A new frequency domain system identification method

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Abstract: A new frequency domain system identification method based on a multi-frequency input signal is proposed. Frequency contents of the oscillating signal are estimated using a modified Kaczmarz algorithm proposed in this paper. Lyapunov stability analysis is performed for this new Kaczmarz algorithm and transient bounds for estimation error are established. Moreover, a new method for estimation of the variance of the measurement noise in Kaczmarz algorithms is also described. A comparison of a transient performance of modified Kaczmarz algorithm and a recursive least-squares algorithm is presented. The results are applied to a frequency domain identification of a DC motor.

Keywords: frequency domain system identification, frequency response, Kaczmarz algorithm, strictly diagonally dominant matrix, persistence of excitation, DC motor, signal processing

1 INTRODUCTION

A property of a linear time-invariant system to be completely characterized by the steady-state response to harmonic input is the basis for the frequency domain system identification methods [1]. A harmonic input (a cosine wave) of a certain frequency applied to a linear system induces oscillations of the output of the same frequency with different magnitude and phase after some transient. The magnitude of the output signal is equal to the magnitude of the input signal multiplied by the module of the transfer function, and the phase shift is equal to the argument of the transfer function.

The frequency response of system can be measured by applying a test harmonic signal of a certain frequency and measuring magnitude and phase of the output signal at that frequency. This procedure is repeated for a number of frequencies in a specified frequency range and only one frequency at a time is used.

This method is time consuming since the response is measured sequentially (only one frequency at a time is used) for a number of frequencies. The identification method proposed in this paper applies a multi-frequency harmonic signal that contains a number of distinct frequencies, and the response is measured simultaneously at each frequency, recovering magnitude and phase. This in turn, essentially reduces a costly time of experiment, providing the same accuracy of estimation.

The model for the output signal is specified in a form of a trigonometric polynomial (a sum of sines and cosines of known frequencies specified in the input signal) with adjustable coefficients.

Many algorithms can be applied for estimation of the frequency contents of oscillating signals. One of such algorithms is a simple Kaczmarz algorithm [2] where the output of the model matches the measured signal exactly in each discrete step and the vector of the parameter mismatch is orthogonal to the regressor vector. The Kaczmarz algorithm was successfully tested for estimation of the frequency contents of the high-resolution engine speed signal [3] in automotive applications. A computational simplicity and robustness against a measurement noise make the Kaczmarz algorithm very attractive for estimation of the frequency contents of complex signals with a large number of frequencies in real-time applications. The Kaczmarz algorithm however, suffers from the following drawbacks: it neither allows the estimation of the variance of measurement noise nor selection of the order of a trigonometric polynomial in the case where new frequencies appear in the signal. A gain update algorithm with a forgetting factor is introduced in this paper aiming to the
performance improvement of a Kaczmarz projection method. Stability analysis is performed for the modified algorithm in this paper unlike in previous papers where similar modifications of the Kaczmarz algorithm were reported [3, 4].

The performance of the modified Kaczmarz algorithm is compared in this paper to the performance of the recursive least-squares (RLS) algorithm. The modified algorithm is applied to the frequency domain identification of a DC motor.

The idea of using broadband excitation signals such as multi-sine signals for a frequency domain system identification was also discussed in [5] (see also references therein). The output of the system which is excited by a broadband signal is analysed via a standard fast Fourier transform (FFT) method that limits the frequencies of the input signal to the set of Fourier frequencies, i.e. a fundamental frequency accompanied by higher-frequency harmonics. Any distinct frequencies of the input signal can be used in the frequency domain system identification method described in this paper.

The contributions of this paper can be summarized as follows.

1. A new frequency domain system identification method based on a multi-frequency input signal applied to a frequency domain identification of DC motor.

This paper is organized as follows. A new frequency domain system identification method is proposed in the next section. A new modified Kaczmarz algorithm is introduced in section 3 and applied to a frequency domain identification of DC motor in section 4. The paper is concluded by a discussion presented in section 5.

2 A NEW FREQUENCY DOMAIN IDENTIFICATION METHOD BASED ON MULTI-FREQUENCY INPUT SIGNAL

A harmonic input $u_k$ of a certain frequency $q$ applied to a linear system with the frequency transfer function $H(iq)$, where $i^2 = -1$ induces oscillations of the output $y_k$ of the same frequency $q$ with different magnitude and phase after some transient. The magnitude of the output signal is equal to the magnitude of the input signal multiplied by the module of the transfer function, and the phase shift is equal to the argument of the transfer function.

The traditional method of the frequency response measurements is time consuming since the response is measured sequentially for a number of frequencies and only one frequency at a time is used. The identification method described in this section applies the composite harmonic signal to the input that contains the number of distinct frequencies, and the response is measured simultaneously at each frequency, recovering the magnitude and the phase.

The multi-frequency test input sequence $u_k$ is chosen as

$$u_k = \sum_{q=1}^{q_n} A_q \cos(qk\Delta)$$

(1)

where $A_q$ is the amplitude of harmonic signal at a frequency $q = q_1, \ldots, q_n$, $n$ is the number of frequencies involved, $k=0,1,2, \ldots$. An example of the multi-frequency test input sequence is shown in Fig. 6.

The principle of superposition implies that a sum of cosine waves as an input forces cosine waves at the output with the same frequencies after a transient and therefore discrete-time measurements of the output result in the sequence

$$y_k = \sum_{q=1}^{q_n} B_{q}, \cos(qk\Delta + \psi_{q_k}) + \xi_k$$

(2)

where magnitude of the response is $B_{q_k} = |H(iq)|A_q$, and the phase $\psi_{q_k} = \arg(H(iq))$ are treated as uncertain parameters, where $|H(iq)| = \frac{b_{q_k}}{a_{q_k}}$ is the module/magnitude and $\arg(H(iq))$ is the argument of the transfer function evaluated at a frequency $q$, and $\xi_k$ is a zero mean white Gaussian measurement noise with a variance $\sigma^2$.

The output sequence (2) can be rewritten as

$$y_k = \sum_{q=1}^{q_n} a_{q_k} \cos(qk\Delta) + b_{q_k} \sin(qk\Delta) + \xi_k$$

(3)

where the relation between equations (2) and (3) is given by magnitude $B_{q_k} = \sqrt{a_{q_k}^2 + b_{q_k}^2}$ and phase $\psi_{q_k} = -\arctan\frac{b_{q_k}}{a_{q_k}}$, where $a_{q_k}$ and $b_{q_k}$ are unknown coefficients of the output trigonometric polynomial.
Finally, the output sequence can be presented in the vector form
\[ y_k = \varphi_k^T \theta_k + \xi_k \]  
(4)
where \( \varphi_k \) is the regressor and \( \theta_k \) is the vector of constant unknown parameters defined as
\[ \varphi_k = \begin{bmatrix} \cos(q_1 k \Delta) \sin(q_1 k \Delta) \\ \sin(q_2 k \Delta) \ldots \cos(q_p k \Delta) \sin(q_p k \Delta) \end{bmatrix} \]  
(5)
and \( \xi_k \) is the regressor defined in (5) and \( \theta_k \) is the following vector of adjustable parameters
\[ \theta_k^T = [a_{q_1}, b_{q_1}, a_{q_2}, b_{q_2}, \ldots, a_{q_n}, b_{q_n}] \]  
(6)
In other words, the output sequence is presented in a form of a trigonometric polynomial with known frequencies and unknown coefficients.

The model of the output sequence can be introduced as
\[ \hat{y}_k = \varphi_k^T \theta_k \]  
(7)
where \( \varphi_k \) is the regressor defined in (5) and \( \theta_k \) is the following vector of adjustable parameters
\[ \theta_k^T = [a_{q_1} b_{q_1}, a_{q_2} b_{q_2}, \ldots, a_{q_n} b_{q_n}] \]  
(8)

A parameter estimation aim is to find an update law for the parameter vector \( \theta_k \) such that \( \theta_k \) converges to \( \theta_* \), i.e. the following holds
\[ E[\theta_k - \theta_*] = 0 \]  
(9)
at steady state, provided that the regressor \( \varphi_k \) is persistently exciting, i.e. there exist positive constants \( p_0, p_1 \) and \( w \) such that the following inequality holds
\[ 0 < p_0 I \leq \sum_{p=k}^{p+k+w} \varphi_p \varphi_p^T \leq p_1 I \]  
(10)
where \( E \) is a mathematical expectation and \( I \) is the identity matrix. Inequalities (10) are valid for any distinct frequencies of the input sequence provided that a zero division in the elements of the information matrix \( \sum_{p=k}^{p+k+w} \varphi_p \varphi_p^T \) is avoided [3].

A frequency domain system identification aim is the convergence of the magnitude and phase
\[ E[B_{qk} - B_q] = 0 \]  
(11)
\[ E[\psi_{qk} - \psi_q] = 0 \]  
(12)
where
\[ B_{qk} = \sqrt{a_{qk}^2 + b_{qk}^2} \]  
(13)
and \( a_{qk} \) and \( b_{qk} \) are adjustable parameters defined in (8), \( q = q_1, \ldots, q_n \) and \( |\tilde{H}(q)| = \frac{B_{qk}}{A_q} \).

The achievement of the system identification aim (11), (12) follows from the achievement of the parameter estimation aim (9) as can be shown using the linearization technique. Hence the system identification problem is reduced to the problem of estimation of the coefficients of a trigonometric polynomial.

Many algorithms can be applied to solve the parameter estimation problem (see for example the algorithms described in [6–11]), where the parameter convergence is granted by the condition of persistency of excitation (10). The faster the parameter convergence, the less time is required for a system identification, leading to the advantage of the proposed approach with respect to a conventional one. Therefore priority in this application should be given to the fast convergent algorithms leading to a rapid frequency domain system identification method.

3 A NEW MODIFIED KACZMARZ ALGORITHM

A new modified Kaczmarz algorithm is described in this section. First, the properties of the least-squares gain update algorithm are established in section 3.1. The modified Kaczmarz algorithm and transient bounds of estimation error are presented in section 3.2. A new method for estimation of a variance of measurement noise is described in section 3.3. Finally, a comparative analysis of the modified Kaczmarz algorithm and the RLS algorithm is performed in section 3.4.

3.1 A least-squares gain update with forgetting factor

Lemma 1

Consider the following gain update law
\[ \Gamma_k^{-1} = \lambda_0 \Gamma_{k-1}^{-1} + \lambda_1 \varphi_k \varphi_k^T, \Gamma_0^{-1} = 0, \gamma_0 > 0 \]  
(14)
where \( k = 1, 2, 3, \ldots \) and \( 0 < \lambda_0 < 1 \) is a forgetting factor and \( \lambda_1 \) is a gain factor with regressor vector. Then there exists a positive constant \( \lambda_0 \) sufficiently close to one such that for all \( \lambda_0, \lambda_0 < 1 \), and for a sufficiently large \( \gamma_0 > 0 \) the matrix \( \Gamma_k^{-1} \) is a strictly diagonally dominant (SDD) matrix if in every row of the matrix, the magnitude

of the diagonal entry in that row is larger than the sum of the magnitudes of all the other (non-diagonal) entries in that row (12), i.e. the following inequalities are valid

\[ |\gamma_{ii}| > \sum_{j=1, j \neq i}^{2n+1} |\gamma_{ij}| \quad i = 1, \ldots, (2n+1) \] (16)

for all admissible frequencies, where \( \gamma_{ij} \), \( i, j = 1, \ldots, (2n+1) \) are the elements of matrix \( \Gamma^{-1}_k \). \( \gamma_{ii} \) are positive diagonal elements of matrix \( \Gamma^{-1}_k \) and \( n \) is the number of frequencies in signal (4).

Moreover, the matrix \( \Gamma^{-1}_k \) is a positive definite matrix, it has positive eigenvalues only for \( k=0,1,2,\ldots \) and the following inequalities are true

\[ 0 < \lambda_{\min}(\Gamma^{-1}_k)I \leq \Gamma^{-1}_k \leq \lambda_{\max}(\Gamma^{-1}_k)I \] (17)

where \( \lambda_{\min}(\Gamma^{-1}_k) > 0 \) and \( \lambda_{\max}(\Gamma^{-1}_k) > 0 \) are minimal and maximal eigenvalues of the matrix \( \Gamma^{-1}_k \) respectively, and \( I \) is the identity matrix.

Finally, the following bounds are valid for minimal and maximal eigenvalues of the matrix \( \Gamma^{-1}_k \)

\[ \lambda_{\min}(\Gamma^{-1}_k) \geq \min_i \{ \gamma_{ii} - \sum_{j=1, j \neq i}^{2n+1} |\gamma_{ij}| \} > 0 \] (18)

\[ \lambda_{\max}(\Gamma^{-1}_k) \leq \max_i \{ \gamma_{ii} + \sum_{j=1, j \neq i}^{2n+1} |\gamma_{ij}| \} \] (19)

for \( i = 1, \ldots, (2n+1) \).

Lemma 1 is proved via explicit evaluation of the elements of matrix \( \Gamma^{-1}_k \). This evaluation is performed in Appendix 2.

**Remark 1**

Here and below extremely low frequencies \( q \) for which \( \cos(q) \) is very close to one are not included in the set of admissible frequencies.

A Kaczmarz update law is modified in the next section with a least-squares gain update described above aiming to the performance improvement. The properties of matrix \( \Gamma^{-1}_k \) reported in Lemma 1 are very useful in stability analysis performed in section 3.2 where the matrix \( \Gamma^{-1}_k \) is used in a Lyapunov function candidate.

### 3.2 Kaczmarz projection algorithm with the gain update

**Lemma 2**

Consider the system (4), (7) bounded measurement noise \( |\xi_k| \leq c_\xi \), \( c_\xi > 0 \) and the following update law for \( \theta_k \)

\[ \dot{\theta}_k = \theta_{k-1} + \frac{\Gamma_{k-1} \phi_k}{\phi_k^T \Gamma_{k-1} \phi_k} (y_k - \theta_{k-1} \phi_k) \] (20)

\[ \Gamma_k = \frac{1}{\lambda_0} [\Gamma_{k-1} - \frac{\Gamma_{k-1} \phi_k \phi_k^T \Gamma_{k-1}}{(\lambda_k + \phi_k^T \Gamma_{k-1} \phi_k)}] \] (21)

\[ \Gamma_0 = \frac{1}{\lambda_0} I, \gamma_0 > 0, \frac{1}{2} < \lambda_0 < 1, \lambda_1 > 0 \]

Then the following transient bound is valid for estimation error \( \dot{\theta}_k = \theta_k - \theta_* \)

\[ \| \dot{\theta}_k \| \leq \sqrt{\frac{\lambda_0^2 V_0 + b(1 - \lambda_0)^2}{\lambda_{\min}(\Gamma^{-1}_k)}} \] (22)

\[ b = \frac{\lambda_{\text{ug}} (1 - \lambda_0)^2 + \lambda_{\text{ug}} c_\xi^2}{(n + 1) (2 \lambda_0 - 1) n + 1} \] (23)

where \( \lambda_{\min}(\Gamma^{-1}_k) \) is a minimal eigenvalue of the matrix \( \Gamma^{-1}_k \) in step \( k=0,1,2,\ldots \) and \( \lambda_{\text{ug}} \) is a global upper bound for \( \lambda_{\max}(\Gamma^{-1}_k) \) that is \( \lambda_{\max}(\Gamma^{-1}_k) \leq \lambda_{\text{ug}} \), \( V_0 = \theta_0^T \Gamma_0^{-1} \theta_0 \), \( k=0,1,2,\ldots \)

The proof of Lemma 2 is given in Appendix 3 and is based on the Lyapunov function \( V_k = \theta_k^T \Gamma_k^{-1} \theta_k \), \( k=0,1,2,\ldots \) where \( \Gamma_k^{-1} \) is a positive definite and symmetric matrix with the properties described in Lemma 1.

**Corollary**

In the noise-free case, where \( \xi_k = 0 \) and \( c_\xi = 0 \) the following transient bound is valid

\[ \| \dot{\theta}_k \| \leq \sqrt{\frac{\lambda_0^2 V_0}{\lambda_{\min}(\Gamma^{-1}_k)}} \] (24)

**Remark 2**

For the case where \( \lambda_0 = 1 \) the minimal eigenvalue of the matrix \( \Gamma^{-1}_k \) tends to infinity, that is \( \lambda_{\min}(\Gamma^{-1}_k) \rightarrow \infty \) whereas \( \lambda_{\min}(\Gamma^{-1}_k) \) remain bounded for \( \lambda_0 < 1 \). Therefore \( \| \dot{\theta}_k \| \rightarrow 0 \) as \( k \rightarrow \infty \) in both cases due to (24).

Transient bound (22) cannot be made arbitrarily small when increasing \( \lambda_{\min}(\Gamma^{-1}_k) \) since \( \lambda_{\text{ug}} \) which is the upper bound for \( \lambda_{\max}(\Gamma^{-1}_k) \) \( (\lambda_{\max}(\Gamma^{-1}_k) \leq \lambda_{\text{ug}}, k=0,1,2,\ldots) \) is also increasing in this case. The ratio \( \frac{\lambda_{\min}(\Gamma^{-1}_k)}{\lambda_{\text{ug}}(\Gamma^{-1}_k)} \) is approximately constant at the steady-state for both \( \lambda_0 = 1 \) and \( \lambda_0 < 1 \).

The parameter mismatch \( \theta_k \) and upper bound (22) are plotted in Fig. 1. This figure shows that the upper bound (22) obtained via the Lyapunov method is unfortunately very conservative.
3.3 Estimation of a variance of measurement noise and an order of trigonometric polynomial

The method for estimation of the variance of the measurement noise $\sigma^2$ is based on relation (53) which relates the regressor vector, the vector of the parameter mismatch and measurement noise. The variance can be calculated due to the fact that the algorithm is unbiased, i.e. a mathematical expectation of parameter vector is equal to a vector of true parameters at steady state [13].

Squaring both sides of (53) and taking mathematical expectation yields

$$E[\hat{\theta}_k^T \varphi_k \varphi_k^T \hat{\theta}_k] = E[\hat{\sigma}^2] = \sigma^2; \quad \hat{\theta}_k = \theta_k - \theta^* \tag{25}$$

The relation above gives the following estimate of the variance of the measurement noise

$$\hat{\sigma}^2 = \frac{1}{N} \sum_{k=1}^{N} (\theta_k - \bar{\theta})^T \varphi_k \varphi_k^T (\theta_k - \bar{\theta}) \tag{26}$$

where $\bar{\theta} = \frac{1}{N} \sum_{k=1}^{N} \theta_k$ is an average value of the parameter vector $\theta_k$ at steady state. The right-hand side of (26) is an estimate of the mathematical expectation $E[\hat{\theta}_k^T \varphi_k \varphi_k^T \hat{\theta}_k]$ where the vector of unknown parameters $\theta_k$ is replaced by the average value $\bar{\theta}$ ($E[\theta_k] = \theta^*$) for a sufficiently large sample size $N$ at steady state.

This variance estimation technique (26) is based on estimation of a mathematical expectation which is well-estimated via a sample average for a sufficiently large sample size only. A histogram of an estimate of variance (26) plotted in Fig. 2 shows a pinpoint accuracy of estimation for a sufficiently large size.

Equation (26) is a useful tool for estimation of the order of trigonometric polynomial (4). The order of the polynomial should be increased (new frequencies should be added to a model) until estimate (26) of the variance of the measurement noise is reduced and this reduction is statistically significant.

3.4 Comparative analysis of the modified Kaczmarz algorithm and recursive least-squares algorithm

The modified Kaczmarz algorithm (20), (21) proposed in this paper is similar to RLS with a forgetting factor algorithm which can be written as follows [1]

$$\theta_k = \theta_{k-1} + \frac{G_{k-1} \varphi_k}{\lambda + \varphi_k^T G_{k-1} \varphi_k} (y_k - \theta_{k-1}^T \varphi_k) \tag{27}$$

$$G_k = \frac{1}{\lambda} G_{k-1} - \frac{G_{k-1} \varphi_k \varphi_k^T G_{k-1}}{\lambda + \varphi_k^T G_{k-1} \varphi_k} \tag{28}$$

Fig. 1 Parameter mismatch $\| \bar{\theta}_k \|$ and upper bound (22) are plotted as functions of a step number. The system (4), (20), (21) with the parameters $\lambda_0 = 0.95, \lambda_1 = 1, \gamma_0 = 10, q = 1, 2, 3, \sigma^2 = 3, \epsilon = 3\sigma$ was simulated.

Fig. 2 Histogram plot which shows the frequencies of data points of an estimate of variance $\hat{\sigma}^2$ falling into discrete intervals (bins). The histogram estimates the probability density function of the variance $\hat{\sigma}^2$. The estimate of the variance $\hat{\sigma}^2$ is calculated using (26) with a sample size of $N = 17000$. The number of data points shown in histogram is equal to 1000, distributed over 20 bins. System (4), (20), (21) with a true variance $\sigma^2 = 5$ was simulated. An average value of the sample variance $\hat{\sigma}^2$ calculated using data shown in the histogram is equal to 4.9593 and a sample standard deviation is equal to 0.0539.
where \(0<\lambda_0\leq 1\) is a forgetting factor. The matrix \(G_k\) is initialized to inverse of the information matrix after several initial steps and then updated recursively using (28).

The modified Kaczmarz algorithm (20), (21) and the RLS algorithm (27), (28) look very similar, but the matrix \(G_k\) plays a different role in these algorithms. This matrix is a gain matrix in the modified Kaczmarz algorithm (20), (21) whereas the matrix \(G_k\) is a recursive estimate of the inverse of the information matrix in the RLS algorithm (27), (28). The inverse of the information matrix can be calculated when this matrix gets a full rank. The modified Kaczmarz algorithm (20), (21) provides estimates which are close to correct values even at the initial steps whereas the RLS algorithm (27), (28) has a delay time until information matrix becomes invertible. The initial steps of two algorithms are shown in Fig. 3.

Notice that the matrix \(G_k\) in RLS algorithm can also be initialized in the same way as in algorithm (20), (21) and switched to the inverse of the information matrix after several initial steps. However, a stability analysis of the overall system is required in this case.

The robustness of the two algorithms against accumulation errors can be compared by adding an equal perturbation term \(0.26I\), where \(I\) is a unity matrix, to the right-hand side of equations for the matrix \(G_k\) in (21) and (28). Estimates provided by the RLS algorithm oscillate whereas the modified Kaczmarz algorithm exhibits a robust behaviour and even a satisfactory performance as can be seen in Fig. 4. The error accumulation problem is well known for RLS algorithms and is discussed for example in reference [14]. As was mentioned earlier, the matrix \(G_k\) in the RLS algorithm is a recursive estimate of the inverse of the information matrix and errors in \(G_k\) destroy optimality and even might have an impact on the stability of the algorithm. In contrast, the matrix \(G_k\) in the modified Kaczmarz algorithm is treated just as a gain matrix which appears to be more tolerant of the error accumulation.

Finally, the three following algorithms: the Kaczmarz algorithm (20) with \(G_k = I\), the modified Kaczmarz algorithm (20), (21), and the RLS algorithm (27), (28) are compared in terms of the variance of the parameter error \(E \| \hat{\theta}_k \|^2\). Figure 5 shows that the modified Kaczmarz algorithm and the Kaczmarz algorithm have the same variance of the parameter error for forgetting factors which are close to one. The modified Kaczmarz algorithm and the RLS algorithm have the same variance of the parameter error for forgetting factors which are smaller than 0.5. The RLS algorithm reduces the variance of the parameter error and shows better performance for forgetting factors which are close to one. However, the modified Kaczmarz algorithm is more robust with respect to accumulation errors and provides estimates which are close to the true values even at the initial steps of the algorithm. It can also be shown that the modified Kaczmarz algorithm converges much faster than the classical
Kaczmarz algorithm. Therefore the modified algorithm is applied in the next section to frequency domain identification of a DC motor. Note that the comparative analysis presented above is valid for a persistently exciting regressor vector for which inequalities (10) are valid. Kaczmarz algorithms might diverge in the absence of the persistence of excitation [15].

4 A FREQUENCY DOMAIN IDENTIFICATION METHOD OF DC MOTOR

DC motors are widely used as actuators in control systems to provide a rotary motion to a variety of electromechanical devices and servo systems. In order to achieve a high-performance regulation, the parameters of the motor should be identified. Newton’s law combined with Kirchhoff’s law yields the following transfer function of a DC motor, where the angle is the output and the voltage is the input

\[ H(p) = \frac{K}{p(T_1 p + 1)(T_2 p + 1)} \]  

(29)

where \( T_1 = 0.1 \) [s] and \( T_2 = 0.02 \) [s] are mechanical and electrical time constants respectively and \( K = 100 \) is a gain.

First, true values of the magnitude and the phase of the transfer function \( H(iq) \) are calculated

\[ |H(iq)| = \frac{K}{q\sqrt{1 + q^2 T_1^2}\sqrt{1 + q^2 T_2^2}} \]  

(30)

\[ \psi_{\Omega} = -90^\circ - \arctan \frac{q(T_1 + T_2)}{1 - q^2 T_1 T_2} \]  

(31)

Fig. 5 A variance of the parameter mismatch \( E \| \hat{\theta}_k \|^2 \) which is estimated at steady state via a sum of sample variances for a sufficiently large sample size. System (4), (7) with \( q = 1,2,3, \sigma^2 = 3 \) was simulated with the Kaczmarz algorithm (20) with \( \Gamma_k = I \), the modified Kaczmarz algorithm (20), (21), and the RLS algorithm (27), (28). The variance is plotted as a function of the two parameters forgetting factor \( \lambda_0 \) and initial value \( \gamma_0 \) with \( \lambda_1 = 1 \).

Fig. 6 The input multi-frequency test sequence \( u_k \) to DC motor is plotted as a function of a step number and specified in a form of trigonometric polynomial

\[ u_k = \sum_{q=0}^{n} A_q \cos(qk \Delta) \]

where \( A_q = 10, 8, 8, 8, 8, 8, 8 \) \([V]\) are the amplitudes at frequencies \( q = 3, 7, 15, 25, 45, 100, 300 \) [Hz] respectively, \( \Delta = 0.0001 \), \( k = 0,1,2,\ldots \)
for the following set of frequencies \( q = 3, 7, 15, 25, 45, 100, 300 \). Alternatively, the magnitude and the phase of the response can be calculated using a function \textit{bode} of a Matlab software.

Second, the multi-frequency input sequence plotted in Fig. 6 is specified in a form of (1) and the model for the output is specified in a form of (7), where the parameters are updated via algorithm (20) and (21). A variance of the measurement noise is \( s^2 = 0.05 \). Finally, the magnitude and the phase of the responses are calculated in Matlab using (13) and (14). The frequency responses for plant (29) are summarized in Table 1, where true values of the magnitude \( |H(iq)| \) and the phase \( \psi_{iq} \), calculated via (30), (31) are compared to the values of the magnitude \( |\tilde{H}(iq)| = \frac{B_{iq}}{A_{iq}} \) and the phase \( \psi_{iqk} \) estimated via average value at steady state for 60000 points using algorithm (20), (21). Table 1 shows very good agreement between estimated and true values of the magnitude and phase. Estimated values of the time constants and the gain for plant (29) can be calculated using the frequency response data listed in Table 1. Since the number of measured points is larger than the number of unknown parameters (two time constants and one gain) the estimated values of these parameters are over-determined and can be found using a least-squares method.

A transient performance of algorithm (20) and (21) is illustrated in Figs 7 and 8.

### 5 DISCUSSION

The methods that are able to rapidly solve the frequency response identification problems are very much in demand in such application areas as communication, instrumentation, civil engineering, bio-medicine and others.
The performance of the transmission channels and cables is identified in communications by sending a test signal and recording the frequency response. The frequency response curves of microphones, amplifiers, loudspeakers, CD players, and other devices can be identified in a similar way in the instrumentation area.

The frequency response functions of such civil engineering structures as buildings, towers, and bridges are used for active vibration mitigation and reduction of damaging effects of such dynamic loads as high winds, extreme waves, and strong earthquakes [16].

Frequency domain analysis can also be used in biomedical signal processing applications for rapid decomposition of rhythmic oscillations of the blood pressure and heart rate into different frequency components and for quantification of the power of these oscillations at each specific frequency [17].

Finally, radar/sonar and seismic signal processing applications should also be mentioned in a frequency domain context.

Simple, fast, and robust algorithms are required for estimation of the frequency components of multi-frequency output measurements in these and many other applications. The performance of the Kaczmarz algorithm is improved in this paper which makes the Kaczmarz projection method applicable to many problems as mentioned earlier.

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APPENDIX 1
Lemma 3
The following relationships are valid

\[ \sum_{r=0}^{w} \lambda_0^{w-r} \cos (rq) = \frac{\lambda_0^{w+1} \cos (\lambda_0 q)}{1 + \lambda_0^2 - 2\lambda_0 \cos (q)} \]

\[ + \frac{-\lambda_0^{w+1} \cos (\lambda_0 q) - \lambda_0 \cos ((w+1)q)}{1 + \lambda_0^2 - 2\lambda_0 \cos (q)} \]  

\[ \sum_{r=0}^{w} \lambda_0^{w-r} \sin (rq) = \frac{\sin (\lambda_0 q) + \lambda_0^{w+1} \sin (q)}{1 + \lambda_0^2 - 2\lambda_0 \cos (q)} \]

\[ + \frac{-\lambda_0 \sin ((w+1)q)}{1 + \lambda_0^2 - 2\lambda_0 \cos (q)}, 0 < \lambda_0 \leq 1 \]

\[ \sum_{r=0}^{w} \lambda_0^{w-r} \cos^2 (rq) = \frac{1}{2} \left[ 1 - \lambda_0^{w+1} \right] \]

\[ + \frac{1}{2} \left[ \lambda_0^{w+2} + \cos (2\lambda_0 q) - \lambda_0^{w+1} \cos (2q) \right] \]

\[ + \frac{-\lambda_0 \cos (2\lambda_0 q)}{1 + \lambda_0^2 - 2\lambda_0 \cos (2q)} \]
\[
\sum_{r=0}^{n} \lambda_0^{w-r} \sin^2 (rq) = \frac{1}{2} \left( 1 - \lambda_0^{w+1} \right) \\
- \frac{1}{2} \lambda_0^{w+2} + \cos (2uq) - \lambda_0^{w+1} \cos (2q) \\
+ \frac{-\lambda_0 \cos (2(u+1)q)}{1 + \lambda_0^2 - 2\lambda_0 \cos (2q)} (35)
\]

\[
\sum_{r=0}^{n} \lambda_0^{w-r} \cos^2 (rq) + \sum_{r=0}^{n} \lambda_0^{w-r} \sin^2 (rq) \\
= (1 - \lambda_0^{w+1}) \\
(1 - \lambda_0), \quad 0 < \lambda_0 < 1
\]

**Remark 3**

Using relations presented above the following sums can easily be calculated

\[
\sum_{r=0}^{n} \lambda_0^{w-r} \cos (rq), \sum_{r=0}^{n} \lambda_0^{w-r} \sin (rq), \sum_{r=0}^{n} \lambda_0^{w-r} \cos (rq), p \neq l
\]

**Remark 4**

The following sums \(\sum_{r=0}^{n} \lambda_0^{w-r} \cos (rq)\), \(\sum_{r=0}^{n} \lambda_0^{w-r} \sin (rq)\) reported in [3], [13] follow immediately from (32) and (33) with \(\lambda_0 = 1\).

**Proof**

Consider the following sums

\[
\sum_{r=0}^{n} \lambda_0^{w-r} e^{iq} = \lambda_0^w + \lambda_0^{w-1} e^{iq} + \ldots + e^{iqn} (36)
\]

\[
\lambda_0 e^{-iq} \left[ \sum_{r=0}^{n} \lambda_0^{w-r} e^{iq} \right] = \\
\lambda_0^{w+1} e^{-iq} + \lambda_0^w + \ldots + \lambda_0 e^{(w-1)iq} (37)
\]

where \(i^2 = -1\). Subtraction of (37) from (36), where all the in-between terms vanish, yields

\[
\sum_{r=0}^{n} \lambda_0^{w-r} e^{iq} = e^{iq} - \lambda_0^{w+1} e^{-iq} \frac{1}{1 - \lambda_0 e^{-iq}} (38)
\]

where \(\lambda_0 e^{-iq} \neq 1\).

Multiplication of the numerator and denominator of the ratio in the right-hand side of (38) by the following complex conjugate to denominator variable \((1 - \lambda_0 e^{iq})^n\) and subsequent decomposition of both sides of (38) into real and imaginary parts gives (32) and (33) when equating real and imaginary parts. Relations (34) and (35) follow directly from (32) and (33).

**APPENDIX 2**

**Proof of Lemma 1**

This lemma is proved via explicit evaluation of the elements of matrix \(\Gamma_k^{-1}\) defined in (15) using trigonometric relations reported in Lemma 3.

Equation

\[
\Gamma_k^{-1} = \lambda_0 \Gamma_{k-1}^{-1} + \lambda_1 \varphi_k \varphi_k^T \\
\Gamma_0^{-1} = \gamma_0 I, \quad \gamma_0 > 0, \quad 0 < \lambda_0 < 1, \lambda_1 > 0
\]

where \(I\) is the identity matrix, has the following solution

\[
\Gamma_k^{-1} = \lambda_0^w \Gamma_n^{-1} + \lambda_1 \varphi_0 \varphi_0^T + \lambda_2 \varphi_2 \varphi_2^T + \ldots + \varphi_{w} \varphi_{w}^T
\]

for \(k = 0, 1, \ldots, w\), where

\[
\varphi_k^T = [1 \cos (q_1 k \Delta) \sin (q_1 k \Delta) \cos (q_2 k \Delta) \ldots \sin (q_n k \Delta) \cos (q_n k \Delta)]
\]

where \(\Delta = 1\) for simplicity.

All the elements of the symmetric matrix \(\Gamma_k^{-1}\) at \(k = w\) are evaluated

\[
\Gamma_w^{-1} = \begin{bmatrix}
\gamma_{11} & \gamma_{12} & \cdots & \gamma_{1m} \\
\gamma_{21} & \gamma_{22} & \cdots & \gamma_{2m} \\
\vdots & \vdots & \ddots & \vdots \\
\gamma_{m1} & \gamma_{m2} & \cdots & \gamma_{mm}
\end{bmatrix}
\]

where \(m = 2n + 1\), \(n\) is the number of frequencies.

Evaluation of diagonal element \(\gamma_{11}\) yields

\[
\gamma_{11} = \gamma_0 \lambda_0^w + \left[ \lambda_0^{w-1} + \lambda_0^{w-2} + \ldots + 1 \right] \lambda_1
\]

\[
= (\gamma_0 - \lambda_1) \lambda_0^w + \lambda_1 \sum_{r=0}^{w} \lambda_0^{w-r} (42)
\]

where \(\gamma_0 > \lambda_1 + \frac{1}{1 - \lambda_0}\) and evaluation of element \(\gamma_{11}\) at steady-state yields

\[
\gamma_{11} \rightarrow \frac{\lambda_1}{(1 - \lambda_0)} \text{ as } w \rightarrow \infty
\]

provided that \(0 < \lambda_0 < 1\).

Notice that element \(\gamma_{11}\) can be made arbitrarily large if \(\lambda_0\) is sufficiently close to one. Non-diagonal \(\gamma_{12}\) can be calculated as follows:
\[
\gamma_{12} \rightarrow \lambda_1 \left[ \sum_{r=0}^{w} \lambda_0^{w-r} \cos (rq) - \lambda_0^w \right] \\
= (32) \lambda_1 \frac{\lambda_0^{w+2} + \cos (wq) - \lambda_0^{w+1} \cos (q)}{1 + \lambda_0^2 - 2\lambda_0 \cos (q)} - \frac{\lambda_0 \cos ((w+1)q)}{1 + \lambda_0^2 - 2\lambda_0 \cos (q)} - \lambda_0^w \tag{44}
\]

where \( q = q_1 \) is the frequency.

Steady-state evaluation gives the relation

\[
\gamma_{12} \rightarrow \lambda_1 \frac{\cos (wq) - \lambda_0 \cos ((w+1)q)}{1 + \lambda_0^2 - 2\lambda_0 \cos (q)} \text{ as } w \rightarrow \infty \tag{45}
\]

provided that \( 0 < \lambda_0 < 1 \). Notice that element \( \gamma_{12} \) is bounded if \( \lambda_0 \) is sufficiently close to one.

The matrix \( \Gamma^{-1}_w \) is proved to be a SDD matrix as \( w \rightarrow \infty \). Notice that a matrix is defined as a SDD matrix, if in every row of the matrix the magnitude of the diagonal entry in that row is larger than the sum of the magnitudes of all the other (non-diagonal) entries in that row. The diagonal dominance is verified row-wise. Consider the first row. The magnitude of the diagonal element in the first row \( \gamma_{11} \) should be larger than the sum of the magnitudes of all the other (non-diagonal) entries in that row. This means that element \( \gamma_{11} \) should be essentially larger in order to accommodate all non-diagonal elements) than the element \( |\gamma_{12}| \) as \( w \rightarrow \infty \). Two elements \( \gamma_{11} \) and \( |\gamma_{12}| \) are plotted for \( w = 150 \) and \( \lambda_1 = 1 \) as functions of the frequency \( q \) and forgetting factor \( \lambda_0 \) and compared in Fig. 9. The elements are calculated using (43) and (45). Diagonal entry \( \gamma_{11} \) is larger than \( |\gamma_{12}| \) for all admissible frequencies and forgetting factors and becomes essentially larger for all forgetting factors which are close to one. Notice that \( \gamma_{11} = \gamma_{12} \) as \( w \rightarrow \infty \) provided that \( q = 0 \) and the diagonal dominance is destroyed in this case.

Non-diagonal element \( \gamma_{13} \) can be evaluated as

\[
\gamma_{13} = \lambda_1 \left[ \sum_{r=0}^{w} \lambda_0^{w-r} \sin (rq) \right] = \frac{\lambda_1 \sin (wq) + \lambda_0^{w+1} \sin (q) - \lambda_0 \sin ((w+1)q)}{1 + \lambda_0^2 - 2\lambda_0 \cos (q)} \tag{46}
\]

Evaluation of element \( \gamma_{13} \) at steady-state gives the relation

\[
\gamma_{13} \rightarrow \lambda_1 \frac{\sin (wq) - \lambda_0 \sin ((w+1)q)}{1 + \lambda_0^2 - 2\lambda_0 \cos (q)} \text{ as } w \rightarrow \infty \tag{47}
\]

provided that \( 0 < \lambda_0 < 1 \).

Using arguments similar to the arguments applied for evaluation of element \( \gamma_{12} \) it can be shown that diagonal element \( \gamma_{11} \) in the first row is significantly larger than the absolute value of \( \gamma_{13} \) if \( \lambda_0 \) is close to one. The next step is evaluation of remaining diagonal elements.

---

**Fig. 9** Elements of the matrix \( \Gamma^{-1}_{k} \). The diagonal entry \( \gamma_{11} \) and non-diagonal entry \( |\gamma_{12}| \) are plotted for \( k = w = 150 \) and \( \lambda_1 = 1 \) as functions of the frequency \( q \) and forgetting factor \( \lambda_0 \). The diagonal entry \( \gamma_{11} \) is calculated using (43) and non-diagonal entry \( |\gamma_{12}| \) is calculated using (45).
Evaluation of diagonal element $\gamma_{22}$ yields

$$\gamma_{22} = (\gamma_0 - \lambda_1)\lambda_0 w + \frac{\lambda_1 (1 - \lambda_0^{w+1})}{2(1 - \lambda_0)}$$

**average part**

$$+ \frac{\lambda_1 \lambda_0 w + \cos (2w) - \lambda_0^{w+1} \cos (2q)}{2} \frac{1 + \lambda_0^2 - 2\lambda_0 \cos (2q)}{2}$$

**periodic part**

provided that $0 < \lambda_0 < 1$. Relation (34) reported in Lemma 3 is used for calculation of elements (48) and (50).

Using similar arguments all other elements of the matrix $\Gamma_{w}^{-1}$ can be calculated. Notice that the diagonal elements $\gamma_{ii}$, $i = 2, 3, \ldots, m$ defined in (48)–(51) have a periodic part and an average part, and all non-diagonal elements have a periodic part only. The average part in turn, consists of a transient part and a steady-state part where $w \to \infty$. Steady-state parts of all the diagonal elements $\gamma_{ii}$, $i = 1, 2, 3, \ldots, m$ can be made sufficiently large as $\lambda_0 \to 1$, and all non-diagonal elements remain bounded as $\lambda_0 \to 1$ for all admissible frequencies $q$, i.e. for all frequencies excepting extremely low ones for which $\cos (q)$ is close to one. Therefore the matrix $\Gamma_{w}^{-1}$ can be made an SDD matrix when $w \to \infty$ and $\lambda_0 \to 1$.

Finally, diagonal element $\gamma_{33}$ can be written as

$$\gamma_{33} = \gamma_0 \lambda_0^w - \frac{\lambda_1 \lambda_0^{w+1}}{2(1 - \lambda_0)} + \frac{\lambda_1}{2(1 - \lambda_0)}$$

**average part**

$$- \frac{\lambda_1 \lambda_0 w + \cos (2w) - \lambda_0^{w+1} \cos (2q)}{2} \frac{1 + \lambda_0^2 - 2\lambda_0 \cos (2q)}{2}$$

**periodic part**

provided that $0 < \lambda_0 < 1$.

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Moreover, the matrix $\Gamma_{k}^{-1}$ can also be made an SDD matrix at the transient for $k = 0, 1, 2, \ldots, w$ for a sufficiently large $\lambda_0 > 0$ since $\gamma_0$ has impact on diagonal elements at transient only and all non-diagonal elements remain bounded. Therefore there exist $\lambda_0$, and a sufficiently large $\lambda_0 > 0$ such that for all $\lambda_0, < \lambda_0 < 1$ the matrix $\Gamma_{k}^{-1}$ is an SDD matrix for all $k = 0, 1, 2, \ldots, w$.

This symmetric SDD matrix $\Gamma_{k}^{-1}$ with positive diagonal entries is positive definite, it has positive eigenvalues only and hence (17) holds due to the Rayleigh–Ritz theorem [12].

Gershgorin’s circle theorem is a useful tool for estimation of the bounds for minimal and maximal eigenvalues of an SDD matrix $\Gamma_{k}^{-1}$. Around every diagonal element $\gamma_{ii}$ of matrix $\Gamma_{k}^{-1}$, a circle with radius that is equal to the sum of the absolute values of the other elements on the same row is drawn. Such circles are called Gershgorin discs. Every eigenvalue $\lambda$ of the matrix $\Gamma_{k}^{-1}$ is located within at least one of the Gershgorin discs $D(\gamma_{ii}, R_i)$

$$|\lambda - \gamma_{ii}| \leq \sum_{j=1, j\neq i}^{n} |\gamma_{ij}| i = 1, \ldots, (2n + 1)$$

where $\gamma_{ij}$ is the centre of the disc, and $R_i = \sum_{j=1, j\neq i}^{2n+1} |\gamma_{ij}|$ is the radius of the disc, $n$ is the number of frequencies. The bounds for minimal and maximal eigenvalues of matrix $\Gamma_{k}^{-1}$ (18) and (19) are defined as the end points of the Gershgorin discs.

Simulation results show that inequalities (18) and (19) provide sufficiently tight bounds for minimal and maximal eigenvalues of matrix $\Gamma_{k}^{-1}$ respectively.
APPENDIX 3

Proof of Lemma 2
First notice that following relations are valid for system (4), (7) and (20), (21)

\[ \phi_k^T \tilde{\theta}_k = \xi_k \]  
(53)

\[ \dot{\theta}_k - \dot{\theta}_{k-1} = -\frac{\Gamma_{k-1} \phi_k \phi_k^T \tilde{\theta}_{k-1}}{\phi_k^T I_{k-1} \phi_k} + \Gamma_{k-1} \phi_k \phi_k^T \xi_k \]  
(54)

\[ \Gamma_k^{-1} = \lambda_0 \Gamma_{k-1}^{-1} + \lambda_1 \phi_k \phi_k^T \]  
(55)

where \( \tilde{\theta}_k = \theta_k - \theta \), is the parameter mismatch and \( \theta \) is the vector of true parameters.

Consider the following Lyapunov function candidate

\[ V_k = \dot{\theta}_k^T \Gamma_k^{-1} \dot{\theta}_k \]  
(56)

which is equal to

\[ V_k = \dot{\theta}_k^T \Gamma_k^{-1} \dot{\theta}_k - (1 - \lambda_0) \dot{\theta}_k^T \Gamma_{k-1}^{-1} \Gamma_{k-1}^T \dot{\theta}_k + \lambda_1 \xi_k^2 \]  
(57)

due to (53) and (55), where \( \Gamma_k^{-1} \) is a positive definite and symmetric matrix whose properties are described in Lemma 1.

Evolution of the first difference yields

\[ V_k - V_{k-1} = \dot{\theta}_k^T \Gamma_k^{-1} \dot{\theta}_k - \dot{\theta}_{k-1}^T \Gamma_{k-1}^{-1} \dot{\theta}_{k-1} \]

\[ - (1 - \lambda_0) \dot{\theta}_k^T \Gamma_{k-1}^{-1} \Gamma_{k-1}^T \dot{\theta}_k + \lambda_1 \xi_k^2 \]

\[ = (\theta_k + \tilde{\theta}_{k-1})^T \Gamma_{k-1}^{-1} \dot{\theta}_{k-1} - (1 - \lambda_0) \dot{\theta}_k^T \Gamma_{k-1}^{-1} \dot{\theta}_k + \lambda_1 \xi_k^2 \]

\[ = - (\theta_k + \tilde{\theta}_{k-1}) \Gamma_{k-1} \phi_k \phi_k^T \tilde{\theta}_{k-1} \]

\[ - (1 - \lambda_0) \dot{\theta}_k \Gamma_{k-1} \phi_k \phi_k^T \tilde{\theta}_k \]

\[ = - (1 - \lambda_0) \dot{\theta}_k \Gamma_{k-1}^{-1} \phi_k \phi_k^T \tilde{\theta}_k + \lambda_1 \xi_k^2 \]

Taking into account (53) the difference \( V_k - V_{k-1} \) is written as

\[ V_k - V_{k-1} = - \dot{\theta}_k^T \Gamma_k^{-1} \frac{\phi_k \phi_k^T}{\phi_k^T I_{k-1} \phi_k} \tilde{\theta}_k \]

\[ - (1 - \lambda_0) \dot{\theta}_k^T \Gamma_{k-1}^{-1} \Gamma_{k-1}^T \dot{\theta}_k + \lambda_1 \xi_k^2 \]  
(58)

Notice that transient bound (67) where \( \lambda_0 = 1 \) follows immediately from (58) with \( \xi_k = 0 \).

Evaluation of the term \(-(1 - \lambda_0) \dot{\theta}_k^T \Gamma_{k-1}^{-1} \Gamma_{k-1}^T \dot{\theta}_k\) using (53) and (54) yields

\[ - (1 - \lambda_0) \dot{\theta}_k^T \frac{\phi_k \phi_k^T}{\phi_k^T I_{k-1} \phi_k} \tilde{\theta}_k = - (1 - \lambda_0) \]  

\[ \left[ \dot{\theta}_k - \Gamma_{k-1} \phi_k \phi_k^T \tilde{\theta}_{k-1} + \Gamma_{k-1} \phi_k \phi_k^T \xi_k \right]^T \]

\[ \Gamma_k^{-1} \left[ \dot{\theta}_k - \Gamma_{k-1} \phi_k \phi_k^T \tilde{\theta}_{k-1} + \Gamma_{k-1} \phi_k \phi_k^T \xi_k \right] \]

\[ = - (1 - \lambda_0) \dot{\theta}_k^T \frac{\phi_k \phi_k^T}{\phi_k^T I_{k-1} \phi_k} \tilde{\theta}_{k-1} + 2 \dot{\theta}_k^T \frac{\phi_k \phi_k^T}{\phi_k^T I_{k-1} \phi_k} \tilde{\theta}_{k-1} \]

\[ + 2 \dot{\theta}_k^T \frac{\phi_k \phi_k^T}{\phi_k^T I_{k-1} \phi_k} \tilde{\theta}_{k-1} + \Gamma_{k-1} \phi_k \phi_k^T \xi_k \]

\[ = - (1 - \lambda_0) \dot{\theta}_k^T \frac{\phi_k \phi_k^T}{\phi_k^T I_{k-1} \phi_k} \tilde{\theta}_{k-1} + 2 \dot{\theta}_k^T \frac{\phi_k \phi_k^T}{\phi_k^T I_{k-1} \phi_k} \tilde{\theta}_{k-1} \]

\[ + 2 \dot{\theta}_k^T \frac{\phi_k \phi_k^T}{\phi_k^T I_{k-1} \phi_k} \tilde{\theta}_{k-1} + \Gamma_{k-1} \phi_k \phi_k^T \xi_k \]  
(59)

Relation (59) contains two cross-terms

\[ -2(1 - \lambda_0) \dot{\theta}_k^T \frac{\phi_k \phi_k^T}{\phi_k^T I_{k-1} \phi_k} \]

and

\[ 2(1 - \lambda_0) \dot{\theta}_k^T \frac{\phi_k \phi_k^T}{\phi_k^T I_{k-1} \phi_k} \]

which depend on noise \( \xi_k \) and should be overbounded. Taking into account the following inequalities

\[ -2(1 - \lambda_0) \dot{\theta}_k^T \frac{\phi_k \phi_k^T}{\phi_k^T I_{k-1} \phi_k} \]

\[ \leq \left( 2 \lambda_0 - 1 \right) \dot{\theta}_k^T \frac{\phi_k \phi_k^T}{\phi_k^T I_{k-1} \phi_k} \tilde{\theta}_{k-1} \]

\[ + \frac{(1 - \lambda_0)^2}{2 \lambda_0 - 1} \xi_k^2 \]

\[ \leq \left( 1 - \lambda_0 \right) \dot{\theta}_k^T \frac{\phi_k \phi_k^T}{\phi_k^T I_{k-1} \phi_k} \tilde{\theta}_{k-1} \]

\[ + \left( 1 - \lambda_0 \right) \xi_k^2 \]

where \( \frac{1}{2} < \lambda_0 < 1 \), and the following identities
\[
[-1 + 2(1 - \lambda_0) + (2\lambda_0 - 1)]\tilde{\theta}_{k-1}^T \Phi_k \tilde{\theta}_{k-1} = 0
\]

and

\[
[(1 - \lambda_0) - (1 - \lambda_0)]\tilde{\theta}_{k-1}^T \Phi_k \tilde{\theta}_{k-1} = 0
\]

evaluation of the first difference \( V_k - V_{k-1} \) yields

\[
V_k = V_{k-1} + \frac{(1 - \lambda_0)^2}{(2\lambda_0 - 1) \phi_k^T \Gamma_{k-1} \phi_k} + \frac{1}{\phi_k^T \Gamma_{k-1} \phi_k} ||\xi_k^2||^2
\]

A solution of inequality (61) gives the following bound

\[
V_k \leq \lambda_0^k V_0 + b ||(1 - \lambda_0^k)/(1 - \lambda_0)||
\]

where \( V_0 = (\theta_0 - \theta_0) I \Gamma_0^{(1)} (\theta_0 - \theta_0) \) and a positive constant \( b \) is given in (62). Calculation of the sum of geometric progression yields

\[
V_k \leq \lambda_0^k V_0 + b \frac{(1 - \lambda_0^k)}{(1 - \lambda_0)}
\]

Finally, taking into account (17) the transient bound for the parameter mismatch \( \tilde{\theta}_k \) can be written as

\[
||\tilde{\theta}_k|| \leq \sqrt{\lambda_0^k V_0 + b \frac{(1 - \lambda_0^k)}{(1 - \lambda_0)}} / \lambda_{\min}(\Gamma_k^{-1})
\]

where \( \lambda_{\min}(\Gamma_k^{-1}) \) is a minimal eigenvalue of the matrix \( \Gamma_k^{-1} \) in step \( k = 0, 1, 2, \ldots \)

In the noise-free case, where \( \xi_k = 0 \) and \( c_\ell = 0 \) the following transient bound is valid

\[
||\tilde{\theta}_k|| \leq \sqrt{\lambda_0^k V_0 / \lambda_{\min}(\Gamma_k^{-1})}
\]