On Models, Bounds, and Estimation Algorithms for Time-Varying Phase Noise

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Abstract—In this paper, a new discrete-time model of phase noise for digital communication systems is presented. The proposed phase noise model is shown to be more accurate than the classical Wiener model. Next, using the proposed discrete-time model, the non-data-aided (NDA) and decision-directed (DD) maximum-likelihood (ML) estimators of time-varying phase noise are derived. To evaluate the performance of the proposed estimators, the Cramér-Rao lower bound (CRLB) for each estimation approach is derived and by using Monte-Carlo simulations it is shown that the mean-square error (MSE) of the proposed estimators converges to the CRLB at moderate signal-to-noise ratios (SNR). Finally, simulation results show that the proposed estimators outperform existing estimation methods as the variance of the phase noise process increases.

I. Introduction

Due to the requirements of high data-rate and spectrum efficiency communications, synchronization has gained more attention in the communication community. Since phase noise adversely affects the performance of communication systems, during the last two decades, there have been numerous studies on the estimation and compensation of phase noise in communication systems, e.g., [1]–[8]. Oscillators are an essential part of wireless communication systems and are used to perform frequency and timing synchronization. However, the output of a non-ideal oscillator is not perfectly periodic and suffers from many imperfections that introduce phase noise to communication systems [9]–[13]. In order to effectively estimate and compensate this phase noise, models that accurately capture the characteristics of non-ideal oscillators are required. The classical oscillator phase noise representation is based on the Wiener-Lévy or random-walk model [5]–[10]. This model is developed according to the Lorentzian portion of the single-sideband (SSB) phase noise spectrum, $L(f)$, which has a $1/f^2$ shape [11]. However, empirical measurements of SSB phase noise spectrum of free-running oscillators show that at smaller frequency offsets the phase noise spectrum deviates from the classical model and has a $1/f^3$ shape [13]–[15]. Recent studies show that these two parts of SSB phase noise spectrum are the results of two independent noise processes in the oscillator circuitry, namely, white and flicker noise [15], [16]. Accordingly, a new model of continuous-time phase noise is proposed in [16], and [15] to explain the shape of SSB phase noise spectrum.

In contrast with the Wiener model which is widely used in the literature, e.g., [5]–[10], the statistical characteristics of this new model in discrete-time domain are not analyzed in detail. Not that knowledge of the statistical characteristics of this new more accurate model can be used to improve the phase noise estimation accuracy in digital communication system which in turn improves the overall system performance.

In [1]–[4], the estimation of constant phase offset using decision-directed (DD)$^2$ and non-data-aided (NDA) methods are analyzed in detail. However, very little information on the estimation of time-varying phase noise is presented, and as shown in this paper the proposed estimators’ performance deteriorate as the variance of the phase noise process increases. In addition, the phase noise model applied in [4] for deriving the maximum-likelihood estimator (MLE) is based on the assumption that the phase fluctuations of each symbol consist of a constant phase plus an independently and identically distributed (i.i.d) Gaussian uncertainty. This model is different from the generally accepted Wiener phase noise model, e.g., the model in [5]–[10]. Thus, as shown in this paper the estimator proposed in [4] cannot be used in the case of Wiener phase noise. In addition, the NDA estimators proposed in [4] and [2] require the knowledge of prior and future received signals to estimate the $n$th symbol’s phase noise, which may introduce significant delays in the phase noise estimation process. This estimators are known as offline estimators. Another offline estimator, that can estimate the phase noise of all observation symbols inside a given observation vector, is proposed in [17]. On the other hand, the estimators proposed in this paper are online estimators and only require the $N$ past received symbols while estimating the current symbol’s phase noise.

In [6] the posterior Cramér-Rao bound (PCRB) [18] and a particle filter based phase noise estimator for the estimation of Wiener phase noise in communication systems are derived. However, the PCRB and estimator in [6] are limited to the case of Binary Phase Shift Keying (BPSK) modulation. Moreover, the PCRB in [6] is derived for the case of Wiener phase noise and hence is not valid for the new model proposed in this
In this paper, the PCRB of [6] is compared with the proposed Cramér-Rao lower bounds (CRLBs) for the case of BPSK and it is shown that two bounds coincide under most practical scenarios of interest. The Bayesian Cramér-Rao bound (BCRB) [19] for evaluation of the estimation performance in the case of Wiener phase noise is proposed in [8]. This study is also limited to the case of BPSK modulation and is only focused on the NDA scenario. No estimator is designed and extending the result to more complex modulation schemes is very difficult.

In [7] the effect of imperfect phase noise estimation on the bit error probability of quadratic phase-shift keying (QPSK) modulated signals is investigated. However, the results are limited and the performance of the proposed estimator is not evaluated. Other phase noise estimation methods, such as iterative methods are also presented in the literature, e.g., [20]–[22]. These iterative algorithms are usually complex to implement.

The contributions of this paper can be summarized as follows:

• A new discrete-time model of the phase noise is proposed which is a generalized version of the discrete-time Wiener model. This model resembles the measured SSB phase noise spectrum of a free-running oscillator more closely and takes into account the effect of both $1/f^2$ and $1/f^3$-shaped portions of the SSB phase noise spectrum. The statistical characteristics of the new model are also derived.

• Based on the proposed model, new NDA and DD MLEs and CRLBs for estimation of the phase noise in $M$-ary PSK modulated signals are derived in closed form. It is also shown that the derived bounds and estimators are applicable to the Wiener phase noise.

In Sec. II, a system model for a point-to-point communication system using $M$-ary PSK modulation is introduced. In Sec. III, first, a continuous-time phase noise model in [15], [16] is briefly studied. Then, the mentioned discrete-time model of the phase noise is proposed based on this model. In Sec. IV, based on the proposed model, new NDA and DD MLEs and CRLBs for estimation of phase noise are derived. In Sec. V, the performance of the derived estimators, for both the proposed phase noise model and the Wiener model, are compared against the CRLB and existing estimators and bounds in the literature.

Notations: italic letters ($x$) are scalar variables, bold letters ($X$) are vectors, bold upper case letters ($X$a,b) denote the $(a,b)th$ entry of matrix $X$, $E[\cdot]$ denotes the statistical expectation, $\Re(\cdot)$, $\Im(\cdot)$, and $\arg(\cdot)$ are real part, imaginary part, and angle of complex values, and $(\cdot)^*$, $(\cdot)^T$, and $(\cdot)^H$ are conjugate, transpose, and conjugate transpose, respectively.

II. SYSTEM MODEL

Fig. 1 depicts the block diagram corresponding to the complex baseband representation of the considered communication system. The received signal, $r_k$, can be written as

$$r_k = e^{j\phi_k} s_k + w_k,$$

where $s_k$ is the $M$-ary PSK modulated symbol transmitted at time instant $k$, $e^{j\phi_k}$ represents the phasor of $\phi_k$, which is the unwanted phase fluctuation of the $k$th received symbol, $r_k$, and $w_k$ is the zero-mean complex additive white Gaussian noise (AWGN) with variance $\sigma_w^2$. The source and statistical model of $\phi_k$ are discussed in detail in Sec. III. Throughout this paper it is assumed that the timing offset and channel gain have been estimated and compensated which is in line with the assumptions in [1]–[8]. As shown in Fig. 1, the received signal is passed through a phase estimator and the estimated phase, $\hat{\phi}_k$, is used to de-rotate the received signal before demodulation.

III. PHASE NOISE MODEL

In this section the phase noise model in [15], [16], which closely resembles measurement results for a free-running oscillator, is briefly introduced. Then we derive the statistical properties of the discrete-time version of this model, given that it is of more interest in digital communication systems.

A. Overview

As illustrated in Fig. 2 and shown in [15], [16], far from the central carrier frequency, $f_c$, the oscillator SSB phase noise spectrum has a $1/f^2$ shape. However, as we move closer to $f_c$, the oscillator SSB phase noise spectrum changes to a $1/f^3$ shape, and for further lower frequency offsets a Gaussian shape of SSB phase noise spectrum can be observed. The classical Wiener phase noise model is motivated by the $1/f^2$ shape portion of the oscillator’s spectrum, also known as Lorentzian spectrum [5]. However, the $1/f^3$ and Gaussian portions of the oscillator’s SSB phase noise spectrum need to be taken into consideration to find a comprehensive phase noise model.

As shown in [15], [16], the output of a noisy oscillator is given by $\zeta(t) = \cos(2\pi f_c t + \phi(t))$, where $\phi(t)$ is the phase fluctuations that is modeled as a real random process (RP). In the continuous-time domain, the phase fluctuation can be expressed as

$$\phi(t) = \int_0^t \Omega(u) \, du,$$

where $\Omega(t)$ is the frequency perturbation, which is a result of different parameters such as thermal noise of circuit elements, noise of transistors, fluctuations in the tuning voltage of voltage controlled oscillators (VCO), etc. [23], [24]. In practice, $\Omega(t)$ is assumed to be a stationary zero-mean Gaussian RP and can be white or colored depending on the source of the noise [11], [15], [24]. As shown in [11], [15], [16] the $1/f^2$-shaped portion is produced by white frequency noise, $\Omega_{\text{white}}(t)$, while the $1/f^3$-shaped plus the Gaussian-shaped portions are due to flicker noise, $\Omega_{\text{flicker}}(t)$, with an approximate power spectral density (PSD) equal to $1/f^{1-\nu}$. Note that, $\nu$ has a small value ($0 < \nu < 1$) and is used to ensure the stationarity of $\Omega_{\text{flicker}}(t)$ [15], [16]. Finally,
the total phase fluctuation, \( \phi(t) \), can be written as
\[
\phi(t) = \phi_{\text{white}}(t) + \phi_{\text{flicker}}(t),
\]
where \( \phi_{\text{white}}(t) \) and \( \phi_{\text{flicker}}(t) \) are two independent phase noise processes, which are produced by \( \Omega_{\text{white}}(t) \) and \( \Omega_{\text{flicker}}(t) \), respectively.

Using (2), the phase noise variation caused by the phase noise process during the time interval \( \tau \) is defined as
\[
\xi(t, \tau) = \phi(t) - \phi(t - \tau) = \int_{t-\tau}^{t} \Omega(u) \, du,
\]
where \( \xi(t, \tau) \) denotes the phase noise innovation and according to the properties of \( \Omega(t) \), it is a zero-mean Gaussian RP. Note that the variance and correlation properties of \( \xi(t, \tau) \) are dependent on the properties of \( \Omega(t) \).

### B. A New Discrete-Time Phase Noise Model

In this section, a new discrete-time phase noise model, consisting of both \( \phi_{\text{white}}(t) \) and \( \phi_{\text{flicker}}(t) \) is proposed, and its statistical properties are derived.

According to (4) and considering a sampling time \( T_s \), the phase innovation between two consecutive samples can be written as
\[
\Delta_n \triangleq \xi(nT_s, T_s) = \phi(nT_s) - \phi(nT_s - T_s) = \int_{(n-1)T_s}^{nT_s} \Omega(u) \, du,
\]
where \( \Delta_n \) is the discrete version of \( \xi(t, \tau) \). Using (5), the phase fluctuation of the \( n \)th sample, \( \phi_n \triangleq \phi(nT_s) \), can be expressed as
\[
\phi_n = \phi_{n-1} + \Delta_n.
\]
According to (3), the total phase noise innovation can be written as addition of two independent phase noise innovations
\[
\Delta_n = \Delta_{\text{white},n} + \Delta_{\text{flicker},n},
\]
where \( \Delta_{\text{white},n} \) and \( \Delta_{\text{flicker},n} \) denote the phase noise innovation corresponding to \( \Omega_{\text{white}}(t) \) and \( \Omega_{\text{flicker}}(t) \), respectively. Note that for the Wiener model, only the white phase noise innovation, \( \Delta_{\text{white},n} \), is considered and \( \Delta_{\text{flicker},n} \) is usually neglected despite its important effect on the final phase noise process.

Based on the above assumptions, the autocorrelation function of \( \Delta_n \) can be calculated as
\[
R(\Delta) = E[\Delta_n \Delta_{n+l}]
= \int_{(n-1)T_s}^{nT_s} \int_{(n+1-1)T_s}^{(n+1)T_s} \Omega(u)\Omega(v) \, du \, dv,
\]
where \( R_{\Omega}(u - v) \) denotes the autocorrelation function of \( \Omega(t) \). Using the Fourier transform of \( R_{\Omega}(u - v) \), \( R(\Delta) \) can be written as
\[
R(\Delta) = \int_{-\infty}^{\infty} \{S_{\Omega}(f) \int_{(n-1)T_s}^{nT_s} \int_{(n+1-1)T_s}^{(n+1)T_s} e^{2\pi i f (u-v)} \, du \, dv \} \, df,
\]
where \( S_{\Omega}(f) \) describes the PSD of \( \Omega(t) \) and \( \Delta_n \) is defined as
\[
\Delta_n = \int_{(n-1)T_s}^{nT_s} \Omega(u) \, du.
\]
Fig. 2: \( L(f) \) of a free-running oscillator. \( k_1 = 4 \times 10^{-4}, k_2 = 0.1, \nu = 0.01, \) if \( T_s = 10^{-6} \Rightarrow \sigma_{\Delta_{\text{white}}}^2 = 4 \times 10^{-10}, \sigma_{\Delta_{\text{flicker}}}^2 = 1.7 \times 10^{-8}. \)

Note that the phase noise process, \( \phi_{\text{flicker}}(t) \), with the autocorrelation function defined by (12) for its innovations, is an fBm [25] with uncorrelated innovations also known as a Wiener Process.

Using (10) and \( S_{\Omega}(f) = \frac{K_2}{f^2} \), the autocorrelation function of \( \Delta_{\text{flicker},n} \) can be determined as
\[
R_{\Delta_{\text{flicker},l}}(f) = \frac{\sigma_{\Delta_{\text{flicker}}}^2}{2} \left( (l-1)^2 - 2l(2^{1-\nu} + l + 1)^2 \nu \right),
\]
where \( \sigma_{\Delta_{\text{flicker}}}^2 \) is the variance, which is given by
\[
\sigma_{\Delta_{\text{flicker}}}^2 = \frac{-K_2 \pi}{(2\pi)^\nu \Gamma(3-\nu) \cos(3\nu/2)} T_s^{2-\nu}.
\]
Note that the phase noise process, \( \phi_{\text{flicker}}(t) \), with the autocorrelation function defined by (12) for its innovations, is an fBm [25] where the variance of the innovations is approximately proportional to \( T_s^2 \).

Finally, according to (7), the autocorrelation function of the total phase noise innovation, \( \Delta_n \), is determined as
\[
R(\Delta) = R_{\Delta_{\text{white},n}}(l) + R_{\Delta_{\text{flicker},n}}(l),
\]
where \( R_{\Delta_{\text{white},n}}(l) \) and \( R_{\Delta_{\text{flicker},n}}(l) \) are defined in (11) and (12), respectively.

### IV. Estimation of Time-Varying Phase Noise

In the following subsections, four algorithms for estimation of the \( k \)th received symbol’s phase noise, \( \phi_k \), are derived. In addition, a CRLB is derived for each case to evaluate the performance of each estimator. The proposed algorithms are based...
on non-data-aided (NDA) and decision-directed (DD) methods. In each method two different schemes, based on the high-SNR and slow-varying phase noise assumptions are implemented.

According to the phase noise model in (6), (7) and Fig. 3, the phase noise of the \((k-i)\)th received symbol is determined as

\[
\phi_{k-i} = (\phi_k - \sum_{m=0}^{i-1} \Delta_m).
\]  

(15)

Using (15) and the system model developed in Sec.II, the \((k-i)\)th received symbol, \(r_{k-i}\), can be written as

\[
r_{k-i} = s_k e^{j(\phi_k - \sum_{m=0}^{i-1} \Delta_m)} + w_{k-i}.
\]  

(16)

Note that all statistical properties of the random phase noise, \(\phi_k\), is translated to the model in (15) and (16). Thus, hereinafter, \(\phi_k\) is assumed as an unknown deterministic parameter over the observation sequence.

A. Non-Data-Aided Estimator

In order to remove the data dependency, the received signal can be passed through a nonlinear function \([2]\). Here, the approach proposed in [4] is used, where the received \(M\)-ary PSK symbols are raised to the power of \(M\). Based on this approach, (16) can be rewritten as

\[
r_{k-i}^M = (s_k e^{j(\phi_k - \sum_{m=0}^{i-1} \Delta_m)} + w_{k-i})^M.
\]  

(17)

Using the binomial theorem, \(r_{k-i}^M\) can be rewritten as

\[
r_{k-i}^M = \sum_{l=0}^{M} \binom{M}{l} (s_k e^{j(\phi_k - \sum_{m=0}^{i-1} \Delta_m)} + w_{k-i})^l w_{k-i}^{M-l}.
\]  

(18)

Assuming that the signal power is much larger than the noise \(w_{k-i}\), the remaining terms after the second term in (18) can be neglected. By defining the \(M\)-ary PSK modulated symbol \(s_k = \sqrt{E_s} e^{j(\frac{2\pi}{B} k)}\), where \(E_s\) is the signal energy and \(L_k \in \{1, \ldots, M\}\) is the index of transmitted message, \(s_k^M\), can be determined as

\[
s_k^M = E_s^M e^{j\frac{2\pi}{B} k M} = E_s^M.
\]  

(19)

Using (19) and by keeping only the first two terms of (18), \(r_{k-i} \triangleq r_{k-i}^M\) can be rewritten as

\[
r_{k-i} = E_s^M e^{jM(\phi_k - \sum_{m=0}^{i-1} \Delta_m)} + M E_s^M e^{j((M-1)(\phi_k - \sum_{m=0}^{i-1} \Delta_m) + \arg(s_k^{M-1}))} w_{k-i}.
\]  

(20)

where \(w_{k-i}\), a rotated and scaled version of \(w_{k-i}\), is still a zero-mean complex Gaussian random variable (RV) with variance \(\sigma_w^2 = M^2 E_s^M(1-\sigma_w^2)\). This is based on the assumption of circularity on the observation noise.

\[
\begin{array}{cccc}
\phi_k & \phi_k - \sum_{m=0}^{i-1} \Delta_m & \phi_k - \Delta_1 & \phi_k - \Delta_2 \\
\phi_k - \sum_{m=0}^{i-1} \Delta_m & \phi_k - \Delta_1 & \phi_k - \Delta_1 & \phi_k - \Delta_1 \\
\phi_k - \Delta_1 & \phi_k - \Delta_1 & \phi_k - \Delta_1 & \phi_k - \Delta_1 \\
\phi_k - \Delta_1 & \phi_k - \Delta_1 & \phi_k - \Delta_1 & \phi_k - \Delta_1 \\
\end{array}
\]

\(N\) symbols

Fig. 3: Vector of \(N\) received symbols and its corresponding phase fluctuation vector.

1) High-SNR: By defining \(\tilde{w}_{k-i} \triangleq w_{k-i} e^{jM(\phi_k - \sum_{m=0}^{i-1} \Delta_m)}\), (20) is rewritten as

\[
\tilde{r}_{k-i} = (E_s^M + \tilde{w}_{k-i}) e^{jM(\phi_k - \sum_{m=0}^{i-1} \Delta_m)}
\]  

(21)

where \(\tilde{w}_{k-i}\), is the rotated version of \(w_{k-i}\) with variance \(\sigma_w^2 = \sigma_w^2\). Next, note that

\[
\arg(E_s^M + \tilde{w}_{k-i}) = \tan^{-1} \left( \frac{\Re(\tilde{w}_{k-i})}{\Im(\tilde{w}_{k-i})} \right).
\]  

(22)

At high SNR, since \(\frac{\Re(\tilde{w}_{k-i})}{\Im(\tilde{w}_{k-i})}\) is small and \(\tan^{-1}(x) \approx x\) for small \(x\), (22) can be rewritten as

\[
\arg(E_s^M + \tilde{w}_{k-i}) \approx \frac{\Im(\tilde{w}_{k-i})}{E_s^M} \triangleq \tilde{w}_{k-i}.
\]  

(23)

The accuracy of this approximation is evaluated in Sec. V by means of numerical simulations. The SNR range in which High-SNR assumption is valid is discussed in Remark 1. Using (21), (22), and (23), \(\tilde{r}_{k-i}\) can be written as

\[
a_{k-i} \triangleq \arg(\tilde{r}_{k-i}) = M \phi_k - M \sum_{m=0}^{i-1} \Delta_m + \tilde{w}_{k-i}.
\]  

(24)

where \(\tilde{w}_{k-i}\), a zero-mean real Gaussian RV with variance \(\sigma_w^2 = \frac{\sigma_w^2}{E_s^M}\), is defined in (23). Since summation of zero-mean real Gaussian RVs is a real Gaussian RV, \(a_{k-i}\) is also a real Gaussian RV. Therefore, the vector \(a \triangleq [a_{k-N+1}, \ldots, a_k]^T\) has an \(N\)-variate Gaussian distribution given by

\[
f_a(\phi_k | a_k) = \frac{1}{\sqrt((2\pi)^N \det(C_a))} e^{-\frac{1}{2}(a_m - m)^T C_a^{-1}(a_m - m)}.
\]  

(25)

where \(m = M \phi_k 1_{N \times 1}\) and \(C_a^{N \times N}\) denote the mean and covariance of \(a\), respectively and \(1 \triangleq [1, 1, \ldots, 1]^T\). The elements of the covariance matrix \(C_a^{N \times N}\) can be determined as

\[
C_{a_m a_n} = E[ (a_k - \bar{a}_{k-N+1}) (a_k - \bar{a}_{k-n})]\n
(26)

\[
E[ (a_k - \bar{a}_{k-n}) (\alpha_k - \bar{a}_k)]
\]

\[
E[ (\alpha_k - \bar{a}_k) (\alpha_k - \bar{a}_m)]
\]

\[
M^2 \sum_{m=0}^{N-1} \sum_{n=0}^{N-1} E[\Delta_m \Delta_n] + E[\Delta_m \Delta_n]
\]

\[
M^2 \sum_{m=0}^{N-1} \sum_{n=0}^{N-1} (R_{\Delta_m \Delta_n} (m-n) + R_{\Delta_m \Delta_n} (m-n))
\]

\[
+ \frac{\delta(x-y)M^2 \sigma_w^2}{2E_s^M}.
\]
where \( x, y \in \{0, \ldots, N-1\} \). The log-likelihood function (LLF) of \( \phi_k \), up to an additive constant is given by
\[
L(\phi_k) = \ln(f_{\phi_k}(\phi_k)) = \frac{1}{2}(a - m_a)^T C_{a}^{-1}(a - m_a).
\]
(27)

In order to find the MLE of \( \phi_k^\star \), the LLF in (27) needs to be maximized, where the derivative of \( L(\phi_k) \) with respect to \( \phi_k \) is determined as
\[
\frac{\partial L(\phi_k)}{\partial \phi_k} = \frac{1}{2} [Ma^T C_{a}^{-1} 1 + M 1^T C_{a}^{-1} a - 2M^2 \phi_k 1^T C_{a}^{-1} 1].
\]
(28)

By setting (28) equal to zero and by carrying out straightforward algebraic manipulations, the MLE for \( \phi_k \) can be derived as
\[
\hat{\phi}_k^{\text{MLE}} = \frac{1}{M} a^T C_{a}^{-1} 1.
\]
(29)

Given that the Cramér-Rao lower bound (CRLB) is defined as [26]

\[
\text{CRLB} = \left( E \left[ -\frac{\partial^2 L(\phi_k)}{\partial \phi_k^2} \right] \right)^{-1},
\]
(30)

the CRLB for the estimation of \( \phi_k \) using the high-SNR assumption, CRLB\(_{\text{HSA}}\), is calculated as

\[
\text{CRLB}_{\text{HSA}} = \frac{1}{M^2 (1^T C_{1}^{-1} 1)}. \quad (31)
\]

2) Slow-Varying Phase Noise: The Taylor series expansion of \( e^x \) for small values of \( x \) can be approximated by \( e^x \approx 1 + x \). Based on the assumption of slow-varying phase noise, the sum of the phase innovations \( M \sum_{m=0}^{\infty} \Delta_m \) can be considered to be small. Thus, \( e^{-jM \sum_{m=0}^{\infty} \Delta_m} \) can be approximated by

\[
e^{-jM \sum_{m=0}^{\infty} \Delta_m} \approx 1 - jM \sum_{m=0}^{\infty} \Delta_m.
\]
(32)

The slow-varying phase noise assumption and approximation in (32) is used and verified in the literature, e.g., [5], [7], [27]-[29]. The results in Sec. V, where the performance of the estimator proposed in this subsection is compared against the CRLB, also validate this assumption (see also Remark 2). Using (32), (20) can be rewritten as
\[
\hat{\phi}_k^{\text{HSA}} = E_{x} e^{jM \phi_k} \left( 1 - jM \sum_{m=0}^{\infty} \Delta_m \right) + \hat{w}_{k-1}.
\]
(33)

\[
= E_{x} e^{jM \phi_k} - jM \sum_{m=0}^{\infty} \Delta_m + \hat{w}_{k-1}.
\]

Given that \( \Delta_m \) and \( \hat{w}_{k-1} \) are Gaussian RVs and based on (33) the vector \( \hat{\phi}_k = [\hat{\phi}_{k-1}^T, \ldots, \hat{\phi}_{k}^T]^T \) has an \( N \)-variate complex Gaussian distribution given by
\[
\hat{\phi}_k | \phi_k \sim \mathcal{CN} \left( \frac{1}{(y)^N \det(C_{\hat{\phi}})} e^{-\frac{1}{2} (\hat{\phi}_k - m_{\phi})^T C_{\phi}^{-1} (\hat{\phi}_k - m_{\phi})} \right),
\]
(34)

where \( m_{\phi} = E_{x} e^{jM \phi_k} 1^T N \times 1 \) is the mean vector. Taking the same approach as (26), elements of the covariance matrix \( C_{\hat{\phi}}^{N \times N} \) can be determined as
\[
C_{\hat{\phi}}^{N+1,y+1} = M^2 E_{x} \sum_{m=0}^{y-1} \sum_{m=0}^{n} (R_{\Delta_{\text{short}}}(m-n) + R_{\Delta_{\text{flicker}}}(m-n)) + M^2 E_{x}^{-1} \delta(x-y) s^2.
\]
(35)
Remark 3: Although the above approach removes the data dependency, it results in a phase ambiguity of $2\pi/M$. In practice training sequences and pilots or differential modulation can be used to solve this problem [4].

Remark 4: Given that the process of removing data dependency in (18) amplifies the AWGN by a factor of $\frac{\pi}{M}$. In practice training sequences and pilots or differential modulation can be used to solve this problem [4].

**B. Decision-Directed Estimator**

The DD scenario is of interest given that in communication systems training sequences or pilot signals are used to facilitate accurate and efficient estimation of synchronization parameters. In this paper the DD scenario refers to the scenario where the $k$th symbol’s phase noise is estimated while assuming that the transmitted symbols prior to the $k$th symbol are known.

After multiplying $r_{k-i}$ by the conjugate of the known transmitted symbol, $s_{k-i}$, one obtains

$$\tilde{r}_{k-i} = E_s e^{j\phi_k} s_{k-i}^* (1 - j \sum_{m=0}^{i-1} \Delta_m) + \tilde{w}_{k-i},$$

(40)

where $\tilde{w}_{k-i}$ is a zero-mean complex Gaussian RV. Similar to the NDA method, two estimators based on the high-SNR and slow-varying phase noise assumptions are derived in the following subsections.

1) High-SNR: Using the same steps as the ones outlined in Sec.IV-A, $\tilde{r}_{k-i}$ can be rewritten as

$$\tilde{r}_{k-i} = |E_s + \tilde{w}_k| \tilde{r}_{k-1} e^{j(\phi_k - \sum_{m=0}^{i-1} \Delta_m)} + \tilde{w}_{k-1},$$

(41)

where $\tilde{w}_{k-1}$ is a rotated version of $\tilde{w}_{k-i}$, and $\tilde{w}_{k-1}$ is a real Gaussian RV. Similar to (40), it is clear that the useful information for the estimation of $\phi_k$ is the angle of $\tilde{r}_{k-1}$. Let us define $\tilde{\phi}_{k-1} \triangleq \arg(\tilde{r}_{k-1})$ and $\tilde{\alpha} \triangleq \{\tilde{\phi}_{k-N}, \ldots, \tilde{\phi}_{k-1}\}^T$, where $\tilde{\alpha}$ has an $N$-variate Gaussian distribution similar to that of (25). The mean vector and covariance matrix of $\tilde{\alpha}$ are given by $\tilde{\mu}_\alpha = \phi_k 1_N^T$ and $\tilde{C}_\alpha = C_N^N$, respectively. Using similar steps as (26), $C_{\alpha}^{N \times N}$ can be calculated as

$$C_{\alpha}^{x,y} = \sum_{m=0}^{i-1} \sum_{n=0}^{i-1} \left( R_{\Delta_{\text{slow}}}(m-n) + R_{\Delta_{\text{nic}}}(m-n) \right)$$

$$+ \delta(x-y) \frac{\sigma_w^2}{2E_s^2},$$

(42)

where $x, y \in \{1, \ldots, N\}$. Analogous to Sec. IV-A, the MLE and CRLB can be determined as

$$\hat{\phi}_{k(D)} = \frac{1}{1^T C^{-1}_\alpha} \tilde{\alpha} \quad \text{CRLB}_{(\text{DD})} = \frac{1}{1^T C^{-1}_\alpha}.$$
estimator based on the slow-varying phase noise assumption converges to CRLB at low SNR. This is anticipated from the analytical results, since according to (22), for the NDA method, the high-SNR assumption is only valid when $M^2 E_s (M-1) \sigma_w^2$ is small. Even with $M = 4$, the results in Fig. 7 show that the assumption in (22) only holds for very high SNR values. In contrast, for the DD estimator, the high-SNR assumption is valid when $E_s \sigma_w^2$ is small which is more feasible at moderate SNRs. Thus, performance in this case is independent of the constellation size and it can be seen in Fig. 8 that the DD estimator based on the high-SNR approach outperforms the NDA method in Fig. 7. In general, the DD method has an error floor compared with the NDA scheme due to the fact that in the case of DD estimation, only the observation sequence up to the $(k-1)$th symbol is used while estimating the $k$th symbol's phase noise.

Fig. 8 depicts the performance of the DD method using the high-SNR assumption. As it can be seen, in each scenario, the estimator’s MSE converges and follows the theoretical CRLB. In this figure, the estimation bounds of phase noise with white and colored innovations are also compared. These results show that colored phase noise innovations can be more accurately estimated in high SNR. This is anticipated as the correlation between phase innovations can be exploited by the estimator to improve estimation accuracy.

In Fig. 9, the performance of the proposed NDA method using slow-varying phase noise assumption is compared against the NDA method of [4] for different SNRs and different white phase noise innovation variances. Block length=11, (Wiener model $\sigma_{\text{flicker}}^2 = 0$).

**VI. CONCLUSIONS**

In this paper a new discrete-time model of time-varying phase noise that more closely resembles the measurement results for a free-running oscillator is proposed. Four different NDA and DD MLEs for estimation of the time-varying phase noise are
derived in closed form. The proposed estimators are not based on an exhaustive search and are shown to converge to the CRLBs over a large range of SNR values. In addition, the assumptions that are used in derivation of the bounds and estimators are validated by comparing the results with the available bounds in the literature. It is also shown that the proposed algorithms significantly outperform some of the available estimators.

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