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Technical Report in Vehicle Dynamics

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Abstract

We consider a linear single-track model of a truck-dolly-semitrailer by deriving Lagrange's equations with the quasi-coordinates chosen as the body fixed longitudinal and lateral velocity of the truck, v_{x1} and v_{y1}, respectively, the yaw angle ψ_1 and the articulation angles θ_1 and θ_2. The linear analysis and simulations are carried out with a constant longitudinal velocity of 80 Kph for the passive vehicle, i.e. no steering on the dolly or semitrailer. The analysis shows an increase in yaw rate gain for the dolly and semitrailer, compared to the truck, for frequencies around 0.4 Hz on the truck steering angle. That is, we have a yaw rate RWA higher than 1 when for frequencies around 0.4 Hz. We also observe a lateral acceleration gain, between 0.1 and 1 Hz, for the dolly and semitrailer. Moreover, the phase angle has a higher frequency dependence for the dolly and semitrailer both for yaw rate and lateral acceleration. Time simulations shows that the dolly and semitrailer experience a time delay and an overshoot when the vehicle combination is subjected to different steering angle maneuvers. This indicates that the vehicle combination is not as stable as the truck alone.

Keywords: Truck-dolly-semitrailer, Linear analysis, Yaw instability, Lateral load transfer

Contents

1 Introduction

The derivation of a linear single-track model of a truck-dolly-semitrailer based on Lagrange's approach is outlined in this report. The derivations are conducted using a powerful tool based on symbolic expressions in Matlab, found in Appendix C. Different analytical expressions can easily be extracted using the tool. Frequency response, free motion and simulations of the linear model for a constant longitudinal velocity are analysed to capture the characteristics of the dynamics and the stability of the vehicle combination. Of special interest is the yaw rates and lateral accelerations for the three units since these are the main factors for yaw instability and load transfer potentially causing roll over, respectively.

2 Lagrange's equations of motion

Consider the truck-dolly-semitrailer vehicle combination depicted in Figure 2.1. The vehicle combination comprises three vehicle units modelled as rigid bodies which are connected with one degree of freedom frictionless cylindrical joints. Only planar motion is considered and since the hinge has no elastic restoring force no potential energy needs to be computed. The system has five degrees of freedom thus five generalised coordinates are needed. An appropriate choice of generalised coordinates are the longitudinal and lateral positions of the truck Center of Gravity (CoG), X_1 and Y_1, respectively, and its yaw angle ψ_1 in an inertial frame and the articulation angles θ_1 and θ_2. The truck has steering angle δ_{11} on the front axle while the tag axle has no steering. The dolly has steering angle δ_2 on both axles and the semitrailer has three rear axles with steering angle δ_3. The tire forces are expressed as

$$\boldsymbol{F}_{1f} = [F_{1fx}, F_{1fy}]^{\mathrm{T}}, \ \boldsymbol{F}_{1r} = [F_{1rx}, F_{1ry}]^{\mathrm{T}}, \ \boldsymbol{F}_{1t} = [F_{1tx}, F_{1ty}]^{\mathrm{T}}, \tag{2.1a}$$
$$\boldsymbol{F}_{21} = [F_{21x}, F_{21y}]^{\mathrm{T}}, \ \boldsymbol{F}_{22} = [F_{22x}, F_{22y}]^{\mathrm{T}}, \tag{2.1b}$$
$$\boldsymbol{F}_{31} = [F_{31x}, F_{31y}]^{\mathrm{T}}, \ \boldsymbol{F}_{32} = [F_{32x}, F_{32y}]^{\mathrm{T}}, \ \boldsymbol{F}_{33} = [F_{33x}, F_{33y}]^{\mathrm{T}}. \tag{2.1c}$$

With the coordinates established Lagrange's equations can be formed to describe the motion of the considered vehicle combination.

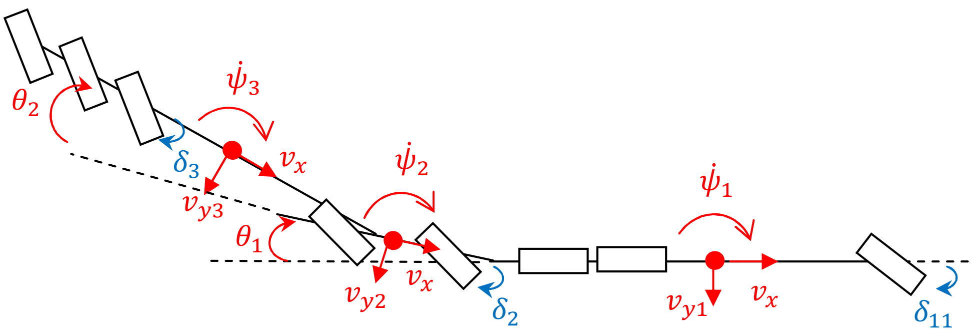


Figure 2.1: Truck-dolly-semitrailer.

2.1 General mechanical system

According to [1], Lagrange's equations are a set of differential equations with the generalised coordinates as dependent variables. It is an alternative, to the Newtonian approach, to form the equations of motion of a mechanical system using energy balance, rather than force balance. When deriving the equations the potential and kinetic energies of the system is differentiated with respect to the generalised coordinates and their time derivatives. The equations for n degree of freedom have the form

$$\frac{d}{dt}\frac{\partial L}{\partial \dot{q}_i} - \frac{\partial L}{\partial q_i} = Q_i, \tag{2.2a}$$

$$L = T - V, \tag{2.2b}$$

where $i = 1, ..., n$, q_i are the generalised coordinates and Q_i the corresponding generalised external forces; T is the kinetic energy and V is the potential energy. The generalised forces are expressed as

$$Q_i = \sum_{k=1}^{N} \boldsymbol{F}_k \cdot \frac{\partial \boldsymbol{r}_k}{\partial q_i}, \tag{2.3}$$

where N is the number of forces and $\mathbf{F}_k$ is the force with position vector $\mathbf{r}_k$. As can be seen in the expression for the generalised forces the acting external forces on the system can be described arbitrarily and can therefore be dealt with after the equations have been stipulated.

2.2 Truck-dolly-semitrailer

A shorter description of the derivation of Lagrange's equations for the truck-dolly-semitrailer is presented here. A more thorough description of the derivations is presented in Appendix A. The approach of the derivations of Lagrange's equations is based on [2].

A suitable choice of generalised coordinates are as stated earlier

$$q = [X_1,\ Y_1,\ \psi_1,\ \theta_1,\ \theta_2]\,. \tag{2.4}$$

When describing the motion of the vehicle it is more convenient to express the velocities in a body fixed coordinate system. A transformation, mapping the inertial coordinates to the body fixed system, can be defined as

$$v_{x1} = \dot{X}_1 \cos(\psi_1) + \dot{Y}_1 \sin(\psi_1), \tag{2.5a}$$

$$v_{y1} = -\dot{X}_1 \sin(\psi_1) + \dot{Y}_1 \cos(\psi_1). \tag{2.5b}$$

Note that the body fixed coordinates x_1 and y_1 can not be obtained by integrating v_{x1} and v_{y1}. The velocities v_{x1} and v_{y1} are thus referred to as quasi-coordinates, see [3]. To form Lagrange's equations for the model, the kinetic energies of the vehicle units (truck, dolly and semitrailer) are determined individually and added together. The kinetic energies T_1, T_2 and T_3 of the truck, dolly and semitrailer, respectively, are nonlinear functions of the quasi-coordinates and can be compactly written as follows

$$T_1 = T_1(v_{x1}, v_{y1}, \dot{\psi}_1), \tag{2.6a}$$

$$T_2 = T_2(v_{x1}, v_{y1}, \dot{\psi}_1, \theta_1, \dot{\theta}_1), \tag{2.6b}$$

$$T_3 = T_3(v_{x1}, v_{y1}, \dot{\psi}_1, \theta_1, \dot{\theta}_1, \theta_2, \dot{\theta}_2). \tag{2.6c}$$

As mentioned earlier the potential energy does not influence the model. The derivatives affected by the transformation from the global coordinates to the quasi-coordinates are

$$\frac{\partial T}{\partial \dot{X}_1} = \frac{\partial T}{\partial v_{x1}} \frac{\partial v_{x1}}{\partial \dot{X}_1} + \frac{\partial T}{\partial v_{y1}} \frac{\partial v_{y1}}{\partial \dot{X}_1} = A \cos(\psi_1) - B \sin(\psi_1), \tag{2.7a}$$

$$\frac{\partial T}{\partial \dot{Y}_1} = \frac{\partial T}{\partial v_{x1}} \frac{\partial v_{x1}}{\partial \dot{Y}_1} + \frac{\partial T}{\partial v_{y1}} \frac{\partial v_{y1}}{\partial \dot{Y}} = A \sin(\psi_1) + B \cos(\psi_1), \tag{2.7b}$$

where

$$A = \frac{\partial T}{\partial v_{x1}}, \tag{2.8a}$$

$$B = \frac{\partial T}{\partial v_{y1}}. \tag{2.8b}$$

Explicit expressions for A and B along with the other derivatives needed can be found in Appendix B. For the truck-dolly-semitrailer there are eight axles; front, rear and tag axle of the truck, two axles on the dolly and three axles on the semitrailer. The generalised forces are calculated for each unit where the position vector to each axle is differentiated with respect to the generalised coordinates

$$Q_i = \sum_{k=1}^{N} \boldsymbol{F}_k \cdot \frac{\partial \boldsymbol{r}_k}{\partial q_i}, \tag{2.9}$$

with q according to (2.4), and $\mathbf{F}_k$ are the tire forces with longitudinal and lateral components F_{kx} and F_{ky}, respectively. The lateral tire force is a function of the vehicle slips and the tire normal force

$$F_{ky} = F_{ky}(\gamma_k, \alpha_k, \kappa_k, F_{kz}). \tag{2.10}$$

More details of Lagrange's equations can be found in Appendix A.

3 Linearisation of Lagrange's equations

Consider the special case of straight driving with constant longitudinal velocity, V. The standard assumptions for derivation of a linear model are used, namely small steering and articulation angles and negligible second and higher order terms. To calculate the generalised external forces Q_i in Equation (2.3), linear tire force characteristics are assumed

$$F_{ky} = C_k \alpha_k, \tag{3.1}$$

where C_k are the cornering stiffness coefficients and α_k are the side slip angles of the tires. We recall that, since no braking is issued, longitudinal tire forces will vanish, i.e., $F_{kx} = 0$. After linearisation Equation (2.2) can be written in the following form

$$\mathrm{M}\ddot{q} + \mathrm{C}\dot{q} + \mathrm{K}q = \mathrm{F}u + \mathrm{H}d, \tag{3.2}$$

where $u = [\delta_2,\ \delta_3]$, $d = \delta_{11}$, M, C and K are obtained by differentiating the linearised forms of Lagrange's equations with $\ddot{q}_i$, $\dot{q}_i$ and q_i respectively, where q is reduced to

$$q = [\mathrm{Y}_1,\ \psi_1,\ \theta_1,\ \theta_2]\,. \tag{3.3}$$

The longitudinal velocity, V, now is a parameter and not a state variable and the vehicle parameters included in the matrices in Equation (3.2) can be found in Appendix D. The mass matrix M only includes terms from the left hand side of Equation (2.2)

$$\mathrm{M} = \begin{bmatrix} m & -(a_2+c_1)m_2-(a_3+c_1+l_2)m_3 & -a_2m_2-(a_3+l_2)m_3 & -a_3m_3 \\ -(a_2+c_1)m_2-(a_3+c_1+l_2)m_3 & J_o & J_r & J_s \\ -a_2m_2-(a_3+l_2)m_3 & J_r & J_p & J_t \\ -a_3m_3 & J_s & J_t & J_q \end{bmatrix}, \tag{3.4}$$

where

$$m = m_1 + m_2 + m_3, \tag{3.5}$$

is the total mass,

$$J = J_1 + J_2 + J_3, \tag{3.6}$$

is the sum of moment of inertia for each unit,

$$J_o = J + (a_2+c_1)^2 m_2 + (a_3+c_1+l_2)^2 m_3, \tag{3.7}$$

is the total moment of inertia about the truck CoG,

$$J_p = J_2 + J_3 + (a_3+l_2)^2 m_3 + a_2^2 m_2, \tag{3.8}$$

is the moment of inertia of the dolly and semitrailer about the articulation point of the truck,

$$J_q = J_3 + a_3^2 m_3, \tag{3.9}$$

is the moment of inertia of the semitrailer about the articulation point of the semitrailer, and the rest of the constants are

$$J_r = J_2 + J_3 + a_2(a_2+c_1)m_2 + (a_3+l_2)(a_3+c_1+l_2)m_3, \tag{3.10a}$$

$$J_s = J_3 + a_3(a_3+c_1+l_2)m_3, \tag{3.10b}$$

$$J_t = J_3 + a_3(a_3+l_2)m_3. \tag{3.10c}$$

The damping matrix, C, includes terms both from the left and right hand side of Equation (2.2)

$$\mathrm{C} = \frac{1}{V}\begin{bmatrix} C & Cs_1 + V^2\ m & C_ts_2 & -C_3l_3 \\ Cs_1 & Cq_1^2 - V^2(a_2m_2+a_3m_3+c_1m_2+c_1m_3+l_2m_3) & Cq_3^2 & C_3(c_1+l_2+l_3)l_3 \\ C_ts_2 & Cq_3^2 - V^2(a_2m_2+a_3m_3+l_2m_3) & C_tq_2^2 & C_3(l_2+l_3)l_3 \\ -C_3l_3 & C_3(c_1+l_2+l_3)l_3 - V^2a_3m_3 & C_3(l_2+l_3)l_3 & C_3l_3^2 \end{bmatrix}, \tag{3.11}$$

where

$$C = C_{1f} + C_{1r} + C_{1t} + C_2 + C_3, \tag{3.12}$$

is the total cornering stiffness,

$$C_t = C_2 + C_3, \tag{3.13}$$

is the combined cornering stiffness for the dolly and semitrailer, and the other constants are

$$Cs_o = C_{1f}a_1 - C_{1r}b_{1r} - C_{1t}b_{1t} - C_2(b_2 + c_1) - C_3(c_1 + l_2 + l_3), \tag{3.14a}$$

$$C_t s_p = -C_2 b_2 - C_3(l_2 + l_3), \tag{3.14b}$$

$$Cq_o^2 = C_{1f}a_1^2 + C_{1r}b_{1r}^2 + C_{1t}b_{1t}^2 + C_2(b2 + c_1)^2 + C_3(c_1 + l_2 + l_3)^2, \tag{3.14c}$$

$$C_t q_p^2 = C_2 b_2^2 + C_3(l_2 + l_3)^2, \tag{3.14d}$$

$$Cq_q^2 = C_2(b_2 + c_1)b_2 + C_3(l_2 + l_3 + c_1)(l_2 + l_3). \tag{3.14e}$$

The stiffness matrix, K, and the F and H matrices will only have contribution from the right hand side of Equation (2.2)

$$\mathrm{K} = \begin{bmatrix} 0 & 0 & -C_t & -C_3 \\ 0 & 0 & C_t(c_1 - s_p) & C_3(c_1 + l_2 + l_3) \\ 0 & 0 & -C_t s_p & C_3(l_2 + l_3) \\ 0 & 0 & C_3 l_3 & C_3 l_3 \end{bmatrix}, \tag{3.15a}$$

$$\mathrm{F} = \begin{bmatrix} C_{1f} \\ C_{1f}a_1 \\ 0 \\ 0 \end{bmatrix}, \tag{3.15b}$$

$$\mathrm{H} = \begin{bmatrix} C_2 & C_3 \\ -C_2(b_2 + c_1) & -C_3(c_1 + l_2 + l_3) \\ -C_2 b_2 & -C_3(l_2 + l_3) \\ 0 & -C_3 l_3 \end{bmatrix}. \tag{3.15c}$$

For convenience Equation (3.2) is rewritten to a set of first order differential equations. The states resulting from the generalised coordinates, for the system sketched in Figure 2.1, are the lateral velocity v_{y1} and yaw rate $\dot{\psi}_1$ of the truck, as well as the articulation angles θ_1 and θ_2,of the dolly and semitrailer, and their time derivatives $\dot{\theta}_1$ and $\dot{\theta}_2$.

$$x = \left[\theta_1,\ \theta_2,\ v_{y1},\ \dot{\psi}_1,\ \dot{\theta}_1,\ \dot{\theta}_2\right]. \tag{3.16}$$

The derived linear parameter varying vehicle model can be described by the following compact system of linear differential equations:

$$\dot{x}(t) = \mathrm{A}x(t) + \mathrm{B}u(t) + \mathrm{E}d(t), \tag{3.17}$$

where the state, input and disturbance vectors are $x = [\theta_1,\ \theta_2,\ v_{y1},\ \dot{\psi}_1,\ \dot{\theta}_1,\ \dot{\theta}_2]$, $u = [\delta_2,\ \delta_3]$ and $d = \delta_{11}$ respectively. The matrices are computed as

$$\mathrm{A} = \begin{bmatrix} 0 & \mathrm{I} \\ \mathrm{M}^{-1}\mathrm{K} & \mathrm{M}^{-1}\mathrm{C} \end{bmatrix},\ \mathrm{B} = \begin{bmatrix} 0 \\ \mathrm{M}^{-1}\mathrm{F} \end{bmatrix},\ \mathrm{E} = \begin{bmatrix} 0 \\ \mathrm{M}^{-1}\mathrm{H} \end{bmatrix}, \tag{3.18}$$

where the equations corresponding to Y_1 and ψ_1 are removed. In addition to the states and control inputs there are other outputs of interest such as the lateral acceleration, yaw rates and vehicle side slips for the three units. For convenience an output vector is defined

$$y(t) = \mathrm{C}x(t) + \mathrm{D}d(t), \tag{3.19}$$

where $y = [a_{y1},\ a_{y2},\ a_{y3},\ \dot{\psi}_1,\ \dot{\psi}_2,\ \dot{\psi}_3,\ \beta_1,\ \beta_2,\ \beta_3,\ \theta_1,\ \theta_2]$. We observe that the yaw rates of the dolly and semitrailer are linear functions of the state variables and are calculated as follows

$$\dot{\psi}_2 = \dot{\psi}_1 + \dot{\theta}_1, \tag{3.20a}$$

$$\dot{\psi}_3 = \dot{\psi}_1 + \dot{\theta}_1 + \dot{\theta}_2. \tag{3.20b}$$

The linear expressions for the lateral accelerations for the dolly and semitrailer can be found in Appendix A.

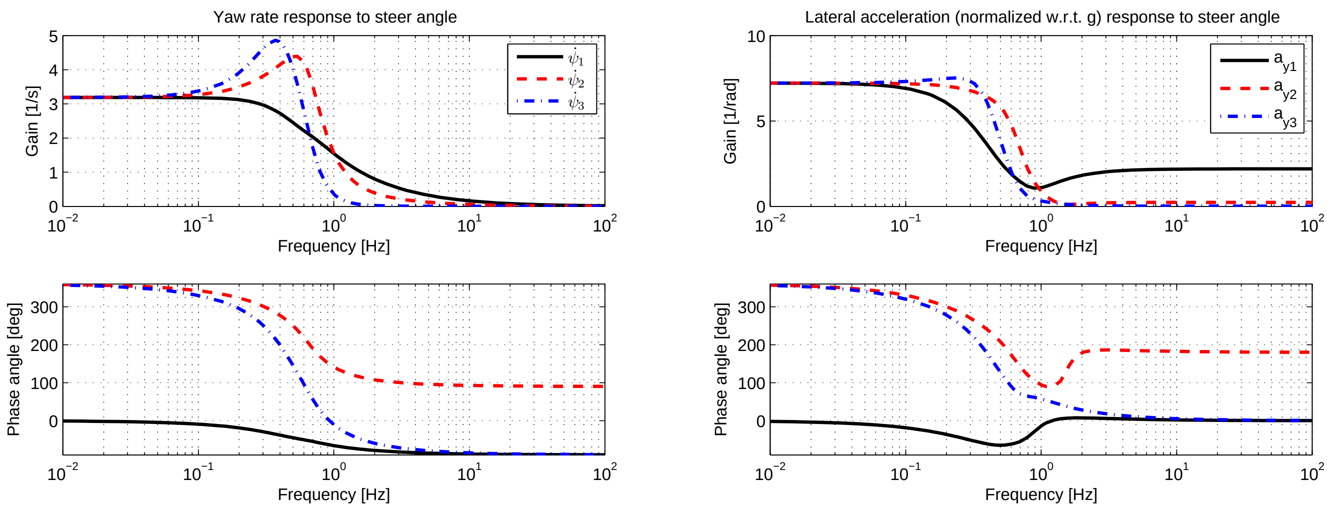
4 Linear analysis

In this section frequency response and the free motion of the vehicle combination are analysed for small deviations from the reference state. Of particular interest is determination of the peak gain frequencies and the rearward amplification (RWA). RWA is defined as the ratio of the maximum value of a motion variable for the rearmost unit to that of the first unit during a specific maneuver, see [4]. In the frequency response the amplitude and the phase angle of the input signal are determined. The amplitude is defined as the magnitude of change of the oscillating variable and the phase angle is defined as the phase shift between the output and input signal.

4.1 Frequency response

In Figure 4.1(a), the frequency responses of the truck, dolly and semitrailer yaw rates are reported for the passive vehicle, i.e., $\delta_2 = \delta_3 = 0$ in Equation (3.17), at the longitudinal speed of 80 Kph. We observe that, compared to the yaw rate of the truck, the dolly and semitrailer yaw rates are amplified around 0.4 and 0.5 Hz, respectively. That is, the considered heavy vehicle combination exhibits a yaw rate RWA higher than 1 when the frequency of the truck steering angle δ_{11} is around 0.4 Hz.

In Figure 4.1(b), the frequency responses of the truck, dolly and semitrailer lateral accelerations are reported with the same set-up as above. We observe that the gain is higher for the dolly and trailer compared to the truck for frequencies between 0.1 and 1 Hz but above 1 Hz the gain of the truck is higher. We also notice that the frequency affects the phase angles of the dolly and semitrailer more in comparison to the truck.

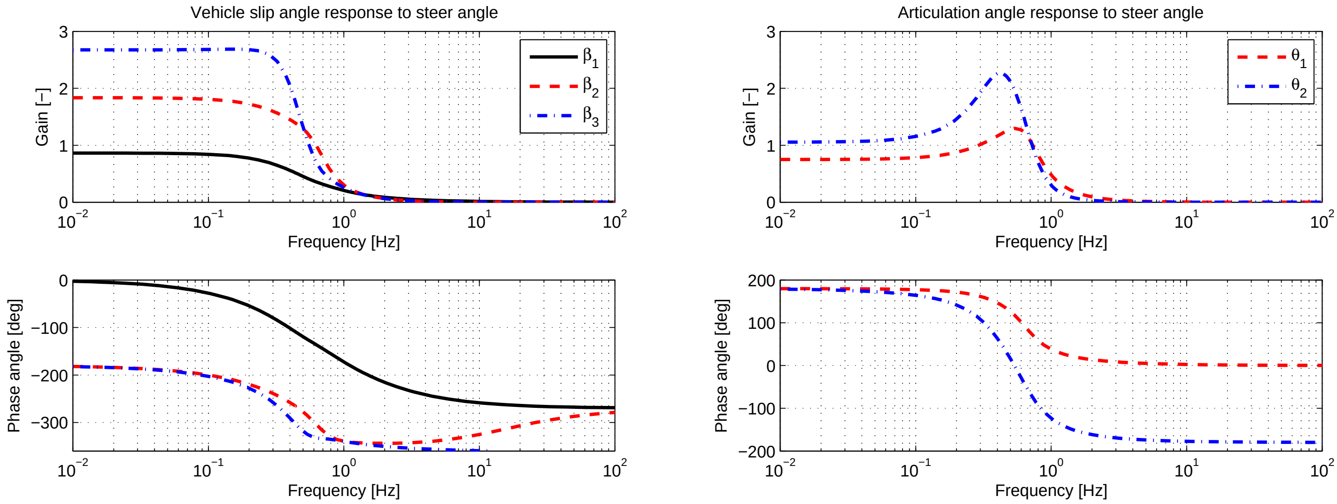


(a) Yaw rate. (b) Lateral acceleration.

Figure 4.1: Frequency responses of the truck (solid line), dolly (dashed) and semitrailer (dash-dotted line) yaw rates and lateral accelerations for the passive vehicle, i.e., $\delta_2 = \delta_3 = 0$ in Equation (3.17).

In Figure 4.2(a), vehicle side slips are reported with the same set-up as above. We observe that the gain for the truck, dolly and semitrailer is roughly 1, 2 and 3 respectively from 0.01 Hz to 0.3 Hz where it then reduces to zero gain for higher frequencies.

In Figure 4.2(b), the frequency responses of the articulation angles are reported where we can observe that the dolly and semitrailer articulation angles are amplified around 0.3 and 0.4 Hz, respectively.



(a) Vehicle slip.

(b) Articulation angle.

Figure 4.2: Frequency responses of the truck (solid line), dolly (dashed) and semitrailer (dash-dotted line) vehicle side slips and articulation angles for dolly and semitrailer, for the passive vehicle, i.e., $\delta_2 = \delta_3 = 0$ in Equation (3.17).

4.2 Free motion

To study the free motion of the vehicle combination, the eigenvalues of Equation (3.17) are investigated by evaluating the determinant of

$$(\mathrm{A} - \lambda\ \mathrm{I}), \tag{4.1}$$

where λ are the eigenvalues and I the identity matrix. The eigenvalues are complex conjugated, where the real parts are all negative which indicates that the system is stable. In Table 4.1, the eigenfrequencies, ω_n, and damping factors, ξ, of the three modes of the system are tabulated for 40 and 80 Kph. The corresponding eigenvectors are shown in Table 4.2. The natural frequency and damping factor are calculated as

$$\omega_n = \sqrt{(\mathrm{Re}[\lambda])^2 + (\mathrm{Im}[\lambda])^2} \tag{4.2a}$$

$$\xi = -\frac{\mathrm{Re}[\lambda]}{\omega_n} \tag{4.2b}$$

40 Kph		
Mode	Natural Frequency, ω_n [Hz]	Damping Factor, ξ [-]
I	0.43	0.94
II	0.65	0.73
III	0.84	0.97

80 Kph		
Mode	Natural Frequency, ω_n [Hz]	Damping Factor, ξ [-]
I	0.43	0.47
II	0.45	0.89
III	0.65	0.37

Table 4.1: Eigenfrequencies for 40 and 80 Kph.

40 Kph			
State \ Mode	I	II	III
θ_1	$0.07 \pm 0.08i$	$-0.01 \pm 0.00i$	$-0.01 \pm 0.01i$
θ_2	$-0.16 \pm 0.15i$	$0.33 \pm 0.12i$	$-0.00 \pm 0.03i$
v_{y1}	$-0.02 \pm 0.05i$	$0.00 \pm 0.00i$	0.96
r	$-0.00 \pm 0.01i$	$0.00 \pm 0.00i$	$-0.01 \pm 0.12i$
$\dot{\theta}_1$	$-0.42 \pm 0.06i$	$0.02 \pm 0.02i$	$-0.08 \pm 0.01i$
$\dot{\theta}_2$	0.87	-0.94	$-0.04 \pm 0.13i$

80 Kph			
State \ Mode	I	II	III
θ_1	$0.03 \pm 0.11i$	$0.00 \pm 0.01i$	$-0.01 \pm 0.00i$
θ_2	$-0.08 \pm 0.20i$	$-0.16 \pm 0.30i$	$-0.01 \pm 0.00i$
v_{y1}	$-0.08 \pm 0.02i$	$0.00 \pm 0.00i$	0.94
r	$0.01 \pm 0.01i$	$0.00 \pm 0.00i$	$0.00 \pm 0.06i$
$\dot{\theta}_1$	$-0.45 \pm 0.04i$	$0.01 \pm 0.01i$	$0.02 \pm 0.01i$
$\dot{\theta}_2$	0.85	0.94	$0.04 \pm 0.01i$

Table 4.2: Eigenvectors for 40 and 80 Kph.

5 Simulation

In this section we present and discuss simulation results for different maneuvers, still using the linear model. Sine and step steer responses are considered.

5.1 Sine response

In the considered testing scenario, the vehicle performs a single lane change maneuver at 80 Kph on dry road. The input steering angle is shown in Figure 5.1. The truck, dolly and semitrailer yaw rates are reported in Figure 5.2(a) and the dolly and semitrailer articulation angles in Figure 5.2(b), where we observe that the dolly and semitrailer have a small delay and a higher gain than the truck. The dolly and semitrailer also experience a yaw rate overshoot of 2 and 4 rad/s, respectively, compared to the steady state value. The overshoot can also be observed for the articulation angles.

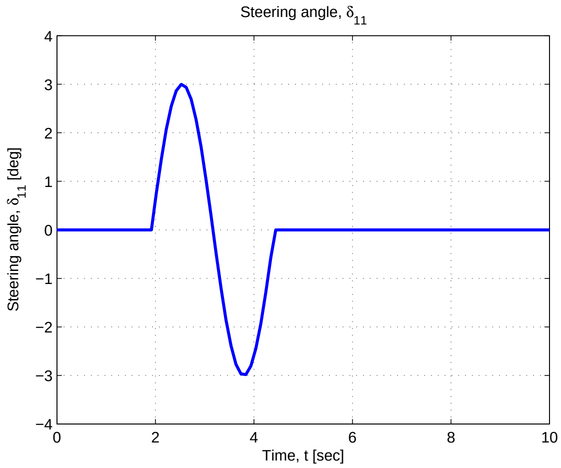
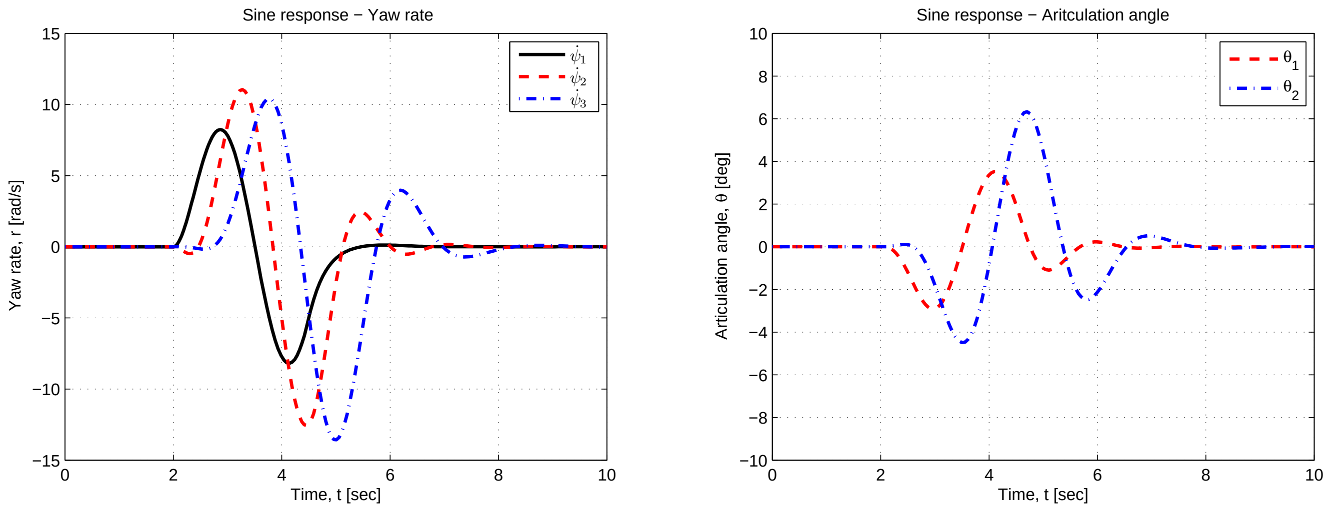


Figure 5.1: Steering angle.



(a) Yaw rate.

(b) Articulation angle.

Figure 5.2: Yaw rate and articulation angle sine response.

In Figure 5.3 the yaw rates of the truck, dolly and semitrailer are reported for a continuous sine input steering angle. The main purpose to include this Figure is for model comparison with [4] and others.

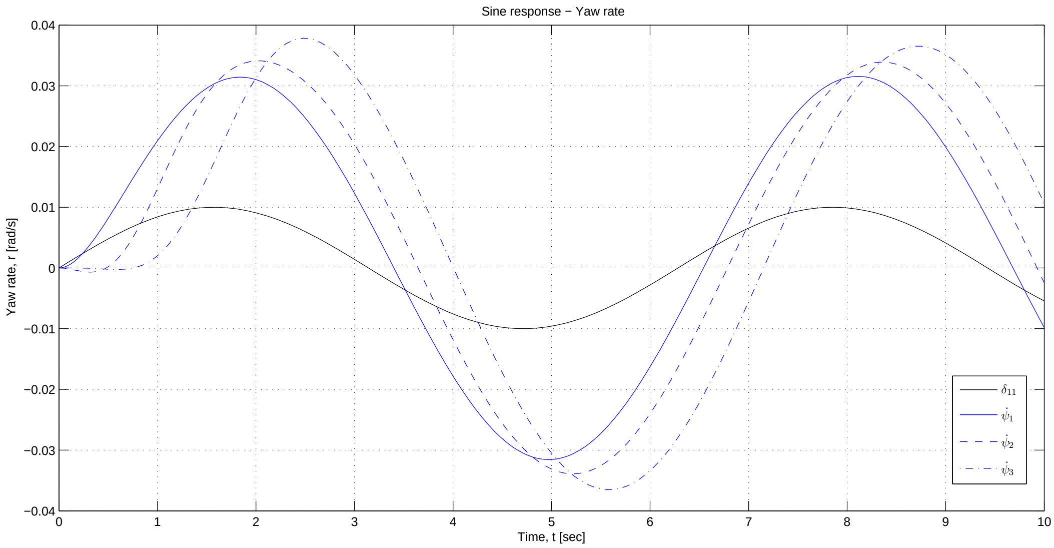
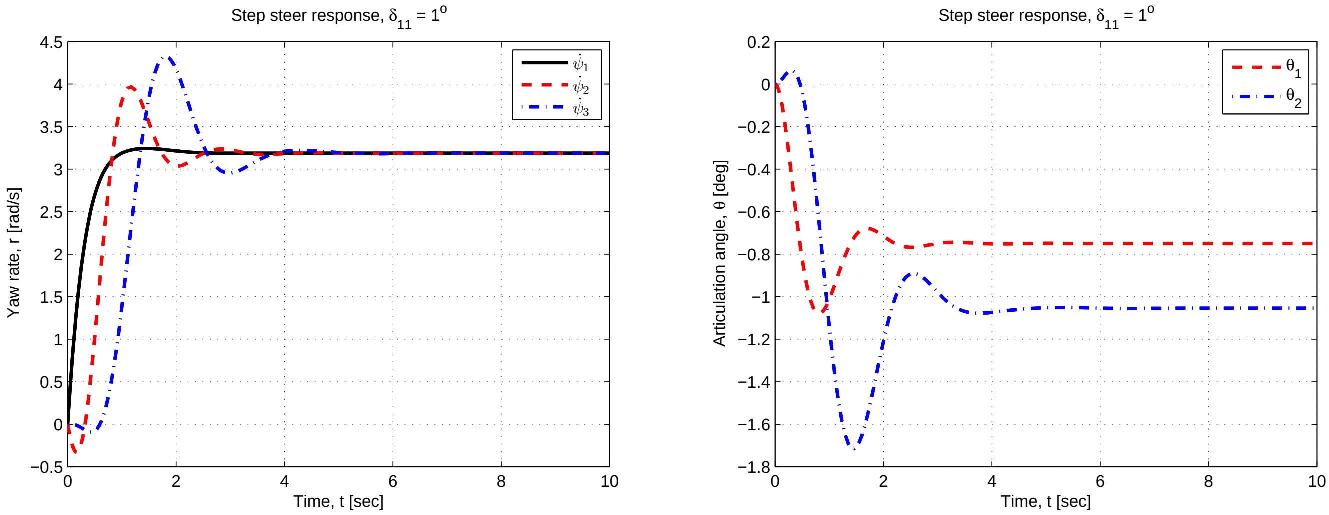


Figure 5.3: Yaw rates of truck (solid blue), dolly (dashed) and semitrailer (dash-dotted line) and the steering angle (solid black).

5.2 Step steer response

In the considered testing scenario, the vehicle performs a step steer input with $\delta_{11} = 1^o$ at 80 Kph on dry road. In Figure 5.4(a) the yaw rates of the truck, dolly and semitrailer and in Figure 5.4(b) the dolly and semitrailer articulation angles are shown. As can be seen the response is quite slow for the dolly and semitrailer compared with the truck and we can observe an overshoot of the yaw rates and articulation angles both for the dolly and the semitrailer.



(a) Yaw rate.

(b) Articulation angle.

Figure 5.4: Yaw rate and articulation angle step response.

6 Conclusions and future work

Reviewing the results from the frequency response and simulation of the linear model, we can observe that the dolly and semitrailer induces an inertia to the system which appears as phase shifts in the frequency response and time delays of the maneuver responses. We can also observe overshoots and an increase in gain for the dolly and semitrailer that indicates the vehicle combination is not as stable as the truck alone which of course is expected. Further efforts should be put on simulation with nonlinear tires, using Magic Formula, [5], or similar. Additional maneuvers, other than sine and step responses, should also be carried out to make the analysis more detailed and widen the scope on the subject. Moreover, analytical expressions of the transfer functions, used for the frequency response, can be extracted using the tool in Appendix C. Additional effort should also be put to analyse the free motion of the vehicle combination.

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A Details of Lagrange's equations

Certain parts of the report that have been simplified or expressed simplistic are presented thoroughly in this section.

A.1 Kinetic energy for the truck-dolly-semitrailer

Lagrange's equations derived for a truck-dolly-semitrailer are described by five generalised coordinates, as seen in Figure 2.1. The generalised coordinates are the longitudinal and lateral positions of the truck Center of Gravity (CoG), X_1 and Y_1, respectively, its yaw angle ψ_1 in an inertial frame and the articulation angles θ_1 and θ_2. The yaw rate $\dot{\psi}_1$ will be referred to as r. The vehicle parameter setup is according to [2].

The velocities of the three units, expressed in the coordinate system of the first unit, are

$$\boldsymbol{v}_1 = \begin{bmatrix} v_{x1}, & v_{y1}, & 0 \end{bmatrix}^{\mathrm{T}}, \tag{A.1a}$$
$$\boldsymbol{v}_2 = \boldsymbol{v}_1 + \boldsymbol{\omega}_1 \times \boldsymbol{r}_{c_1} + \boldsymbol{\omega}_2 \times \boldsymbol{r}_{a_2}, \tag{A.1b}$$
$$\boldsymbol{v}_3 = \boldsymbol{v}_1 + \boldsymbol{\omega}_1 \times \boldsymbol{r}_{c_1} + \boldsymbol{\omega}_2 \times \boldsymbol{r}_{l_2} + \boldsymbol{\omega}_3 \times \boldsymbol{r}_{a_3}, \tag{A.1c}$$

where

$$\boldsymbol{\omega}_1 = \begin{bmatrix} 0, & 0, & r \end{bmatrix}^{\mathrm{T}}, \tag{A.2a}$$
$$\boldsymbol{\omega}_2 = \begin{bmatrix} 0, & 0, & r + \dot{\theta}_1 \end{bmatrix}^{\mathrm{T}}, \tag{A.2b}$$
$$\boldsymbol{\omega}_3 = \begin{bmatrix} 0, & 0, & r + \dot{\theta}_1 + \dot{\theta}_2 \end{bmatrix}^{\mathrm{T}}, \tag{A.2c}$$

are the rotational velocities for each unit and

$$\boldsymbol{r}_{c_1} = \begin{bmatrix} -c_1, & 0, & 0 \end{bmatrix}^{\mathrm{T}}, \tag{A.3a}$$
$$\boldsymbol{r}_{a_2} = \begin{bmatrix} -a_1, & 0, & 0 \end{bmatrix}^{\mathrm{T}}, \tag{A.3b}$$
$$\boldsymbol{r}_{l_2} = \begin{bmatrix} -l_2, & 0, & 0 \end{bmatrix}^{\mathrm{T}}, \tag{A.3c}$$
$$\boldsymbol{r}_{a_3} = \begin{bmatrix} -a_3, & 0, & 0 \end{bmatrix}^{\mathrm{T}}, \tag{A.3d}$$

are position vectors.

The kinetic energy of the system is the sum of the kinetic energies of the three units

$$T = \frac{1}{2} m_1 \boldsymbol{v}_1 \cdot \boldsymbol{v}_1 + \frac{1}{2} m_2 \boldsymbol{v}_2 \cdot \boldsymbol{v}_2 + \frac{1}{2} m_3 \boldsymbol{v}_3 \cdot \boldsymbol{v}_3 + \frac{1}{2} J_1 r^2 + \frac{1}{2} J_2 \left(r + \dot{\theta}_1 \right)^2 + \frac{1}{2} J_3 \left(r + \dot{\theta}_1 + \dot{\theta}_2 \right)^2, \tag{A.4}$$

where m_1, m_2 and m_3 are the masses and J_1, J_2 and J_3 are the moments of inertia for the truck, dolly and semitrailer respectively.

A.2 The left hand side of Lagrange's equations

As described in the report, Lagrange's equations have the form

$$\frac{d}{dt} \left(\frac{\partial T}{\partial \dot{q}_i} \right) - \frac{\partial T}{\partial q_i} = Q_i. \tag{A.5}$$

The derivatives affected by the transformation from the global to the body fixed coordinate system are

$$\frac{\partial T}{\partial \dot{X}_1} = \frac{\partial T}{\partial v_{x1}} \frac{\partial v_{x1}}{\partial \dot{X}_1} + \frac{\partial T}{\partial v_{y1}} \frac{\partial v_{y1}}{\partial \dot{X}} = A \cos(\psi_1) - B \sin(\psi_1), \tag{A.6a}$$
$$\frac{\partial T}{\partial \dot{Y}_1} = \frac{\partial T}{\partial v_{x1}} \frac{\partial v_{x1}}{\partial \dot{Y}_1} + \frac{\partial T}{\partial v_{y1}} \frac{\partial v_{y1}}{\partial \dot{Y}} = A \sin(\psi_1) + B \cos(\psi_1), \tag{A.6b}$$

where

$$A = \frac{\partial T}{\partial v_{x1}}, \tag{A.7a}$$

$$B = \frac{\partial T}{\partial v_{y1}}. \tag{A.7b}$$

The expressions for A and B along with the derivatives $\frac{\partial T}{\partial \dot{\theta}_1}$, $\frac{\partial T}{\partial \dot{\theta}_2}$, $\frac{\partial T}{\partial r}$, $\frac{\partial T}{\partial \theta_1}$, $\frac{\partial T}{\partial \theta_2}$, $\frac{\partial T}{\partial \psi_1}$, $\frac{\partial T}{\partial X_1}$ and $\frac{\partial T}{\partial Y_1}$, can be found in Appendix B.

The time derivatives needed are

$$\begin{bmatrix} \frac{\partial}{\partial t}\left(\frac{\partial T}{\partial \dot{X}_1}\right) \\ \frac{\partial}{\partial t}\left(\frac{\partial T}{\partial \dot{Y}_1}\right) \\ \frac{\partial}{\partial t}\left(\frac{\partial T}{\partial \dot{\psi}_1}\right) \\ \frac{\partial}{\partial t}\left(\frac{\partial T}{\partial \dot{\theta}_1}\right) \\ \frac{\partial}{\partial t}\left(\frac{\partial T}{\partial \dot{\theta}_2}\right) \end{bmatrix} = \begin{bmatrix} \left(\dot{A} - Br\right)\ \cos(\psi_1) - \left(\dot{B} + Ar\right)\ \sin(\psi_1) \\ \left(\dot{A} - Br\right)\ \sin(\psi_1) + \left(\dot{B} + Ar\right)\ \cos(\psi_1) \\ \frac{\partial}{\partial t}\left(\frac{\partial T}{\partial \dot{\psi}_1}\right) \\ \frac{\partial}{\partial t}\left(\frac{\partial T}{\partial \dot{\theta}_1}\right) \\ \frac{\partial}{\partial t}\left(\frac{\partial T}{\partial \dot{\theta}_2}\right) \end{bmatrix}. \tag{A.8}$$

After rotating to the body fixed coordinate system the left hand side of Lagrange's equations become

$$\begin{bmatrix} \dot{A} - Br \\ \dot{B} + Ar \\ \frac{\partial}{\partial t}\left(\frac{\partial T}{\partial r}\right) - \frac{\partial T}{\partial \psi_1} \\ \frac{\partial}{\partial t}\left(\frac{\partial T}{\partial \dot{\theta}_1}\right) - \frac{\partial T}{\partial \theta_1} \\ \frac{\partial}{\partial t}\left(\frac{\partial T}{\partial \dot{\theta}_2}\right) - \frac{\partial T}{\partial \theta_2} \end{bmatrix}. \tag{A.9}$$

A.3 Generalised forces

For the truck-dolly-semitrailer there are eight axles; front, rear and tag axle on the truck, two axles on the dolly and three axles on the semitrailer. For simplicity the axles on the dolly and semitrailer are lumped and the average distances to the axles are used. The generalised forces can then be calculated as

$$Q_i = \sum_{k=1}^{n} \boldsymbol{F}_k \cdot \frac{\partial \boldsymbol{r}_k}{\partial q_i}. \tag{A.10}$$

The position vector for each axle is

$$\boldsymbol{r}_{1f} = \begin{bmatrix} a_1, & 0, & 0 \end{bmatrix}^{\mathrm{T}}, \tag{A.11a}$$

$$\boldsymbol{r}_{1r} = \begin{bmatrix} -b_{1r}, & 0, & 0 \end{bmatrix}^{\mathrm{T}}, \tag{A.11b}$$

$$\boldsymbol{r}_{1t} = \begin{bmatrix} -b_{1t}, & 0, & 0 \end{bmatrix}^{\mathrm{T}}, \tag{A.11c}$$

$$\boldsymbol{r}_2 = -\begin{bmatrix} c_1 + b_2\ \sin(\theta_1), & b_2\ \sin(\theta_1), & 0 \end{bmatrix}^{\mathrm{T}}, \tag{A.11d}$$

$$\boldsymbol{r}_3 = -\begin{bmatrix} c_1 + l_2\ \cos(\theta_1) + l_3\ \cos(\theta_1 + \theta_2), & l_2\ \sin(\theta_1) + l_3\ \sin(\theta_1 + \theta_2), & 0 \end{bmatrix}^{\mathrm{T}}. \tag{A.11e}$$

The position vectors are then differentiated with respect to the generalised coordinates, according to Equation (A.10). The resulting generalised forces are

$$Q_{x_1} = F_{1rx} + F_{1tx} + \sum_{i=1}^{3}(F_{ix}\ \cos(\varphi_i) - F_{iy}\ \sin(\varphi_i)), \tag{A.12a}$$

$$Q_{y_1} = F_{1ry} + F_{1ty} + \sum_{i=1}^{3}(F_{ix}\ \sin(\varphi_i) + F_{iy}\ \cos(\varphi_i)), \tag{A.12b}$$

$$\begin{aligned} Q_{\psi_1} &= F_{1ry}b_{1r} + F_{1ty}b_{1t} - c_1 \sum_{i=2}^{3}(F_{ix}\ \cos(\varphi_i) + F_{iy}\ \sin(\varphi_i)) \\ &\quad - \sum_{i=2}^{3} l_i(F_{3x}\ \sin(\xi_i) + F_{3y}\ \cos(\xi_i)) + \sum_{i=1}^{2} d_i\ (-1)^{i+1}(F_{ix}\ \sin(\delta_i) + F_{iy}\ \cos(\delta_i)), \end{aligned} \tag{A.12c}$$

$$Q_{\theta_1} = -b_2(F_{2x}\ \sin(\delta_2) + F_{2y}\ \cos(\delta_2)) - \sum_{i=2}^{3} l_i(F_{3x}\ \sin(\xi_i) + F_{3y}\ \cos(\xi_i)), \tag{A.12d}$$

$$Q_{\theta_2} = -l_3(+F_{3x}\ \sin(\delta_2) + F_{3y}\ \cos(\delta_2)), \tag{A.12e}$$

where

$$\varphi_1 = \delta_1,\ \varphi_2 = \delta_2 + \theta_1,\ \varphi_3 = \delta_2 + \theta_1 + \theta_2, \tag{A.13a}$$

$$\xi_1 = \delta_2 + \theta_2,\ \xi_2 = \delta_2, \tag{A.13b}$$

$$d_1 = a_1,\ d_2 = b_2, \tag{A.13c}$$

$$F_{1x} = F_{1fx},\ F_{1y} = F_{1fy}. \tag{A.13d}$$

A.4 Linear expressions of lateral tire forces

In linear analysis the lateral tire forces are expressed as $F_{ky} = C_k\alpha_k$, where C_k are the cornering stiffness coefficients and α_k are the side slip angles of the tires. A geometrical relation between the side slip angle and the velocity and possible steering angle of a tire can easily be described as

$$\tan(\alpha_k + \delta_k) = \frac{v_{ky}}{v_{kx}}. \tag{A.14}$$

Solving for the side slip angle

$$\alpha_k = -\arctan(\frac{v_{ky}}{v_{kx}}) + \delta_k. \tag{A.15}$$

The velocities for the front, rear and tag axle of the truck and the lumped axles for the dolly and semitrailer are

$$\boldsymbol{v}_{1f} = \boldsymbol{v}_1 + \boldsymbol{\omega}_1 \times \boldsymbol{r}_{1f}, \tag{A.16a}$$

$$\boldsymbol{v}_{1r} = \boldsymbol{v}_1 + \boldsymbol{\omega}_1 \times \boldsymbol{r}_{1r}, \tag{A.16b}$$

$$\boldsymbol{v}_{1t} = \boldsymbol{v}_1 + \boldsymbol{\omega}_1 \times \boldsymbol{r}_{1t}, \tag{A.16c}$$

$$\boldsymbol{v}_{2l} = \boldsymbol{v}_1 + \boldsymbol{\omega}_1 \times \boldsymbol{r}_{c_1} + \boldsymbol{\omega}_2 \times \boldsymbol{r}_{b_2}, \tag{A.16d}$$

$$\boldsymbol{v}_{3l} = \boldsymbol{v}_1 + \boldsymbol{\omega}_1 \times \boldsymbol{r}_{c_1} + \boldsymbol{\omega}_2 \times \boldsymbol{r}_{l_2} + \boldsymbol{\omega}_3 \times \boldsymbol{r}_{l_3}, \tag{A.16e}$$

where

$$\boldsymbol{r}_{b_2} = \begin{bmatrix} -b_2, & 0, & 0 \end{bmatrix}^{\mathrm{T}}, \tag{A.17a}$$

$$\boldsymbol{r}_{l_3} = \begin{bmatrix} -l_3, & 0, & 0] \end{bmatrix}^{\mathrm{T}}. \tag{A.17b}$$

A.5 Lateral acceleration for the dolly and semitrailer

The lateral acceleration for the dolly and semitrailer are derived as

$$\boldsymbol{a}_{c1} = \boldsymbol{a}_{CG1} + \dot{\boldsymbol{\omega}}_1 \times \boldsymbol{r}_{c1} + \boldsymbol{\omega_1} \times (\boldsymbol{\omega}_1 \times \boldsymbol{r}_{c1}), \tag{A.18a}$$
$$\boldsymbol{a}_{CG2} = \boldsymbol{a}_{c1} + \dot{\boldsymbol{\omega}}_2 \times \boldsymbol{r}_{a2} + \boldsymbol{\omega_2} \times (\boldsymbol{\omega}_2 \times \boldsymbol{r}_{a2}), \tag{A.18b}$$
$$\boldsymbol{a}_{c2} = \boldsymbol{a}_{C1} + \dot{\boldsymbol{\omega}}_2 \times \boldsymbol{r}_{l2} + \boldsymbol{\omega_2} \times (\boldsymbol{\omega}_2 \times \boldsymbol{r}_{l2}), \tag{A.18c}$$
$$\boldsymbol{a}_{CG2} = \boldsymbol{a}_{C2} + \dot{\boldsymbol{\omega}}_3 \times \boldsymbol{r}_{a3} + \boldsymbol{\omega_3} \times (\boldsymbol{\omega}_3 \times \boldsymbol{r}_{a3}). \tag{A.18d}$$

After linearization the expressions for the lateral accelerations of the dolly and semitrailer become

$$a_{y,2} = \dot{v}_{y1} + Vr - a_2\ddot{\theta}_1 - (a_2 + c_1)\dot{r}, \tag{A.19a}$$
$$a_{y,3} = \dot{v}_{y1} + Vr - (a_3 + l_2)\ddot{\theta}_1 - a_3\ddot{\theta}_2 - (a_3 + l_2 + c_1)\dot{r}. \tag{A.19b}$$

The non-linear expressions can be found in Appendix B.

B Matlab output

To get the following output, run truck_dolly_semitrailer.m (Appendix C), and type the variable name of interest in the Matlab Command Window.

```
% Velocity of CoG of the truck
v1 =

[ vx1(t), vy1(t)]

% Velocity of CoG of the dolly
v2 =

[ vx1(t) + a2*sin(theta1(t))*(r(t) + theta1dot(t)), vy1(t) - c1*r(t) - a2*cos(theta1(t))*(r(t) + theta1dot(t))]

% Velocity of CoG of the trailer
v3 =

[ vx1(t) + l2*sin(theta1(t))*(r(t) + theta1dot(t)) + a3*sin(theta1(t) + theta2(t))*(r(t) + theta1dot(t) + theta2dot(t)),
  vy1(t) - c1*r(t) - l2*cos(theta1(t))*(r(t) + theta1dot(t)) - a3*cos(theta1(t) + theta2(t))*(r(t) + theta1dot(t) + theta2dot(t))]

% Kinetic energy, T = T1 + T2 + T3
T =

(m3*((c1*r(t) - vy1(t) + l2*cos(theta1(t))*(r(t) + theta1dot(t)) + a3*cos(theta1(t) + theta2(t))*(r(t) + theta1dot(t)...
 + theta2dot(t)))^2 + (vx1(t) + l2*sin(theta1(t))*(r(t) + theta1dot(t)) + a3*sin(theta1(t) + theta2(t))*(r(t)...
 + theta1dot(t) + theta2dot(t)))^2))/2 + (m2*((c1*r(t) - vy1(t) + a2*cos(theta1(t))*(r(t) + theta1dot(t)))^2 + (vx1(t)...
 + a2*sin(theta1(t))*(r(t) + theta1dot(t)))^2))/2 + (J3*(r(t) + theta1dot(t) + theta2dot(t))^2)/2 + (J2*(r(t)...
 + theta1dot(t))^2)/2 + (m1*(vx1(t)^2 + vy1(t)^2))/2 + (J1*r(t)^2)/2

% A
A =

m1*vx1(t) + m3*vx1(t) + (m2*(2*vx1(t) + 2*a2*sin(theta1(t))*(r(t) + theta1dot(t))))/2 + l2*m3*sin(theta1(t))*(r(t)...
+ theta1dot(t)) + a3*m3*sin(theta1(t) + theta2(t))*(r(t) + theta1dot(t) + theta2dot(t))

% B
B =

m1*vy1(t) - (m2*(2*c1*r(t) - 2*vy1(t) + 2*a2*cos(theta1(t))*(r(t) + theta1dot(t))))/2 - (m3*(2*c*r(t) - 2*vy1(t)...
+ 2*l2*cos(theta1(t))*(r(t) + theta1dot(t)) + 2*a3*cos(theta1(t) + theta2(t))*(r(t) + theta1dot(t) + theta2dot(t))))/2

% dTdtheta1dot
dTdtheta1dot =

(m2*(2*a2*cos(theta1(t))*(c1*r(t) - vy1(t) + a2*cos(theta1(t))*(r(t) + theta1dot(t))) + 2*a2*sin(theta1(t))*(vx1(t)...
+ a2*sin(theta1(t))*(r(t) + theta1dot(t)))))/2 + (J2*(2*r(t) + 2*theta1dot(t)))/2 + (m3*(2*(l2*cos(theta1(t))...
+ a3*cos(theta1(t) + theta2(t)))*(c1*r(t) - vy1(t) + l2*cos(theta1(t))*(r(t) + theta1dot(t)) + a3*cos(theta1(t)...
+ theta2(t))*(r(t) + theta1dot(t) + theta2dot(t))) + 2*(l2*sin(theta1(t)) + a3*sin(theta1(t) + theta2(t)))*(vx1(t)...
+ l2*sin(theta1(t))*(r(t) + theta1dot(t)) + a3*sin(theta1(t) + theta2(t))*(r(t) + theta1dot(t) + theta2dot(t)))))/2...
+ (J3*(2*r(t) + 2*theta1dot(t) + 2*theta2dot(t)))/2

% dTdtheta2dot
dTdtheta2dot =

J3*r(t) + J3*theta1dot(t) + J3*theta2dot(t) + a3^2*m3*r(t) + a3^2*m3*theta1dot(t) + a3^2*m3*theta2dot(t)...
- a3*m3*cos(theta1(t) + theta2(t))*vy1(t) + a3*m3*sin(theta1(t) + theta2(t))*vx1(t) + a3*c1*m3*cos(theta1(t)...
+ theta2(t))*r(t) + a3*l2*m3*cos(theta2(t))*r(t) + a3*l2*m3*cos(theta2(t))*theta1dot(t)

% dTdr
dTdr =

(m3*(2*(l2*sin(theta1(t)) + a3*sin(theta1(t) + theta2(t)))*(vx1(t) + l2*sin(theta1(t))*(r(t) + theta1dot(t))...
 + a3*sin(theta1(t) + theta2(t))*(r(t) + theta1dot(t) + theta2dot(t))) + 2*(c1 + l2*cos(theta1(t))...
 + a3*cos(theta1(t) + theta2(t)))*(c1*r(t) - vy1(t) + l2*cos(theta1(t))*(r(t) + theta1dot(t)) + a3*cos(theta1(t)...
 + theta2(t))*(r(t) + theta1dot(t) + theta2dot(t)))))/2 + J1*r(t) + (m2*(2*(c1 + a2*cos(theta1(t)))*(c1*r(t)...
 - vy1(t) + a2*cos(theta1(t))*(r(t) + theta1dot(t))) + 2*a2*sin(theta1(t))*(vx1(t) + a2*sin(theta1(t))*(r(t)...
 + theta1dot(t)))))/2 + J3*(r(t) + theta1dot(t) + theta2dot(t)) + J2*(r(t) + theta1dot(t))

% dTdtheta1
dTdtheta1 =

(m3*(2*(l2*cos(theta1(t))*(r(t) + theta1dot(t)) + a3*cos(theta1(t) + theta2(t))*(r(t) + theta1dot(t)...
 + theta2dot(t)))*(vx1(t) + l2*sin(theta1(t))*(r(t) + theta1dot(t)) + a3*sin(theta1(t) + theta2(t))*(r(t)...
 + theta1dot(t) + theta2dot(t))) - 2*(l2*sin(theta1(t))*(r(t) + theta1dot(t)) + a3*sin(theta1(t) + theta2(t))*(r(t)...
 + theta1dot(t) + theta2dot(t)))*(c1*r(t) - vy1(t) + l2*cos(theta1(t))*(r(t) + theta1dot(t)) + a3*cos(theta1(t)...
 + theta2(t))*(r(t) + theta1dot(t) + theta2dot(t)))))/2 + a2*m2*(r(t) + theta1dot(t))*(cos(theta1(t))*vx1(t)...
 + sin(theta1(t))*vy1(t) - c1*sin(theta1(t))*r(t))

% dTdtheta2
dTdtheta2 =
```

```
-a3*m3*(r(t) + theta1dot(t) + theta2dot(t))*(l2*sin(theta2(t))*r(t) - sin(theta1(t) + theta2(t))*vy1(t) - cos(theta1(t)...
+ theta2(t))*vx1(t) + l2*sin(theta2(t))*theta1dot(t) + c1*sin(theta1(t) + theta2(t))*r(t))

% dTdpsi1
dTdpsi1 =

vy1(t)*((m3*(2*vx1(t) + 2*l2*sin(theta1(t))*(r(t) + theta1dot(t)) + 2*a3*sin(theta1(t) + theta2(t))*(r(t) + theta1dot(t)...
+ theta2dot(t))))/2 + m1*vx1(t) + (m2*(2*vx1(t) + 2*a2*sin(theta1(t))*(r(t) + theta1dot(t))))/2) + vx1(t)*((m2*(2*c1*r(t)...
- 2*vy1(t) + 2*a2*cos(theta1(t))*(r(t) + theta1dot(t))))/2 - m1*vy1(t) + (m3*(2*c1*r(t) - 2*vy1(t) + 2*l2*cos(theta1(t))*(r(t)...
+ theta1dot(t)) + 2*a3*cos(theta1(t) + theta2(t))*(r(t) + theta1dot(t) + theta2dot(t))))/2)

% dTdX1
dTdX1 =

0

% dTdY1
dTdY1 =

0

% Lateral acceleration of the dolly (non-linear)
ay2 =

vdot - a2*(rdot + theta1dotdot) - c1*rdot + r*vx1

% Lateral acceleration of the semitrailer (non-linear)
ay3 =

vdot - l2*(rdot + theta1dotdot) - c1*rdot + r*vx1 - a3*(rdot + theta1dotdot + theta2dotdot)

% Eigenvalues for 40 Kph
eval_40 =

  -2.9748 + 2.7718i        0                  0                  0                  0                  0
        0           -2.9748 - 2.7718i        0                  0                  0                  0
        0                  0           -2.5469 + 0.9185i        0                  0                  0
        0                  0                  0           -2.5469 - 0.9185i        0                  0
        0                  0                  0                  0           -5.0864 + 1.2959i        0
        0                  0                  0                  0                  0           -5.0864 - 1.2959i

% Eigenvalues for 80 Kph
eval_80 =

  -1.4877 + 3.7839i        0                  0                  0                  0                  0
        0           -1.4877 - 3.7839i        0                  0                  0                  0
        0                  0           -1.2823 + 2.3954i        0                  0                  0
        0                  0                  0           -1.2823 - 2.3954i        0                  0
        0                  0                  0                  0           -2.5341 + 1.2988i        0
        0                  0                  0                  0                  0           -2.5341 - 1.2988i
```

C Matlab program

```
%% =======================================================================
%
% truck_dolly_semitrailer_model
% - - - - - - - - - - - - - - -
%
% Program deriving linearized forms of Lagrange's equations for a
% truck-dolly-semitrailer, using Symbolic Math Toolbox. A one-track model
% is used to describe the vehicles.
%
%
% Using non-standard function(s):   R2
%                                   R3
%                                   fdiff
%                                   mtaylor
%                                   odefunc_#
%
% Written by:   Anders Sjblom
%               Michael Levn
%
% Last edit:    2011-06-30
%
% =======================================================================
clear all; close all; clc

% define (real) variables
syms m1 m2 m3 J1 J2 J3 c1 a2 a3 l2  t real

% define time-dependent variables
vx1 = sym('vx1(t)');                   % longitudinal velocity
vy1 = sym('vy1(t)');                   % lateral velocity

syms r theta1 theta2 psii1 theta1dot theta2dot real
omega1 = [0,0,r];                      % yaw rate, first unit
omega2 = [0,0,r+theta1dot];            % yaw rate, second unit
omega3 = [0,0,r+theta1dot+theta2dot];  % yaw rate, third unit

% explicit expression for the velocity of CoG of the truck
v1 = [vx1,vy1];

% explicit expression for the velocity of CoG of the dolly
wxr=cross(omega1,([-c1,0,0]')')+cross(omega2,(R3(theta1)*[-a2,0,0]')');
wxr=wxr(1,1:end-1)';
v2 = v1 + wxr.';

% explicit expression for the velocity of CoG of the trailer
wxr=cross(omega1,([-c1,0,0]')')+cross(omega2,(R3(theta1)*[-l2,0,0]')')...
    +cross(omega3,(R3(theta1+theta2)*[-a3,0,0]')');
wxr=wxr(1,1:end-1)';
v3 = v1 + wxr.';

% re-define variables (time-dependence)
psii1 = sym('psii1(t)');               % yaw angle, psi1
r = sym('r(t)');                       % yaw rate, r (dpsi1/dt)
theta1 = sym('theta1(t)');             % articulation angle between truck and dolly
theta1dot = sym('theta1dot(t)');       % derivative of the articulation angle
theta2 = sym('theta2(t)');             % articulation angle between dolly and semitrailer
theta2dot = sym('theta2dot(t)');       % derivative of the articulation angle

v1=subs(v1,'r','r(t)');
v1=subs(v1,'psii1','psii1(t)');
v1=subs(v1,'theta1','theta1(t)');
v1=subs(v1,'theta2','theta2(t)');
v1=subs(v1,'theta1dot','theta1dot(t)');
v1=subs(v1,'theta2dot','theta2dot(t)');
v2=subs(v2,'r','r(t)');
v2=subs(v2,'psii1','psii1(t)');
v2=subs(v2,'theta1','theta1(t)');
v2=subs(v2,'theta2','theta2(t)');
v2=subs(v2,'theta1dot','theta1dot(t)');
v2=subs(v2,'theta2dot','theta2dot(t)');
v3=subs(v3,'r','r(t)');
v3=subs(v3,'psii1','psii1(t)');
v3=subs(v3,'theta1','theta1(t)');
v3=subs(v3,'theta2','theta2(t)');
v3=subs(v3,'theta1dot','theta1dot(t)');
v3=subs(v3,'theta2dot','theta2dot(t)');



%% -== LAGRANGE'S EQUATIONS ==-

% kinetic energy, T = T1 + T2 + T3
T = 1/2*m1*(v1*v1.') + 1/2*m2*(v2*v2.') + 1/2*m3*(v3*v3.') + 1/2*J1*r^2 + 1/2*J2*(r+theta1dot)^2  + 1/2*J3*(r + theta1dot + theta2dot)^2;

Aa = fdiff(T,'vx1(t)');                                                      % A = dT/dvx1
Bb = fdiff(T,'vy1(t)');                                                      % B = dT/dvy1

Cc = [(Aa*cos(psii1)-Bb*sin(psii1)),(Aa*sin(psii1)+Bb*cos(psii1))];          % C = [dT/dX1dot, dT/dY1dot] = [A*dvx1/dX1dot + B*dvy1/dX1dot , A*dvx1/dY1dot + B*dvy1/dY1dot]
Dd = diff(Cc,'t');                                                           % D = d/dt*([dT/dX1dot, dT/dY1dot])
Ee = Dd*R2(psii1);                                                           % E = D*R2(psi1) (tranformed to body-fixed system)

% Ee =
% Adot - Bb*r ( = Qx) (1)
% Bdot + Aa*r ( = Qy) (2)

dTdr = fdiff(T,'r(t)');                                                      % dT/dr
ddtdTdr = diff(dTdr,'t');                                                    % d/dt*(dT/dr)
dTdpsii1 = vy1*Aa - vx1*Bb;                                                  % dT/dpsii1 = dT/dvx1*dvx1/dpsi1 + dT/dvy1*dvy1/dpsi1 = A*vy1 - B*vx1
% d/dt*(dT/dr) - vy1*Aa + vx1*Bb ( = Qr) (3)

dTdtheta1dot = fdiff(T,'theta1dot(t)');
ddtdTdtheta1dot = diff(dTdtheta1dot,'t');
dTdtheta1 = fdiff(T,'theta1(t)');
% d/dt*(dT/dtheta1dot) - dT/dtheta1dot ( = Qt) (4)

dTdtheta2dot = fdiff(T,'theta2dot(t)');
```

```
ddtdTdtheta2dot = diff(dTdtheta2dot,'t');
dTdtheta2 = fdiff(T,'theta2(t)');
% d/dt*(dT/dtheta1dot) - dT/dtheta1dot ( = Qt2) (5)

% arranging Lagrange's equations in 'L',
L = [Ee(1); Ee(2); ddtdTdr - dTdpsi1; ddtdTdtheta1dot - dTdtheta1; ddtdTdtheta2dot - dTdtheta2]; % L = [(1); (2); (3); (4); (5)]

% substituting variable names
syms vx1 vx1dot vy1 vy1dot r rdot theta1 theta1dot theta1dotdot theta2 theta2dot theta2dotdot V real
L = subs(L,'diff(vx1(t), t)', vx1dot);
L = subs(L,'diff(vy1(t), t)', vy1dot);
L = subs(L,'diff(r(t), t)', rdot);
L = subs(L,'diff(psii1(t), t)', r);
L = subs(L,'diff(theta1(t), t)', theta1dot);
L = subs(L,'diff(theta1dot(t), t)', theta1dotdot);
L = subs(L,'diff(theta2(t), t)', theta2dot);
L = subs(L,'diff(theta2dot(t), t)', theta2dotdot);
L = subs(L, 'vx1(t)' , vx1);
L = subs(L, 'vy1(t)' , vy1);
L = subs(L, 'r(t)' , r);
L = subs(L, 'theta1(t)' , theta1);
L = subs(L, 'theta2(t)' , theta2);
L = subs(L, 'theta1dot(t)' , theta1dot);
L = subs(L, 'theta2dot(t)' , theta2dot);

% linearize the equations using 1st order Taylor expansion ('mtaylor')
for i = 1:length(L)
    L_l(i) = mtaylor(L(i), [vx1 vx1dot vy1 vy1dot r rdot theta1 theta1dot theta1dotdot theta2 theta2dot theta2dotdot], [V 0 0 0 0 0 0 0 0 0 0 0]);
end

% Write the equations on state form, with q = [Y1 psi1 theta1 theta2]
% M*q'' + C*q' + K*q = F

% Mass Matrix, M (Contribution from the left hand side)
M11 = diff(L_l(2),vy1dot);
M12 = diff(L_l(2),rdot);
M13 = diff(L_l(2),theta1dotdot);
M14 = diff(L_l(2),theta2dotdot);
M21 = diff(L_l(3),vy1dot);
M22 = diff(L_l(3),rdot);
M23 = diff(L_l(3),theta1dotdot);
M24 = diff(L_l(3),theta2dotdot);
M31 = diff(L_l(4),vy1dot);
M32 = diff(L_l(4),rdot);
M33 = diff(L_l(4),theta1dotdot);
M34 = diff(L_l(4),theta2dotdot);
M41 = diff(L_l(5),vy1dot);
M42 = diff(L_l(5),rdot);
M43 = diff(L_l(5),theta1dotdot);
M44 = diff(L_l(5),theta2dotdot);

M = [M11,M12,M13 M14; M21,M22,M23 M24; M31,M32,M33,M34; M41,M42,M43,M44];
M = simple(M);

% Stiffness Matrix, C1 (Contribution from the left hand side)
C111 = diff(L_l(2),vy1);
C112 = diff(L_l(2),r);
C113 = diff(L_l(2),theta1dot);
C114 = diff(L_l(2),theta2dot);
C121 = diff(L_l(3),vy1);
C122 = diff(L_l(3),r);
C123 = diff(L_l(3),theta1dot);
C124 = diff(L_l(3),theta2dot);
C131 = diff(L_l(4),vy1);
C132 = diff(L_l(4),r);
C133 = diff(L_l(4),theta1dot);
C134 = diff(L_l(4),theta2dot);
C141 = diff(L_l(5),vy1);
C142 = diff(L_l(5),r);
C143 = diff(L_l(5),theta1dot);
C144 = diff(L_l(5),theta2dot);

C_1 = [C111,C112,C113,C114; C121,C122,C123,C124; C131,C132,C133,C134; C141,C142,C143,C144];



%% -== GENERALIZED FORCES ==-

% define (real) variables
syms X1 Y1 psii1 theta1 theta2 real                                              % generalized coordinates
syms vx1 vx1dot vy1 vy1dot r rdot theta1dot theta1dotdot theta2dot theta2dotdot real % derivatives of the generalized coordinates
syms a1 a2 a3 b1r b1t b2 l3 c1 l2 real                                           % length parameters
syms F1fx F1rx F1tx F2x F3x real                                                 % longitudinal forces
syms C1f C1r C1t C2 C3 real                                                      % cornering stiffnesses
syms delta11 delta2 delta3 real                                                  % steering angles
syms V real                                                                      % linearization point for the longitudinal velocity

% neglect longitudinal forces
F1fx=0;F1rx=0;F1tx=0;F2x=0;F3x=0;

% Velocities of the tires
% Truck (front, rear, tag)
wxr=cross(omega1,(R3(psii1)*[a1,0,0].').');wxr=wxr(1,1:end-1);
v1f = [vx1,vy1]*R2(psii1).' + wxr;
wxr=cross(omega1,(R3(psii1)*[-b1r,0,0].').');wxr=wxr(1,1:end-1);
v1r = [vx1,vy1]*R2(psii1).' + wxr;
wxr=cross(omega1,(R3(psii1)*[-b1t,0,0].').');wxr=wxr(1,1:end-1);
v1t = [vx1,vy1]*R2(psii1).' + wxr;

% Dolly (lumped)
wxr=cross(omega1,(R3(psii1)*[-c1,0,0].').')+cross(omega2,(R3(psii1+theta1)*[-b2,0,0].').');
wxr=wxr(1,1:end-1);
v21 = [vx1,vy1]*R2(psii1).' + [c1*r*sin(psii1),-c1*r*cos(psii1)]...
    + [b2*(r + theta1dot)*sin(psii1 + theta1), -b2*(r + theta1dot)*cos(psii1 + theta1)];

% Semi-trailer (lumped)
v31 = [vx1,vy1]*R2(psii1).' + [c1*r*sin(psii1),-c1*r*cos(psii1)] + [l2*(r + theta1dot)*sin(psii1 + theta1), -l2*(r + theta1dot)*cos(psii1 + theta1)]...
      + [l3*(r + theta1dot + theta2dot)*sin(psii1 + theta1 + theta2), -l3*(r + theta1dot + theta2dot)*cos(psii1 + theta1 + theta2)];

% rotate to the body-fixed system
v1f = v1f*R2(psii1);
v1r = v1r*R2(psii1);
```

```
v1t = v1t*R2(psii1);
v2l = v2l*R2(psii1+theta1);
v3l = v3l*R2(psii1+theta1+theta2);

% slip angles, alpha
alpha1f = atan(-v1f(2)/v1f(1)) + delta11;
alpha1r = atan(-v1r(2)/v1r(1));
alpha1t = atan(-v1t(2)/v1t(1));
alpha2 = atan(-v2l(2)/v2l(1)) + delta2;
alpha3 = atan(-v3l(2)/v3l(1)) + delta3;

% lateral forces (as functions of slip angle, alpha, and cornering stiffness, C)
F1fy = C1f*alpha1f;
F1ry = C1r*alpha1r;
F1ty = C1t*alpha1t;
F2y = C2*alpha2;
F3y = C3*alpha3;

% Forces and corresponding position vectors
% Truck
% front tire
F1f = [F1fx,F1fy]*R2(delta11+psii1).';
r1f = [X1 + a1*cos(psii1), Y1 + a1*sin(psii1)]; % position vector of the forces on front tire

% differentiate the position vector w.r.t. the generalized coordinates
dr1fdX1 = diff(r1f,X1);
dr1fdY1 = diff(r1f,Y1);
dr1fdpsi1 = diff(r1f,psii1);
dr1fdtheta1 = diff(r1f,theta1);
dr1fdtheta2 = diff(r1f,theta2);

% rear tire
F1r=[F1rx,F1ry]*R2(psii1).';
r1r = [X1 - b1r*cos(psii1), Y1 - b1r*sin(psii1)]; % position vector of the forces on rear tire

% differentiate the position vector w.r.t. the generalized coordinates
dr1rdX1 = diff(r1r,X1);
dr1rdY1 = diff(r1r,Y1);
dr1rdpsi1 = diff(r1r,psii1);
dr1rdtheta1 = diff(r1r,theta1);
dr1rdtheta2 = diff(r1r,theta2);

% tag tire
F1t=[F1tx,F1ty]*R2(psii1).';
r1t = [X1 - b1t*cos(psii1), Y1 - b1t*sin(psii1)]; % position vector of the forces on tag tire

dr1tdX1 = diff(r1t,X1);
dr1tdY1 = diff(r1t,Y1);
dr1tdpsi1 = diff(r1t,psii1);
dr1tdtheta1 = diff(r1t,theta1);
dr1tdtheta2 = diff(r1t,theta2);

% generalized forces for the truck
Qx1 = F1f*dr1fdX1.' + F1r*dr1rdX1.' + F1t*dr1tdX1.';
Qy1 = F1f*dr1fdY1.' + F1r*dr1rdY1.' + F1t*dr1tdY1.';
Qp1 = F1f*dr1fdpsi1.' + F1r*dr1rdpsi1.'+ F1t*dr1tdpsi1.';
Qt11 = F1f*dr1fdtheta1.' + F1r*dr1rdtheta1.' + F1t*dr1tdtheta1.';
Qt12 = F1f*dr1fdtheta2.' + F1r*dr1rdtheta2.' + F1t*dr1tdtheta2.';

% Dolly
% 1st tire
F2 = [F2x,F2y]*R2(psii1 + delta2 + theta1).';
r2 = [X1 - c1*cos(psii1)-b2*cos(psii1+theta1), Y1 - c1*sin(psii1)-b2*sin(psii1+theta1)];

% differentiate the position vector w.r.t. the generalized coordinates
dr2dX1 = diff(r2,X1);
dr2dY1 = diff(r2,Y1);
dr2dpsi1 = diff(r2,psii1);
dr2dtheta1 = diff(r2,theta1);
dr2dtheta2 = diff(r2,theta2);

% generalized forces for the dolly
Qx2 = F2*dr2dX1.';
Qy2 = F2*dr2dY1.';
Qp2 = F2*dr2dpsi1.';
Qt21 = F2*dr2dtheta1.';
Qt22 = F2*dr2dtheta2.';

%Semi-trailer
% 1st tire
F3 = [F3x,F3y]*R2(delta2+psii1+theta1+theta2).';
r3 = [X1 - c1*cos(psii1)-l2*cos(psii1+theta1) - l3*cos(psii1+theta1+theta2), Y1 - c1*sin(psii1)-l2*sin(psii1+theta1) - l3*sin(psii1+theta1+theta2)];

% differentiate the position vector w.r.t. the generalized coordinates
dr3dX1 = diff(r3,X1);
dr3dY1 = diff(r3,Y1);
dr3dpsi1 = diff(r3,psii1);
dr3dtheta1 = diff(r3,theta1);
dr3dtheta2 = diff(r3,theta2);

% generalized forces for the semi-trailer
Qx3 = F3*dr3dX1.';
Qy3 = F3*dr3dY1.';
Qp3 = F3*dr3dpsi1.';
Qt31 = F3*dr3dtheta1.';
Qt32 = F3*dr3dtheta2.';

% assembly of the generalized forces
Qx = Qx1 + Qx2 + Qx3;
Qy = Qy1 + Qy2 + Qy3;
Qp = Qp1 + Qp2 + Qp3;
Qt1 = Qt11 + Qt21 + Qt31;
Qt2 = Qt12 + Qt22 + Qt32;

% arranging generalized foces in 'Q' and linearize with taylor
Q =[Qx,Qy,Qp]*R3(psii1);
Q =[Q,Qt1,Qt2];

% linearize the equations using 1st order Taylor expansion ('mtaylor')
for i=1:length(Q)
Q_l(i) = mtaylor(Q(i),[vx1 vx1dot vy1 vy1dot r rdot theta1 theta1dot theta1dotdot theta2 theta2dot theta2dotdot delta11 delta2 delta3],[V 0 0 0 0 0 0 0 0 0 0 0 0 0 0]);
end
```

```
Q_l = Q_l.';

% Stiffness matrix C2 (Contribution from right hand side)
C211 = -diff(Q_l(2),vy1);
C212 = -diff(Q_l(2),r);
C213 = -diff(Q_l(2),theta1dot);
C214 = -diff(Q_l(2),theta2dot);
C221 = -diff(Q_l(3),vy1);
C222 = -diff(Q_l(3),r);
C223 = -diff(Q_l(3),theta1dot);
C224 = -diff(Q_l(3),theta2dot);
C231 = -diff(Q_l(4),vy1);
C232 = -diff(Q_l(4),r);
C233 = -diff(Q_l(4),theta1dot);
C234 = -diff(Q_l(4),theta2dot);
C241 = -diff(Q_l(5),vy1);
C242 = -diff(Q_l(5),r);
C243 = -diff(Q_l(5),theta1dot);
C244 = -diff(Q_l(5),theta2dot);

C_2 = [C211,C212,C213,C214; C221,C222,C223,C224; C231,C232,C233,C234; C241,C242,C243,C244];

% assembling stiffness matrix C
C = C_1 + C_2;
C = simple(C);

% K (Contribution from right hand side)
K11 = -diff(Q_l(2),Y1);
K12 = -diff(Q_l(2),psii1);
K13 = -diff(Q_l(2),theta1);
K14 = -diff(Q_l(2),theta2);

K21 = -diff(Q_l(3),Y1);
K22 = -diff(Q_l(3),psii1);
K23 = -diff(Q_l(3),theta1);
K24 = -diff(Q_l(3),theta2);

K31 = -diff(Q_l(4),Y1);
K32 = -diff(Q_l(4),psii1);
K33 = -diff(Q_l(4),theta1);
K34 = -diff(Q_l(4),theta2);

K41 = -diff(Q_l(5),Y1);
K42 = -diff(Q_l(5),psii1);
K43 = -diff(Q_l(5),theta1);
K44 = -diff(Q_l(5),theta2);

K = [K11,K12,K13,K14; K21,K22,K23,K24; K31,K32,K33,K34; K41,K42,K43,K44];
K = simple(K);

% F (Contribution from the right hand side)
F1 = Q_l(2) + K11*Y1 + K12*psii1 + K13*theta1 + K14*theta2 + C211*vy1 + C212*r + C213*theta1dot + C214*theta2dot;
F2 = Q_l(3) + K21*Y1 + K22*psii1 + K23*theta1 + K24*theta2 + C221*vy1 + C222*r + C223*theta1dot + C224*theta2dot;
F3 = Q_l(4) + K31*Y1 + K32*psii1 + K33*theta1 + K34*theta2 + C231*vy1 + C232*r + C233*theta1dot + C234*theta2dot;
F4 = Q_l(5) + K41*Y1 + K42*psii1 + K43*theta1 + K44*theta2 + C241*vy1 + C242*r + C243*theta1dot + C244*theta2dot;

F = [F1;F2;F3;F4];
F = simple(F);

% ... = Fu + Hd, as in the report (not used here)
FF(1,1) = diff(F(1),delta11);
FF(2,1) = diff(F(2),delta11);
FF(3,1) = diff(F(3),delta11);
FF(4,1) = diff(F(4),delta11);

HH(1,1) = diff(F(1),delta2);
HH(2,1) = diff(F(2),delta2);
HH(3,1) = diff(F(3),delta2);
HH(4,1) = diff(F(4),delta2);
HH(1,2) = diff(F(1),delta3);
HH(2,2) = diff(F(2),delta3);
HH(3,2) = diff(F(3),delta3);
HH(4,2) = diff(F(4),delta3);



%% -== REDUCE TO 1ST ORDER DE ==-

% x' = Ax + Bu + Ed, where
% x = [y1 psi1 theta1 theta2 vy1 r theta1dot theta2dot],
% u = [delta2 delta3],
% d = [delta11]

invM=inv(M); % inverse of the mass matrix

A = -[zeros(4) -eye(4); invM*K invM*C];
G = [zeros(4,1); invM*F];

B11 = diff(G(1),delta2);
B12 = diff(G(1),delta3);
B21 = diff(G(2),delta2);
B22 = diff(G(2),delta3);
B31 = diff(G(3),delta2);
B32 = diff(G(3),delta3);
B41 = diff(G(4),delta2);
B42 = diff(G(4),delta3);
B51 = diff(G(5),delta2);
B52 = diff(G(5),delta3);
B61 = diff(G(6),delta2);
B62 = diff(G(6),delta3);
B71 = diff(G(7),delta2);
B72 = diff(G(7),delta3);
B81 = diff(G(8),delta2);
B82 = diff(G(8),delta3);

B = [B11,B12; B21,B22; B31,B32; B41,B42; B51,B52; B61,B62; B71,B72; B81,B82];

E1 = diff(G(5),delta11);
E2 = diff(G(6),delta11);
E3 = diff(G(7),delta11);
E4 = diff(G(8),delta11);
```

```
E = [0;0;0;0;E1;E2;E3;E4];
E = simple(E);



%% -== INSERT NUMERICAL VALUES ==-

global An Bn En

% run vehicle parameter file
vehicle_parameters

C2 = C21+C22;
C3 = C31+C32+C33;
b2 = (b2f+b2r)/2;
l3 = l32;

Vx = 80/3.6; % 80 kph

An = A;
An = subs(An , 'V', Vx);
An = subs(An , 'C1f', C1f);
An = subs(An , 'C1r', C1r);
An = subs(An , 'C1t', C1t);
An = subs(An , 'C2', C2);
An = subs(An , 'C3', C3);
An = subs(An , 'm1', m1);
An = subs(An , 'm2', m2);
An = subs(An , 'm3', m3);
An = subs(An , 'J1', J1);
An = subs(An , 'J2', J2);
An = subs(An , 'J3', J3);
An = subs(An , 'a1', a1);
An = subs(An , 'a2', a2);
An = subs(An , 'a3', a3);
An = subs(An , 'b1r', b1r);
An = subs(An , 'b1t', b1t);
An = subs(An , 'b2', b2);
An = subs(An , 'l3', l3);
An = subs(An , 'c1', c1);
An = subs(An , 'l2', l2);

Bn = B;
Bn = subs(Bn , 'V', Vx);
Bn = subs(Bn , 'C1f', C1f);
Bn = subs(Bn , 'C1r', C1r);
Bn = subs(Bn , 'C1t', C1t);
Bn = subs(Bn , 'C2', C2);
Bn = subs(Bn , 'C3', C3);
Bn = subs(Bn , 'm1', m1);
Bn = subs(Bn , 'm2', m2);
Bn = subs(Bn , 'm3', m3);
Bn = subs(Bn , 'J1', J1);
Bn = subs(Bn , 'J2', J2);
Bn = subs(Bn , 'J3', J3);
Bn = subs(Bn , 'a1', a1);
Bn = subs(Bn , 'a2', a2);
Bn = subs(Bn , 'a3', a3);
Bn = subs(Bn , 'b1r', b1r);
Bn = subs(Bn , 'b1t', b1t);
Bn = subs(Bn , 'b2', b2);
Bn = subs(Bn , 'l3', l3);
Bn = subs(Bn , 'c1', c1);
Bn = subs(Bn , 'l2', l2);

En = E;
En = subs(En , 'V', Vx);
En = subs(En , 'C1f', C1f);
En = subs(En , 'C1r', C1r);
En = subs(En , 'C1t', C1t);
En = subs(En , 'C2', C2);
En = subs(En , 'C3', C3);
En = subs(En , 'm1', m1);
En = subs(En , 'm2', m2);
En = subs(En , 'm3', m3);
En = subs(En , 'J1', J1);
En = subs(En , 'J2', J2);
En = subs(En , 'J3', J3);
En = subs(En , 'a1', a1);
En = subs(En , 'a2', a2);
En = subs(En , 'a3', a3);
En = subs(En , 'b1r', b1r);
En = subs(En , 'b1t', b1t);
En = subs(En , 'b2', b2);
En = subs(En , 'l3', l3);
En = subs(En , 'c1', c1);
En = subs(En , 'l2', l2);

% output
% y = Cx + Dd, where
% y = [ay1 r beta1 theta1dot theta2dot r2 r3 ay2 ay3 vy2 vy3...
%      psi1 theta1 theta2]

a61 = -(a2+c1);
a71 = -a2;
a62 = -(a3+c1+l2);
a72 = -(a3+l2);
a82 = -a3;

CC = [An(5,1) An(5,2) An(5,3) An(5,4) An(5,5) (An(5,6)+Vx) An(5,7) An(5,8); % ay1
    0 0 0 0 0 1 0 0;                                                     % r
    0 0 0 0 -1/Vx 0 0 0;                                                 % beta1
    0 0 0 0 0 0 1 0;                                                     % theta1dot
    0 0 0 0 0 0 0 1;                                                     % theta2dot
    0 0 0 0 0 1 1 0;                                                     % r2
    0 0 0 0 0 1 1 1;                                                     % r3

    (An(5,1)+An(6,1)*a61+An(7,1)*a71) (An(5,2)+An(6,2)*a61+An(7,2)*a71) (An(5,3)+An(6,3)*a61+An(7,3)*a71) (An(5,4)+An(6,4)*a61+An(7,4)*a71)...
    (An(5,5)+An(6,5)*a61+An(7,5)*a71) (An(5,6)+An(6,6)*a61+An(7,6)*a71+Vx) (An(5,7)+An(6,7)*a61+An(7,7)*a71) (An(5,8)+An(6,8)*a61+An(7,8)*a71);     % ay2

    (An(5,1)+An(6,1)*a62+An(7,1)*a72+An(8,1)*a82) (An(5,2)+An(6,2)*a62+An(7,2)*a72+An(8,2)*a82) (An(5,3)+An(6,3)*a62+An(7,3)*a72+An(8,3)*a82)...
    (An(5,4)+An(6,4)*a62+An(7,4)*a72+An(8,4)*a82) (An(5,5)+An(6,5)*a62+An(7,5)*a72+An(8,5)*a82) (An(5,6)+An(6,6)*a62+An(7,6)*a72+An(8,6)*a82+Vx)...
```

```
    (An(5,7)+An(6,7)*a62+An(7,7)*a72+An(8,7)*a82) (An(5,8)+An(6,8)*a62+An(7,8)*a72+An(8,8)*a82);                          % ay3

    0 0 0 0 1 -(a2+c1) -a2 0;                                               % vy2
    0 0 0 0 1 -(a3+c1+l2) -(a3+l2) -a3;                                     % vy3
    0 1 0 0 0 0 0 0;                                                        % psi1
    0 0 1 0 0 0 0 0;                                                        % theta1
    0 0 0 1 0 0 0 0];                                                       % theta2


D = [En(5);0;0;0;0;0;0;En(5)+En(6)*a61+En(7)*a71;En(5)+En(6)*a62+En(7)*a72+En(8)*a82;0;0;0;0;0];



% -== FREQUENCY RESPONSE & SIMULATION ==-

%% Frequency response
close all

% state space
SYS = ss(An,En,CC,D);
% transfer function
tfb = tf(SYS);
% calculate gain & phase angle
[amp,phase,w]=bode(tfb);
% delete extra dimension
amp = squeeze(amp);
phase = squeeze(phase);

% Yaw Rate
figure(1)
a1 = subplot(2,1,1);
graph1 = semilogx(w/(2*pi),amp(2,:),'k','LineWidth',2);
set(get(a1,'XLabel'),'String','Frequency [Hz]');
set(get(a1,'YLabel'),'String','Gain [1/s]');
set(get(a1,'Title'),'String','Yaw rate response to steer angle');
axis([0.01 100 0 5]);
grid on
hold on

a2 = subplot(2,1,2);
graph2 = semilogx(w/(2*pi),phase(2,:),'k','LineWidth',2);
set(get(a2,'XLabel'),'String','Frequency [Hz]');
set(get(a2,'YLabel'),'String','Phase angle [deg]')
axis([0.01 100 -90 360]);
grid on
hold on

a1 = subplot(2,1,1);
graph1 = semilogx(w/(2*pi),amp(6,:),'r--','LineWidth',2);
a2 = subplot(2,1,2);
graph2 = semilogx(w/(2*pi),phase(6,:),'r--','LineWidth',2);

a1 = subplot(2,1,1);
graph1 = semilogx(w/(2*pi),amp(7,:),'b-.','LineWidth',2);
h = legend('$\dot{\psi}_1$','$\dot{\psi}_2$','$\dot{\psi}_3$');
set(h,'Interpreter','latex')
a2 = subplot(2,1,2);
graph2 = semilogx(w/(2*pi),phase(7,:),'b-.','LineWidth',2);

% Lateral Acceleration
figure(2)
a1 = subplot(2,1,1);
graph1 = semilogx(w/(2*pi),amp(1,:)/9.81,'k','LineWidth',2);
set(get(a1,'XLabel'),'String','Frequency [Hz]');
set(get(a1,'YLabel'),'String','Gain [1/rad]');
set(get(a1,'Title'),'String','Lateral acceleration (normalized w.r.t. g) response to steer angle');
axis([0.01 100 0 10]);
grid on
hold on

a2 = subplot(2,1,2);
graph2 = semilogx(w/(2*pi),phase(1,:),'k','LineWidth',2);
set(get(a2,'XLabel'),'String','Frequency [Hz]');
set(get(a2,'YLabel'),'String','Phase angle [deg]')
axis([0.01 100 -90 360]);
grid on
hold on

a1 = subplot(2,1,1);
graph1 = semilogx(w/(2*pi),amp(8,:)/9.81,'r--','LineWidth',2);
a2 = subplot(2,1,2);
graph2 = semilogx(w/(2*pi),phase(8,:),'r--','LineWidth',2);

a1 = subplot(2,1,1);
graph1 = semilogx(w/(2*pi),amp(9,:)/9.81,'b-.','LineWidth',2);
legend('a_{y1}','a_{y2}','a_{y3}')
a2 = subplot(2,1,2);
graph2 = semilogx(w/(2*pi),phase(9,:),'b-.','LineWidth',2);

% Vehicle Slip
figure(3)
a1 = subplot(2,1,1);
graph1 = semilogx(w/(2*pi),amp(3,:),'k','LineWidth',2);
set(get(a1,'XLabel'),'String','Frequency [Hz]');
set(get(a1,'YLabel'),'String','Gain [-]');
set(get(a1,'Title'),'String','Vehicle slip angle response to steer angle');
axis([0.01 100 0 3]);
grid on
hold on

a2 = subplot(2,1,2);
graph2 = semilogx(w/(2*pi),phase(3,:)-360,'k','LineWidth',2);
set(get(a2,'XLabel'),'String','Frequency [Hz]');
set(get(a2,'YLabel'),'String','Phase angle [deg]')
axis([0.01 100 -360 0]);
grid on
hold on

a1 = subplot(2,1,1);
graph1 = semilogx(w/(2*pi),amp(10,:)/Vx,'r--','LineWidth',2);
a2 = subplot(2,1,2);
graph2 = semilogx(w/(2*pi),phase(10,:)-360,'r--','LineWidth',2);
```

```
a1 = subplot(2,1,1);
graph1 = semilogx(w/(2*pi),amp(11,:)/Vx,'b-.','LineWidth',2);
legend('\beta_1','\beta_2','\beta_3')
a2 = subplot(2,1,2);
graph2 = semilogx(w/(2*pi),phase(11,:)-360,'b-.','LineWidth',2);

% Articulation Angles
figure(4)
a1 = subplot(2,1,1);
graph1 = semilogx(w/(2*pi),amp(13,:),'r--','LineWidth',2);
set(get(a1,'XLabel'),'String','Frequency [Hz]');
set(get(a1,'YLabel'),'String','Gain [-]');
set(get(a1,'Title'),'String','Articulation angle response to steer angle');
axis([0.01 100 0 3]);
grid on
hold on

a2 = subplot(2,1,2);
graph2 = semilogx(w/(2*pi),phase(13,:),'r--','LineWidth',2);
set(get(a2,'XLabel'),'String','Frequency [Hz]');
set(get(a2,'YLabel'),'String','Phase angle [deg]')
axis([0.01 100 -200 200]);
grid on
hold on

a1 = subplot(2,1,1);
graph1 = semilogx(w/(2*pi),amp(14,:),'b-.','LineWidth',2);
legend('\theta_1','\theta_2')
a2 = subplot(2,1,2);
graph2 = semilogx(w/(2*pi),phase(14,:),'b-.','LineWidth',2);

%% Simulation, sine (1)

% illustratetion of input signal to sine response
t = linspace(0,10);
delta = zeros(size(t));
omega = 0.4*2*pi;

for i=1:length(t)
    if i<20;
        delta(i)=0;
    elseif i>=20 && i<45
        delta(i) = (3*pi)/180.*sin(omega*(t(i)-t(20)));
    elseif i>=45
        delta(i) = 0;
    end
end
figure(10)
plot(t,delta*180/pi,'b','LineWidth',2)
axis([0 10 -4 4])
title('Steering angle, \delta_{11}')
xlabel('Time, t [sec]')
ylabel('Steering angle, \delta_{11} [deg]')
grid on

% sine response
clear delta t
[q,Y] = ode45(@odefunc_sine1,[0,10],zeros(1,8));
figure(5)
plot(q,Y(:,6)*180/pi,'k','LineWidth',2)
grid on
hold on
plot(q,(Y(:,6)+Y(:,7))*180/pi,'r--','LineWidth',2)
axis([0 10 -15 15])
plot(q,(Y(:,6)+Y(:,7)+Y(:,8))*180/pi,'b-.','LineWidth',2)
xlabel('Time, t [sec]')
ylabel('Yaw rate, r [rad/s]')
title('Sine response - Yaw rate')
h = legend('$\dot{\psi}_1$','$\dot{\psi}_2$','$\dot{\psi}_3$');
set(h,'Interpreter','latex')

figure(6)
plot(q,Y(:,3)*180/pi,'r--','LineWidth',2)
grid on
hold on
plot(q,Y(:,4)*180/pi,'b-.','LineWidth',2)
axis([0 10 -10 10])
xlabel('Time, t [sec]')
ylabel('Articulation angle, \theta [deg]')
title('Sine response - Aritculation angle')
legend('\theta_1','\theta_2')

%% Simulation, sine (2)

% steering angle
t = linspace(0,10);
delta = 0.01*sin(t);
figure(7)
plot(t,delta,'k')
grid on
hold on

% sine response
clear delta t
[q,Y] = ode45(@odefunc_sine2,[0,10],zeros(1,8));
plot(q,Y(:,6),'b')
plot(q,(Y(:,6)+Y(:,7)),'b--')
plot(q,(Y(:,6)+Y(:,7)+Y(:,8)),'b-.')
axis([0 10 -0.04 0.04])
xlabel('Time, t [sec]')
ylabel('Yaw rate, r [rad/s]')
title('Sine response - Yaw rate')
h = legend('$\delta_{11}$','$\dot{\psi}_1$','$\dot{\psi}_2$','$\dot{\psi}_3$');
set(h,'Interpreter','latex')
set(h,'Location','Best')

%% Simulation, step

% sine response
clear delta t
[q,Y] = ode45(@odefunc_step,[0,10],zeros(1,8));
```

```matlab
figure(8)
plot(q,Y(:,6)*180/pi,'k','LineWidth',2)
grid on
hold on
plot(q,(Y(:,6)+Y(:,7))*180/pi,'r--','LineWidth',2)
plot(q,(Y(:,6)+Y(:,7)+Y(:,8))*180/pi,'b-.','LineWidth',2)
xlabel('Time, t [sec]')
ylabel('Yaw rate, r [rad/s]')
title('Step steer response, \delta_{11} = 1^o')
h = legend('$\dot{\psi}_1$','$\dot{\psi}_2$','$\dot{\psi}_3$');
set(h,'Interpreter','latex')

figure(9)
plot(q,Y(:,3)*180/pi,'r--','LineWidth',2)
grid on
hold on
plot(q,Y(:,4)*180/pi,'b-.','LineWidth',2)
xlabel('Time, t [sec]')
ylabel('Articulation angle, \theta [deg]')
title('Step steer response, \delta_{11} = 1^o')
legend('\theta_1','\theta_2')



%% -== EIGENFREQUENCIES ==-

% run vehicle parameter file
vehicle_parameters

An2 = A;
An2 = subs(An2 , 'C1f', C1f);
An2 = subs(An2 , 'C1r', C1r);
An2 = subs(An2 , 'C1t', C1t);
An2 = subs(An2 , 'C2', C2);
An2 = subs(An2 , 'C3', C3);
An2 = subs(An2 , 'm1', m1);
An2 = subs(An2 , 'm2', m2);
An2 = subs(An2 , 'm3', m3);
An2 = subs(An2 , 'J1', J1);
An2 = subs(An2 , 'J2', J2);
An2 = subs(An2 , 'J3', J3);
An2 = subs(An2 , 'a1', a1);
An2 = subs(An2 , 'a2', a2);
An2 = subs(An2 , 'a3', a3);
An2 = subs(An2 , 'b1r', b1r);
An2 = subs(An2 , 'b1t', b1t);
An2 = subs(An2 , 'b2', b2);
An2 = subs(An2 , 'l3', l3);
An2 = subs(An2 , 'c1', c1);
An2 = subs(An2 , 'l2', l2);
An2_10 = subs(An2,V,10/3.6); % 10 kph
An2_40 = subs(An2,V,40/3.6); % 40 kph
An2_80 = subs(An2,V,80/3.6); % 80 kph

[evec_40,eval_40]=eig(An2_40);
[evec_80,eval_80]=eig(An2_80);

evec_40 = evec_40(3:end,3:end) % eigenvectors for 40 kph
eval_40 = eval_40(3:end,3:end) % eigenvalues for 40 kph

evec_80 = evec_80(3:end,3:end) % eigenvectors for 80 kph
eval_80 = eval_80(3:end,3:end) % eigenvalues for 80 kph



% ======================================================================
%
% R2
% - -
%
% 2x2 Rotational matrix [cos(x) -sin(x) ; sin(x) cos(x)] between
% coordinate frames.
%
% SYNTAX:   R = R2(x)
%
% INPUT:    x - angle of rotation, e.g. psii, delta
%
% OUTPUT:   R - rotational matrix [cos(x) -sin(x) ; sin(x) cos(x)]
%
%
% Written by:   Anders Sjblom
%               Michael Levn
%
% Last edit:    2010-12-09
%
% ======================================================================

function R = R2(x)

R = [cos(x) -sin(x) ; sin(x) cos(x)];

end



% ======================================================================
%
% R3
% - -
%
% 3x3 Rotational matrix [cos(x) -sin(x) 0 ; sin(x) cos(x) 0 ; 0 0 1]
% between coordinate frames.
%
% SYNTAX:   R = R3(x)
%
% INPUT:    x - angle of rotation, e.g. psii, delta
%
% OUTPUT:   R - rotational matrix [cos(x) -sin(x) 0 ; sin(x) cos(x) 0;...]
%                                 ... 0 0 1]
%
% Written by:   Anders Sjblom
```

```matlab
%               Michael Levn
%
% Last edit:    2010-12-09
%
% =======================================================================

function R = R3(x)

R = [cos(x) -sin(x) 0 ; sin(x) cos(x) 0 ; 0 0 1];

end
```

```matlab
% =======================================================================
%
% fdiff
% - - - -
%
% Function that differentiate symbolic expressions with respect to
% time-dependent variables.
%
%
% SYNTAX:   M = fdiff(T,'f(t)')
%
% INPUT:    T - symbolic expression
%           f - time-dependent variable, e.g. u(t)
%
% OUTPUT:   M - the derivative of T w.r.t. f(t), i.e. diff(T,f(t))
%
%
% Written by:   Anders Sjblom
%               Michael Levn
%
% Last edit:    2010-12-09
%
% =======================================================================

function M = fdiff(T,f)
s = char(f);        % converts the symbolic variable to a string
syms t ft           % define time, t, and dummy variable ft

eval(['T = subs(T, '' ' s ' '', ft);']);    % substitute time-dependent variable to dummy variable ft
M = diff(T,ft);                             % differentiate symbolic expression T w.r.t. ft
eval(['M = subs(M, ft, '' ' s ' '');']);    % substitute back ft to f(t)
end
```

```matlab
% =======================================================================
%
% mtaylor
% - - - -
%
% Multi-variable first order taylor expansion.
%
% SYNTAX:   T = mtaylor(L,x,p)
%
% INPUT:    L - expression to be linearized
%           x - vector with variables to be linearized
%           p - corresponding points of linearization for the variables in x
%
% OUTPUT:   T - linearized expression
%
%
% Written by:   Anders Sjblom
%               Michael Levn
%
% Last edit:    2010-12-09
%
% =======================================================================

function T = mtaylor(L,x,p)
T = 0;
C = L;

for i=1:length(x)
    C = subs(C,x(i),p(i)); % constant term, f(x0,y0,...)
end

for i=1:length(x)
    M(i) = diff(L,x(i));
    for j=1:length(x)
        M(i) = subs(M(i),x(j),p(j)); % first order terms, df(x0,y0,...)/dx + df(x0,y0,...)/dy + ...
    end
end

for i=1:length(x)
    T = T + M(i)*(x(i) - p(i)); % assembly; T = df(x0,y0,...)/dx * (x-x0) + df(x0,y0,...)/dy * (y-y0) + ...
end

T = T + C; % add constant term; T = T + f(x0,y0,...)
```

```matlab
% Function used in ode45, sine response
function dy = odefunc_sine1(t,Y)
global An Bn En

omega = 0.4*2*pi;
if t < 2
    delta = 0;
elseif    t >=2 && t<4.5
    delta = (3*pi)/180*sin(omega*(t-2));
elseif t >= 4.5
    delta = 0;
end

dy = An*Y + En*delta;
```

```
% Function used in ode45, sine response
function dy = odefunc_step(t,Y)
global An Bn En

delta = 0.01*sin(t);
dy = An*Y + En*delta;
```

```
% Function used in ode45, step response
function dy = odefunc_step(t,Y)
global An Bn En

delta = 1*pi/180;
dy = An*Y + En*delta;
```

```
% vehicle_parameters
% input data for truck-dolly-semitrailer,
% taken from BradModelSetup.m

C1f = 4.0741e5;
C1r = 3.3066e5;
C1t = 3.3066e5;
C21 = 3.6885e5;
C22 = 3.6885e5;
C31 = 4.0375e5;
C32 = 4.0375e5;
C33 = 4.0375e5;
m1 = 19000;
m2 = 2070;
m3 = 31910;
J1 = 120000;
J2 = 1100;
J3 = 413707;
a1 = 3;
a2 = 3.275;
a3 = 5.118;
b1r = 1.6;
b1t = 2.97;
b2f = 3.2750;
b2r = 4.7250;
l31=  6.27;
l32 = 7.7;
l33 = 9.13;
c1 = 3.5;
l2 = 4;
```

D Vehicle parameters

Here the numerical data used to obtain the results in the report are presented. Only parameters actually used in the computations are tabulated. The cornering stiffness coefficients for the dolly and semitrailer are lumped as C_2 and C_3, respectively. Open vehicle_parameters.m (Appendix C) to change the numerical values.

Parameter	Description	Value	Unit
m_1	Mass of truck	19000	kg
m_2	Mass of dolly	2070	kg
m_3	Mass of semitrailer	31910	kg
J_1	Moment of inertia of truck about vertical axis through its CoG	120000	kg m^2
J_2	Moment of inertia of dolly about vertical axis through its CoG	1100	kg m^2
J_3	Moment of inertia of semitrailer about vertical axis through its CoG	413707	kg m^2
C_{1f}	Truck front axle cornering stiffness	407410	N/rad
C_{1r}	Truck rear axle cornering stiffness	330660	N/rad
C_{1t}	Truck tag axle cornering stiffness	330660	N/rad
C_2	Dolly axle cornering stiffness truck tag axle cornering stiffness	737700	N/rad
C_3	Semitrailer axle cornering stiffness	1211250	N/rad
a_1	Distance from front axle of truck to truck CoG	3	m
a_2	Distance from articulation point of truck to dolly CoG	3.275	m
a_3	Distance from articulation point of semitrailer to semitrailer CoG	5.118	m
b_{1r}	Distance from truck CoG to first rear axle	1.6	m
b_{1t}	Distance from truck CoG to tag axle	2.97	m
b_2	Distance from articulation point of truck to dolly axle	4	m
c_1	Distance from tractor CoG to tractor articulation point	3.5	m
l_2	Length of dolly (distance between front and rear articulation points)	4	m
l_3	Distance from articulation point of semitrailer to semitrailer axle	7.7	m

E Parameter transformation

The vehicle parameters used for the considered vehicle model have been expressed according to [2], which is a distance-based setup (all lengths are positive). An additional vehicle parameter setup, according to [4], have been considered which is a position-based setup (both positive and negative lengths occurs). Below an explanation on how to transform between the distance- and position-based setup is presented.

Note: The length b_3 is here referred to the distance from dolly CoG to dolly axle, and not the distance from articulation point of truck to dolly axle as in the report.

First unit parameters		
Position-based	Description	Distance-based
m1	Unit mass	m1
L11	1st axle distance from CoG	a1
L12	2nd axle distance from CoG	b1r (abs(L12))
L13	3rd axle distance from CoG	b1t (abs(L13))
wb1	Wheel base	l1
d1r	Rear coupling distance from CoG	c1 (abs(d1r))
Iz1	Moment of inertia	J1
C11	Cornering stiffness at 1st axle	C1f
C12	Cornering stiffness at 2nd axle	C1r
C13	Cornering stiffness at 3rd axle	C1t
	LL	Lightest Loaded
Second unit parameters		
Position-based	Description	Distance-based
d2f	Front coupling distance from CoG	a2
d2r	Rear coupling distance from CoG	c2 (abs(d2r))
Iz2	Moment of inertia	J2
C21	Cornering stiffness at 1st axle	C21
C22	Cornering stiffness at 2nd axle	C22
l2	Distance from front to rear articulation point	d2f+abs(d2r)
b2	Distance from front articulation point to axles (for simplicity)	abs((L21+L22)/2)
C2	Lumped stiffnesses	C21+C22+C23
Third unit parameters		
Position-based	Description	Distance-based
d3f	Front coupling distance from CoG	a2
L32	Rear coupling distance from CoG	c2 (abs(d2r))
Iz3	Moment of inertia	J2
C31	Cornering stiffness at 1st axle	C31
C32	Cornering stiffness at 2nd axle	C32
C33	Cornering stiffness at 3rd axle	C33
b3	Axle distances from CoG (for simplicity)	abs(L32)
C3	Lumped stiffnesses	C31+C32+C33

The transformation matrix from position-based to distance-based parameters.

$$
\begin{pmatrix}
1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & \\
0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & -1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & -1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & -1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1/2 & 1/2 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & -1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 1 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & -1 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1
\end{pmatrix}
\begin{pmatrix}
C11 \\ C12 \\ C13 \\ L11 \\ L12 \\ L13 \\ d1r \\ d2f \\ L21 \\ L22 \\ d2r \\ a2 \\ c2 \\ d3f \\ L32 \\ Iz1 \\ Iz2 \\ Iz3
\end{pmatrix}
=
\begin{pmatrix}
C1f \\ C1r \\ C1t \\ a1 \\ b1r \\ b1t \\ c1 \\ a2 \\ b2 \\ c2 \\ l2 \\ a3 \\ b3 \\ J1 \\ J2 \\ J3
\end{pmatrix}
$$