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Enhanced gauge groups in $\mathcal{N} = 4$ topological amplitudes and Lorentzian Borcherds algebras

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We continue our study of algebraic properties of $\mathcal{N} = 4$ topological amplitudes in heterotic string theory compactified on $\mathbb{T}^2$, initiated in arXiv:1102.1821. In this work we evaluate a particular one-loop amplitude for any enhanced gauge group $\mathfrak{h} \subset \mathfrak{e}_8 \oplus \mathfrak{e}_8$, i.e. for arbitrary choice of Wilson line moduli. We show that a certain analytic part of the result has an infinite product representation, where the product is taken over the positive roots of a Lorentzian Kac-Moody algebra $\mathfrak{g}^{++}$. The latter is obtained through double extension of the complement $\mathfrak{g} = (\mathfrak{e}_8 \oplus \mathfrak{e}_8)/\mathfrak{h}$. The infinite product is automorphic with respect to a finite index subgroup of the full T-duality group $SO(2, 18; \mathbb{Z})$ and, through the philosophy of Borcherds-Gritsenko-Nikulin, this defines the denominator formula of a generalized Kac-Moody algebra $\mathcal{G}(\mathfrak{g}^{++})$, which is an 'automorphic correction' of $\mathfrak{g}^{++}$. We explicitly give the root multiplicities of $\mathcal{G}(\mathfrak{g}^{++})$ for a number of examples.

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I. INTRODUCTION

This work is a continuation of our previous analysis [1] of a particular class of BPS-saturated higher string-loop amplitudes $\mathcal{F}_g$, arising in type II string theory compactified on $\mathrm{K3} \times \mathbb{T}^2$, which are captured by correlation functions of the $\mathcal{N} = 4$ topological string [2]. For any loop order g, the dual amplitudes in heterotic string theory compactified on $\mathbb{T}^6$ receive contributions at all loop orders in (heterotic) perturbation theory. However, the leading contribution in the weak coupling limit is a one-loop expression which can be studied in detail. In particular, following a similar analysis of Harvey and Moore in the $\mathcal{N} = 2$ setting [3,4], one can explicitly perform the worldsheet torus integral and analyze its algebraic and modular properties. The latter are expected as a consequence of the familiar worldsheet $SL(2, \mathbb{Z})$-invariance of the integrand. Mathematically, heterotic one-loop amplitudes fall into the category of so-called "singular theta correspondences," as analyzed in detail by Borcherds [5,6]. This means that the worldsheet $SL(2, \mathbb{Z})$ and the target space T-duality group $SO(6, 22; \mathbb{Z})$ form a dual reductive pair within a larger metaplectic group, and integrating over the fundamental domain of $SL(2, \mathbb{Z})$ thus induces good modular properties under the T-duality group.

We focus on an analytic piece of the one-loop integral which is characterized by the fact that it does not violate a particular class of supersymmetric Ward identities discussed in [2,7] (see also [8,9]). Upon splitting $\mathbb{T}^6 = \mathbb{T}^4 \times \mathbb{T}^2$, we evaluate $\mathcal{F}_1^{\text{analy}}$ explicitly in the large-volume limit of $\mathbb{T}^4$ using the method of orbits [3,10] (or in mathematical parlance, the Rankin-Selberg method), for any choice of the unbroken gauge group $\mathfrak{h} \subset \mathfrak{e}_8 \oplus \mathfrak{e}_8$. In particular, we allow for $\mathfrak{h}$ to be semisimple, thus extending previous results in the literature where always one $\mathfrak{e}_8$-factor remained unbroken [3,11–15]. Equivalently, we allow for Wilson lines to be embedded in any of the factors in the sum $\mathfrak{e}_8 \oplus \mathfrak{e}_8$. As we will see, in this setting the theta correspondence is generalized to subgroups of $SL(2, \mathbb{Z})$ and the T-duality group, respectively.

Taking the above mentioned modular properties into account, we show further that part of the analytic integral $\mathcal{F}_1^{\text{analy}}$ can be written as an infinite product over the positive root lattice of the Lorentzian Kac-Moody algebra $\mathfrak{g}^{++}$, which is a double extension of the complement $\mathfrak{g} = (\mathfrak{e}_8 \oplus \mathfrak{e}_8)/\mathfrak{h}$. In the spirit of [16,17], the infinite product so obtained then defines an "automorphic correction" $\mathcal{G}(\mathfrak{g}^{++})$ of $\mathfrak{g}^{++}$. The correction $\mathcal{G}(\mathfrak{g}^{++})$ is a Borcherds-Kac-Moody (BKM) algebra with real root lattice coinciding with the root lattice of $\mathfrak{g}^{++}$. The BPS-integral $\mathcal{F}_1^{\text{analy}}$ can then be related to the denominator formula of the BKM algebra $\mathcal{G}(\mathfrak{g}^{++})$.

This paper is structured as follows. In Sec. II, we discuss some relevant features of Wilson line moduli in Narain compactifications, with special emphasis on how the splitting of the Narain-lattice $\Gamma^{6,22}$ depends on the enhanced gauge group $\mathfrak{h} \subset \mathfrak{e}_8 \oplus \mathfrak{e}_8$. Then, in Sec. III, we first recall the structure of the $\mathcal{N} = 4$ topological amplitudes $\mathcal{F}_g$, focusing on the role of harmonicity, which allows us to single out the analytic part $\mathcal{F}_g^{\text{analy}}$ of the full amplitude $\mathcal{F}_g$. In Sec. III B, we evaluate the one-loop integral $\mathcal{F}_1^{\text{analy}}$ explicitly, and in Sec. III C, we show how to write part of the result in terms of an infinite product, from which we extract the denominator formula $\Phi_{\mathfrak{g}}(\mathbf{y})$ of the BKM algebra $\mathcal{G}(\mathfrak{g}^{++})$. We show that the automorphic properties of the denominator formula with respect to a subgroup

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$\mathfrak{G} \subset SO(2, 18; \mathbb{Z})$ can be made manifest through an integral representation of $\Phi_{\mathfrak{g}}(\mathbf{y})$, corresponding to a theta correspondence for certain congruence subgroups $\mathfrak{G} \times \Gamma_{[0]} \subset SO(2, 18; \mathbb{Z}) \times SL(2, \mathbb{Z})$. In Sec. IV, we further analyze some particular examples in detail. Finally, we end in Sec. V with a discussion of our results and suggestions for future work. Various calculational details and some relevant mathematical background are relegated to the three Appendices A, B, and C.

II. WILSON LINE MODULI IN NARAIN COMPACTIFICATIONS

We start by reviewing some basic facts about toroidal compactifications of the $E_8 \times E_8$ heterotic string. The classical moduli space for heterotic string theory on $\mathbb{T}^6$ is described by the coset space

$$\mathcal{M} = (SL(2, \mathbb{R})/U(1)) \times (SO(6, 22)/(SO(6) \times SO(22))), \tag{2.1}$$

where the first factor encodes the heterotic 'axio-dilaton', while the second factor accounts for the remaining Narain moduli of the torus. In order to make contact with the topological amplitudes analyzed in Sec. III, we only need to consider the perturbative string spectrum. The latter consists of the states that are created from a momentum ground state labeled by $(p^L, \vec{p}; p^R, \vec{p})$ by the action of the oscillators. Here, the compactified (and internal) momenta take values in the Narain-lattice,[1] $(p^L, p^R) \in \Gamma^{6,22}$, while $\vec{p}$ describes the space-time momentum, i.e. the uncompactified 4-dimensional theory. The T-duality group, which leaves the Narain-lattice invariant, is $SO(6, 22; \mathbb{Z})$, and thus the quantum moduli space is the quotient of (2.1) by this arithmetic group.

For the one-loop string amplitudes it will be important to realize that the BPS-states come in two different classes: Since the compactification of heterotic string theory on $\mathbb{T}^6$ preserves $\mathcal{N} = 4$ supersymmetry we may distinguish between $1/2$ BPS-states associated with short multiplets, and $1/4$ BPS-states associated with intermediate multiplets. As has been discussed in [18], all $1/4$ BPS-states are nonperturbative and only the $1/2$ BPS-states are perturbative. Thus, perturbative topological amplitudes only receive contributions from $1/2$ BPS-states.

A. The sublattice $\Gamma_{\mathfrak{g}} \subset \Gamma^{6,22}$

In the following, we shall be interested in a certain sublattice $\Gamma_{\mathfrak{g}}$ of the full momentum lattice $\Gamma^{6,22}$. In order to describe this sublattice, we proceed in two steps. First, we split $\mathbb{T}^6 = \mathbb{T}^2 \times \mathbb{T}^4$, and take the large-volume limit of the $\mathbb{T}^4$, effectively setting the $\mathbb{T}^4$ momenta to zero. This corresponds to restricting ourselves to momentum ground states in the even self-dual lattice $\Gamma^{2,18}$ of signature (2, 18), which is obtained by splitting

$$\Gamma^{6,22} = \Gamma^{2,18} \oplus \Gamma^{4,4}, \tag{2.2}$$

where $\Gamma^{4,4}$ describes the momenta of the $\mathbb{T}^4$. Notice that we are, therefore, effectively considering $E_8 \times E_8$ heterotic string theory compactified on $\mathbb{T}^2$, for which the components in $\Gamma^{2,18}$ characterize the momentum ground states. The moduli space of such compactifications is described by the Kähler (T) and complex structure (U) moduli of $\mathbb{T}^2$, as well as by two real Wilson lines $\vec{v}_\alpha \in \mathbb{R}^{16}$, $\alpha = 1, 2$. At a generic point in this moduli space, a general element of the momentum lattice $\Gamma^{2,18}$ can be parametrized as $x = (m_1, n_1; m_2, n_2; \vec{\ell})$, where (m_1, m_2) and (n_1, n_2) are the momentum and winding numbers along $\mathbb{T}^2$, while $\vec{\ell} \in \Lambda_{\mathfrak{e}_8} \oplus \Lambda_{\mathfrak{e}_8}$, where $\Lambda_{\mathfrak{e}_8}$ is the root lattice of the Lie algebra $\mathfrak{e}_8$. The inner product on $\Gamma^{2,18}$ is defined by

$$\langle x|x'\rangle = -m_1 n_1' - n_1 m_1' - m_2 n_2' - n_2 m_2' + \vec{\ell} \cdot \vec{\ell}', \tag{2.3}$$

where the first four terms represent the Lorentzian inner product on $\Gamma^{2,2} \simeq \Pi^{1,1} \oplus \Pi^{1,1}$, and the last term is the standard Euclidean inner product inherited from $\mathbb{R}^{16} \supset \Lambda_{\mathfrak{e}_8} \oplus \Lambda_{\mathfrak{e}_8}$. For a given vector $x = (m_1, n_1; m_2, n_2; \vec{\ell}) \in \Gamma^{2,18}$, the actual internal momentum is then a vector in $\mathbb{R}^{16}$:

$$\vec{P}(x) = n_1 \vec{v}_1 + n_2 \vec{v}_2 + \vec{\ell}. \tag{2.4}$$

In the following, we want to consider the subspace of the moduli space where the Wilson lines $\vec{v}_\alpha$ break the $\mathfrak{e}_8 \oplus \mathfrak{e}_8$ gauge symmetry to a fixed unbroken gauge symmetry $\mathfrak{h}$,

$$\mathfrak{e}_8 \oplus \mathfrak{e}_8 \longrightarrow \mathfrak{h} \quad \text{with} \quad \mathfrak{h} \oplus \mathfrak{g} \subset \mathfrak{e}_8 \oplus \mathfrak{e}_8, \tag{2.5}$$

where $\mathfrak{g}$ is the maximal commuting subalgebra in $\mathfrak{e}_8 \oplus \mathfrak{e}_8$. To describe this compactly, we first combine $\vec{v}_1$ and $\vec{v}_2$ into a complex Wilson line $\vec{V} = \vec{v}_1 + i\vec{v}_2$. Given $\vec{V}$, we then denote by $\Lambda_{\mathfrak{h}}$ the sublattice of $\Lambda_{\mathfrak{e}_8} \oplus \Lambda_{\mathfrak{e}_8}$ consisting of all vectors that are orthogonal to the complex Wilson line $\vec{V}$ (or equivalently to both real Wilson lines $\vec{v}_\alpha$),

$$\Lambda_{\mathfrak{h}} = \{\vec{d} \in \Lambda_{\mathfrak{e}_8} \oplus \Lambda_{\mathfrak{e}_8} : \vec{d} \cdot \vec{V} = 0\}. \tag{2.6}$$

The vectors of length squared two in $\Lambda_{\mathfrak{h}}$ are the roots of the unbroken Lie algebra $\mathfrak{h}$. The commutant of $\mathfrak{h}$ in $\mathfrak{e}_8 \oplus \mathfrak{e}_8$ defines the Lie algebra $\mathfrak{g}$, whose root lattice is spanned by the roots of $\mathfrak{e}_8 \oplus \mathfrak{e}_8$ that are orthogonal to $\Lambda_{\mathfrak{h}}$ [see Eq. (2.5)]. Adding to the corresponding root lattice the $\mathbb{T}^2$ torus directions in $\Gamma^{2,2}$ leads to the sublattice $\Gamma_{\mathfrak{g}} \subseteq \Gamma^{2,18}$. More formally, $\Gamma_{\mathfrak{g}}$ is defined as

$$\Gamma_{\mathfrak{g}} = \{x \in \Gamma^{2,18} : \vec{P}(x) \in \Lambda_{\mathfrak{h}}^{\perp}\}, \tag{2.7}$$

where $\vec{P}(x)$ was defined in (2.4). Note that the root lattice $\Lambda_{\mathfrak{g}}$ is the sublattice of $\Gamma_{\mathfrak{g}}$ generated by the vectors of the form $(0, 0; 0, 0; \vec{\ell}) \in \Gamma_{\mathfrak{g}}$, where $\vec{\ell}$ is orthogonal to $\Lambda_{\mathfrak{h}}$. If $\mathfrak{g}$ is

[1] In our conventions, the left-movers are "supersymmetric", while the right-movers are "bosonic".

semisimple (i.e. $\mathfrak{g} = \bigoplus_{i=1}^{n} \mathfrak{g}_{(i)}$ with $n > 1$ and $\mathfrak{g}_{(i)}$ simple Lie algebras), we will adopt the notation

$$\vec{\ell} = (\vec{\ell}_1, \ldots, \vec{\ell}_n), \quad \text{with} \quad \vec{\ell}_i \in \Lambda_{\mathfrak{g}_{(i)}}, \tag{2.8}$$

and the inner product inherited from $\Gamma^{2,18}$ becomes

$$\langle x|x' \rangle = -m_1 n_1' - m_1' n_1 - m_2 n_2' - m_2' n_2 + \sum_{i=1}^{n} \vec{\ell}_i \cdot \vec{\ell}_i', \quad x, x' \in \Gamma_{\mathfrak{g}}. \tag{2.9}$$

Furthermore, by definition of $\Lambda_{\mathfrak{h}}$, $n_1 \vec{v}_1 + n_2 \vec{v}_2 \in \Lambda_{\mathfrak{h}}^{\perp}$, and hence $\Gamma_{\mathfrak{g}}$ has signature $(2, 2 + k)$ where $k = \mathrm{rk}(\mathfrak{g})$. Since $\Lambda_{\mathfrak{h}}$ is naturally a sublattice of $\Gamma^{2,18}$ (corresponding to choosing $n_i = m_j = 0$), we have

$$\Gamma_{\mathfrak{g}} \oplus \Lambda_{\mathfrak{h}} \subseteq \Gamma^{2,18}, \tag{2.10}$$

and the sublattice on the left hand side is of maximal rank. Generically, $\Gamma_{\mathfrak{g}} \oplus \Lambda_{\mathfrak{h}}$ is a proper sublattice of $\Gamma^{2,18}$ with index s. We can write the decomposition

$$\Gamma^{2,18} = (\Gamma_{\mathfrak{g}} \oplus \Lambda_{\mathfrak{h}}) \oplus \bigoplus_{\mu=1}^{s-1} (\lambda_\mu + \Gamma_{\mathfrak{g}} \oplus \Lambda_{\mathfrak{h}}), \tag{2.11}$$

where λ_μ denotes the different cosets. More precisely, this construction can be understood as follows. While the lattice $\Gamma_{\mathfrak{g}} \oplus \Lambda_{\mathfrak{h}}$ is integral Euclidean it is in general not self-dual and the (finite) quotient group $(\Gamma_{\mathfrak{g}} \oplus \Lambda_{\mathfrak{h}})^* / (\Gamma_{\mathfrak{g}} \oplus \Lambda_{\mathfrak{h}})$ is typically nontrivial. We therefore choose a set of generators λ_μ to write the coset representatives of the glue group as (see e.g. [19,20])

$$(\Gamma_{\mathfrak{g}} \oplus \Lambda_{\mathfrak{h}})^* / (\Gamma_{\mathfrak{g}} \oplus \Lambda_{\mathfrak{h}}) = \{\lambda_\mu\}, \quad \text{with} \quad \lambda_\mu \in \Gamma_{\mathfrak{g}} \oplus \Lambda_{\mathfrak{h}}. \tag{2.12}$$

Here, the conjugacy classes μ are called the glue classes and λ_μ is sometimes referred to as the glue vector. The union of all glue vectors in all glue classes μ forms the integral lattice $\Gamma^{2,18}$. The order of the glue group (i.e. the number of different glue classes $\mu = 0, \ldots, s-1$) is given by $|\Lambda_{\mathfrak{g}}| = s$. In some cases we also introduce $\lambda_0 \equiv 0$, and include $\mu = 0$ in (2.11).

In order to get an intuition for these various sublattices it is useful to consider the "extremal" cases. For generic Wilson line moduli $\vec{V}$, then $\Lambda_{\mathfrak{h}} = \{0\}$, and the condition for $\vec{P}(x)$ to be orthogonal to $\Lambda_{\mathfrak{h}}$ is empty; in this case, $\mathfrak{g} = \mathfrak{e}_8 \oplus \mathfrak{e}_8$, and the lattice $\Gamma_{\mathfrak{g}}$ has signature $(2, 18)$. This is the case discussed at length in [1]. The other extremal case arises if $\vec{V} = \vec{0}$, in which case $\Lambda_{\mathfrak{h}} = \Lambda_{\mathfrak{e}_8} \oplus \Lambda_{\mathfrak{e}_8}$. Then the condition to be orthogonal to $\Lambda_{\mathfrak{h}}$ means that $\vec{P}(x) = 0$, and $\Gamma_{\mathfrak{g}} \cong \Pi^{2,2}$ has signature $(2, 2)$ and is generated by n_i and m_j. For suitable intermediate choices of Wilson lines, however, we can also get lattices that lie in between.

B. The Wilson line moduli

As is clear from this discussion, the lattice decomposition (2.10) depends on the choice of Wilson lines. In the following, we want to study the submanifold of the moduli space (which we shall call $\mathcal{M}_{2,2+k}$, where k is the rank of $\mathfrak{g}$) along which this decomposition is constant. One way to guarantee this is to fix the "direction" of $\vec{V}$ in the following way. Let us introduce a basis $\vec{e}_i$, $i = 1, \ldots, 16$ for $\mathbb{R}^{16}$ consisting of the simple roots of $\mathfrak{e}_8 \oplus \mathfrak{e}_8$. (Thus, in particular, $\vec{e}_i \cdot \vec{e}_j = \mathcal{C}_{ij}$, with $\mathcal{C}_{jk}$ the Cartan matrix of $\mathfrak{e}_8 \oplus \mathfrak{e}_8$.) We denote the dual basis by $\vec{f}^l$, $l = 1, \ldots, 16$ so that $\vec{e}_j \cdot \vec{f}^l = \delta_j^l$, and write $\vec{V}$ in this basis, i.e. $\vec{V} = V_j \vec{f}^j$. If $\vec{V}$ has precisely k nonzero coefficients V_j, say $V_{l(1)}, \ldots, V_{l(k)}$, then the decomposition (2.10) is generically independent of the actual values of these coefficients. Indeed, $\Lambda_{\mathfrak{h}}$ is then a (16-k)-dimensional lattice generated by

$$\Lambda_{\mathfrak{h}} = \mathrm{span}_{\mathbb{Z}}(e_j | j \notin \{l(1), \ldots, l(k)\}). \tag{2.13}$$

We parametrize an arbitrary point in this moduli space $\mathcal{M}_{2,2+k}$ by

$$\mathbf{y} = (U, T; \vec{V}) \in \mathbb{C}^{1,1+k}, \tag{2.14}$$

where $\vec{V} = (V_1, \ldots, V_k)$ and V_j is the component with respect to $\vec{f}^{l(j)}$, $j = 1, \ldots, k$. On the space $\mathbb{C}^{1,1+k}$, we have the inner product $(\mathbf{y}|\mathbf{y}') = -TU' - T'U + \vec{V} \cdot \vec{V}'$ such that

$$(\mathbf{y}|\mathbf{y}) = -2TU + \vec{V}^2. \tag{2.15}$$

For the following, it is also useful to define the map (see [3])

$$u: \mathbb{C}^{1,1+k} \to \mathbb{C}^{2,2+k}, \quad \mathbf{y} = (U, T; \vec{V}) \mapsto u(\mathbf{y}) = \left(U, T; \frac{(\mathbf{y}|\mathbf{y})}{2}, 1; \vec{V}\right), \tag{2.16}$$

which associates to every element $\mathbf{y} \in \mathbb{C}^{1,1+k}$ a lightlike vector $u(\mathbf{y}) \in \mathbb{C}^{2,2+k}$. Here, the inner product on $\mathbb{C}^{2,2+k}$ is defined by $\langle \cdot | \cdot \rangle$ as in (2.9). With this notation, an arbitrary momentum state $x \in \Gamma_{\mathfrak{g}}$ parametrized by $x = (m_1, n_1; m_2, n_2; \vec{\ell})$ has $|p^L|^2 = -2|\langle x|u(\mathbf{y})\rangle|^2 / Y$, where $Y = (\Im \mathbf{y} | \Im \mathbf{y})$ and $\Im \mathbf{y} = (U_2, T_2, \Im \vec{V})$ is the imaginary part of $\mathbf{y}$. We further have $\langle x|x \rangle = (|p^R|^2 - |p^L|^2)$.

III. TOPOLOGICAL AMPLITUDES AND DENOMINATOR FORMULAS

In this section, we introduce and analyze a particular class of topological $\mathcal{N} = 4$ amplitudes $\mathcal{F}_g$ in heterotic string theory compactified on $\mathbb{T}^6$ (see [2,7]). We evaluate the one-loop integral corresponding to a particular analytic part $\mathcal{F}_1^{\text{analy}}$ of the amplitude, for a generic unbroken gauge algebra $\mathfrak{h}$. We show that part of the result can be identified with the infinite product side of the denominator formula for a certain Borcherds extension of $\mathfrak{g}^{++}$ (for details on Borcherds algebras see Appendix A). To verify the automorphic properties of the denominator formula, we find an explicit integral representation of (the logarithm of) the infinite product, which corresponds to a theta correspondence for congruence subgroups of $SO(2, 2+k;\mathbb{Z}) \times SL(2,\mathbb{Z})$.

A. $\mathcal{N} = 4$ topological amplitudes

In the naive field theory limit, the couplings $\mathcal{F}_g$ only receive contributions from perturbative $1/2$ BPS-states. However, in string theory additional nonanalytic terms appear as well. We shall make use of the key observation from our previous work [1], namely, that one may use the harmonicity equations satisfied by $\mathcal{F}_g$ to isolate an analytic part $\mathcal{F}_g^{\text{analy}}$. In a sense, this represents the $\mathcal{N} = 4$ analogue of the "threshold corrections" in $\mathcal{N} = 2$ theories.

In [2,7] (see also [21]), a particular class of $\mathcal{N} = 4$ topological string amplitudes has been discovered. These amplitudes appear at the g-loop level in type II string theory compactified on $K3 \times \mathbb{T}^2$, while their dual counterparts in heterotic string theory compactified on $\mathbb{T}^6$ start receiving contributions at the one-loop level. We will focus on the case $g = 1$ for which the latter amplitude takes the following form

$$\mathcal{F}_1(\mathbf{y}) = \int_{\mathbb{F}} \frac{d^2\tau}{\bar{\eta}^{24}} \tau_2 G_2(\tau,\bar{\tau}) \Theta^{(6,22)}(\tau,\bar{\tau},\mathbf{y}), \tag{3.1}$$

where the integral is over the fundamental domain $\mathbb{F} := \mathbb{F}(\Gamma) = \mathbb{H}/\Gamma$ of $\Gamma = SL(2,\mathbb{Z})$, where $\mathbb{H}$ is the standard upper half plane. Moreover, the expression

$$\Theta^{(6,22)}(\tau,\bar{\tau},\mathbf{y}) = \sum_{\substack{p\in\Gamma^{6,22}\\ p\neq 0}} q^{(1/2)|p^L|^2} \bar{q}^{(1/2)|p^R|^2}, \tag{3.2}$$

is a Siegel-Narain theta function (without momentum insertions) of the even unimodular lattice $\Gamma^{6,22}$. Notice that we do not sum over $p = 0$ in the definition of $\Theta^{(6,22)}(\tau,\bar{\tau},\mathbf{y})$. As was explained in [1], this is a particular choice of regularization which removes an overall singularity of $\mathcal{F}_1$. The object $G_2(\tau,\bar{\tau})$ in (3.1) is a weight 4 nonantiholomorphic modular form. The explicit expression was computed in [22] (see also [23]) and is given by

$$G_2(\tau,\bar{\tau}) = \zeta(4)(\bar{E}_4(\tau) + 5\hat{\bar{E}}_2^2(\tau,\bar{\tau})), \quad \text{with}$$
$$\hat{\bar{E}}_2(\tau,\bar{\tau}) = E_2(\bar{\tau}) - \frac{3}{\pi\tau_2}, \tag{3.3}$$

where $E_{2k}(\tau)$ is the weight $2k$ Eisenstein series. Notice that E_2 is a "quasimodular form" [24,25], which means that, in addition to a weight factor, it also receives an anomalous shift-term under modular transformations. Therefore, following standard practice, we have introduced the quantity $\hat{\bar{E}}_2$, which is an honest weight 2 modular form, but nonantiholomorphic in τ. It is natural to decompose $G_2(\tau,\bar{\tau})$ into an analytic (antiholomorphic) and nonanalytic part

$$G_2(\tau,\bar{\tau}) = G_2^{\text{analy}}(\bar{\tau}) + G_2^{\text{non-analy}}(\tau,\bar{\tau}), \quad \text{with}$$
$$G_2^{\text{analy}}(\bar{\tau}) = \zeta(4)\bar{E}_4(\tau), \qquad G_2^{\text{non-analy}}(\tau,\bar{\tau}) = 5\zeta(4)\hat{\bar{E}}_2^2(\tau,\bar{\tau}). \tag{3.4}$$

In [1], this splitting was proposed based on the fact that (for generic g) the nonanalytic part is responsible for an anomalous violation of particular supersymmetric Ward-identities [7] ("harmonicity relations") satisfied by the amplitudes $\mathcal{F}_g$ at the string quantum level.

At particular points where the gauge group is enhanced due to the presence of additional massless bosons, some additional care is needed, since the harmonicity relations require regularization of certain singular contributions. However, once this subtlety has been properly addressed, we will continue using the definition of $G_g^{\text{analy}}(\bar{\tau})$ and discard the remaining nonantiholomorphic terms.

As the main object of study, we thus introduce the analytic one-loop integral

$$\mathcal{F}_1^{\text{analy}}(\mathbf{y}) = \int_{\mathbb{F}} \frac{d^2\tau}{\bar{\eta}^{24}} \tau_2 G_2^{\text{analy}}(\bar{\tau}) \Theta^{(6,22)}(\tau,\bar{\tau},\mathbf{y}). \tag{3.5}$$

We recall that the decomposition (3.4) does not break modular invariance, and therefore the integral $\mathcal{F}_1^{\text{analy}}(\mathbf{y})$ is well defined.

As in Sec. II, we will consider the internal six-torus to be factorized as $\mathbb{T}^6 = \mathbb{T}^4 \times \mathbb{T}^2$, and take the large-volume limit of $\mathbb{T}^4$. This implies that the Siegel-Narain theta function of the original $\Gamma^{6,22}$ Narain-lattice decomposes according to

$$\frac{G_2^{\text{analy}}(\bar{\tau})\tau_2^2}{\bar{\eta}^{24}} \Theta^{(6,22)} \sim \text{Vol}\, \frac{G_2^{\text{analy}}(\bar{\tau})}{\bar{\eta}^{24}} \Theta^{(2,18)}, \tag{3.6}$$

where Vol is the volume of $\mathbb{T}^4$ and $\Theta^{(2,18)}$ the Siegel-Narain theta function of the lattice $\Gamma^{2,18}$ appearing in (2.2). As we shall see in Sec. III, due to the choice of Wilson line $\vec{V}$ described in Sec. II B, the $\Gamma^{2,18}$ lattice will be decomposed even further. For the time being, however, we will study more closely (3.6), which will already teach us some valuable lessons about the algebraic properties of $\mathcal{F}_1^{\text{analy}}$.

For example, one can show that the integral $\mathcal{F}_1^{\text{analy}}$ develops singularities at complex codimension one submanifolds of $SO(2,2+k)/(SO(2)\times SO(k))$, which coincide with the walls of the (complexified) fundamental Weyl chamber of the hyperbolic extension $\mathfrak{g}^{++}$ of the broken part $\mathfrak{g}$ of the gauge algebra $\mathfrak{e}_8 \oplus \mathfrak{e}_8$. As a consequence, the singularity behavior of the BPS-spectrum is controlled by the hyperbolic Weyl group $\mathcal{W}(\mathfrak{g}^{++})$, similarly as for the nonperturbative $1/4$ BPS dyon spectrum [26]. In order to better understand the underlying algebraic structure, we will now proceed to evaluate the integral $\mathcal{F}_1^{\rm analy}$ explicitly.

B. One-Loop integral for any choice of gauge group

The first step to explicitly perform the analytic one-loop integral (3.6) for arbitrary choices of the gauge group is to implement the splitting (2.5) at the level of the Siegel-Narain theta function $\Theta^{(2,18)}(\tau,\bar\tau,\mathbf{y})$. Since we are interested in the submanifold $\mathcal{M}_{2,2+k}$ of the moduli space, along which only some components of $\vec V$ are nonzero, we can use the same decomposition as in (2.11) of the lattice $\Gamma^{2,18}$ and write

$$\frac{G_2^{\rm analy}(\bar\tau)}{\bar\eta^{24}}\Theta^{(2,18)}(\tau,\bar\tau,\mathbf{y}) \equiv \sum_{\mu=0}^{s-1}\mathcal{P}_\mu^{(k)}(\bar\tau)\Theta_\mu^{(2,2+k)}(\tau,\bar\tau,\mathbf{y}). \quad (3.7)$$

Here, $\Theta_\mu^{(2,2+k)}(\tau,\bar\tau,\mathbf{y})$ is the theta function associated to the $\Gamma_{\mathfrak{g}}$ coset λ_μ (see (2.11)),

$$\Theta_\mu^{(2,2+k)}(\mathbf{y}) = \sum_{x\in\Gamma_{\mathfrak{g}}+\lambda_\mu^{\mathfrak{g}}} \bar q^{(1/2)\langle x|x\rangle} e^{2\pi\tau_2(|\langle x|u(\mathbf{y})\rangle|^2/(\Im\mathbf{y}|\Im\mathbf{y})^2)}, \quad (3.8)$$

while $\mathcal{P}_\mu^{(k)}(\bar\tau)$ captures the contributions from $G_2^{\rm analy}/\bar\eta^{24}$ and the theta constants from the different $\Lambda_{\mathfrak{h}}$ cosets

$$\mathcal{P}_\mu^{(k)}(\bar\tau) = \frac{G_2^{\rm analy}}{\bar\eta^{24}}\Theta_\mu^{\mathfrak{h}}(\bar\tau) = \frac{G_2^{\rm analy}}{\bar\eta^{24}}\sum_{\vec\ell\in\Lambda_{\mathfrak{h}}+\lambda_\mu^{\mathfrak{h}}} \bar q^{(1/2)\vec\ell\cdot\vec\ell}. \quad (3.9)$$

Since $\Lambda_{\mathfrak{h}}$ is a sublattice of $\mathfrak{e}_8\oplus\mathfrak{e}_8$ that is orthogonal to $\vec V$, $\mathcal{P}_\mu^{(k)}(\bar\tau)$ does not depend on the moduli $\mathbf{y} = (U, T; \vec V)$. In (3.8) and (3.9) μ labels the s conjugacy classes of the lattices as in (2.11), while $\lambda_\mu^{\mathfrak{g}}$ and $\lambda_\mu^{\mathfrak{h}}$ are the projections of λ_μ onto $\mathfrak{g}$ and $\mathfrak{h}$, respectively. We will parametrize the summation in (3.8) by $x = (m_1, n_1; m_2, n_2; \vec\ell)$ with $\vec\ell\in\Lambda_{\mathfrak{g}}+\lambda_\mu^{\mathfrak{g}}$, and similarly for (3.9).

Putting things together, the analytic integral can be written in the following form

$$\mathcal{F}_1^{\rm analy}(\mathbf{y}) = \int_{\mathbb{F}} d^2\tau\tau_2\sum_{\mu=0}^{s-1}\mathcal{P}_\mu^{(k)}(\bar\tau)\Theta_\mu^{(2,2+k)}(\tau,\bar\tau,\mathbf{y}). \quad (3.10)$$

An integral of this type has already been computed in [1] by splitting the integral in different orbits with respect to $SL(2,\mathbb{Z})$ (this method was first developed in [10] and further extended in [3,14,27–30]). Generalizing the result of [1] to the case of arbitrary gauge groups we arrive at the following explicit expression

$$\begin{aligned}\mathcal{F}_1^{\rm analy}(\mathbf{y}) = \sum_{\mu=0}^{s-1}\Bigg\{&\sum_{\vec\ell\in\Lambda_{\mathfrak{g}}+\lambda_\mu^{\mathfrak{g}}}\Bigg[\frac{2\pi Y}{3U_2}(c_\mu(0,\vec\ell) - 24c_\mu(-1,\vec\ell)) + 2\log|1-e^{2\pi i\vec\ell\odot\vec V}|^{c_\mu(0,\vec\ell)}\\ &+ 2\log\prod_{\substack{n',r\in\mathbb{Z}\\ r>0}}|1-e^{2\pi i(rT+n'U+\vec\ell\odot\vec V)}|^{c_\mu(n'r,\vec\ell)} + 2\log\prod_{n=1}^{\infty}|1-e^{2\pi i(nU+\vec\ell\odot\cdot\vec V)}|^{c_\mu(0,\vec\ell)}\Bigg]\\ &+ c_\mu(0,\vec 0)\left(\frac{\pi U_2}{3} - \ln Y + K\right) + 2\log\prod_{n=1}^{\infty}|1-e^{2\pi i nU}|^{c_\mu(0,0)}\Bigg\} + \frac{2U_2}{3\pi} + \frac{2\pi}{U_2}(\vec\ell\odot\Im\vec V)((\vec\ell\odot\Im\vec V)+U_2), \end{aligned} \quad (3.11)$$

where $K = \gamma_E - 1 - \ln\frac{8\pi}{3\sqrt3}$, with γ_E being the Euler-Mascheroni constant. Furthermore, we have introduced the shorthand notation for the modified scalar product: $\vec\ell\odot\vec V = \ell\cdot\Re\vec V + i|\vec\ell\cdot\Im\vec V|$. The coefficients $c_\mu(n'r,\vec\ell)$ arise from the Fourier expansion

$$\begin{aligned}\sum_{\mu=0}^{s-1}\mathcal{P}_\mu^{(k)}(\bar\tau)&\sum_{\vec\ell\in\Lambda_{\mathfrak{g}}+\lambda_\mu^{\mathfrak{g}}}\bar q^{(1/2)\vec\ell\cdot\vec\ell}e^{2\pi i\vec\ell\cdot\vec z}\\ &= \sum_{\mu=0}^{s-1}\sum_{n=-1}^{\infty}\sum_{\vec\ell\in\Lambda_{\mathfrak{g}}+\lambda_\mu^{\mathfrak{g}}} c_\mu(n,\vec\ell)\bar q^n e^{2\pi i\vec\ell\cdot\vec z}.\end{aligned} \quad (3.12)$$

For some simple examples, explicit expressions for $c_\mu(n,\vec\ell)$ will be given in Sec. IV.

By construction, (3.12) transforms as a weak Jacobi form under $SL(2,\mathbb{Z})$, and thus the coefficients $c_\mu(n,\vec\ell)$ only depend on $(n,\vec\ell)$ through the combination $(n-\frac12\vec\ell\cdot\vec\ell)$ [31]. Moreover, by inspection of (3.9) it is clear that the integrand in (3.10) has a simple pole at $\tau\to i\infty$, hence

$$c_\mu\left(n-\frac12\vec\ell\cdot\vec\ell\right) = 0 \quad \forall\ n-\frac12\vec\ell\cdot\vec\ell < -1. \quad (3.13)$$

In the following, we shall mainly be interested in the contribution of the trivial conjugacy class labeled by $\mu = 0$. For this, the only terms in the sum over $\vec\ell$ with $\vec\ell\neq\vec 0$ come from the degenerate orbit and have $\vec\ell\cdot\vec\ell = 2$. We can then choose to work in a chamber of the moduli space where $\Im\vec V\in\Lambda_{\mathfrak{g}}^+\otimes\mathbb{C}$, for which the condition $\vec\ell\cdot(\Im\vec V) > 0$ can be equivalently written as $\vec\ell\in\Lambda_{\mathfrak{g}}^+$.

C. Borcherds lift and denominator formula

In the following, we shall restrict the analysis to a particular part of the full analytic amplitude (3.11), corresponding to the contribution of the trivial conjugacy class $\mu = 0$. We will show that this may be identified with the infinite product side of the denominator formula for the Borcherds extension $\mathcal{G}(\mathfrak{g}^{++})$, where $\mathfrak{g}^{++}$ is the double extension of the unbroken gauge algebra $\mathfrak{g}$.[2] From this point of view, the reason for restricting to $\mu = 0$ becomes clear: only for the zero conjugacy class does the sum over $\vec{\ell}$ correspond to a sum over roots of $\mathfrak{g}$. Indeed, the higher conjugacy classes give rise to sums over weights of $\mathfrak{g}$ rather than roots.

1. Automorphic product

The zero conjugacy class contribution to (3.11) can be written as

$$\mathcal{F}_1^{\rm analy}(\mathbf{y})|_{\mu=0} = \log\|\Phi_{\mathfrak{g}}(\mathbf{y})\|^2 + c_0(0,\vec{0})\Big(\frac{\pi U_2}{3} - \ln Y + K\Big) + \ldots, \tag{3.14}$$

where we have defined

$$\Phi_{\mathfrak{g}}(\mathbf{y}) = e^{-2\pi i(\rho|\mathbf{y})} \prod_{(r,n';\vec{\ell})>0} (1 - e^{2\pi i(rT+n'U+\vec{\ell}\cdot\vec{V})})^{c_0(n'r,\vec{\ell})}, \tag{3.15}$$

and the norm $\|\cdot\|$ in (3.14) takes into account the contribution with $(r,n';\vec{\ell})<0$. We will give a more precise description of this norm below, once we have analyzed the modular properties of $\Phi_{\mathfrak{g}}(\mathbf{y})$ in detail. For the moment we just remark that the exact range of $(r,n';\vec{\ell})$ needs to be discussed separately for the cases when $\mathfrak{g}$ is simple or semisimple. Since this discussion is mostly technical and somewhat tedious, we have relegated it to Appendices B 1 and B 2 respectively. There we show that the product can be written to range over the elements $\alpha \in \Lambda^+_{\mathfrak{g}^{++}}$ with norm $\alpha^2 \leq 2$. Because of this, we can then write $\Phi_{\mathfrak{g}}(\mathbf{y})$ as the following product

$$\Phi_{\mathfrak{g}}(\mathbf{y}) = e^{-2\pi i(\rho|\mathbf{y})} \prod_{\alpha\in\Lambda^+_{\mathfrak{g}^{++}}} (1 - e^{2\pi i(\alpha|\mathbf{y})})^{c_0(-\alpha^2/2)}, \tag{3.16}$$

where we used $c_0(n) = 0$ for $n < -1$. The idea is now to identify $\Phi_{\mathfrak{g}}(\mathbf{y})$ with the denominator formula (A1) for a BKM algebra which we shall call $\mathcal{G}(\mathfrak{g}^{++})$ to indicate that it is an 'automorphic correction' (in the terminology of [16]) of $\mathfrak{g}^{++}$. Indeed, we would identify ρ as the Weyl-vector of $\mathcal{G}(\mathfrak{g}^{++})$ (see Appendix A) and the multiplicities of all (real and imaginary) roots of $\mathcal{G}(\mathfrak{g}^{++})$ could then be conveniently read off as the Fourier coefficients $c_0(n,\vec{\ell})$ as defined by the seed function

$$\psi_{\mathfrak{g}}(\bar{\tau},\vec{z}) = \sum_{n=-1}^{\infty}\sum_{\vec{\ell}\in\Lambda_{\mathfrak{g}}} c_0\Big(n - \frac{1}{2}\vec{\ell}\cdot\vec{\ell}\Big)\bar{q}^n e^{2\pi i\vec{\ell}\cdot\vec{z}}, \tag{3.17}$$

with Fourier coefficients arising from the zeroth conjugacy class $\mu = 0$ in (3.12). However, to justify the identification of $\Phi_{\mathfrak{g}}(\mathbf{y})$ with a denominator formula for $\mathcal{G}(\mathfrak{g}^{++})$, we must show that $\Phi_{\mathfrak{g}}(\mathbf{y})$ extends to an automorphic form on

$$\mathfrak{G}\backslash\mathcal{M}_{2,2+k} = \mathfrak{G}\backslash SO(2,2+k)/(SO(2)\times SO(2+k)), \tag{3.18}$$

for some discrete subgroup $\mathfrak{G} \subset SO(2,2+k)$. To show this, we note that although the seed function $\psi_{\mathfrak{g}}(\tau,\vec{z})$ in (3.17) no longer transforms nicely under the full mapping class group Γ of the original string worldsheet torus, it is nevertheless a weak Jacobi form with respect to a congruence subgroup $\Gamma_{[0]} \subset \Gamma$.[3] Realizing, moreover, that (3.10) has structurally the form of a "multiplicative" (Borcherds) lift, one might suspect that the modular properties of $\psi_{\mathfrak{g}}(\bar{\tau},\vec{z})$ with respect to $\Gamma_{[0]}$ directly translate into modular properties of $\Phi_{\mathfrak{g}}(\mathbf{y})$ with respect to some subgroup $\mathfrak{G}$ of the T-duality group $SO(2,2+k;\mathbb{Z})$. We shall now verify that this is indeed the case.

2. Theta correspondence and modular properties

As already mentioned, the integral representation (3.10) of the amplitude $\mathcal{F}_1^{\rm analy}$ provides an example of a so-called theta correspondence. Since this notion will play an important role in what follows, we begin this section with a brief review of the key features (see, e.g, [6,32,33] for more details).

Let (G_1, G_2) be a dual reductive pair of Lie groups in the sense of Howe [34]. This means that the product $G_1 \times G_2$ is a subgroup of (the universal cover of) a symplectic group $Sp(W)$, with W a symplectic vector space, such that G_1 (resp. G_2) is the centralizer of G_2 (respectively, G_1) inside $Sp(W)$. The standard example is when $G_1 = SL(2,\mathbb{R})$ and $G_2 = SO(m,n)$ such that $SL(2,\mathbb{R})\times SO(m,n) \subset Sp(2(m+n))$. Automorphic forms correspond to irreducible components in the decomposition of $L^2(G_1(\mathbb{Z})\backslash G_1)$ and $L^2(G_2(\mathbb{Z})\backslash G_2)$, where $(G_1(\mathbb{Z}), G_2(\mathbb{Z}))$ are discrete subgroups. In a nutshell, the theta correspondence is then an integral transform from automorphic representations of G_1 to automorphic representations of G_2. For the reductive pair $(SL(2,\mathbb{R}), SO(m,n))$, the kernel of this integral transform is a Siegel-Narain theta series $\Theta^{(m,n)}(\tau,\bar{\tau},\mathbf{y})$, as exemplified by (3.1) for $(m,n) = (6,22)$.

The purpose of this section is to determine the modular properties of the infinite product $\Phi_{\mathfrak{g}}(\mathbf{y})$ in (3.16). We shall do this by utilizing the theta correspondence outlined above. We thus seek an integral transform from a $\Gamma_{[0]} \subset SL(2,\mathbb{Z})$ modular form to an automorphic form for $\mathfrak{G} \subset SO(2,2+k;\mathbb{Z})$ which can be identified with $\Phi_{\mathfrak{g}}(\mathbf{y})$. To this end, we assume that $\Gamma_{[0]}$ has finite index N (we will see that this is indeed the case in all examples discussed in Sec IV),

[2]For details on double extensions of Lie algebras in this context see Appendix A of [1].

[3]This is, in fact, true for every individual μ in the right hand side of (3.7): each summand is a weak Jacobi form of zero weight under a particular congruence subgroup $\Gamma_{[\mu]} \subset \Gamma$. We notice,, in particular, that every single summand is invariant under the generator $T \in SL(2,\mathbb{Z})$ which acts as T: $\tau\tau + 1$.

for which we can choose coset representatives $\gamma_1, \ldots, \gamma_N$ such that Γ can be written as the disjoint union (see e.g. [35])

$$\Gamma = \gamma_1 \Gamma_{[0]} \cup \ldots \cup \gamma_N \Gamma_{[0]}. \tag{3.19}$$

We can then construct a fundamental domain of $\Gamma_{[0]}$ by

$$\mathbb{F}_{[0]} := \gamma_1 \mathbb{F} \cup \ldots \cup \gamma_N \mathbb{F} = \mathbb{H}/\Gamma_{[0]}, \tag{3.20}$$

where $\mathbb{F} = \mathbb{H}/\Gamma$ is the fundamental domain of Γ. Using this result, we can rewrite $\mathcal{F}_1^{\text{analy}}$ in the following manner

$$\mathcal{F}_1^{\text{analy}}(\mathbf{y}) = \frac{1}{N}\int_{\mathbb{F}_{[0]}} d^2\tau \tau_2 \mathcal{P}_0^{(k)}(\bar{\tau})\Theta_0^{(2,2+k)}(\tau, \bar{\tau}, \mathbf{y}) + \frac{1}{N}\int_{\mathbb{F}_{[0]}} d^2\tau\tau_2 \sum_{\mu=1}^{s-1} \mathcal{P}_\mu^{(k)}(\bar{\tau})\Theta_\mu^{(2,2+k)}(\tau, \bar{\tau}, \mathbf{y}). \tag{3.21}$$

Notice that this is indeed a consistent splitting of the integral, since the integrands of both terms separately are modular invariant under $\Gamma_{[0]}$ in the fundamental domain $\mathbb{F}_{[0]}$. We can now use similar methods as developed in [10] (and further extended in [3,14,28–30,36])[4] to evaluate the first term of (3.21) separately. To this end, we first perform a Poisson resummation to obtain

$$\int_{\mathbb{F}_{[0]}} d^2\tau\tau_2 \mathcal{P}_0^{(k)}(\bar{\tau})\Theta_0^{(2,2+k)}(\tau, \bar{\tau}, \mathbf{y}) = \int_{\mathbb{F}_{[0]}} \frac{d^2\tau}{\tau_2^2}\frac{Y}{U_2}\mathcal{P}_0^{(k)}(\bar{\tau}) \sum_{\substack{(p_1,n_1;p_2,n_2)\\ \vec{\ell}\in\Gamma_{\mathfrak{g}}}} \bar{q}^{(1/2)\vec{\ell}\cdot\vec{\ell}} e^{F(A,\mathbf{y})} \tag{3.22}$$

with the shorthand notation

$$F(A, \mathbf{y}) = 2\pi i \vec{\ell}\cdot\vec{z} - \frac{\pi Y}{U_2^2\tau_2}|\mathcal{A}|^2 - 2\pi i T \det A - \frac{\pi n_2(\vec{V}^2\tilde{\mathcal{A}} - \bar{\vec{V}}^2\mathcal{A})}{U_2} + \frac{2\pi i(\Im\vec{V})^2}{U_2^2}(n_1 + n_2\bar{U})\mathcal{A}$$

where $p_1, p_2, n_1, n_2 \in \mathbb{Z}$ such that $(p_1, n_1; p_2, n_2; \vec{\ell}) \neq (0, 0; 0, 0; \vec{0})$ and the matrices $(A, \mathcal{A}, \tilde{\mathcal{A}})$ are the same as in [1]. We have also used the shorthand expression $\vec{z} = \frac{i}{2U_2}(\vec{V}\tilde{\mathcal{A}} - \bar{\vec{V}}\mathcal{A})$. In this form, following [15,29,38,39], we can use modular invariance of the integrand under $\Gamma_{[0]}$, and trade a modular $\Gamma_{[0]}$-transformation $\tau \mapsto \frac{a\tau+b}{c\tau+d}$ for a transformation of the matrix A. This allows us to extend the domain of integration to images of $\mathbb{F}_{[0]}$ under $\Gamma_{[0]}$, while simultaneously restricting the summation over A to inequivalent $\Gamma_{[0]}$-orbits with an appropriate choice of representative matrices. To make this more precise, we use the result of [10] that a generic matrix A lies in exactly one out of three inequivalent $SL(2, \mathbb{Z})$ orbits, with representatives denoted by $A = 0$, A_0^{ND}, A_0^{D} which we take to be the same as in [1].

[4]See also e.g. [37] for a recent treatment of such integrals in the mathematics literature.

We now obtain

$$\begin{aligned}\int_{\mathbb{F}_{[0]}} d^2\tau\tau_2\mathcal{P}_0^{(k)}(\bar{\tau})\Theta_0^{(2,2+k)}(\tau, \bar{\tau}, \mathbf{y}) &= \int_{\mathbb{F}_{[0]}} \frac{d^2\tau}{\tau_2^2}\frac{Y}{U_2}\mathcal{P}_0^{(k)}(\bar{\tau})\sum_{\vec{\ell}\in\Gamma_{\mathfrak{g}}} \bar{q}^{(1/2)\vec{\ell}\cdot\vec{\ell}}e^{F(0,\mathbf{y})} \\ &+ \int_{\mathbb{F}_{[0]}} \frac{d^2\tau}{\tau_2^2}\frac{Y}{U_2}\mathcal{P}_0^{(k)}(\bar{\tau})\sum_{\substack{V\in\Gamma\\ \vec{\ell}\in\Gamma_{\mathfrak{g}}}} \bar{q}^{(1/2)\vec{\ell}\cdot\vec{\ell}}e^{F(A_0^{\text{ND}}V,\mathbf{y})} \\ &+ \int_{\mathbb{F}_{[0]}} \frac{d^2\tau}{\tau_2^2}\frac{Y}{U_2}\mathcal{P}_0^{(k)}(\bar{\tau})\sum_{\substack{V\in\Gamma\\ \vec{\ell}\in\Gamma_{\mathfrak{g}}}} \bar{q}^{(1/2)\vec{\ell}\cdot\vec{\ell}}e^{F(A_0^{\text{D}}V,\mathbf{y})}.\end{aligned} \tag{3.23}$$

In order to apply this result to the case of $\Gamma_{[0]} \subset \Gamma$, we note that (after choosing an appropriate representative matrix A_0) each of these Γ-orbits can be decomposed into several (inequivalent) $\Gamma_{[0]}$ orbits by writing

$$A = A_0 V = \sum_{i=1}^N A_0\gamma_i\hat{V}, \quad \text{with} \quad V \in \Gamma, \quad \hat{V} \in \Gamma_{[0]}. \tag{3.24}$$

Note moreover, that the integration over the fundamental domain of $\Gamma_{[0]}$ allows us to write

$$\begin{aligned}\int_{\mathbb{F}_{[0]}} d^2\tau\tau_2\mathcal{P}_0^{(k)}(\bar{\tau})\Theta_0^{(2,2+k)}(\tau, \bar{\tau}, \mathbf{y}) &= \sum_{i=1}^N\Bigg[\int_{\mathbb{F}_{[0]}} \frac{d^2\tau}{\tau_2^2}\frac{Y}{U_2}\mathcal{P}_0^{(k)}(\bar{\tau})\sum_{\vec{\ell}\in\Gamma_{\mathfrak{g}}} \bar{q}^{(1/2)\vec{\ell}\cdot\vec{\ell}}e^{F(0,\mathbf{y})} \\ &+ \int_{\mathbb{F}_{[0]}} \frac{d^2\tau}{\tau_2^2}\frac{Y}{U_2}\mathcal{P}_0^{(k)}(\bar{\tau})\sum_{\substack{\hat{V}\in\Gamma_{[0]}\\ \vec{\ell}\in\Gamma_{\mathfrak{g}}}} \bar{q}^{(1/2)\vec{\ell}\cdot\vec{\ell}}e^{F(A_0^{\text{ND}}\gamma_i\hat{V},\mathbf{y})} \\ &+ \int_{\mathbb{F}_{[0]}} \frac{d^2\tau}{\tau_2^2}\frac{Y}{U_2}\mathcal{P}_0^{(k)}(\bar{\tau})\sum_{\substack{\hat{V}\in\Gamma_{[0]}\\ \vec{\ell}\in\Gamma_{\mathfrak{g}}}} \bar{q}^{(1/2)\vec{\ell}\cdot\vec{\ell}}e^{F(A_0^{\text{D}}\gamma_i\hat{V},\mathbf{y})}\Bigg] \\ &:= \sum_{i=1}^N (I_0^{[0]}(\gamma_i) + I_{\text{ND}}^{[0]}(\gamma_i) + I_{\text{D}}^{[0]}(\gamma_i)).\end{aligned} \tag{3.25}$$

For simplicity, we can focus on the trivial coset representative γ_1 which by itself results in a well defined expression. Choosing the representative matrices A_0^{ND} and A_0^{D} in the same way as in [10] one can work out explicit expressions for the contributions to the individual orbits.

For the zero orbit $I_0^{[0]}(\gamma_1)$, one has $A = 0$ and the integral over $\mathbb{F}_{[0]}$ can be solved using standard methods due to modular covariance with respect to $\Gamma_{[0]}$, and the integral can be reduced to an integral over $\tau_1 \in [-1/2, 1/2)$ at $\tau_2 \to \infty$. This contribution, however, is the same as the $\mu = 0$ part of the zeroth orbit contribution to (3.11).

In the nondegenerate orbit, the representative A_0^{ND} can be parametrized in the standard way in terms of upper-triangular matrices with integer entries (r, j, p), satisfying $r > j \geq 0$ and $p \in \mathbb{Z}\backslash\{0\}$. The integration domain can be unfolded to the full upper half plane:

$$\mathcal{I}_{\mathrm{ND}}^{[0]}(\gamma_1) \sim \int_{\mathbb{H}} \frac{d^2\tau}{\tau_2^2} \frac{Y}{U_2} \mathcal{P}_0^{(k)} \sum_{\substack{p\neq 0 \\ r>j\geq 0 \\ \vec{b}\in\Gamma_{\mathfrak{g}}}} \bar{q}^{(1/2)\vec{b}\cdot\vec{b}} e^{F(A_0^{\mathrm{ND}},\mathbf{y})}.$$

Up to a factor of $1/N$ this yields the $\mu = 0$ contribution of nondegenerate part of (3.11).

Finally, we turn to the degenerate orbit. Realizing that transformations of the form T^m for integer m leave A_0 invariant provided we choose $c = 0$ and $d = 1$, we can further restrict the domain of integration to the semi-infinite strip $\mathbb{S}$ parametrized by $(\tau_1, \tau_2) \in [-1/2, 1/2) \times [0, \infty)$, such that we may write

$$\mathcal{I}_{\mathrm{D}}^{[0]}(\gamma_1) = \int_{\mathbb{S}} \frac{d^2\tau}{\tau_2^2} \frac{Y}{U_2} \mathcal{P}_0^{(k)} \sum_{\substack{j,p\in\mathbb{Z} \\ (j,p)\neq(0,0) \\ \vec{b}\in\Gamma_{\mathfrak{g}}}} \bar{q}^{(1/2)\vec{b}\cdot\vec{b}} e^{F(A_0^{\mathrm{D}},\mathbf{y})}.$$

Up to a factor of $1/N$, this is again the $\mu = 0$ contribution of the degenerate part of (3.11).

To conclude, we have found that (up to an irrelevant factor of $1/N$ which stems from the fact that there are exactly N inequivalent choices of the coset representatives $\gamma_1, \ldots, \gamma_N$) the zero conjugacy class contribution to (3.11) can be expressed as follows

$$\mathcal{F}_1^{\mathrm{analy}}(\mathbf{y})|_{\mu=0} \sim \mathcal{I}_0^{[0]}(\gamma_1) + \mathcal{I}_{\mathrm{D}}^{[0]}(\gamma_1) + \mathcal{I}_{\mathrm{ND}}^{[0]}(\gamma_1). \quad (3.26)$$

Hence, up to an overall N-dependent factor, we can write the infinite product $\Phi_{\mathfrak{g}}(\mathbf{y})$ in terms of an explicit integral theta lift:

$$\log\|\Phi_{\mathfrak{g}}(\mathbf{y})\|^2 \sim \int_{\mathbb{F}_{[0]}} d^2\tau \tau_2 \mathcal{P}_0^{(k)}(\bar{\tau}) \Theta_0^{(2,2+k)}(\tau, \bar{\tau}, \mathbf{y}) + \cdots \quad (3.27)$$

where the ellipsis represent irrelevant terms, c.f. (3.14).[5] This expression makes the modular properties of $\Phi_{\mathfrak{g}}(\mathbf{y})$ with respect to $\mathfrak{G} \subset SO(2, 2+k)$ manifest. Indeed, $\mathfrak{G}$ is generically only a subgroup of the full T-duality group $SO(2, 2+k; \mathbb{Z})$. More precisely, $\Phi_{\mathfrak{g}}(\mathbf{y})$ retains the invariance under lattice shifts $\mathbf{y} \to \mathbf{y} + v$, $v \in \Lambda_{\mathfrak{g}^{++}}$, and under $w \in SO(1, 1+k; \mathbb{Z})$, while the symmetry under $\mathcal{S} \in SO(2, 2+k; \mathbb{Z})$ (see Appendix A) is generically broken. These statements are consistent with Theorem 2.23 in [40] (which in turn builds upon earlier work by Borcherds [5,6]). Borcherds projects arising from lifts of Jacobi forms for $\Gamma_0(N)$ have also been constructed recently in [41]; it would be interesting to understand if there is a relation to our work.[6]

3. Denominator formula and automorphic correction

With the modular properties under $\mathfrak{G} \subset SO(2, 2+k)$ now manifest, we can indeed identify $\Phi_{\mathfrak{g}}(\mathbf{y})$ with the denominator formula of a new algebra $\mathcal{G}(\mathfrak{g}^{++})$. Thus, we can reinterpret the infinite product over $\Lambda_{\mathfrak{g}^{++}}^+$ in (3.16) as a product over the positive roots $\Delta_{\mathcal{G}}^+$ of $\mathcal{G}(\mathfrak{g}^{++})$

$$\Phi_{\mathfrak{g}}(\mathbf{y}) = e^{-2\pi i(\rho|\mathbf{y})} \prod_{\alpha\in\Delta_{\mathcal{G}}^+} (1 - e^{2\pi i(\alpha|\mathbf{y})})^{c_0(-\alpha^2/2)}. \quad (3.28)$$

Following [5,42], due to the appearance of simple imaginary roots, the automorphic correction $\mathcal{G}(\mathfrak{g}^{++})$ indeed falls in the class of generalized Kac-Moody algebras. While the multiplicity of all real positive roots is given by $c_0(-1) = 1$, the multiplicities of the imaginary roots are encoded via (3.16) in the remaining Fourier coefficients $c_0(-\alpha^2/2) = c_0(n'r - \vec{\ell}\cdot\vec{\ell}/2)$. The norm $\|\cdot\|^2$ in (3.14) can now also be interpreted as splitting the infinite product (3.16) into contributions from positive and negative roots. More abstractly and in view of its modular properties, $\Phi_{\mathfrak{g}}(\mathbf{y})$ can be interpreted as a meromorphic section of the line bundle $\mathcal{L} \to \mathfrak{G}\backslash\mathcal{M}_{2,2+k}$ of weight $c_0(0)/2$ modular forms on $\mathcal{M}_{2,2+k}$. In this language, the norm $\|\cdot\|$ corresponds to the invariant Petersson metric on $\mathcal{L}$ [5,6,32].

IV. EXPLICIT EXAMPLES

We shall now illustrate the general discussion of the previous sections by a few explicit examples for different choices of simple Lie algebras $\mathfrak{g}$. Our prime example will be the case $\mathfrak{g} = \mathfrak{a}_1$, which we will discuss in quite some detail. We will then proceed to study a list of further examples to show that our approach works in great generality. For many of these examples the decomposition of the full $\Gamma^{2,18}$ Siegel-Narain theta function for nontrivial $\vec{V}$ has already been considered previously in the literature using the so-called "sequential Higgs mechanism" [13,43]. Our method, however, is more flexible and allows a quick adaptation also for more general cases. One very particular class of examples corresponding to semisimple $\mathfrak{g}$, which have not previously been discussed in the literature, will be presented in Sec IV B.

A. Simple Sequence: $\mathfrak{g} = \mathfrak{a}_k$

The first series of examples we wish to study are $\mathfrak{g} = \mathfrak{a}_k$ with $k = 1, \ldots, 4$. We will be fairly explicit for the case $k = 1$, which acts to demonstrate the methods we have discussed in the previous sections, and will only state the relevant results in the other cases.

[5] As a side-remark we would like to comment that the choice of the representative γ_1 was merely due to convenience. Although we have not checked this explicitly, we expect that the contributions from each of the remaining terms in (3.25) in fact yield similar results.

[6] We thank Boris Pioline for pointing out this reference.

TABLE I. Fourier coefficients of (3.17) for various algebras $\mathfrak{g}$ which can be deduced using the work of [20,44].

$\mathfrak{g}$	Fourier coefficients $c_0(n-\frac{1}{2}\vec{\ell}\cdot\vec{\ell})$			
$\mathfrak{a}_2$	$c_0(-1)=1$	$c_0(0)=576$	$c_0(1)=110\,322$	$c_0(2)=8\,142\,848$
$\mathfrak{a}_3$	$c_0(-1)=1$	$c_0(0)=544$	$c_0(1)=94\,014$	$c_0(2)=5\,691\,200$
$\mathfrak{a}_4$	$c_0(-1)=1$	$c_0(0)=524$	$c_0(1)=83\,874$	$c_0(2)=4\,185\,500$

1. The case $\mathfrak{g}=\mathfrak{a}_1$

Our first example is the case $\mathfrak{g}=\mathfrak{a}_1$ such that the unbroken gauge algebra is $\mathfrak{h}=\mathfrak{e}_7\oplus\mathfrak{e}_8$. The Wilson line $\vec{V}$ is proportional to the single root of $\mathfrak{a}_1$ and we will call the coefficient V in the following. Following the work of [20,44] on theta series of Lie algebra lattices, we can immediately extract the relevant part of the integrand in (3.7):

$$\begin{aligned}&\mathcal{P}_0^{(\mathfrak{a}_1)}(\bar\tau)\Theta_0^{(\mathfrak{a}_1)}(\tau,\bar\tau,V)\\ &=\frac{E_4(\bar\tau)^2}{\eta(\bar\tau)^{24}}(\vartheta_3(2\bar\tau)^7+7\vartheta_3(2\bar\tau)^3\vartheta_2(2\bar\tau)^4)\vartheta_3(2V,2\bar\tau)\\ &:=\frac{E_4(\bar\tau)^2}{\eta(\bar\tau)^{24}}f(V,\bar\tau)=\sum_{n=-1}^{\infty}\sum_{\ell\in\mathbb{Z}}c_0(n,\ell)\bar q^n e^{2\pi i\ell V},\end{aligned}\tag{4.1}$$

where $f(V,\bar\tau)$ was defined by the last equality on the first line. Explicit evaluation yields the following values for the first few Fourier coefficients

$$\begin{aligned}&c_0(-1)=1\qquad c_0(0)=630\\ &c_0(1)=138024\qquad c_0(2)=9987360.\end{aligned}\tag{4.2}$$

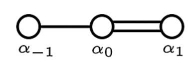
If we want to use (4.1) to define a Borcherds extension of $\mathfrak{a}_1^{++}$ we first need to show that it transforms well under (a congruence subgroup of) $SL(2,\mathbb{Z})$. We indeed claim that (4.1) transforms as a weak Jacobi form of weight 0 under $\Gamma_0(4)$ (see Appendix C). To see this, we first notice that the overall factor $\bar{E}_4^2/\bar\eta^{24}$ transforms with weight -4 under the full $SL(2,\mathbb{Z})$. Thus, all we have to consider are the modular properties of the function $f(V,\bar\tau)$ defined in (4.1). We can make the latter manifest by expanding the combination of theta series in a basis of modular forms of $\Gamma_0(4)$ (for details of the notation see Appendix C)

$$\begin{aligned}f(V,\bar\tau)=&\left(\frac{2E_4(\bar\tau)}{45}-\frac{E_4(2\bar\tau)}{20}+\frac{4E_4(4\bar\tau)}{45}\right)\phi_{0,1}(\bar\tau,V)\\ &-\left(\frac{8E_6(\bar\tau)}{189}-\frac{11E_6(2\bar\tau)}{252}+\frac{16E_6(4\bar\tau)}{189}\right.\\ &\left.+26h_6(\bar\tau)\right)\phi_{-2,1}(\bar\tau,V),\end{aligned}\tag{4.3}$$

which indeed proves that (4.1) is invariant under $\Gamma_0(4)$. This implies that the infinite product $\Phi_{\mathfrak{a}_1}(\mathbf{y})$, which is computed from (3.15) by inserting the coefficients (4.2), is also automorphic with respect to a finite index subgroup $\mathfrak{G}\subset SO(2,3;\mathbb{Z})$ which is induced by $\Gamma_0(4)$ through the theta correspondence [40] (see also section 13 of [5] for a discussion of modular products induced from modular forms for $\Gamma_0(N)$). Therefore, as explained before, $\Phi_{\mathfrak{a}_1}(\mathbf{y})$ defines the denominator formula for a BKM algebra $\mathcal{G}(\mathfrak{a}_1^{++})$, i.e.

$$\Phi_{\mathfrak{a}_1}(\mathbf{y})=e^{-2\pi i(\rho|\mathbf{y})}\prod_{\alpha\in\Delta^+_{\mathcal{G}(\mathfrak{a}_1^{++})}}(1-e^{2\pi i(\alpha|\mathbf{y})})^{c_0(-(\alpha|\alpha)/2)}.\tag{4.4}$$

The root multiplicities of $\mathcal{G}(\mathfrak{a}_1^{++})$ are simply given by $\mathrm{mult}(\alpha)=c_0(-\alpha^2/2)=c_0(n'r-\frac{1}{2}\vec{\ell}\cdot\vec{\ell})$, encoded in (4.2). In particular, as we can read off, the simple positive roots all have squared length 2, and thus appear with multiplicity $c_0(-1)=1$. The corresponding hyperbolic subalgebra $\mathfrak{a}_k^{++}$ is characterized by the 3×3 Cartan matrix whose Dynkin diagram is:



As a final comment, we would like to remark that $\mathcal{G}(\mathfrak{a}_1^{++})$ constructed here differs from the automorphic completion $\mathfrak{g}_{1,0}$ of $\mathfrak{a}_1^{++}$ considered in [16] since we have not added any odd roots; in other words, $\mathcal{G}(\mathfrak{a}_1^{++})$ is not a ''super BKM algebra'', in contrast to $\mathfrak{g}_{1,0}$.

2. The cases $\mathfrak{g}=\mathfrak{a}_2,\mathfrak{a}_3,\mathfrak{a}_4$

Let us now also briefly sketch the remaining members of this series of Lie algebras, i.e. the examples $\mathfrak{g}=\mathfrak{a}_k\cong\mathfrak{sl}(k+1,\mathbb{R})$ for $k=2,3,4$ with the algebras $\mathfrak{h}$ given by $\mathfrak{e}_6\oplus\mathfrak{e}_8$, $\mathfrak{d}_5\oplus\mathfrak{e}_8$, and $\mathfrak{a}_4\oplus\mathfrak{e}_8$ respectively. Furthermore, we will use the notation $y=(U,T;V_{i(k)})$.[7] We are again interested in the contribution of the zero conjugacy class in the integrand (3.7). The latter can be derived in a straightforward manner using the results of [20,44,45] for the theta series of the root lattices of $\mathfrak{a}_k$, $\mathfrak{d}_5$ and $\mathfrak{e}_6$.

Indeed, with this information, we can immediately compute the Fourier coefficients c_μ, introduced in (3.12). For the reader's convenience, we have compiled the first few of them in Table I. The explicit expressions of the theta series also imply modular invariance of the $\mu=0$ contribution under some particular congruence subgroup of $SL(2,\mathbb{Z})$, which is again necessary for an interpretation as the

[7]In order to avoid cluttering the notation, we will from now on denote $V_{i(k)}$ simply by V_i.

TABLE II. Fourier coefficients $c_0(n)$ for various algebras $\mathfrak{g} = \mathfrak{g}_{(1)} \oplus \mathfrak{g}_{(2)}$. Note that the coefficients are symmetric under the exchange of $\mathfrak{g}_{(1)}$ and $\mathfrak{g}_{(2)}$.

	n	$\mathfrak{a}_1$	$\mathfrak{a}_2$	$\mathfrak{a}_3$	$\mathfrak{a}_4$
$\mathfrak{a}_1$	-1	1	1	1	1
	0	516	462	430	410
	1	92 160	70 614	57 954	50 094
	2	7 002 096	4 528 948	3 105 820	2 236 960
$\mathfrak{a}_2$	-1		1	1	1
	0		408	376	356
	1		51 984	41 052	34 272
	2		2 878 112	1 936 448	1 365 668
$\mathfrak{a}_3$	-1			1	1
	0			344	324
	1			31 144	25 004
	2			1 276 640	880 340
$\mathfrak{a}_3$	-1				1
	0				304
	1				19 264
	2				592 040

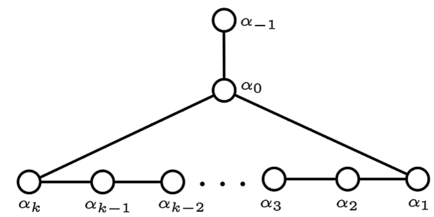
denominator formula for a BKM algebra. To be specific, the groups are given in the following table:

group	$\mathfrak{a}_1$	$\mathfrak{a}_2$	$\mathfrak{a}_3$	$\mathfrak{a}_4$
$\Gamma_{[0]}$	$\Gamma_0(4)$	$\Gamma_0(6)$	$\Gamma_0(8)$	$\Gamma_0(10)$

As explained before, this implies that $\Phi_{\mathfrak{a}_k}(\mathbf{y})$, which is defined using the coefficients of Table I, has good modular properties under a finite index subgroup $\mathfrak{G} \subset SO(2, 2+k; \mathbb{Z})$ and can be interpreted as the denominator formula for the BKM algebra $\mathcal{G}(\mathfrak{a}_k^{++})$

$$\Phi_{\mathfrak{a}_k}(\mathbf{y}) = e^{-2\pi i(\rho|\mathbf{y})} \prod_{\alpha\in\Delta^+_{\mathcal{G}(\mathfrak{a}_k^{++})}} (1 - e^{2\pi i(\alpha|\mathbf{y})})^{c_0(-(\alpha|\alpha)/2)}. \quad (4.6)$$

Thus the root multiplicities of $\mathcal{G}(\mathfrak{a}_k^{++})$ are simply given by the Fourier coefficients in Table I. In particular, the simple positive roots all have length 2, and thus appear with multiplicity $c_0(-1) = 1$. The corresponding hyperbolic subalgebra $\mathfrak{a}_k^{++}$ is characterized by the $(k+2) \times (k+2)$ Cartan matrix whose Dynkin diagram is of the form



B. Semisimple Sequence: $\mathfrak{g} = \mathfrak{a}_{k_1} \oplus \mathfrak{a}_{k_2}$

We also want to discuss examples in which $\mathfrak{g}$ is no longer a simple group. As an illustrative series of examples, let us consider the case where

$$\mathfrak{g} = \mathfrak{a}_{k_1} \oplus \mathfrak{a}_{k_2}$$
$$\mathfrak{h} = \mathfrak{h}_{k_1} \oplus \mathfrak{h}_{k_2} \quad \text{with} \quad \mathfrak{h}_{k_i} = \begin{cases} \mathfrak{e}_7 & \text{if } k_i = 1 \\ \mathfrak{e}_6 & \text{if } k_i = 2 \\ \mathfrak{d}_5 & \text{if } k_i = 3 \\ \mathfrak{a}_4 & \text{if } k_i = 4 \end{cases}$$
$$(i = 1, 2). \quad (4.8)$$

In order to be able to make use of the results of Sec. III B, we need to find the equivalent of the Fourier coefficients introduced in (3.12). To this end, we perform a Poisson resummation, after which we can write for the integral the following sum over conjugacy classes

$$\mathcal{F}_1^{\text{analy}}(\mathbf{y}) = \int_{\mathbb{F}} \frac{d^2\tau}{\tau_2^2} \frac{Y}{U_2} \sum_{\mu=0}^{s-1} \frac{G_2^{\text{analy}}}{\bar{\eta}^{24}} \sum_{\mu=0}^{s-1} \Theta_\mu^{\mathfrak{h}_{k_1}}(\bar{\tau}) \Theta_\mu^{\mathfrak{h}_{k_2}}(\bar{\tau}) \sum_{(p_1,n_1;p_2,n_2)} \sum_{\substack{\vec{\ell}_1\in\Lambda_1+\lambda_\mu^1 \\ \vec{\ell}_2\in\Lambda_2+\lambda_\mu^2}} \bar{q}^{(1/2)(\vec{\ell}_1\cdot\vec{\ell}_1+\vec{\ell}_2\cdot\vec{\ell}_2)}$$
$$\cdot\, e^{2\pi i\vec{\ell}_1\cdot\vec{z}_1+2\pi i\vec{\ell}_2\cdot\vec{z}_2-(\pi Y/U_2^2\tau_2)|\mathcal{A}|^2-2\pi iT\det A-(\pi n_2(\vec{V}^2\tilde{\mathcal{A}}-\vec{\bar{V}}^2\mathcal{A})/U_2)+(2\pi i(\Im\vec{V})^2/U_2^2)(n_1+n_2\bar{U})\mathcal{A}}. \quad (4.9)$$

Here, λ_μ^1 and λ_μ^2 are the projections of the glue vector on the root lattices Λ_1 and Λ_2 of $\mathfrak{a}_{k_1}$ and $\mathfrak{a}_{k_2}$, respectively, while $\Theta_\mu^{\mathfrak{h}_{k_{1,2}}}(\bar{\tau})$ are the theta series of the various $\Lambda_{\mathfrak{h}_{k_{1,2}}}$ cosets. The latter are obtained from the projections $\lambda_\mu^{\mathfrak{h}_{1,2}}$ of the glue vector λ_μ onto $\mathfrak{h}_{k_1}$ and $\mathfrak{h}_{k_2}$, respectively

$$\Theta_\mu^{\mathfrak{h}_{k_a}}(\bar{\tau}) = \sum_{\vec{\ell}\in\Lambda_{\mathfrak{h}_{k_a}}+\lambda_\mu^{\mathfrak{h}_{k_a}}} \bar{q}^{(1/2)\vec{\ell}\cdot\vec{\ell}}, \quad \forall\ a = 1, 2. \quad (4.10)$$

As before, $\mu = 0, \ldots, s-1$ labels the various conjugacy classes. The Fourier expansion (3.12) for the case at hand can then be written more explicitly as

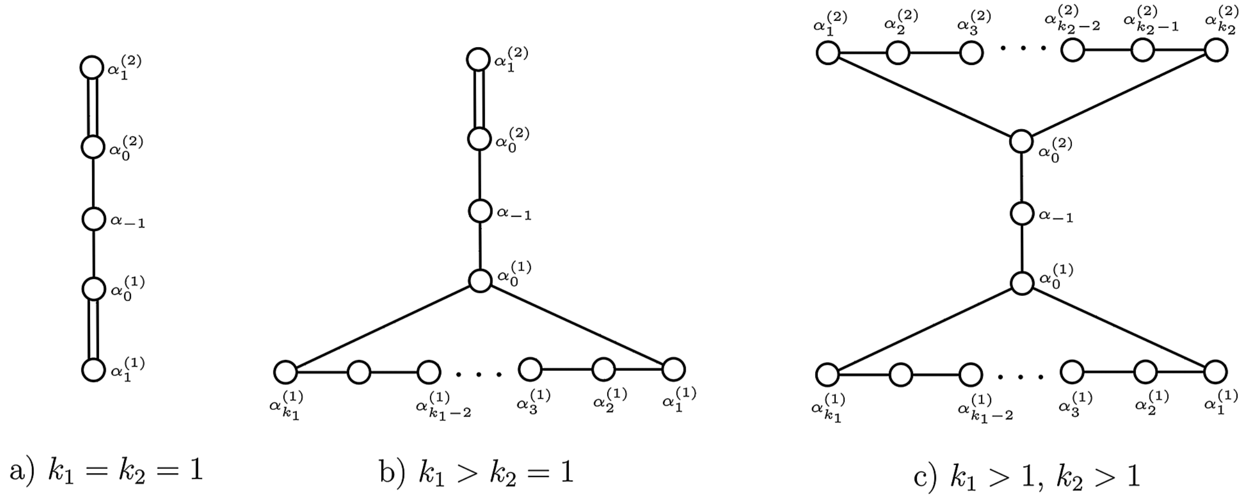


FIG. 1. Dynkin diagrams of the double extensions $(\mathfrak{a}_{k_1} \oplus \mathfrak{a}_{k_2})^{++}$ for various (k_1, k_2).

$$\frac{G_2^{\mathrm{analy}}}{\bar{\eta}^{24}} \sum_{\mu=0}^{s-1} \Theta_\mu^{\mathfrak{h}_{k_1}}(\bar{\tau}) \Theta_\mu^{\mathfrak{h}_{k_2}}(\bar{\tau}) \sum_{(p_1,n_1;p_2,n_2)} \sum_{\substack{\vec{\ell}_1 \in \Lambda_1 + \lambda_\mu^1 \\ \vec{\ell}_2 \in \Lambda_2 + \lambda_\mu^2}} \bar{q}^{(1/2)(\vec{\ell}_1\cdot\vec{\ell}_1 + \vec{\ell}_2\cdot\vec{\ell}_2)} e^{2\pi i \vec{\ell}_1\cdot\vec{z}_1 + 2\pi i \vec{\ell}_2\cdot\vec{z}_2}$$
$$= \sum_{\mu=0}^{s-1} \sum_{n=-1}^{\infty} \sum_{\substack{\vec{\ell}_1 \in \Lambda_1 + \lambda_\mu^1 \\ \vec{\ell}_2 \in \Lambda_2 + \lambda_\mu^2}} c_\mu\left[n - \frac{1}{2}(\vec{\ell}_1 \cdot \vec{\ell}_1 + \vec{\ell}_2 \cdot \vec{\ell}_2)\right] \bar{q}^n e^{2\pi i(\vec{\ell}_1\cdot\vec{z}_1 + \vec{\ell}_2\cdot\vec{z}_2)}. \tag{4.11}$$

The modular properties, particularly of the $\mu = 0$ contribution, follow this time already from our analysis of Sec. IVA 1, and we can thus immediately proceed to interpretation in terms of a BKM algebra. According to Sec. III C, the important information about root multiplicities (i.e. the denominator formula) of $\mathcal{G}(\mathfrak{g}^{++})$ is encoded in the Fourier coefficients of the trivial conjugacy class $\mu = 0$. We have tabulated the first few such coefficients for different choices of k_1 and k_2 in Table II. These coefficients are sufficient to obtain the full denominator formula from (3.14). Let us consider this result also from a more algebraic perspective, i.e. from the point of view of Eq. (3.16). The $(2 + k_1 + k_2) \times (2 + k_1 + k_2)$ Cartan matrix is encoded in the Dynkin diagrams in Fig. 1. This matrix has a single zero eigenvalue, which implies that the associated simple roots are not all linearly independent. Indeed one can check that

$$\alpha_0^{(2)} = \alpha_0^{(1)} - \sum_{i=1}^{k_1} \alpha_i^{(1)} + \sum_{m=1}^{k_2} \alpha_m^{(2)}. \tag{4.12}$$

Following the general discussion of Sec. III C, the Fourier coefficients c_0 will correspond to the root multiplicities of the algebra[8] $\mathcal{G}((\mathfrak{a}_{k_1} \oplus \mathfrak{a}_{k_2})^{++})$. As required, the simple positive roots all have length 2 and therefore have multiplicity $c_0(-1) = 1$.

[8]Our conventions for double extensions of semisimple Lie algebras follow the philosophy of [46]; see also Appendix A of [1] for details on our precise conventions.

V. CONCLUSIONS AND DISCUSSION

In this work, we have analyzed a particular $\mathcal{N} = 4$ topological one-loop amplitude $\mathcal{F}_1$ in heterotic string theory on $\mathbb{T}^2$. We evaluated the integral $\mathcal{F}_1$ explicitly for arbitrary enhanced semisimple gauge group $\mathfrak{h} \subset \mathfrak{e}_8 \oplus \mathfrak{e}_8$, i.e. for any choice of Wilson lines. The analytic part $\mathcal{F}_1^{\mathrm{analy}}(\mathbf{y})$ can be written in terms of an infinite product over a Lorentzian lattice, identified with the root lattice of the Lorentzian extension $\mathfrak{g}^{++}$ of the complement $\mathfrak{g} = (\mathfrak{e}_8 \oplus \mathfrak{e}_8)/\mathfrak{h}$. Using the Borcherds-Gritsenko-Nikulin philosophy of "automorphic correction", this gives rise to a class of Borcherds algebras $\mathcal{G}(\mathfrak{g}^{++})$, of which the root multiplicities are explicitly calculable in terms of the Fourier coefficients of certain modular forms.

As a by-product of our analysis, we have provided explicit expressions for this class of one-loop integrals in heterotic string theory on $\mathbb{T}^2$ for an arbitrary breaking of the gauge group. In particular, our method does not require the factors $\mathfrak{h}$ and $\mathfrak{g}$ in (2.5) to be simple. These results generalize previous work, which had been restricted to specific choices of Wilson lines, notably always keeping one of the $\mathfrak{e}_8$-factors unbroken (see for instance [3,13,14]).

The present work arose as a continuation of our previous analysis [1], where a certain universal "algebra of BPS states'' $\mathcal{G}$ for heterotic string theory on $\mathbb{T}^2$ was constructed using an auxiliary bosonic conformal field theory. It is an interesting open question whether there is a similar ''microscopic'' CFT construction of the class of ''automorphically corrected'' Borcherds algebras $\mathcal{G}(\mathfrak{g}^{++})$ uncovered herein. Although these algebras appear not to be subalgebras of the BPS-algebra of [1], it is conceivable that they can be obtained as quotients of $\mathcal{G}$.

A natural extension of our analysis would be to go away from the large-volume limit of $\mathbb{T}^4$ and consider the full $\mathcal{N}=4$ amplitude on $\mathbb{T}^6$ for which the Narain moduli space is enlarged to $SO(6,22;\mathbb{Z})\backslash\mathcal{M}_{6,22}$. Since this is no longer a Hermitian symmetric domain, one might recover a complex structure by treating the harmonic superspace amplitude $\mathcal{F}_g(\mathbf{y},u,\bar{u})$ as an automorphic function on the extended moduli space $\mathcal{M}_{6,22}\times SU(4)/(SU(2)\times SU(2)\times U(1))\cong SO(6,22)/(SO(4)\times SO(2)\times SO(22))$, similarly to the twistor space construction of [47].[9] This point of view might also shed light on the geometric meaning of the harmonicity and second order equations satisfied by $\mathcal{F}_g(\mathbf{y},u,\bar{u})$.

ACKNOWLEDGMENTS

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APPENDIX A: WEYL VECTORS AND DENOMINATOR FORMULAS

Our conventions for Lie algebras and Borcherds-Kac-Moody algebras can be found in Appendix A of [1]. Below, we just briefly recall some of the essential features which are needed for the present analysis. We denote by $\mathfrak{g}$ a finite Lie algebra, $\mathfrak{g}^{++}$ its Lorentzian extension, and by $\mathcal{G}$ a general BKM algebra.

Similarly, as for finite Lie algebras, BKM algebras have a Weyl vector ρ, satisfying $(\rho|\alpha)\leq -\frac{1}{2}(\alpha|\alpha)$, with equality if and only if α is a simple root. By restricting the general Weyl-Kac-Borcherds character formula to the trivial representation one obtains the so-called denominator formula

$$\sum_{w\in\mathcal{W}}\epsilon(w)w(S)=e^{\rho}\prod_{\alpha\in\Delta_+}(1-e^{\alpha})^{\mathrm{mult}\,\alpha}. \tag{A1}$$

[9] We thank Boris Pioline for suggesting this possibility.

This formula relates a sum over the Weyl group $\mathcal{W}(\mathcal{G})$ to an infinite product over all positive roots Δ_+ of $\mathcal{G}$. The factor $S(w)$ is a correction due to the imaginary simple roots [42]:

$$S=e^{\lambda+\rho}\sum_{\alpha\in\Lambda^+_{\mathcal{G}}}\xi(\alpha)e^{\alpha}, \tag{A2}$$

where $\xi(\alpha)=(-1)^m$ if α is a sum of m distinct pairwise orthogonal imaginary simple roots which are orthogonal to λ, and $\xi(\alpha)=0$ otherwise.

A key point of this paper is the fact that BKM algebras $\mathcal{G}$ can be constructed from Lorentzian Kac-Moody algebras $\mathfrak{g}^{++}$ through a so-called automorphic correction [5,17], as we now recall.

Suppose we are given a weak Jacobi form $\psi(\tau,z)$ with expansion coefficients

$$\psi_{\mathfrak{g}}(\tau,z)=\sum_{\lambda\in\Lambda_{\mathfrak{g}^{++}}}c(\lambda)q^{-(1/2)(\lambda|\lambda)}e^{2\pi i(z|\lambda)}, \tag{A3}$$

where $\Lambda_{\mathfrak{g}^{++}}$ is the root lattice of $\mathfrak{g}^{++}$. Then we consider the modular product [5]

$$\Phi_{\mathfrak{g}}(\mathbf{y})=e^{-2\pi i(\rho|\mathbf{y})}\prod_{\lambda\in\Lambda^+_{\mathfrak{g}^{++}}}(1-e^{-2\pi i(\lambda|\mathbf{y})})^{c(\lambda)},$$
$$\mathbf{y}\in\Lambda_{\mathfrak{g}^{++}}\otimes\mathbb{C}, \tag{A4}$$

where ρ is the lattice Weyl-vector of $\mathfrak{g}^{++}$. Upon identifying (A4) with (A1) we interpret the additional terms in (A4) with additional roots—beyond those already in $\Delta^+_{\mathfrak{g}^{++}}$. Because of the crucial minus sign in the exponent of q in (A3), these additional roots are generically imaginary.[10] It was shown in [5,42] that there exists indeed a BKM $\mathcal{G}$ with these roots. In order to emphasize that the latter was constructed from $\mathfrak{g}^{++}$, we will in many cases write $\mathcal{G}\equiv\mathcal{G}(\mathfrak{g}^{++})$. It is, however, important to realize that the extension of $\mathfrak{g}^{++}$ is not unique, since different modular products will lead to different algebras $\mathcal{G}$.

By construction, the product $\Phi_{\mathfrak{g}}(\mathbf{y})$ is an automorphic function for $SO(1,1+k;\mathbb{Z})$. However, Borcherds shows [5] that in fact $\Phi_{\mathfrak{g}}(\mathbf{y})$ extends to an automorphic form of weight $c(0)/2$ for the full T-duality group $SO(2,2+k;\mathbb{Z})$. To be precise, it is invariant under shifts

$$\Phi_{\mathfrak{g}}(\mathbf{y}+v)=\Phi_{\mathfrak{g}}(\mathbf{y}),\quad \text{with}\quad v\in\Lambda_{\mathfrak{g}^{++}}, \tag{A5}$$

and arbitrary transformations under $SO(1,1+k;\mathbb{Z})$ (maybe even extended by a nontrivial multiplier system, see e.g. [5])

$$\Phi_{\mathfrak{g}}(w(\mathbf{y}))=\Phi_{\mathfrak{g}}(\mathbf{y}),\quad \text{with}\quad w\in SO(1,1+k;\mathbb{Z}). \tag{A6}$$

However, it transforms with weight $c(0)/2$ under the following transformation

[10] We do not consider the case in this paper that also additional simple real roots are added in this way. See [48] for examples where this happens.

$$\Phi_{\mathfrak{g}}(\mathcal{S}(\mathbf{y})) = \left[\frac{(\mathbf{y}|\mathbf{y})}{2}\right]^{c(0)/2} \Phi_{\mathfrak{g}}(\mathbf{y}), \quad \text{with} \\ SO(2,2+k;\mathbb{Z}) \ni \mathcal{S}: \ \mapsto \mathbf{y}\frac{2\mathbf{y}}{(\mathbf{y}|\mathbf{y})}. \tag{A7}$$

More generally, if $\Phi_{\mathfrak{g}}(\mathbf{y})$ is a modular form for a subgroup $\mathfrak{G} \subset SO(2, 2+k; \mathbb{Z})$, the weight under the corresponding $\mathcal{S}$-transformation is given by a character of $\mathfrak{G}$ (see, e.g. [40]).

APPENDIX B: POSITIVE ROOT CONDITION

1. Proof of positive root condition: simple $\mathfrak{g}$

When $\mathfrak{g}$ is simple, the range of the product $(r, n'; \vec{\ell}) > 0$ in (3.15) is defined by

$$n'r - \frac{1}{2}\vec{\ell}\cdot\vec{\ell} \geq -1, \quad \text{and} \\ \begin{cases} r > 0, n' \in \mathbb{Z}, & \vec{\ell} \in \Lambda_{\mathfrak{g}} \quad \text{or} \\ r = 0, n' > 0, & \vec{\ell} \in \Lambda_{\mathfrak{g}} \quad \text{or} \\ r = n' = 0, & \vec{\ell} \in \Lambda_{\mathfrak{g}}^{+}. \end{cases} \tag{B1}$$

As mentioned above, the norm $\|\cdot\|^2$ in (3.14) takes into account that there are contributions with $(r, n'; \vec{\ell}) > 0$ and contributions with $(r, n'; \vec{\ell}) < 0$. It remains to show that (B1) are the conditions that characterize the elements of $\Lambda^{+}_{\mathfrak{g}^{++}}$ with $\alpha^2 \leq 2$. Let us work with the set of simple roots α_I of $\mathfrak{g}^{++}$ in the following basis

$$\alpha_{-1} = (1, -1; \vec{0}) \tag{B2}$$

$$\alpha_0 = (-1, 0; -\vec{\theta}) \tag{B3}$$

$$\alpha_i = (0, 0; \vec{e}_i), \qquad i = 1, \ldots, k, \tag{B4}$$

where $\vec{\theta}$ is the highest root of $\mathfrak{g}$ (which exists for every simple Lie algebra), and $\vec{e}_i$, $i = 1, \ldots, k$ are the simple roots of $\mathfrak{g}$. An arbitrary positive root of $\mathfrak{g}^{++}$ may be written as a linear combination of the simple roots α_{-1}, α_0, α_i

$$\alpha = \sum_{I=-1}^{k} x_I \alpha_I \in \Lambda^{+}_{\mathfrak{g}^{++}}, \quad \text{with} \quad x_I \in \mathbb{Z}_+, \tag{B5}$$

where $\mathbb{Z}_+$ denotes the non-negative integers. Using the definition of the inner product (2.15), we find that the scalar product of α with $\mathbf{y}$ is given by

$$(\alpha|\mathbf{y}) = x_{-1}T + (x_0 - x_{-1})U + (x_i\vec{e}_i - x_0\vec{\theta})\cdot\vec{V}. \tag{B6}$$

Since the exponent in (3.16) is $c_0(-\alpha^2/2)$, it is natural to identify

$$x_{-1} = r, \quad x_0 = n' + r, \quad x_i\vec{e}_i = \vec{\ell} + (n'+r)\vec{\theta}. \tag{B7}$$

Contracting the last identity with the fundamental weights $\vec{f}^i$ of $\mathfrak{g}$, we can write the coefficients x_i as $x_i = \vec{\ell}\cdot\vec{f}^i + (n'+r)\vec{\theta}\cdot\vec{f}^i$. The proof then reduces to a case-by-case analysis. For example, if $r > 0$, we obviously have $x_{-1} > 0$, but then in order for $n'r - \frac{1}{2}\vec{\ell}\cdot\vec{\ell} \geq -1$, we need that $n' \geq -1$, thus leading to $x_0 \geq 0$. In order to understand the condition for x_i we consider the different possibilities for n' separately. If $n' = -1$, then $r = 1$ and $\vec{\ell} = \vec{0}$, and thus $x_i = 0$. Similarly, for $n' = 0$, $\vec{\ell}\cdot\vec{\ell} \leq 2$, which means that either $\vec{\ell}$ is a root of $\mathfrak{g}$ or $\vec{\ell} = \vec{0}$. In the latter case it follows immediately that $x_i \geq 0$, while in the former case

$$x_i = \vec{\ell}\cdot\vec{f}^i + r\vec{\theta}\cdot\vec{f}^i \geq (r-1)\vec{\theta}\cdot\vec{f}^i \geq 0. \tag{B8}$$

Finally, for $n' \geq 1$ we use the Cauchy-Schwarz inequality, following a similar discussion in [4], to conclude that

$$|\vec{\ell}\cdot\vec{f}^i|^2 \leq (\vec{f}^i\cdot\vec{f}^i)(\vec{\ell}\cdot\vec{\ell}) \leq (\vec{f}^i\cdot\vec{f}^i)(2 + 2n'r) \\ \leq (n'+r)^2(\vec{f}^i\cdot\theta)^2. \tag{B9}$$

Since $\vec{\theta}\cdot\vec{f}^i \geq 0$ for all fundamental weights, it then follows that also $x_i \geq 0$. The other cases work similarly, and it follows that (B1) characterizes indeed the elements of $\Lambda^{+}_{\mathfrak{g}^{++}}$ with $\alpha^2 \leq 2$.

2. Proof of positive root condition: semisimple $\mathfrak{g}$

Let us now repeat the discussion for the case that the broken gauge group $\mathfrak{g}$ is semisimple. For simplicity of presentation, we shall restrict to the case when $\mathfrak{g}$ decomposes into a sum of two simple factors, $\mathfrak{g} = \mathfrak{g}_{(1)} \oplus \mathfrak{g}_{(2)}$, of rank k_1 and k_2, respectively. The generalization to more factors is straightforward. It is still possible to write the integral in terms of an infinite product (3.14), but now the condition on $(r, n'; \vec{\ell})$ is replaced by the conditions

$$n'r - \frac{1}{2}\vec{\ell}\cdot\vec{\ell} \geq -1 \quad \text{and either} \\ \begin{cases} r > 0, n' \in \mathbb{Z}, & \vec{\ell}_{(1)} \in \Lambda_{\mathfrak{g}_{(1)}}\vec{\ell}_{(2)} \in \Lambda_{\mathfrak{g}_{(2)}} \\ r = 0, n' > 0, & \vec{\ell}_{(1)} \in \Lambda_{\mathfrak{g}_{(1)}}, \vec{\ell}_{(2)} \in \Lambda_{\mathfrak{g}_{(2)}}, \\ r = n' = 0, & \vec{\ell}\cdot\Im\vec{V} > 0, \end{cases} \tag{B10}$$

where $\vec{\ell}\cdot\vec{\ell} = \vec{\ell}_{(1)}\cdot\vec{\ell}_{(1)} + \vec{\ell}_{(2)}\cdot\vec{\ell}_{(2)}$. Here, we work in a chamber of the moduli space where

$$\Im\vec{V} \in (\Lambda^{+}_{\mathfrak{g}_{(1)}} \oplus \Lambda^{+}_{\mathfrak{g}_{(2)}}) \otimes \mathbb{C}, \tag{B11}$$

such that the only contribution to the degenerate orbit with $\vec{\ell} \neq \vec{0}$ comes from vectors $\vec{\ell}$ which correspond to simple roots of either $\mathfrak{g}_{(1)}$ or $\mathfrak{g}_{(2)}$, both of which have length squared two. We will now show that (B10) are just the conditions which characterize "positive" elements of the root lattice of $\mathfrak{g}^{++}$ of norm $\alpha^2 \leq 2$, which—just as in the simple case—will allow us to reinterpret $\Phi_{\mathfrak{g}}$ as an infinite product of the form (3.16) over the positive roots of $\mathcal{G}((\mathfrak{g}_{(1)} \oplus \mathfrak{g}_{(2)})^{++})$.[11] Here, the "positive" elements of the root lattice of $\mathfrak{g}^{++}$ are those that have positive scalar product with a fixed vector β of the underlying vector space

$$\Lambda^+_{\mathfrak{g}^{++}} = \{x \in \Lambda_{\mathfrak{g}^{++}}: (x|\beta) > 0\}. \tag{B12}$$

For further convenience, we will choose the vector $\vec{\beta}$ to be of the form

$$\beta = (u+2, u+1; \vec{w}_1; \vec{w}_2) \quad \text{with}$$
$$u = \vec{\theta}_1 \cdot \vec{w}_1 + \vec{\theta}_2 \cdot \vec{w}_2 > 0, \tag{B13}$$

where $\vec{w} = (\vec{w}_1, \vec{w}_2) \in \Lambda^+_{\mathfrak{g}_{(1)}} \oplus \Lambda^+_{\mathfrak{g}_{(2)}}$. In the following, we will find it useful to introduce $\vec{\theta} = (\vec{\theta}_{(1)}, \vec{\theta}_{(2)}) \in \Lambda_{\mathfrak{g}}$. Let us also introduce a basis of simple roots $\tilde{\alpha}_I$ for $\tilde{\mathfrak{g}}^{++}$

$$\tilde{\alpha}_{-1} = (1, -1; \vec{0}; \vec{0}) \qquad \tilde{\alpha}_0^{(1)} = (-1, 0; -\vec{\theta}_{(1)}; \vec{0})$$
$$\tilde{\alpha}_0^{(2)} = (-1, 0; \vec{0}; -\vec{\theta}_{(2)}) \tag{B14}$$

$$\tilde{\alpha}_i^{(1)} = (0, 0; \vec{e}_i^{(1)}; \vec{0})\tilde{\alpha}_m^{(2)} = (0, 0; \vec{0}; \vec{e}_m^{(2)}), \tag{B15}$$

where $i = 1, \ldots, k_1, m = 1, \ldots, k_2$, and $\vec{e}^{(1)}, \vec{e}^{(2)}$ are simple roots of $\mathfrak{g}_{(1)}, \mathfrak{g}_{(2)}$, with $\vec{\theta}_{(1)}, \vec{\theta}_{(2)}$ the corresponding highest roots. The roots (B14) and (B15) define an overcomplete basis for the root lattice $\Lambda_{\mathfrak{g}^{++}} = \Pi^{1,1} \oplus \Lambda_{\mathfrak{g}_{(1)}} \oplus \Lambda_{\mathfrak{g}_{(2)}}$. In fact, there is one relation (generating the center $\mathfrak{r}$ of $\tilde{\mathfrak{g}}^{++}$, see [1] for more details) which we may use to express $\tilde{\alpha}_0^{(2)}$ in terms of the other roots

$$\tilde{\alpha}_0^{(2)} = \tilde{\alpha}_0^{(1)} + \sum_{i=1}^{k_1} (\vec{\theta}_{(1)} \cdot \vec{f}^i)\tilde{\alpha}_i^{(1)} - \sum_{m=1}^{k_2} (\vec{\theta}_{(2)} \cdot \vec{f}^m)\tilde{\alpha}_m^{(2)}. \tag{B16}$$

Here $\vec{f}^i$ and $\vec{f}^m$ are the fundamental weights of $\mathfrak{g}_{(1)}$ and $\mathfrak{g}_{(2)}$, respectively. With this relation, we can then write for any $\alpha \in \Lambda_{\mathfrak{g}^{++}}$

$$\alpha = x_{-1}\tilde{\alpha}_{-1} + x_0\tilde{\alpha}_0^{(1)} + \sum_{i=1}^{k_1} x_i^{(1)}\tilde{\alpha}_i^{(1)} + \sum_{m=1}^{k_2} x_m^{(2)}\tilde{\alpha}_m^{(2)}. \tag{B17}$$

Using the same inner product as in (2.15) we find that the product between α and a moduli vector $\mathbf{y} = (U, T; \vec{V}_{(1)}, \vec{V}_{(2)})$ reads

$$(\alpha|\mathbf{y}) = x_{-1}T + (x_0 - x_{-1})U + \left(\sum_{i=1}^{k_1} x_i^{(1)}\vec{e}_i - x_0\vec{\theta}_{(1)}\right) \cdot \vec{V}_{(1)}$$
$$+ \sum_{m=1}^{k_2} x_m^{(2)}\vec{e}_m \cdot \vec{V}_{(2)}. \tag{B18}$$

With these preparations, the scalar product of a generic vector $\alpha \in \Lambda_{\mathfrak{g}^{++}}$, parametrized as in (B17), with β is given by

$$(\alpha|\beta) = x_{-1} + x^0(u+1) + \left(\sum_{i=1}^{k_1} x_i^{(1)}\vec{e}_i - x_0\vec{\theta}_{(1)}\right) \cdot \vec{w}_1$$
$$+ \sum_{m=1}^{k_2} x_m^{(2)}\vec{e}_m \cdot \vec{w}_2. \tag{B19}$$

Comparing (B18) to the exponent of the denominator formula (3.16) suggests the identification

$$x_{-1} = r \qquad x_0 = n' + r$$
$$\sum_{i=1}^{k_1} x_i^{(1)}\vec{e}_i - \vec{\theta}_{(1)}(n'+r) = \vec{\ell}_{(1)} \qquad \sum_{m=1}^{k_2} x_m^{(2)}\vec{e}_m = \vec{\ell}_{(2)},$$

in terms of which the scalar product (B19) becomes

$$(\alpha|\beta) = r + (n'+r)(u+1) + (\vec{\ell} \cdot \vec{w}). \tag{B20}$$

In order to show that (B10) indeed characterizes vectors of $\Lambda^+_{\mathfrak{g}^{++}}$ with norm ≤ 2, we first have to show that (B20) is positive for all three cases in (B10). This can again be done by a case-by-case analysis which is rather similar to that in Sec. III C. For example, for $r > 0$ the first equation implies $n' \geq -1$. If $n' = -1$, the first equation furthermore implies that $\vec{\ell} = \vec{0}$, and thus $(\alpha|\beta) > 0$. For $n' = 0$, we have instead $\vec{\ell} \cdot \vec{\ell} \leq 2$, which means that either $\vec{\ell} = \vec{0}$ or $\vec{\ell}$ is one of the roots of $\mathfrak{g}_{(1)}$ or $\mathfrak{g}_{(2)}$. In the former case, it immediately follows that $(\alpha|\beta) = r(u+2) > 0$, while in the latter case,

$$(\alpha|\beta) \geq r(u+2) - \max(\vec{\theta}_{(1)} \cdot \vec{w}_1, \vec{\theta}_{(2)} \cdot \vec{w}_2) > 0. \tag{B21}$$

Finally, for $n' > 0$ we can estimate

$$\begin{aligned}(\alpha|\beta) &\geq 2r + n' + (n'+r)(\vec{\theta} \cdot \vec{w}) - |\vec{\ell} \cdot \vec{w}| \\ &\geq 2r + n' + (n'+r)(\vec{\theta} \cdot \vec{w}) - \sqrt{(\vec{w} \cdot \vec{w})(\vec{\ell} \cdot \vec{\ell})} \\ &\geq 2r + n' + (n'+r)(\vec{\theta} \cdot \vec{w}) - \sqrt{(2 + 2n'r)(\vec{w} \cdot \vec{w})} > 0,\end{aligned} \tag{B22}$$

where the last inequality follows from expanding $\vec{w}$ into weights $\vec{f}_i$ for $\mathfrak{g}_{(1)}$ and $\mathfrak{g}_{(2)}$, respectively, and using the estimate $(\vec{\theta} \cdot \vec{f}_i)^2(\vec{\theta} \cdot \vec{f}_j)^2 \geq (\vec{f}_i \cdot \vec{f}_i)(\vec{f}_j \cdot \vec{f}_j) \geq (\vec{f}_i \cdot \vec{f}_j)^2$. All other cases follow in a similar fashion and we will not explicitly write them down here.

[11] See Appendix A of [1] for our conventions for the double extension $(\mathfrak{g}_{(1)} \oplus \mathfrak{g}_{(2)})^{++}$.

Conversely, one can also show that if (B10) is not satisfied, the corresponding α is not an element of $\Lambda^+_{\mathfrak{g}^{++}}$ since $(\alpha|\beta) < 0$. Thus, also in the case $\mathfrak{g}$ being semisimple, (3.14) can be written in the form of (3.16), which is identified with the infinite product part of the denominator formula for the Borcherds algebra $\mathcal{G}(\mathfrak{g}^{++})$.

APPENDIX C: JACOBI FORMS FOR $\Gamma_0(4)$

For the explicit computations in Sec. IVA 1, we require some terminology of weak Jacobi forms (in the framework of the congruence subgroup $\Gamma_0(4)$). Any weak Jacobi form of index 1 can be expanded in terms of a basis of Jacobi forms of weight 0 and -2 respectively [31]

$$f_{w,1}(\tau, z) = h_w^{(1)}(\tau)\phi_{0,1}(\tau, z) + h_{w+2}^{(2)}(\tau)\phi_{-2,1}(\tau, z), \quad \text{(C1)}$$

where we have the definitions

$$\phi_{0,1}(\tau, z) := 4\sum_{i=2}^{4}\frac{\vartheta_i(\tau, z)^2}{\vartheta_i(\tau, 0)^2}, \qquad \phi_{-2,1}(\tau, z) := -\frac{\vartheta_1(\tau, z)^2}{\eta(\tau)^6}, \quad \text{(C2)}$$

and $h^{(1,2)}$ are modular forms of weight w and $w + 2$ respectively. In Sec. IVA 1, we will be interested in the case where the latter are not modular forms under the full $SL(2, \mathbb{Z})$, but rather one of its congruence subgroups $\Gamma_0(N)$, where we define for $N \in \mathbb{N}$

$$\Gamma_0(N) := \left\{\begin{pmatrix} a & b \\ c & d \end{pmatrix} \in SL(2, \mathbb{Z})\colon\, c = 0 \bmod N\right\}. \quad \text{(C3)}$$

Specifically, we will be interested in the case $w = 4$ and $N = 4$. A (for our purposes convenient) basis for the spaces $M_w(\Gamma_0(4))$ of modular forms of weight w under $\Gamma_0(4)$ is given by (for further details see [49])

$$M_4(\Gamma_0(4))\colon \{E_4(\tau), E_4(2\tau), E_4(4\tau)\}, \quad \text{(C4)}$$

$$M_6(\Gamma_0(4))\colon \{E_6(\tau), E_6(2\tau), E_6(4\tau), h_6(\tau)\}, \quad \text{(C5)}$$

where $h_6(\tau)$ is an element of the space of cusp forms (i.e. forms which vanish at all cusps of $\overline{\mathbb{H}/\Gamma_0(N)}$). Its Fourier expansion is given by[12]

$$h_6(\tau) = q - 12q^3 + 54q^5 - 88q^7 - 99q^9 + 540q^{11} - 418q^{13} - 648q^{15} + 594q^{17} + \mathcal{O}(q^{19}). \quad \text{(C6)}$$

[12] This can be extracted from http://modi.countnumber.de/.

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