# FINITE ELEMENT APPROXIMATION OF THE LINEARIZED CAHN-HILLIARD-COOK EQUATION

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Abstract. The linearized Cahn-Hilliard-Cook equation is discretized in the spatial variables by a standard finite element method. Strong convergence estimates are proved under suitable assumptions on the covariance operator of the Wiener process, which is driving the equation. The backward Euler time stepping is also studied. The analysis is set in a framework based on analytic semigroups. The main effort is spent on proving detailed error bounds for the corresponding deterministic Cahn-Hilliard equation. The results should be interpreted as results about approximation of the stochastic convolution, which is a part of the mild solution of the nonlinear Cahn-Hilliard-Cook equation.

## 1. Introduction

When the Cahn-Hilliard equation (cf. [2, 3]) is perturbed by noise, we obtain the so-called Cahn-Hilliard-Cook equation (cf. [1, 5])

$$\begin{aligned} &\mathrm{d}u - \Delta v\,\mathrm{d}t = \mathrm{d}W, && \text{for } x \in \mathcal{D},\ t > 0,\\ &v = -\Delta u + f(u), && \text{for } x \in \mathcal{D},\ t > 0,\\ &\frac{\partial u}{\partial n} = 0,\ \frac{\partial \Delta u}{\partial n} = 0, && \text{for } x \in \partial\mathcal{D},\ t > 0,\\ &u(\cdot,0) = u_0, \end{aligned} \tag{1.1}$$

where $u = u(x,t)$, $\Delta = \sum_{i=1}^{d} \frac{\partial^2}{\partial x_i^2}$, and $\frac{\partial}{\partial n}$ denotes the outward normal derivative on $\partial\mathcal{D}$. We assume that $\mathcal{D}$ is a bounded domain in $\mathbf{R}^d$ for $d \le 3$ with sufficiently smooth boundary. A typical $f$ is $f(s) = s^3 - s$. The purpose of this work is to study numerical approximation by the finite element method of the linearized Cahn-Hilliard-Cook equation, where $f = 0$.

We use the semigroup framework of [12] in order to give (1.1) a rigorous meaning. Let $\|\cdot\|$ and $(\cdot,\cdot)$ denote the usual norm and inner product in the Hilbert space $H = L_2(\mathcal{D})$ and let $H^s = H^s(\mathcal{D})$ be the usual Sobolev space with norm $\|\cdot\|_s$. We also let $\dot{H}$ be the subspace of $H$, which is orthogonal

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to the constants, that is, $\dot{H} = \{v \in H : (v,1) = 0\}$, and we let $P\colon H \to \dot{H}$ be the orthogonal projector.

We define the linear operator $A = -\Delta$ with domain of definition

$$D(A) = \Big\{ v \in H^2 : \frac{\partial v}{\partial n} = 0 \text{ on } \partial\mathcal{D} \Big\}.$$

Then $A$ is a selfadjoint, positive definite, unbounded linear operator on $\dot{H}$ with compact inverse. When it is considered as an unbounded operator on $H$, it is positive semidefinite with an orthonormal eigenbasis $\{\varphi_j\}_{j=0}^\infty$ and corresponding eigenvalues $\{\lambda_j\}_{j=0}^\infty$ such that

$$0 = \lambda_0 < \lambda_1 \le \lambda_2 \le \cdots \le \lambda_j \le \cdots, \quad \lambda_j \to \infty.$$

The first eigenfunction is constant, $\varphi_0 = |\mathcal{D}|^{-\frac{1}{2}}$. We also define

$$|v|_s = \|A^{\frac{s}{2}} P v\| = \Big( \sum_{j=1}^\infty \lambda_j^s (v, \varphi_j)^2 \Big)^{1/2}, \quad s \in \mathbf{R}, \tag{1.2}$$

and $\dot{H}^s = \{v \in \dot{H} : |v|_s < \infty\}$ for $s \ge 0$. For $s < 0$ we define $\dot{H}^s$ to be the closure of $\dot{H}$ with respect to $|\cdot|_s$. Then $\dot{H}^0 = \dot{H}$ and $\|v\|^2 = |v|_0^2 + (v, \varphi_0)^2$. It is well known that, for integer $s \ge 0$, $\dot{H}^s$ is a subspace of $H^s \cap \dot{H}$ characterized by certain boundary conditions and that the norms $|\cdot|_s$ and $\|\cdot\|_s$ are equivalent on $\dot{H}^s$. In particular, we have $\dot{H}^1 = H^1 \cap \dot{H}$ and the norm $|v|_1 = \|A^{\frac{1}{2}} v\| = \|\nabla v\|$ is equivalent to $\|v\|_1$ on $\dot{H}^1$.

For $v \in H$ we define

$$\mathrm{e}^{-tA^2} v = \sum_{j=0}^\infty \mathrm{e}^{-t\lambda_j^2} (v, \varphi_j) \varphi_j.$$

Then $\{E(t)\}_{t \ge 0} = \{\mathrm{e}^{-tA^2}\}_{t\ge 0}$ is the analytic semigroup on $H$ generated by $-A^2$. We note that

$$E(t)v = \sum_{j=1}^\infty \mathrm{e}^{-t\lambda_j^2} (v, \varphi_j)\varphi_j + (v, \varphi_0)\varphi_0 = E(t)Pv + (I - P)v,$$

where $(I - P)v = |\mathcal{D}|^{-1} \int_{\mathcal{D}} v \,\mathrm{d}x$ is the average of $v$.

Let $(\Omega, \mathcal{F}, \mathbf{P}, \{\mathcal{F}_t\}_{t\ge0})$ be a filtered probability space, let $Q$ be a selfadjoint, positive semidefinite, bounded linear operator on $H$, and let $\{W(t)\}_{t\ge0}$ be an $H$-valued $Q$-Wiener process adapted to the filtration $\{\mathcal{F}_t\}_{t\ge0}$.

Now the Cahn-Hilliard-Cook equation (1.1) may be written formally

$$\mathrm{d}X(t) + A^2 X(t)\,\mathrm{d}t + Af(X(t))\,\mathrm{d}t = \mathrm{d}W(t), \quad t > 0; \quad X(0) = X_0. \tag{1.3}$$

The semigroup framework of [12] gives a rigorous meaning to this in terms of the mild solution, which satisfies the integral equation

$$X(t) = E(t)X_0 - \int_0^t E(t-s)Af(X(s))\,\mathrm{d}s + \int_0^t E(t-s)\,\mathrm{d}W(s),$$

where $\int_0^t \cdots \,\mathrm{d}W(s)$ denotes the $H$-valued Itô integral. Existence and uniqueness of solutions is proved in [6]. This is based on the natural splitting of the solution as $X(t) = Y(t) + W_A(t)$, where

$$W_A(t) = \int_0^t E(t-s)\,\mathrm{d}W(s)$$

is a stochastic convolution, and where

$$Y(t) = E(t)X_0 - \int_0^t E(t-s)Af(X(s))\,\mathrm{d}s$$

satisfies the random evolution problem

$$\dot{Y}(t) + A^2Y(t) + Af\big(Y(t) + W_A(t)\big) = 0, \quad t > 0; \quad Y(0) = X_0.$$

The study of the stochastic convolution $W_A(t)$ is thus a first step towards the study of the nonlinear problem.

In this work we therefore study numerical approximation of the linearized Cahn-Hilliard-Cook equation

$$\mathrm{d}X + A^2X\,\mathrm{d}t = \mathrm{d}W, \quad t > 0; \quad X(0) = X_0, \tag{1.4}$$

with the mild solution

$$X(t) = E(t)X_0 + \int_0^t E(t-s)\,\mathrm{d}W(s). \tag{1.5}$$

The nonlinear equation is studied in a forthcoming paper [11]. We remark that a linearized equation of the form (1.4), but with $A^2$ replaced by $A^2 + A$ is studied by numerical simulation in the physics literature [7, 9].

For the approximation of the Cahn-Hilliard equation we follow the framework of [8]. We assume that we have a family $\{S_h\}_{h>0}$ of finite-dimensional approximating subspaces of $H^1$. Let $P_h\colon H \to S_h$ denote the orthogonal projector. We then define $\dot{S}_h = \{\chi \in S_h : (\chi, 1) = 0\}$. The operator $A_h\colon S_h \to \dot{S}_h$ (the "discrete Laplacian") is defined by

$$(A_h\chi, \eta) = (\nabla\chi, \nabla\eta), \quad \forall \chi \in S_h,\, \eta \in \dot{S}_h,$$

The operator $A_h$ is selfadjoint, positive definite on $\dot{S}_h$, positive semidefinite on $S_h$, and $A_h$ has an orthonormal eigenbasis $\{\varphi_{h,j}\}_{j=0}^{N_h}$ with corresponding eigenvalues $\{\lambda_{h,j}\}_{j=0}^{N_h}$. We have

$$0 = \lambda_{h,0} < \lambda_{h,1} \le \cdots \le \lambda_{h,j} \le \cdots \le \lambda_{h,N_h},$$

and $\varphi_{h,0} = \varphi_0 = |\mathcal{D}|^{-\frac{1}{2}}$. Moreover, we define $E_h(t)\colon S_h \to S_h$ by

$$E_h(t)v_h = \mathrm{e}^{-tA_h^2}v_h = \sum_{j=1}^{N_h} \mathrm{e}^{-t\lambda_{h,j}}(v_h, \varphi_{h,j})\varphi_{h,j} + (v_h, \varphi_0)\varphi_0.$$

Then $\{E_h(t)\}_{t\ge0}$ is the analytic semigroup generated by $-A_h^2$. Clearly, $P_h\colon \dot{H} \to \dot{S}_h$ and

$$E_h(t)P_hv = E_h(t)P_hPv + (I - P)v.$$

The finite element approximation of the linearized Cahn-Hilliard-Cook equation (1.4) is: Find $X_h(t) \in S_h$ such that,

$$\mathrm{d}X_h + A_h^2 X_h \,\mathrm{d}t = P_h \,\mathrm{d}W, \quad t > 0; \quad X_h(0) = P_h X_0. \tag{1.6}$$

The mild solution of (1.6) is

$$X_h(t) = E_h(t) P_h X_0 + \int_0^t E_h(t-s) P_h \,\mathrm{d}W(s). \tag{1.7}$$

We note that

$$\int_0^t E(t-s)(I-P)\,\mathrm{d}W(s) = (I-P)\int_0^t \mathrm{d}W(s) = (I-P)W(t),$$

so that (1.5) can be written

$$\begin{aligned} X(t) &= E(t)PX_0 + (I-P)X_0 \\ &\quad + \int_0^t E(t-s)P\,\mathrm{d}W(s) + (I-P)W(t), \end{aligned} \tag{1.8}$$

and similarly, for (1.7),

$$\begin{aligned} X_h(t) &= E_h(t)P_hPX_0 + (I-P)X_0 \\ &\quad + \int_0^t E_h(t-s)P_hP\,\mathrm{d}W(s) + (I-P)W(t). \end{aligned}$$

Therefore, the error analysis can be based on the formula

$$\begin{aligned} X_h(t) - X(t) &= \big(E_h(t)P_h - E(t)\big)PX_0 \\ &\quad + \int_0^t \big(E_h(t-s)P_h - E(t-s)\big)P\,\mathrm{d}W(s), \end{aligned} \tag{1.9}$$

and it is sufficient to work in the spaces $\dot{H}$ and $\dot{S}_h$. Note that the numerical computations are carried out in $S_h$ and that $\dot{S}_h$ is only used in the analysis.

Let $k = \delta t$ be a timestep, $t_n = nk$, $\delta X_{h,n} = X_{h,n} - X_{h,n-1}$, $\delta W_n = W(t_n) - W(t_{n-1})$, and apply Euler's method to (1.6) to get

$$\delta X_{h,n} + A_h^2 X_{h,n}\,\delta t = P_h\,\delta W_n, \quad n \geq 1; \quad X_{h,0} = P_h X_0. \tag{1.10}$$

With $E_{kh} = (I + kA_h^2)^{-1}$ we obtain a discrete variant of the mild solution

$$X_{h,n} = E_{kh}^n P_h X_0 + \sum_{j=1}^n E_{kh}^{n-j+1} P_h\,\delta W_j.$$

In Section 2 we assume that $\{S_h\}_{h>0}$ admits an error estimate of order $\mathcal{O}(h^r)$ as the mesh parameter $h \to 0$ for some integer $r \geq 2$. Then we show error estimates for the semigroup $E_h(t)$ with minimal regularity requirement. More precisely, in Theorem 2.1 we show, for $\beta \in [1, r]$ and all $t \geq 0$,

$$\begin{aligned} &\|F_h(t)v\| \leq Ch^\beta\,|v|_\beta, \quad v \in \dot{H}^\beta, \\ &\Big(\int_0^t \|F_h(\tau)v\|^2\,\mathrm{d}\tau\Big)^{\frac{1}{2}} \leq C|\log h|h^\beta\,|v|_{\beta-2}, \quad v \in \dot{H}^{\beta-2}, \end{aligned}$$

where $F_h(t) = E_h(t)P_h - E(t)$ is the error operator in (1.9).

Analogous estimates are obtained for the implicit Euler approximation in Theorem 2.2.

In Section 3 we follow the technique developed in [14, 13] and use these estimates to prove strong convergence estimates for approximation of the linear Cahn-Hilliard-Cook equation. Let $L_2(\Omega, \dot{H}^\beta)$ be the space of square integrable $\dot{H}^\beta$-valued random variables with norm

$$\|X\|_{L_2(\Omega,\dot{H}^\beta)} = \Big(\mathbf{E}\big\{|X|_\beta^2\big\}\Big)^{\frac{1}{2}} = \Big(\int_\Omega |X(\omega)|_\beta^2 \,\mathrm{d}\mathbf{P}(\omega)\Big)^{\frac{1}{2}}, \tag{1.11}$$

and let $\|T\|_{\mathrm{HS}}$ denote the Hilbert-Schmidt norm of bounded linear operators on $H$, $\|T\|_{\mathrm{HS}}^2 = \sum_{j=1}^\infty \|T\phi_j\|^2$, where $\{\phi_j\}_{j=1}^\infty$ is an arbitrary orthonormal basis for $H$. In Theorem 3.1 we study the spatial regularity of the mild solution (1.5) and show

$$\|X(t)\|_{L_2(\Omega,\dot{H}^\beta)} \le C\big(\|X_0\|_{L_2(\Omega,\dot{H}^\beta)} + \|A^{\frac{\beta-2}{2}}Q^{\frac{1}{2}}\|_{\mathrm{HS}}\big) \quad \text{for } \beta \ge 0.$$

Moreover, in Theorem 3.2 we show strong convergence for the mild solution $X_h$ in (1.7):

$$\begin{aligned}&\|X_h(t) - X(t)\|_{L_2(\Omega,H)}\\ &\qquad \le Ch^\beta\big(\|X_0\|_{L_2(\Omega,\dot{H}^\beta)} + |\log h|\|A^{\frac{\beta-2}{2}}Q^{\frac{1}{2}}\|_{\mathrm{HS}}\big), \quad \beta \in [1, r].\end{aligned}$$

In Theorem 3.3 for the fully discrete case we obtain similarly, for $\beta \in [1, \min(r, 4)]$,

$$\begin{aligned}&\|X_{h,n} - X(t_n)\|_{L_2(\Omega,H)}\\ &\qquad \le \big(C|\log h|h^\beta + C_{k,\beta}k^{\frac{\beta}{4}}\big)\big(\|X_0\|_{L_2(\Omega,\dot{H}^\beta)} + \|A^{\frac{\beta-2}{2}}Q^{\frac{1}{2}}\|_{\mathrm{HS}}\big),\end{aligned}$$

where $C_{\beta,k} = \frac{C}{4-\beta}$ for $\beta < 4$ and $C_{\beta,k} = C|\log k|$ for $\beta = 4$.

Note that these bounds are uniform with respect to $t \ge 0$ and $t_n \ge 0$.

Our results require that $\|A^{\frac{\beta-2}{2}}Q^{\frac{1}{2}}\|_{\mathrm{HS}} < \infty$. In order to see what this means we compute two special cases. For $Q = I$ (spatially uncorrelated noise, or space-time white noise), by using the asymptotics $\lambda_j \sim j^{\frac{2}{d}}$, we have

$$\|A^{\frac{\beta-2}{2}}Q^{\frac{1}{2}}\|_{\mathrm{HS}}^2 = \|A^{\frac{\beta-2}{2}}\|_{\mathrm{HS}}^2 = \sum_{j=1}^\infty \lambda_j^{\beta-2} \sim \sum_{j=1}^\infty j^{(\beta-2)\frac{2}{d}} < \infty,$$

if $\beta < 2 - \frac{d}{2}$. Hence, for example, $\beta < \frac{1}{2}$ if $d = 3$. On the other hand, if $Q$ is of trace class, $\mathrm{Tr}(Q) = \|Q^{\frac{1}{2}}\|_{\mathrm{HS}}^2 < \infty$, then we may take $\beta = 2$.

There are few studies of numerical methods for the Cahn-Hilliard-Cook equation. We are only aware of [4], in which convergence in probability was proved for a difference scheme for the nonlinear equation in multiple dimensions, and [10], where strong convergence was proved for the finite element method for the linear equation in 1-D.

## 2. Error estimates for the Cahn-Hilliard semigroup

We start this section with some necessary inequalities. Let $\{E(t)\}_{t\geq 0} = \{\mathrm{e}^{-tA^2}\}_{t\geq 0}$ and $\{E_h(t)\}_{t\geq 0} = \{\mathrm{e}^{-tA_h^2}\}_{t\geq 0}$ be the semigroups generated by $-A^2$ and $-A_h^2$, respectively. By the smoothing property there exist positive constants $c, C$, independent of $h$ and $t$, such that

$$\|A_h^{2\beta} E_h(t) P_h P v\| + \|A^{2\beta} E(t) P v\| \leq C t^{-\beta} \mathrm{e}^{-ct} \|v\|, \quad \beta \geq 0, \tag{2.1}$$

$$\int_0^t \|A_h E_h(s) P_h P v\|^2 \,\mathrm{d}s + \int_0^t \|A E(s) P v\|^2 \,\mathrm{d}s \leq C \|v\|^2. \tag{2.2}$$

Let $R_h \colon \dot{H}^1 \to \dot{S}_h$ be the Ritz projector defined by

$$(\nabla R_h v, \nabla \chi) = (\nabla v, \nabla \chi), \quad \forall \chi \in \dot{S}_h.$$

It is clear that $R_h = A_h^{-1} P_h A$. We assume that for some integer $r \geq 2$, we have the error bound, with the norm defined in (1.2),

$$\|R_h v - v\| \leq C h^\beta |v|_\beta, \quad v \in \dot{H}^\beta,\ 1 \leq \beta \leq r. \tag{2.3}$$

This holds with $r = 2$ for the standard piecewise linear Lagrange finite element method in a bounded convex polygonal domain $\mathcal{D}$. For higher order elements the situation is more complicated and we refer to standard texts on the finite element method. In the next theorem we prove error estimates for the Cahn-Hilliard semigroup in the semidiscrete case.

**Theorem 2.1.** *Set $F_h(t) = E_h(t) P_h - E(t)$. Then there are $h_0$ and $C$, such that for $h \leq h_0$, $1 \leq \beta \leq r$ and $t \geq 0$, we have*

$$\|F_h(t) v\| \leq C h^\beta \, |v|_\beta, \quad v \in \dot{H}^\beta, \tag{2.4}$$

$$\Big( \int_0^t \|F_h(\tau) v\|^2 \,\mathrm{d}\tau \Big)^{\frac{1}{2}} \leq C |\log h| h^\beta \, |v|_{\beta-2}, \quad v \in \dot{H}^{\beta-2}. \tag{2.5}$$

Note that $F_h(t) v = F_h(t) P v$ for $v \in H$, so that it is sufficient to take $v \in \dot{H}$. The reason why we assume $\beta \geq 1$ is that in (2.5) we need at least $v \in \dot{H}^{-1}$ for $E_h(t) P_h v$ to be defined. The bound (2.4) can be found in [8], but (2.5) is new. We adapt the technique from [8] to provide a self-contained proof.

*Proof.* Let $u(t) = E(t) v$, $u_h(t) = E_h(t) P_h v$ be the solutions of

$$u_t + A^2 u = 0, \quad t > 0; \quad u(0) = v, \tag{2.6}$$

$$u_{h,t} + A_h^2 u_h = 0, \quad t > 0; \quad u_h(0) = P_h v. \tag{2.7}$$

Here $u_t$ denotes the time derivative. Set $e(t) = u_h(t) - u(t)$. We want to prove that

$$\|e(t)\| \leq C h^\beta \, |v|_\beta, \quad v \in \dot{H}^\beta,$$

$$\Big( \int_0^t \|e(\tau)\|^2 \,\mathrm{d}\tau \Big)^{\frac{1}{2}} \leq C |\log h| h^\beta \, |v|_{\beta-2}, \quad v \in \dot{H}^{\beta-2}.$$

Let $G = A^{-1}P$ and $G_h = A_h^{-1}P_hP$. Apply $G$ to (2.6) to get $Gu_t + Au = 0$, and apply $G_h^2$ to (2.7) to get $G_h^2 u_{h,t} + u_h = 0$. Hence,

$$G_h^2 e_t + e = -G_h^2 u_t - u + G_h(Gu_t + Au) = (G_hA - I)u - G_h(G_hA - I)Gu_t,$$

that is,

$$G_h^2 e_t + e = \rho + G_h\eta, \tag{2.8}$$

where $\rho = (R_h - I)u, \eta = -(R_h - I)Gu_t$. Take the inner product of (2.8) by $e_t$ to get

$$\|G_he_t\|^2 + \frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\|e\|^2 = (\rho, e_t) + (\eta, G_he_t),$$

Since $(\eta, G_he_t) \leq \|\eta\|\|G_he_t\| \leq \frac{1}{2}\|\eta\|^2 + \frac{1}{2}\|G_he_t\|^2$, we obtain

$$\|G_he_t\|^2 + \frac{\mathrm{d}}{\mathrm{d}t}\|e\|^2 \leq 2(\rho, e_t) + \|\eta\|^2.$$

Multiply this inequality by $t$ to get

$$t\|G_he_t\|^2 + t\frac{\mathrm{d}}{\mathrm{d}t}\|e\|^2 \leq 2t(\rho, e_t) + t\|\eta\|^2.$$

For compactness of notation we write $t\|\eta\|^2$ instead of $t\|\eta(t)\|^2$ and similarly for the other terms. Note that

$$t\frac{\mathrm{d}}{\mathrm{d}t}\|e\|^2 = \frac{\mathrm{d}}{\mathrm{d}t}\big(t\|e\|^2\big) - \|e\|^2, \quad t(\rho, e_t) = \frac{\mathrm{d}}{\mathrm{d}t}\big(t(\rho, e)\big) - (\rho, e) - t(\rho_t, e),$$

so that

$$t\|G_he_t\|^2 + \frac{\mathrm{d}}{\mathrm{d}t}\big(t\|e\|^2\big) \leq 2\frac{\mathrm{d}}{\mathrm{d}t}\big(t(\rho, e)\big) + 2|(\rho, e)| + 2|t(\rho_t, e)| + t\|\eta\|^2 + \|e\|^2.$$

But

$$\begin{aligned}|(\rho, e)| &\leq \|\rho\|\|e\| \leq \frac{1}{2}\|\rho\|^2 + \frac{1}{2}\|e\|^2,\\ |t(\rho_t, e)| &\leq t\|\rho_t\|\|e\| \leq \frac{1}{2}t^2\|\rho_t\|^2 + \frac{1}{2}\|e\|^2.\end{aligned}$$

Hence,

$$t\|G_he_t\|^2 + \frac{\mathrm{d}}{\mathrm{d}t}\big(t\|e\|^2\big) \leq 2\frac{\mathrm{d}}{\mathrm{d}t}\big(t(\rho, e)\big) + \|\rho\|^2 + t^2\|\rho_t\|^2 + t\|\eta\|^2 + 3\|e\|^2.$$

Integrate over $[0, t]$ and use Young's inequality to get

$$\begin{aligned}\int_0^t \tau\|G_he_t\|^2\,\mathrm{d}\tau + t\|e\|^2 \leq 2t\|\rho\|^2 + \frac{1}{2}t\|e\|^2 + \int_0^t \|\rho\|^2\,\mathrm{d}\tau + \int_0^t \tau^2\|\rho_t\|^2\,\mathrm{d}\tau\\ + \int_0^t \tau\|\eta\|^2\,\mathrm{d}\tau + 3\int_0^t \|e\|^2\,\mathrm{d}\tau.\end{aligned}$$

Hence,

$$t\|e\|^2 \leq Ct\|\rho\|^2 + C\int_0^t \big(\|\rho\|^2 + \tau^2\|\rho_t\|^2 + \tau\|\eta\|^2 + \|e\|^2\big)\,\mathrm{d}\tau. \tag{2.9}$$

We must bound $\int_0^t \|e\|^2\,\mathrm{d}\tau$. Multiply (2.8) by $e$ to get

$$\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\|G_he\|^2+\|e\|^2 \le \|\rho\|\,\|e\|+\|\eta\|\,\|G_he\| \le \frac{1}{2}\|\rho\|^2+\frac{1}{2}\|e\|^2+\|\eta\|\max_{0\le\tau\le t}\|G_he\|,$$

so that

$$\frac{\mathrm{d}}{\mathrm{d}t}\|G_he\|^2+\|e\|^2 \le \|\rho\|^2+2\|\eta\|\max_{0\le\tau\le t}\|G_he\|. \tag{2.10}$$

Integrate (2.10), note that $G_he(0)=A_h^{-1}P_h(P_h-I)v=0$, to get

$$\|G_he\|^2+\int_0^t\|e\|^2\,\mathrm{d}\tau \le \int_0^t\|\rho\|^2\,\mathrm{d}\tau+\max_{0\le\tau\le t}\|G_he\|^2+\Big(\int_0^t\|\eta\|\,\mathrm{d}\tau\Big)^2.$$

Hence, since $t$ is arbitrary,

$$\int_0^t\|e\|^2\,\mathrm{d}\tau \le \int_0^t\|\rho\|^2\,\mathrm{d}\tau+\Big(\int_0^t\|\eta\|\,\mathrm{d}\tau\Big)^2. \tag{2.11}$$

We insert (2.11) in (2.9) and conclude

$$\begin{aligned} t\|e\|^2 &\le Ct\|\rho\|^2+C\int_0^t\big(\|\rho\|^2+\tau^2\|\rho_t\|^2+\tau\|\eta\|^2\big)\,\mathrm{d}\tau \\ &\quad +C\Big(\int_0^t\|\eta\|\,\mathrm{d}\tau\Big)^2. \end{aligned} \tag{2.12}$$

We compute the terms in the right hand side. With $v\in\dot H^\beta$, recalling $\rho=(R_h-I)u$ and using (2.3), we have

$$\|\rho(t)\| \le Ch^\beta|u(t)|_\beta \le Ch^\beta\|E(t)A^{\frac{\beta}{2}}v\| \le Ch^\beta\|A^{\frac{\beta}{2}}v\| \le Ch^\beta|v|_\beta, \tag{2.13}$$

so that,

$$t\|\rho\|^2 \le Ch^{2\beta}t|v|_\beta^2,\quad \int_0^t\|\rho\|^2\,\mathrm{d}\tau \le Ch^{2\beta}t|v|_\beta^2.$$

Similarly, by (2.1),

$$\|\rho_t(t)\| \le Ch^\beta|u_t(t)|_\beta \le Ch^\beta\|A^2E(t)A^{\frac{\beta}{2}}v\| \le Ch^\beta t^{-1}|v|_\beta,$$

so that

$$\int_0^t\tau^2\|\rho_t\|^2\,\mathrm{d}\tau \le Ch^{2\beta}t|v|_\beta^2. \tag{2.14}$$

Moreover, since $\eta=-(R_h-I)Gu_t=(R_h-I)GA^2E(t)v$,

$$\|\eta(t)\| \le Ch^\beta|Gu_t(t)|_\beta \le Ch^\beta\|AE(t)A^{\frac{\beta}{2}}v\| \le Ch^\beta t^{-\frac{1}{2}}|v|_\beta,$$

so that

$$\Big(\int_0^t\|\eta\|\,\mathrm{d}\tau\Big)^2 \le Ch^{2\beta}t|v|_\beta^2,\quad \int_0^t\tau\|\eta\|^2\,\mathrm{d}\tau \le Ch^{2\beta}t|v|_\beta^2.$$

By inserting these in (2.12) we conclude

$$t\|e\|^2 \le Ch^{2\beta}t|v|_\beta^2,$$

which proves (2.4).

To prove (2.5) we recall (2.11) and let $v \in \dot{H}^{\beta-2}$. By using (2.3) and (2.2) we obtain

$$\begin{aligned}\int_0^t \|\rho\|^2 \,\mathrm{d}\tau \le Ch^{2\beta}\int_0^t |u|_\beta^2 \,\mathrm{d}\tau = Ch^{2\beta}\int_0^t \|AE(\tau)A^{\frac{\beta-2}{2}}v\|^2 \,\mathrm{d}\tau \\ \le Ch^{2\beta}|v|_{\beta-2}^2.\end{aligned} \tag{2.15}$$

Now we compute $\int_0^t \|\eta\| \,\mathrm{d}\tau$. To this end we assume first $1 < \beta \le r$ and let $1 \le \gamma < \beta$. By using (2.1) and (2.3) we get

$$\begin{aligned}\int_0^t \|\eta\| \,\mathrm{d}\tau &\le Ch^\gamma \int_0^t |Gu_t|_\gamma \,\mathrm{d}\tau = Ch^\gamma \int_0^t \|A^{2-\frac{\beta-\gamma}{2}}E(\tau)A^{\frac{\beta-2}{2}}v\| \,\mathrm{d}\tau \\ &\le Ch^\gamma \int_0^t \tau^{-1+\frac{\beta-\gamma}{4}}\mathrm{e}^{-c\tau}\,\mathrm{d}\tau\, |v|_{\beta-2},\end{aligned}$$

where, since $0 < \beta-\gamma \le r-1$,

$$\int_0^t \tau^{-1+\frac{\beta-\gamma}{4}}\mathrm{e}^{-c\tau}\,\mathrm{d}\tau = \frac{4}{\beta-\gamma}\int_0^{t^{\frac{\beta-\gamma}{4}}} \mathrm{e}^{-cs^{\frac{4}{\beta-\gamma}}}\,\mathrm{d}s \le \frac{C}{\beta-\gamma}\int_0^\infty \mathrm{e}^{-cs^{\frac{4}{r-1}}}\,\mathrm{d}s.$$

Hence, with $C$ independent of $\beta$,

$$\int_0^t \|\eta\| \,\mathrm{d}\tau \le \frac{Ch^\gamma}{\beta-\gamma}|v|_{\beta-2}. \tag{2.16}$$

Now let $\frac{1}{\beta-\gamma} = |\log h| = -\log h$, so $\gamma \to \beta$ as $h \to 0$, and

$$\gamma \log h = (\gamma-\beta+\beta)\log h = 1 + \beta\log h.$$

Therefore we have

$$\frac{h^\gamma}{\beta-\gamma} = |\log h|\mathrm{e}^{\gamma\log h} = |\log h|\mathrm{e}^{1+\beta\log h} \le C|\log h|h^\beta.$$

Put this in (2.16) to get, for $1 < \beta \le r$,

$$\int_0^t \|\eta\| \,\mathrm{d}\tau \le Ch^\beta|\log h||v|_{\beta-2}, \tag{2.17}$$

and hence also for $1 \le \beta \le r$, because $C$ is independent of $\beta$. Finally, we put (2.15) and (2.17) in (2.11) to get

$$\Big(\int_0^t \|e\|^2 \,\mathrm{d}\tau\Big)^{\frac{1}{2}} \le C|\log h|h^\beta|v|_{\beta-2},$$

which is (2.5). □

Now we turn to the fully discrete case. The backward Euler method applied to

$$u_{h,t} + A_h^2 u_h = 0, \quad t > 0; \quad u_h(0) = P_h v,$$

defines $U_n \in S_h$ by

$$\partial U_n + A_h^2 U_n = 0, \quad n \ge 1; \quad U_0 = P_h v, \tag{2.18}$$

where $\partial U_n = \frac{1}{k}(U_n - U_{n-1})$. Denoting $E_{kh}^n = (I + kA_h^2)^{-n}$, we have $U_n = E_{kh}^n v$. The next theorem provides error estimates for the Euler approximation of the Cahn-Hilliard semigroup.

**Theorem 2.2.** *Set $F_n = E_{kh}^n P_h - E(t_n)$. Then there are $h_0, k_0$ and $C$, such that for $h \le h_0$, $k \le k_0$, $1 \le \beta \le \min(r,4)$, and $n \ge 1$, we have*

$$\|F_n v\| \le C(h^\beta + k^{\frac{\beta}{4}})|v|_\beta, \quad v \in \dot{H}^\beta, \tag{2.19}$$

$$\Big(k\sum_{j=1}^n \|F_j v\|^2\Big)^{\frac{1}{2}} \le \big(C|\log h| h^\beta + C_{\beta,k} k^{\frac{\beta}{4}}\big)|v|_{\beta-2}, \quad v \in \dot{H}^{\beta-2}, \tag{2.20}$$

*where $C_{\beta,k} = \frac{C}{4-\beta}$ for $\beta < 4$ and $C_{\beta,k} = C|\log k|$ for $\beta = 4$.*

*Proof.* Let $G$ and $G_h$ be as in the proof of Theorem 2.1. With $e_n = U_n - u_n = E_{kh}^n P_h v - E(t_n)v$, we get

$$G_h^2 \partial e_n + e_n = \rho_n + G_h \eta_n + G_h \delta_n, \tag{2.21}$$

where $u_n = u(t_n)$, $u_{t,n} = u_t(t_n)$ and

$$\rho_n = (R_h - I)u_n, \quad \eta_n = -(R_h - I)G\partial u_n, \quad \delta_n = -G(\partial u_n - u_{t,n}).$$

Multiply (2.21) by $\partial e_n$ and note that

$$(\eta_n, G_h \partial e_n) \le \|\eta_n\|^2 + \frac{1}{4}\|G_h \partial e_n\|^2, \ (\delta_n, G_h \partial e_n) \le \|\delta_n\|^2 + \frac{1}{4}\|G_h \partial e_n\|^2,$$

to get

$$\|G_h \partial e_n\|^2 + 2(e_n, \partial e_n) \le 2(\rho_n, \partial e_n) + 2\|\eta_n\|^2 + 2\|\delta_n\|^2. \tag{2.22}$$

We have the following identities

$$\partial(a_n b_n) = (\partial a_n) b_n + a_{n-1}(\partial b_n) \tag{2.23}$$

$$= (\partial a_n) b_n + a_n(\partial b_n) - k(\partial a_n)(\partial b_n). \tag{2.24}$$

By using (2.24) we have

$$\begin{aligned} 2(e_n, \partial e_n) &= \partial\|e_n\|^2 + k\|\partial e_n\|^2, \\ (\rho_n, \partial e_n) &= \partial(\rho_n, e_n) - (\partial\rho_n, e_n) + k(\partial\rho_n, \partial e_n). \end{aligned}$$

Put these in (2.22) and cancel $k\|\partial e_n\|^2$ to get

$$\|G_h \partial e_n\|^2 + \partial\|e_n\|^2 \le 2\partial(\rho_n, e_n) - 2(\partial\rho_n, e_n) + k\|\partial\rho_n\|^2 + 2\|\eta_n\|^2 + 2\|\delta_n\|^2.$$

Multiply this by $t_{n-1}$, and note that $k \le t_{n-1}$ for $n \ge 2$, so that for $n \ge 1$ we have

$$\begin{aligned} &t_{n-1}\|G_h \partial e_n\|^2 + t_{n-1}\partial\|e_n\|^2 \\ &\quad \le 2t_{n-1}\partial(\rho_n, e_n) - 2t_{n-1}(\partial\rho_n, e_n) + t_{n-1}^2\|\partial\rho_n\|^2 \\ &\qquad + 2t_{n-1}\|\eta_n\|^2 + 2t_{n-1}\|\delta_n\|^2. \end{aligned} \tag{2.25}$$

By (2.23) we have

$$t_{n-1}\partial\|e_n\|^2 = \partial(t_n\|e_n\|^2) - \|e_n\|^2,$$
$$2t_{n-1}\partial(\rho_n, e_n) = 2\partial(t_n(\rho_n, e_n)) - 2(\rho_n, e_n).$$

Put these in (2.25) to get

$$\begin{aligned} t_{n-1}\|G_h\partial e_n\|^2 &+ \partial(t_n\|e_n\|^2) \\ &\le C\big(\partial(t_n(\rho_n, e_n)) + \|\rho_n\|^2 + t_{n-1}^2\|\partial\rho_n\|^2 + \|e_n\|^2\big) \\ &\quad + C\big(t_{n-1}\|\eta_n\|^2 + t_{n-1}\|\delta_n\|^2\big). \end{aligned} \tag{2.26}$$

Note that

$$k\sum_{j=1}^n \partial\big(t_j\|e_j\|^2\big) = t_n\|e_n\|^2, \quad k\sum_{j=1}^n \partial\big(t_j(\rho_j, e_j)\big) = t_n(\rho_n, e_n). \tag{2.27}$$

By summation in (2.26) and using (2.27) we get

$$\begin{aligned} k\sum_{j=1}^n t_{j-1}\|G_h\partial e_j\|^2 + t_n\|e_n\|^2 &\le Ct_n\|\rho_n\|^2 \\ &\quad + Ck\sum_{j=1}^n \big(\|\rho_j\|^2 + t_{j-1}^2\|\partial\rho_j\|^2 + \|e_j\|^2\big) \\ &\quad + Ck\sum_{j=1}^n \big(t_{j-1}\|\eta_j\|^2 + t_{j-1}\|\delta_j\|^2\big). \end{aligned} \tag{2.28}$$

Now we estimate $k\sum_{j=1}^n \|e_j\|^2$. Multiply (2.21) by $e_n$ to get

$$2(G_h^2\partial e_n, e_n) + \|e_n\|^2 \le \|\rho_n\|^2 + 2\big(\|\eta_n\| + \|\delta_n\|\big)\|G_he_n\|. \tag{2.29}$$

By (2.24) we have

$$2(G_h^2\partial e_n, e_n) = 2(\partial G_he_n, G_he_n) = \partial\|G_he_n\|^2 + k\|\partial G_he_n\|^2. \tag{2.30}$$

By summation in (2.29) and using $G_he_0 = 0$, we get

$$\begin{aligned} \|G_he_n\|^2 + k\sum_{j=1}^n \|e_j\|^2 \le k\sum_{j=1}^n \|\rho_j\|^2 &+ \frac{1}{2}\max_{j\le n}\|G_he_j\|^2 \\ &+ 2\Big(k\sum_{j=1}^n \big(\|\eta_j\| + \|\delta_j\|\big)\Big)^2. \end{aligned}$$

Hence,

$$k\sum_{j=1}^n \|e_j\|^2 \le k\sum_{j=1}^n \|\rho_j\|^2 + 2\Big(k\sum_{j=1}^n \big(\|\eta_j\| + \|\delta_j\|\big)\Big)^2. \tag{2.31}$$

By putting (2.31) in (2.28) we get

$$
\begin{aligned}
t_n\|e_n\|^2 \leq & Ct_n\|\rho_n\|^2 \\
& + Ck\sum_{j=1}^{n}\Big(\|\rho_j\|^2 + t_{j-1}^2\|\partial\rho_j\|^2 + t_{j-1}\|\eta_j\|^2 + t_{j-1}\|\delta_j\|^2\Big) \\
& + C\Big(k\sum_{j=1}^{n}\big(\|\eta_j\| + \|\delta_j\|\big)\Big)^2.
\end{aligned} \tag{2.32}
$$

We compute these terms. With $v \in \dot{H}^\beta$ we have by (2.13),

$$
\|\rho_n\|^2 \leq Ch^{2\beta}|v|_\beta^2, \quad k\sum_{j=1}^{n}\|\rho_j\|^2 \leq Ch^{2\beta}t_n|v|_\beta^2. \tag{2.33}
$$

By using the Cauchy-Schwartz inequality we have

$$
\begin{aligned}
k\sum_{j=1}^{n} t_{j-1}^2\|\partial\rho_j\|^2 &= k\sum_{j=2}^{n} t_{j-1}^2\Big\|\frac{1}{k}\int_{t_{j-1}}^{t_j}\rho_t\,\mathrm{d}\tau\Big\|^2 \\
&\leq \sum_{j=2}^{n}\Big(t_{j-1}^2\frac{1}{k}\int_{t_{j-1}}^{t_j}\tau^{-2}\mathrm{d}\tau\int_{t_{j-1}}^{t_j}\tau^2\|\rho_t(\tau)\|^2\,\mathrm{d}\tau\Big) \leq \int_0^{t_n}\tau^2\|\rho_t\|^2\mathrm{d}\tau.
\end{aligned}
$$

Hence, by (2.14),

$$
k\sum_{j=1}^{n} t_{j-1}^2\|\partial\rho_j\|^2 \leq Ch^{2\beta}t_n|v|_\beta^2. \tag{2.34}
$$

By using (2.3) and (2.1) we have

$$
\begin{aligned}
\|\eta_j\| &\leq Ch^\beta|G\partial u_j|_\beta \leq \frac{Ch^\beta}{k}\Big\|\int_{t_{j-1}}^{t_j}AE(\tau)A^{\frac{\beta}{2}}v\,\mathrm{d}\tau\Big\| \\
&\leq \frac{Ch^\beta}{k}\int_{t_{j-1}}^{t_j}\tau^{-\frac{1}{2}}\,\mathrm{d}\tau\|A^{\frac{\beta}{2}}v\| \leq \frac{Ch^\beta}{k}(\sqrt{t_j}-\sqrt{t_{j-1}})|v|_\beta \leq \frac{Ch^\beta}{\sqrt{t_j}}|v|_\beta.
\end{aligned}
$$

So

$$
k\sum_{j=1}^{n} t_{j-1}\|\eta_j\|^2 \leq Ch^{2\beta}t_n|v|_\beta^2, \quad k\sum_{j=1}^{n}\|\eta_j\| \leq Ch^\beta t_n^{\frac{1}{2}}|v|_\beta. \tag{2.35}
$$

By using (2.1) we have, for $j \geq 2$,

$$
\begin{aligned}
\|\delta_j\| &\leq \Big\|\frac{1}{k}\int_{t_{j-1}}^{t_j}(\tau - t_{j-1})Gu_{tt}(\tau)\mathrm{d}\tau\Big\| \leq \int_{t_{j-1}}^{t_j}\|A^{3-\frac{\beta}{2}}E(\tau)A^{\frac{\beta}{2}}v\|\,\mathrm{d}\tau \\
&\leq C\int_{t_{j-1}}^{t_j}\tau^{\frac{-6+\beta}{4}}\,\mathrm{d}\tau|v|_\beta,
\end{aligned}
$$

so that, by Hölder's inequality with $p=\frac{4}{\beta}$ and $q=\frac{4}{4-\beta}$, $1\le\beta<4$,

$$\begin{aligned}\int_{t_{j-1}}^{t_j}\tau^{\frac{-6+\beta}{4}}\,\mathrm{d}\tau &\le Ck^{\frac{\beta}{4}}\Big(\int_{t_{j-1}}^{t_j}\big(\tau^{\frac{-6+\beta}{4}}\big)^{\frac{4}{4-\beta}}\,\mathrm{d}\tau\Big)^{\frac{4-\beta}{4}}\\ &\le Ck^{\frac{\beta}{4}}\Big(\frac{\beta-4}{2}\big(t_{j-1}^{-\frac{2}{4-\beta}}-t_j^{-\frac{2}{4-\beta}}\big)\Big)^{\frac{4-\beta}{4}}\le Ck^{\frac{\beta}{4}}t_{j-1}^{-\frac{1}{2}}.\end{aligned}$$

The same result is obtained with $\beta=4$. For $j=1$ we have

$$\begin{aligned}\|\delta_1\| &\le \Big\|\frac{1}{k}\int_0^k\tau G u_{tt}(\tau)\,\mathrm{d}\tau\Big\|\le C\frac{1}{k}\int_0^k\tau^{\frac{-2+\beta}{4}}\,\mathrm{d}\tau|v|_\beta\\ &\le C\frac{4}{2+\beta}k^{\frac{-2+\beta}{4}}|v|_\beta\le Ck^{\frac{\beta}{4}}t_1^{-\frac{1}{2}}|v|_\beta.\end{aligned}$$

So we have, for $j\ge1$,

$$\|\delta_j\|\le Ck^{\frac{\beta}{4}}t_j^{-\frac{1}{2}}|v|_\beta.$$

Hence,

$$k\sum_{j=1}^n\|\delta_j\|\le ck^{\frac{\beta}{4}}t_n^{\frac{1}{2}}|v|_\beta,\quad k\sum_{j=1}^n t_{j-1}\|\delta_j\|^2\le Ck^{\frac{\beta}{2}}t_n|v|_\beta^2. \tag{2.36}$$

Put (2.33), (2.34), (2.35), and (2.36) in (2.32), to get

$$\|e_n\|\le C(h^\beta+k^{\frac{\beta}{4}})|v|_\beta.$$

This completes the proof (2.19).

To prove (2.20) we recall (2.31) and let $v\in\dot H^{\beta-2}$. For the first term we write $k\sum_{j=1}^n\|\rho_j\|^2=k\|\rho_1\|^2+k\sum_{j=2}^n\|\rho_j\|^2$, where by (2.1)

$$k\|\rho_1\|^2\le kCh^{2\beta}\|AE(k)A^{\frac{\beta-2}{2}}v\|^2\le Ch^{2\beta}|v|_{\beta-2},$$

and

$$\begin{aligned}k\sum_{j=2}^n\|\rho_j\|^2 &= \sum_{j=2}^n\int_{t_{j-1}}^{t_j}\Big\|\rho(s)+\int_s^{t_j}\rho_t(\tau)\,\mathrm{d}\tau\Big\|^2\,\mathrm{d}s\\ &\le 2\sum_{j=2}^n\int_{t_{j-1}}^{t_j}\|\rho(s)\|^2\,\mathrm{d}s+2\sum_{j=2}^n\int_{t_{j-1}}^{t_j}\Big\|\int_s^{t_j}\rho_t(\tau)\,\mathrm{d}\tau\Big\|^2\,\mathrm{d}s\\ &\le 2\int_{t_1}^{t_n}\|\rho(s)\|^2\,\mathrm{d}s+2\sum_{j=2}^n\int_{t_{j-1}}^{t_j}(t_j-s)\int_{t_{j-1}}^{t_j}\|\rho_t(\tau)\|^2\,\mathrm{d}\tau\,\mathrm{d}s\\ &\le 2\int_0^{t_n}\|\rho\|^2\,\mathrm{d}\tau+2k\int_{t_1}^{t_n}\tau\|\rho_t\|^2\,\mathrm{d}\tau,\end{aligned}$$

since $t_j-s\le k\le\tau$ and where, by (2.15),

$$\int_0^{t_n}\|\rho\|^2\,\mathrm{d}\tau\le Ch^{2\beta}|v|_{\beta-2}^2,$$

and

$$k\int_{t_1}^{t_n}\tau\|\rho_t\|^2\,\mathrm{d}\tau\le Ch^{2\beta}k\int_{t_1}^{t_n}\tau\|A^3E(\tau)A^{\frac{\beta-2}{2}}v\|^2\,\mathrm{d}\tau$$
$$\le Ch^{2\beta}k\int_k^{t_n}\tau^{-2}\,\mathrm{d}\tau\,|v|^2_{\beta-2}\le Ch^{2\beta}k(k^{-1}-t_n^{-1})|v|^2_{\beta-2}\le Ch^{2\beta}|v|^2_{\beta-2}.$$

So

$$k\sum_{j=1}^{n}\|\rho_j\|^2\le Ch^{2\beta}|v|^2_{\beta-2}. \tag{2.37}$$

Now we compute $k\sum_{j=1}^n\|\eta_j\|$. Recall that $\eta_j=-(R_h-I)G\partial u_j$ and $\eta=-(R_h-I)Gu_t$, so

$$\|\eta_j\|=\Big\|(R_h-I)G\frac{1}{k}\int_{t_{j-1}}^{t_j}u_t\,\mathrm{d}\tau\Big\|\le\frac{1}{k}\int_{t_{j-1}}^{t_j}\|(R_h-I)Gu_t\|\,\mathrm{d}\tau$$
$$\le\frac{1}{k}\int_{t_{j-1}}^{t_j}\|\eta\|\,\mathrm{d}\tau,$$

and hence by (2.17) we have

$$k\sum_{j=1}^{n}\|\eta_j\|\le\int_0^{t_n}\|\eta\|\,\mathrm{d}\tau\le Ch^{\beta}|\log h||v|_{\beta-2}. \tag{2.38}$$

For computing $k\sum_{j=1}^n\|\delta_j\|$ we use (2.1) and obtain for $1\le\beta<4$,

$$\|\delta_j\|\le\frac{1}{k}\int_{t_{j-1}}^{t_j}(\tau-t_{j-1})\|Gu_{tt}(\tau)\|\,\mathrm{d}\tau\le\int_{t_{j-1}}^{t_j}\|A^{4-\frac{\beta}{2}}E(\tau)A^{\frac{\beta-2}{2}}v\|\,\mathrm{d}\tau$$
$$\le C\int_{t_{j-1}}^{t_j}\tau^{-2+\frac{\beta}{4}}\,\mathrm{d}\tau\,|v|_{\beta-2}.$$

Hence,

$$k\sum_{j=2}^{n}\|\delta_j\|\le Ck\int_k^{t_n}\tau^{-2+\frac{\beta}{4}}\,\mathrm{d}\tau|v|_{\beta-2}\le Ck\frac{4}{4-\beta}\Big(k^{-1+\frac{\beta}{4}}-t_n^{-1+\frac{\beta}{4}}\Big)|v|_{\beta-2}$$
$$\le\frac{C}{4-\beta}k^{\frac{\beta}{4}}|v|_{\beta-2}$$

and

$$k\|\delta_1\|\le\int_0^k\tau\|Gu_{tt}(\tau)\|\,\mathrm{d}\tau\le\int_0^k\tau\|A^{4-\frac{\beta}{2}}E(\tau)A^{\frac{\beta-2}{2}}v\|\,\mathrm{d}\tau$$
$$\le C\int_0^k\tau^{\frac{\beta}{4}-1}\,\mathrm{d}\tau\,|v|_{\beta-2}\le\frac{C}{4-\beta}k^{\frac{\beta}{4}}|v|_{\beta-2}.$$

Therefore, for $1 \leq \beta < 4$,

$$k \sum_{j=1}^{n} \|\delta_j\| \leq \frac{C}{4-\beta} k^{\frac{\beta}{4}} |v|_{\beta-2}.$$

If we put $\frac{1}{4-\beta} = |\log k|$, we also have

$$\begin{aligned} k \sum_{j=1}^{n} \|\delta_j\| &\leq \frac{C}{4-\beta} k^{1-\frac{4-\beta}{4}} |v|_{\beta-2} = C|\log k| k \mathrm{e}^{-\frac{4-\beta}{4}\log k} |v|_{\beta-2} \\ &\leq Ck|\log k||v|_{\beta-2} = C|\log k||v|_{\beta-2}. \end{aligned}$$

Therefore, for $1 \leq \beta \leq 4$, we have

$$k \sum_{j=1}^{n} \|\delta_j\| \leq C_{\beta,k} k^{\frac{\beta}{4}} |v|_{\beta-2}. \tag{2.39}$$

where $C_{\beta,k} = \frac{C}{4-\beta}$ for $\beta < 4$ and $C_{\beta,k} = C|\log k|$ for $\beta = 4$. Finally we put (2.37), (2.38) and (2.39) in (2.31), to get

$$\Big(k \sum_{j=1}^{n} \|e_j\|^2\Big)^{\frac{1}{2}} \leq \Big(Ch^{\beta}|\log h| + C_{\beta,k} k^{\frac{\beta}{4}}\Big)|v|_{\beta-2}.$$

□

## 3. Approximation of the linear Cahn-Hilliard-Cook equation

Consider the linear Cahn-Hilliard-Cook equation (1.4) with mild solution

$$X(t) = E(t)X_0 + \int_0^t E(t-s)\,\mathrm{d}W(s). \tag{3.1}$$

We recall the isometry of the Itô integral,

$$\mathbf{E}\Big\{\Big\|\int_0^t B(s)\,\mathrm{d}W(s)\Big\|^2\Big\} = \mathbf{E}\Big\{\int_0^t \|B(s)Q^{\frac{1}{2}}\|_{\mathrm{HS}}^2\,\mathrm{d}s\Big\}, \tag{3.2}$$

where the Hilbert-Schmidt norm is defined by

$$\|T\|_{\mathrm{HS}}^2 = \sum_{l=1}^{\infty} \|T\phi_l\|^2. \tag{3.3}$$

Here $\{\phi_l\}_{l=1}^{\infty}$ is an arbitrary orthonormal basis for $H$. In the next theorem we consider the regularity of the mild solution (3.1). The $L_2(\Omega, \dot{H}^{\beta})$-norm is defined in (1.11).

**Theorem 3.1.** *Let $X(t)$ be the mild solution (3.1) with $X_0 \in L_2(\Omega, \dot{H}^{\beta})$ and $\|A^{\frac{\beta-2}{2}}Q^{\frac{1}{2}}\|_{\mathrm{HS}} < \infty$ for some $\beta \geq 0$. Then*

$$\|X(t)\|_{L_2(\Omega,\dot{H}^{\beta})} \leq C\Big(\|X_0\|_{L_2(\Omega,\dot{H}^{\beta})} + \|A^{\frac{\beta-2}{2}}Q^{\frac{1}{2}}\|_{\mathrm{HS}}\Big), \quad t \geq 0.$$

*Moreover, if $\beta = 0$, then for the norm in $H$ we have*

$$\|X(t)\|_{L_2(\Omega,H)} \le C\Big(\|X_0\|_{L_2(\Omega,H)} + \|A^{-1}Q^{\frac{1}{2}}\|_{\mathrm{HS}} + t^{\frac{1}{2}}\Big), \quad t \ge 0.$$

*Proof.* Recall the definition of $|\cdot|_\beta$ in (1.2). By using the isometry (3.2), the definition of the Hilbert-Schmidt norm (3.3), and (2.1), (2.2) we get, for $\beta \ge 0$, see (1.8),

$$\begin{aligned}
\|X(t)\|^2_{L_2(\Omega,\dot H^\beta)} &= \mathbf{E}\Big\{\Big|E(t)X_0 + \int_0^t E(t-s)\,\mathrm{d}W(s)\Big|^2_\beta\Big\} \\
&\le C\Big(\mathbf{E}\big\{\big\|A^{\frac{\beta}{2}}PE(t)X_0\big\|^2\big\} + \mathbf{E}\Big\{\Big\|\int_0^t A^{\frac{\beta}{2}}PE(t-s)\,\mathrm{d}W(s)\Big\|^2\Big\}\Big) \\
&\le C\Big(\|X_0\|^2_{L_2(\Omega,\dot H^\beta)} + \int_0^t \|A^{\frac{\beta}{2}}E(s)PQ^{\frac{1}{2}}\|^2_{\mathrm{HS}}\,\mathrm{d}s\Big) \\
&\le C\Big(\|X_0\|^2_{L_2(\Omega,\dot H^\beta)} + \sum_{l=1}^\infty \int_0^t \|A^{\frac{\beta}{2}}E(s)PQ^{\frac{1}{2}}\phi_l\|^2\,\mathrm{d}s\Big) \\
&\le C\Big(\|X_0\|^2_{L_2(\Omega,\dot H^\beta)} + \sum_{l=1}^\infty \|A^{\frac{\beta-2}{2}}Q^{\frac{1}{2}}\phi_l\|^2\Big) \\
&= C\Big(\|X_0\|^2_{L_2(\Omega,\dot H^\beta)} + \|A^{\frac{\beta-2}{2}}Q^{\frac{1}{2}}\|^2_{\mathrm{HS}}\Big).
\end{aligned}$$

For $\beta = 0$ and the $H$-norm, there are additional terms

$$\begin{aligned}
&\mathbf{E}\{\|(I-P)X_0\|^2\} = \mathbf{E}\{(X_0,\varphi_0)^2\} \le \|X_0\|^2_{L_2(\Omega,H)}, \\
&\mathbf{E}\{\|(I-P)W(t)\|^2\} = \mathbf{E}\{(W(t),\varphi_0)^2\} \le Ct.
\end{aligned}$$

□

The finite element approximation of the linear Cahn-Hilliard-Cook equation is: Find $X_h(t) \in S_h$ such that

$$\mathrm{d}X_h + A_h^2 X_h\,\mathrm{d}t = P_h\,\mathrm{d}W, \quad t > 0; \quad X_h(0) = P_h X_0, \tag{3.4}$$

with the mild solution

$$X_h(t) = E_h(t)P_hX_0 + \int_0^t E_h(t-s)P_h\,\mathrm{d}W(s). \tag{3.5}$$

**Theorem 3.2.** *Let $X_h$ and $X$ be the mild solutions* (3.5) *and* (3.1) *with $X_0 \in L_2(\Omega,\dot H^\beta)$ and $\|A^{\frac{\beta-2}{2}}Q^{\frac{1}{2}}\|_{\mathrm{HS}} < \infty$ for some $\beta \in [1,r]$. Then there are $h_0$ and $C$, such that, for $h \le h_0$ and $t \ge 0$,*

$$\|X_h(t) - X(t)\|_{L_2(\Omega,H)} \le Ch^\beta\big(\|X_0\|_{L_2(\Omega,\dot H^\beta)} + |\log h|\|A^{\frac{\beta-2}{2}}Q^{\frac{1}{2}}\|_{\mathrm{HS}}\big).$$

*Proof.* Use (3.1) and (3.5) and set $F_h(t) = E_h(t)P_h - E(t)$ to get

$$\|X_h(t) - X(t)\|_{L_2(\Omega,H)} \le \|e_1(t)\|_{L_2(\Omega,H)} + \|e_2(t)\|_{L_2(\Omega,H)},$$

where, see (1.9),

$$e_1(t) = F_h(t)X_0 = F_h(t)PX_0,$$
$$e_2(t) = \int_0^t F_h(t-s)\,\mathrm{d}W(s) = \int_0^t F_h(t-s)P\,\mathrm{d}W(s).$$

By using Theorem 2.1 we get

$$\|e_1(t)\|_{L_2(\Omega,H)} = \big(\mathbf{E}\|F_h(t)X_0\|^2\big)^{\frac12} \le Ch^\beta\big(\mathbf{E}|X_0|_\beta^2\big)^{\frac12} = Ch^\beta\|X_0\|_{L_2(\Omega,\dot H^\beta)}.$$

For the second term we use the isometry (3.2), the definition of the Hilbert-Schmidt norm (3.3), and Theorem 2.1,

$$\begin{aligned}\|e_2(t)\|^2_{L_2(\Omega,H)} &= \mathbf{E}\Big(\Big\|\int_0^t F_h(t-s)\,\mathrm{d}W(s)\Big\|^2\Big)\\ &= \int_0^t \|F_h(t-s)Q^{\frac12}\|^2_{\mathrm{HS}}\,\mathrm{d}s = \sum_{l=1}^\infty \int_0^t \|F_h(s)Q^{\frac12}\phi_l\|^2\,\mathrm{d}s\\ &\le C|\log h|^2h^{2\beta}\sum_{l=1}^\infty |Q^{\frac12}\phi_l|^2_{\beta-2} = C|\log h|^2h^{2\beta}\|A^{(\beta-2)/2}Q^{\frac12}\|^2_{\mathrm{HS}}.\end{aligned}$$

□

Now we consider the fully discrete Cahn-Hilliard-Cook equation (1.10) with mild solution

$$X_{h,n} = E^n_{kh}P_hX_0 + \sum_{j=1}^n E^{n-j+1}_{kh}P_h\,\delta W_j, \quad \text{where } E_{kh} = (I+kA_h^2)^{-1}. \tag{3.6}$$

**Theorem 3.3.** *Let $X_{h,n}$ and $X$ be given by (3.6) and (3.1) with $X_0 \in L_2(\Omega,\dot H^\beta)$ and $\|A^{\frac{\beta-2}{2}}Q^{\frac12}\|_{\mathrm{HS}} < \infty$ for some $\beta \in [1,\min(r,4)]$. Then there are $h_0, k_0$ and $C$, such that, for $h \le h_0$, $k \le k_0$, and $n \ge 1$,*

$$\begin{aligned}&\|X_{h,n} - X(t_n)\|_{L_2(\Omega,H)}\\ &\qquad\le \big(C|\log h|h^\beta + C_{\beta,k}k^{\frac{\beta}{4}}\big)\big(\|X_0\|_{L_2(\Omega,\dot H^\beta)} + \|A^{\frac{\beta-2}{2}}Q^{\frac12}\|_{\mathrm{HS}}\big),\end{aligned}$$

*where $C_{\beta,k} = \frac{C}{4-\beta}$ for $\beta < 4$ and $C_{\beta,k} = C|\log k|$ for $\beta = 4$.*

*Proof.* By using (3.1) and (3.6) we get, with $F_n = E^n_{kh}P_h - E(t_n)$,

$$\begin{aligned}e_n &= F_nX_0 + \sum_{j=1}^n \int_{t_{j-1}}^{t_j} F_{n-j+1}\,\mathrm{d}W(s)\\ &\quad + \sum_{j=1}^n \int_{t_{j-1}}^{t_j} \big(E(t_n-t_{j-1}) - E(t_n-s)\big)\,\mathrm{d}W(s) = e_{n,1}+e_{n,2}+e_{n,3}.\end{aligned}$$

By using Theorem 2.2 we have

$$\|e_{n,1}\|_{L_2(\Omega,H)} = \big(\mathbf{E}\|F_nX_0\|^2\big)^{\frac12} \le C(h^\beta + k^{\frac{\beta}{4}})\|X_0\|_{L_2(\Omega,\dot H^\beta)}. \tag{3.7}$$

By using the isometry (3.2) and Theorem 2.2 we get

$$\begin{aligned}
\|e_{n,2}\|^2_{L_2(\Omega,H)} &= \mathbf{E}\Big(\Big\|\sum_{j=1}^{n}\int_{t_{j-1}}^{t_j} F_{n-j+1}\,\mathrm{d}W(s)\Big\|^2\Big) \\
&= \sum_{j=1}^{n}\int_{t_{j-1}}^{t_j} \|F_{n-j+1}Q^{\frac12}\|^2_{\mathrm{HS}}\,\mathrm{d}s = k\sum_{l=1}^{\infty}\sum_{j=1}^{n}\|F_{n-j+1}Q^{\frac12}\phi_l\|^2 \\
&\le \sum_{l=1}^{\infty}\big(C|\log h|h^{\beta}+C_{\beta,k}k^{\frac{\beta}{4}}\big)^2|Q^{\frac12}\phi_l|^2_{\beta-2} \\
&= \big(C|\log h|h^{\beta}+C_{\beta,k}k^{\frac{\beta}{4}}\big)^2\|A^{\frac{\beta-2}{2}}Q^{\frac12}\|^2_{\mathrm{HS}}.
\end{aligned}$$

By using the isometry property (3.2) again we have

$$\begin{aligned}
\|e_{n,3}\|^2_{L_2(\Omega,H)} &\le \mathbf{E}\Big(\Big\|\sum_{j=1}^{n}\int_{t_{j-1}}^{t_j}(E(t_n-t_{j-1})-E(t_n-s))\,\mathrm{d}W(s)\Big\|^2\Big) \\
&= \sum_{j=1}^{n}\int_{t_{j-1}}^{t_j}\|(E(t_n-t_{j-1})-E(t_n-s))Q^{\frac12}\|^2_{\mathrm{HS}}\,\mathrm{d}s \\
&= \sum_{l=1}^{\infty}\sum_{j=1}^{n}\int_{t_{j-1}}^{t_j}\|A^{-\frac{\beta}{2}}(E(s-t_{j-1})-I)AE(t_n-s)A^{\frac{\beta-2}{2}}Q^{\frac12}\phi_l\|^2\,\mathrm{d}s.
\end{aligned}$$

Using the well-known inequality

$$\|A^{\frac{-\beta}{2}}\big(E(t)-I\big)w\| \le Ct^{\frac{\beta}{4}}\|w\|,$$

with $t=s-t_j, w=AE(t_n-s)A^{\frac{\beta-2}{2}}Q^{\frac12}\phi_l$, together with (2.2), we get

$$\begin{aligned}
\|e_{n,3}\|^2_{L_2(\Omega,H)} &\le Ck^{\frac{\beta}{2}}\sum_{l=1}^{\infty}\int_0^{t_n}\|AE(t_n-s)A^{\frac{\beta-2}{2}}Q^{\frac12}\phi_l\|^2\,\mathrm{d}s \\
&\le Ck^{\frac{\beta}{2}}\sum_{l=1}^{\infty}\|A^{\frac{\beta-2}{2}}Q^{\frac12}\phi_l\|^2 = Ck^{\frac{\beta}{2}}\|A^{\frac{\beta-2}{2}}Q^{\frac12}\|^2_{\mathrm{HS}}.
\end{aligned}$$

Putting these together proves the desired result. □

## 4. Conclusion

We have studied numerical approximation of the linearized Cahn-Hilliard-Cook equation by a spatially semidiscrete finite element method and a completely discrete method based on the implicit Euler time-stepping. We have proved strong convergence estimates of optimal order except for a logarithmic factor.

By means of the Itô isometry the proofs are reduced to proving error estimates for the corresponding deterministic problem, that is, error estimates for approximations of the Cahn-Hilliard semigroup $\mathrm{e}^{-tA^2}$. This is where the main effort has been spent. Since the finite element method is based on

$(A_h)^2$ rather than $(A^2)_h$ this analysis is more difficult than for the linear heat equation in [13].

Our results should be interpreted as results about approximation of the stochastic convolution $\int_0^t \mathrm{e}^{-(t-s)A^2}\,\mathrm{d}W(s)$, which is a part of the mild solution of the nonlinear Cahn-Hilliard-Cook equation. The remaining part, which solves a nonlinear random evolution problem, is studied in the sequel [11], where strong convergence is proved for the spatially semidiscrete approximation of the nonlinear Cahn-Hilliard-Cook equation, but without known rate of convergence. To obtain the optimal rate of convergence remains a challenge. Another open problem is to study weak convergence.

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