fact that both permutations are the same.

Running the simulation with a noiseless measurement, we get the correct estimates for both of the angles of arrival, whereas the MMSE estimates are \(-0.1\) for both targets – a point that lies between the angles– and are useless. Running the simulation with noise generated according to the model for 500 Monte Carlo runs, we get a MOSPA error of 0.006 for the estimates of \(u\).

V. Conclusion

We reviewed literature regarding the exact computation of the MMOSPA estimate in the particle filter and an approximation in an MHT, expanding upon previous work to show the relationship between the solution and the S-D assignment and quadratic semi assignment problems and provide an explicit solution for scalar states. We then demonstrated that MMOSPA estimation can be used to reduce jitter in track display, as well as to provide useful estimates in radar superresolution scenarios where MMSE estimation fails.

REFERENCES


APPENDIX I

AN EXPLICIT MMOSPA SOLUTION FOR SCALAR STATES

In [8], an explicit solution for the MMOSPA estimate of two targets having scalar states was presented. We shall generalize the results to PDFs having an arbitrary number of targets having scalar states.

The MMOSPA solution is not unique, because the ordering of the targets can always be switched. We shall thus define a unique ordering of the elements in the MMOSPA estimate. For \(N_T\) targets, we shall say that \(\hat{x}_1 \geq \hat{x}_2 \geq \ldots \geq \hat{x}_{N_T}\). Let us consider when each ordering of the measure is the minimum in \([7]\). Suppose that we have costs corresponding to two orderings, \(d_a\) and \(d_b\), as defined in \([8]\) that differ only in that targets \(x_i\) and \(x_j\) have been swapped. \(d_a\) is the dominant ordering when

\[ d_a \leq d_b \iff \|x - \hat{x}\|^2 \leq \|x - \chi \hat{x}\|^2 \tag{31} \]

where \(\chi\) is a permutation matrix such that \(\chi \hat{x}\) switches \(x_1, x_2, \ldots, x_{N_T}\). Letting \(x\) be the remaining target states whose ordering is the same in \(d_a\) and \(d_b\), we will break the orders.
The optimization in (38) can also be formulated as an multiframe assignment problem, as used for tracking in [15]. The optimization in (38) can be rewritten as

$$\{a_1, \ldots, a_{N_H}\} = \arg \max_{a_1 \ldots a_{N_H}} \sum_{i=2}^{N_H} w_i \tilde{x}'_{i,a} \sum_{j=1}^{i-1} w_j \tilde{x}_j,a_i$$

(41)

The simplification comes from eliminating all terms that are independent of the ordering of the states (i.e., $\tilde{x}'_{i,a}, \tilde{x}_j,a_i$ terms), grouping all identical products, and eliminating a multiplicative coefficient of 2.

Define $\hat{x}_k(i_k)$ to be the state of target $i_k$ in the $k$th hypothesis. In other words, we are indexing the state vectors in $\tilde{x}_k$. From (41), we can see that

$$c_k(i_1, \ldots, i_{N_H}, n) = \Pr\{u_{N_H} \} \tilde{x}_N (i_{N_H})' \sum_{j=1}^{N_H-1} w_j \tilde{x}_j(i_j)$$

(42)

Now, let us define a binary association variable $\rho_{i_1, i_2, \ldots, i_{N_H}}$ that takes value 1 if $(i_1, i_2, \ldots, i_{N_H})$ is associated to track $n$ and value 0 otherwise. The optimal assignments can then be obtained by maximizing the cost

$$\hat{\rho} = \arg \max_{\rho} \sum_{n=1}^{N_T} \sum_{i_1=1}^{N_T} \sum_{i_2=1}^{N_T} \ldots \sum_{i_{N_H}=1}^{N_T} \rho_{i_1, i_2, \ldots, i_{N_H}} c_{i_1, i_2, \ldots, i_{N_H}}$$

subject to the constraints

$$\forall \rho \in \{0,1\}$$

$$\sum_{i_1=0}^{N_T} \sum_{i_2=0}^{N_T} \ldots \sum_{i_{N_H}=0}^{N_T} \rho_{i_1, i_2, \ldots, i_{N_H}} = 1 \quad n = 1, 2, \ldots, N_T$$

$$\sum_{n=1}^{N_T} \sum_{i_1=0}^{N_T} \sum_{i_2=0}^{N_T} \ldots \sum_{i_{N_H}=0}^{N_T} \rho_{i_1, i_2, \ldots, i_{N_H}} = 1 \quad i_1 = 1, 2, \ldots, N_T$$

$$\sum_{n=1}^{N_T} \sum_{i_1=0}^{N_T} \sum_{i_2=0}^{N_T} \ldots \sum_{i_{N_H}=0}^{N_T} \rho_{i_1, i_2, \ldots, i_{N_H}} = 1 \quad i_2 = 1, 2, \ldots, N_T$$

$$\vdots$$

$$\sum_{n=1}^{N_T} \sum_{i_1=0}^{N_T} \sum_{i_2=0}^{N_T} \ldots \sum_{i_{N_H-1}=0}^{N_T} \rho_{i_1, i_2, \ldots, i_{N_H-1}} = 1 \quad i_{N_H-1} = 1, 2, \ldots, N_T$$

The $\rho$ are binary association variables such that $\rho_{i_1, i_2, \ldots, i_{N_H}} = 1$ if target $n$ is assigned the positions in the hypotheses defined by the $N_H$-tuple $\{i_1, i_2, \ldots, i_{N_H}\}$. For example, if there are 3 hypotheses and two targets, if $\rho_{2,1,1,1} = 1$ then that means that target 1 is assigned position 1 in hypotheses 1 and 3 and in the sum in Equation (15) and position 2 in hypothesis 2. From the constraints, we also know that $\rho_{2,1,2,2} = 1$ and all other $\rho$ terms must be zero.

11The multiframe assignment problem is sometimes called the S-D assignment problem, where S corresponds to $N_H$ in our problem. Solutions to the S-D assignment problem have been considered in [15].