An elementary approach to elementary operators on $\mathcal{B}(H)$

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1. Introduction

In 1975 J. Anderson and F. Foias wrote a paper [2] whose title could be used (with modifications) in many later papers on elementary operators. In the present paper we discuss properties which operators on a Hilbert space H share with elementary operators on $\mathcal{B}(H)$. The reason of the search of common properties is that an elementary operator $\Delta = \sum_{k=1}^{n} L_{a_k} R_{b_k}$ has a formal adjoint $\tilde{\Delta} = \sum_{k=1}^{n} L_{a_k^*} R_{b_k^*}$ which turns into a proper adjoint if restricted to the ideal $\mathcal{S}_2$ of Hilbert-Schmidt operators, and so it is natural to expect that their adjointness on $\mathcal{B}(H)$ is not absolutely formal. After discussion of the general case we come to normal elementary operators (that is those whose coefficient families are commutative and consists of normal operators) and study which properties of normal operators on Hilbert space they share.

Our paper is a kind of review but some formulations of the results seem to be new. Some statements of the paper can be extended to a more wide class of multiplication operators, but we prefer to restrict ourselves to elementary operators which allows us to avoid more complicated technique of Varopoulos algebras, Haagerup tensor products, spectral synthesis and so on. Moreover we hope that such approach makes the subject really elementary and gives the possibility to present the results with ideas of their proofs in a text of reasonable length. The reader can find transparent proofs and various extensions of the main part of our results in [19, 22, 23].

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1.1. Notations

Let H be a Hilbert space, $\mathcal{B}(H)$ be the space of bounded linear operators on H. We denote by $\mathcal{S}_p$, $1 \le p < \infty$, a Schatten-von-Neumann ideal and write $||\cdot||_p$ for the corresponding norm and let $\mathcal{S}_\infty$ to denote the space of compact operators.

Let us say that an operator (or a family of operators) is *multicyclic* or *has finite multicyclicity* if there is a finite set of vectors which is not contained in its proper closed invariant subspace.

For a subset M of a metric space we say that its Hausdorff dimension does not exceed a number $r > 0$ if there exist $C > 0$ such that for $\epsilon > 0$ there is a covering $\mathcal{B} = \{\beta_j\}$ of M by pairwise disjoint Borel sets with $\mathrm{diam}\beta_j < \epsilon$ and $|\mathcal{B}|_r := (\sum_j (\mathrm{diam}\beta_j)^r \le C$.

If $\mathbb{A} = (A_1, ..., A_n)$ is a family of commuting normal operators then by $\sigma(\mathbb{A})$ we denote the joint spectrum, and by $E_{\mathbb{A}}(\cdot)$ the spectral measure of $\mathbb{A}$.

We say that the *essential dimension* of $\mathbb{A}$ does not exceed $r > 0$ (and write ess-dim $\mathbb{A} \le r$) if there is a subset D of $\sigma(\mathbb{A})$ such that $E_{\mathbb{A}}(\sigma(\mathbb{A}) \setminus D) = 0$ and $\dim(D) \le r$.

2. Approximate inverse intertwinings

The proofs of many further statements are based on several results of very general nature which we gather in the present section. The reader can consider them as exercises in functional analysis, (s)he can also find their solutions in Section 6 of [22].

Let $\mathfrak{X}$ and $\mathfrak{Y}$ be topological vector spaces, $\Phi : \mathfrak{X} \to \mathfrak{Y}$ a continuous imbedding with dense range, and let S and T be operators acting in $\mathfrak{X}$ and $\mathfrak{Y}$, respectively, intertwined by the mapping Φ: $T\Phi = \Phi S$. We write in this case that we are given an intertwining triple (or just an intertwining) (Φ, S, T).

A net of linear mappings $F_\alpha : \mathfrak{Y} \to \mathfrak{X}$ is called an *approximate inverse intertwining* (AII) for the intertwining (Φ, S, T) if

(a) $F_\alpha \Phi \to 1_{\mathfrak{X}}$,

(b) $\Phi F_\alpha \to 1_{\mathfrak{Y}}$ and

(c) $F_\alpha T - SF_\alpha \to 0_{\mathfrak{X}}$

in the topology of simple convergence.

Denote by Φ^{-1} the full inverse image under the mapping Φ: $\Phi^{-1}(M) = \{x \in \mathfrak{X} \mid \Phi(x) \in M\}$ for any $M \subset \mathfrak{Y}$ (non-necessarily $M \subset \Phi(\mathfrak{X})$). As usual the range of a map X is denoted by $\mathrm{im}\, X$.

Theorem 2.1. *If the intertwinings (Φ, S_i, T_i), $1 \le i \le n$, have a common AII, then*

$$\Phi^{-1}(\sum_i \mathrm{im}\, T_i) \subset \overline{\sum_i \mathrm{im}\, S_i}.$$

Let $\mathcal{H}$ be a Hilbert space equipped with the weak operator topology.

Corollary 2.2. *If $\mathfrak{X} = \mathcal{H}$ and (Φ, S, T) has an AII, then*

$$\Phi(\ker S^*) \cap \operatorname{im} T = \{0\}.$$

Theorem 2.3. *Let Φ intertwine pairs S_i, T_i $(i = 1, 2)$. Suppose that $\mathfrak{X}$ is a Banach space equipped with a weak topology, and $||S_2 x|| \le ||S_1 x||$ for any $x \in \mathfrak{X}$. If (Φ, S_1, T_1) has AII then*

$$T_1^{-1}(\operatorname{im} \Phi) \subset T_2^{-1}(\operatorname{im} \Phi)$$

and

$$||\Phi^{-1} T_2 y|| \le ||\Phi^{-1} T_1 y|| \tag{2.1}$$

for any $y \in T_1^{-1}(\operatorname{im} \Phi)$.

The following result is an immediate consequence of Theorem 2.3.

Corollary 2.4. *Let $\mathfrak{X} = \mathcal{H}$. Suppose that S is a normal operator on $\mathcal{H}$ and that the intertwinings (Φ, S, T_1), (Φ, S^*, T_2) have AII's (not necessarily coinciding). Then $T_1^{-1}(\operatorname{im} \Phi) = T_2^{-1}(\operatorname{im} \Phi)$ and*

$$||\Phi^{-1} T_2 y|| = ||\Phi^{-1} T_1 y||$$

for any $y \in T_1^{-1}(\operatorname{im} \Phi) = T_2^{-1}(\operatorname{im} \Phi)$.

In particular, $\ker T_1 = \ker T_2$.

Let (Φ, S, T) be an intertwining. If $\mathfrak{X}$ is a dual Banach space with the weak-* topology (for example if $\mathfrak{X} = \mathcal{H}$) then to obtain an AII it suffices to construct a net of operators $F_\alpha : \mathfrak{Y} \to \mathfrak{X}$ which satisfies a weakened version of (AII)-conditions:

(a$'$) $F_\alpha \Phi x$ is bounded for each $x \in \mathfrak{X}$,

(b$'$) $\Phi F_\alpha \to 1_{\mathfrak{Y}}$ and

(c$'$) $(F_\alpha T - S F_\alpha) y$ is bounded for each $y \in Y$.

In general a net satisfying ((a$'$),(b$'$),(c$'$)) is called *an approximate inner semi-intertwining* for (Φ, S, T) (AIS, for short).

Denote by $\mathfrak{X}^*$ the space of continuous antilinear functionals on $\mathfrak{X}$, endowed with the weak-* topology (in particular, $\mathcal{H}^* = \mathcal{H}$). The adjoint operators (on $\mathcal{X}^*$ or between $\mathcal{X}^*$ and $\mathcal{Y}^*$) are defined in the usual way. In particular, the adjoint of an operator on $\mathcal{H}$ has the usual meaning.

It is not difficult to see that if $\{F_\alpha\}$ is an AII for (Φ, S, T) then $\{F_\alpha^*\}$ is an AII for (Φ^*, T^*, S^*).

Let $\Phi : \mathcal{H} \to \mathfrak{Y}$ intertwine operators S, S^* with T_1, T_2. Let $\{F_\alpha\} : \mathfrak{Y} \to \mathcal{H}$ be an AII for the intertwining (Φ, S, T_1). It is called a $*$-*approximate inverse intertwining* ($*$-AII) for the ordered pair $((\Phi, S, T_1), (\Phi, S^*, T_2))$ if $\{F_\alpha^* F_\alpha\}$ is an AII for $(\Phi\Phi^*, T_1^*, T_2)$.

A $*$-*approximate inverse semiintertwining* ($*$-AIS) is defined in a similar way: it is an AIS (= AII) $\{F_\alpha\}$ for (Φ, S, T_1) such that $\{F_\alpha^* F_\alpha\}$ is an AIS for $(\Phi\Phi^*, T_1^*, T_2)$.

Theorem 2.5. *(i) If the pair $((\Phi, S, T_1), (\Phi, S^*, T_2))$ has $*$-AIS, then*

$$(\operatorname{im} T_1) \cap T_2^{-1}(\Phi\Phi^*(\mathfrak{Y}^*)) \subset \Phi(\mathcal{H}).$$

(ii) If $((\Phi, S, T_1),\ (\Phi, S^*, T_2))$ *has* $*$*-AII, then*

$$||\Phi^{-1}(T_1 y)||^2 = \langle (\Phi\Phi^*)^{-1}(T_2T_1 y), y\rangle$$

for any $y \in (T_2T_1)^{-1}(\Phi\Phi^*(\mathcal{Y}^*))$.

Corollary 2.6. *If* $((\Phi, S, T_1),\ (\Phi, S^*, T_2))$ *has a* $*$*-AIS then* $\operatorname{im} T_1 \cap \ker T_2 = \{0\}$.

We will finish by a result which has some similarity to Theorem 2.3 but is not related to AII's.

Theorem 2.7. *Let* $\Phi : \mathcal{H} \to \mathfrak{Y}$ *intertwine commuting normal operators* S_1, S_2 *with operators* T_1 *and* T_2 *respectively. Suppose that*

$$\ker(S_1) \cap \Phi^{-1}(T_2\mathfrak{Y}) = \{0\}.$$

Let $\|S_2 h\| \leq \|S_1 h\|$ *for each* $h \in \mathcal{H}$. *Then the inequality (*2.1*) holds for each* $y \in \mathfrak{Y}$ *such that* $T_1 y \in \Phi(\mathcal{H})$ *and* $T_2 y \in \Phi(\mathcal{H})$.

Corollary 2.8. *Let* $\Phi : \mathcal{H} \to \mathfrak{Y}$ *intertwine a normal operator* S *with* T_1 *and its adjoint* S^* *with* T_2. *Suppose that* $\ker(S) \cap \Phi^{-1}(T_2\mathfrak{Y}) = \{0\}$. *Then the inequality (*2.1*) holds for each* $y \in \mathfrak{Y}$ *such that* $T_1 y \in \Phi(\mathcal{H})$ *and* $T_2 y \in \Phi(\mathcal{H})$.

3. General elementary operators: the range

Here by Δ we denote an elementary operator $\sum_{i=1}^n L_{A_i} R_{B_i}$ on $\mathcal{B}(H)$:

$$\Delta(X) = \sum_{i=1}^n A_i X B_i$$

and set

$$\widetilde{\Delta}(X) = \sum_{i=1}^n A_i^* X B_i^*.$$

We denote by Δ_p and $\widetilde{\Delta}_p$ the restriction of Δ and respectively $\widetilde{\Delta}$ to the ideal $\mathcal{S}_p$, $1 \leq p \leq \infty$ (by $\mathcal{S}_\infty$ as usually we denote the ideal $\mathcal{K}(H)$ of all compact operators).

3.1. The intersection with the kernel of adjoint

For each operator acting on a Hilbert space, the closure of its range has zero intersection with the kernel of its adjoint. We will discuss related conditions for elementary operators, that is

$$\overline{\Delta(\mathcal{B}(H))} \cap \ker \tilde{\Delta} = \{0\} \tag{3.1}$$

or more strong one

$$\|\Delta(X) + Y\| \geq C\|Y\| \text{ for all } X \in \mathcal{B}(H), Y \in \ker \tilde{\Delta} \text{ and some } C > 0, \tag{3.2}$$

(which means that the angle between $\overline{\Delta(\mathcal{B}(H))}$ and $\ker \tilde{\Delta}$ is non-zero), or more weak ones

$$\Delta(\mathcal{B}(H)) \cap \ker \tilde{\Delta} = \{0\} \tag{3.3}$$

and

$$\Delta(\mathcal{S}_p) \cap \ker \tilde{\Delta}_p = \{0\}. \tag{3.4}$$

The last conditions can be rewritten as follows:

$$\ker \tilde{\Delta}\Delta = \ker \Delta. \tag{3.5}$$

and

$$\ker \tilde{\Delta}_p \Delta_p = \ker \Delta_p \tag{3.6}$$

respectively. Note that the validity of (3.5) for a class of operators can have strong consequences. For example if it is true for an operator Δ which commutes with $\tilde{\Delta}$ then $\ker \tilde{\Delta} \subset \ker \Delta$. On this way one can obtain various extensions of the Fuglede-Putnam theorem (see Section 4).

Problem 1. *Is (3.5) true for inner derivations, that is for operators $\Delta = L_A - R_A$?*

The positive answer was known for the case that A is a weighted shift [11] or a subnormal operator with a cyclic vector [11]. Moreover it was shown in [11] that in the latter case the condition (3.1) holds. If A is normal then even (3.2) holds with $C = 1$ [2]. But in general (3.1) is not true for inner derivations: Anderson [3] has shown that the closure of the image of $L_A - R_A$ can contain 1_H.

Note that the intersection of the left part of (3.3) with $\mathcal{S}_1$ is trivial; moreover the following much stronger condition holds:

$$\overline{\Delta(\mathcal{B}(H))}^{w^*} \cap \ker \tilde{\Delta}_1 = \{0\}. \tag{3.7}$$

Here $\overline{\Delta(\mathcal{B}(H))}^{w^*}$ denotes the closure of $\Delta(\mathcal{B}(H))$ in the weak-* topology. To verify (3.7) note that if $X \in \ker \tilde{\Delta}_1$ then $tr(X^*\Delta(Y)) = tr(\tilde{\Delta}(X)^*Y) = 0$ for each $Y \in \mathcal{B}(H)$. Hence $tr(X^*Z) = 0$ for each $Z \in \overline{\Delta(\mathcal{B}(H))}^{w^*}$. It follows that $tr(X^*X) = 0$ for $X \in \overline{\Delta(\mathcal{B}(H))}^{w^*} \cap \ker \tilde{\Delta}_1$.

It would be important to prove the triviality of the intersection of the left part of (3.3) with $\mathcal{S}_2$:

Problem 2. *For which elementary operators Δ does the equality*

$$\Delta(\mathcal{B}(H)) \cap \ker \widetilde{\Delta}_2 = 0 \tag{3.8}$$

hold?

We conjecture that for all. But could it be proved at least for inner derivations?

It will be shown in Section 4 that (3.8) is true (in a more strong version) for all normal elementary operators.

Some conditions which are sufficient for (3.5) can be written in terms close in spirit to Voiculescu's notion of quasidiagonality with respect to a symmetrically normed ideal [26]. We restrict ourselves to Schatten ideals $\mathcal{S}_p$ for simplicity.

Let us say that a family $\mathbb{A} = \{A_k : 1 \le k \le m\}$ of operators is p-semidiagonal if there exists a sequence of projections P_n of finite rank such that $P_n \to 1$ in the strong operator topology, and

$$\sup_n ||[A_k, P_n]||_p < \infty, \text{ for each } k \le m.$$

It is clear that if $p_1 < p_2$ then each p_1-semidiagonal family is p_2-semidiagonal. In particular, 1-semidiagonality is the strongest of these conditions. Clearly, any family is ∞-semidiagonal, where $\|\cdot\|_\infty$ is the operator norm.

Let us list some examples of p-semidiagonal families (see [22] for transparent proofs or references).

1) Any family of operators with matrices (with respect to some basis) supported by a finite number of diagonals (i.e. $a_{ij} = 0$ if $|i-j| > m$ for some $m \in \mathbb{N}$) is 1-semidiagonal. One of the simplest classes of such examples consists of families of weighted shifts.

This class of examples can be considerably extended as follows.

If (a_{ij}) is a matrix of an operator A in a basis $\{e_n\}_{n=1}^\infty$, let us set $|A|_n = \sup_{|i-j|=n} |a_{ij}|$ and $|A|_{diag} = \sum_{n=1}^\infty n|A|_n$. We say that A is diagonally bounded (with respect to the basis $\{e_n\}_{n=1}^\infty$) if $|A|_{diag} < \infty$. Then:

2) Any family of diagonally bounded (with respect to the same basis) operators is 1-semidiagonal.

As a consequence we obtain the following result of Voiculescu [26]:

3) Any family which belongs to the algebra of operators on $L_2(\mathbb{T})$ generated by shifts $u(t) \mapsto u(t-\theta)$ and multiplication operators by twice differentiable functions $f \in C^2(\mathbb{T})$ (we will call the elements of these algebras "generalized Bishop's operatos"), is 1-semidiagonal.

To deduce this from 2), it suffices to calculate the matrices of shift and multiplication operators in the standard basis $e_k = \exp(ikt)$, $k \in \mathbb{Z}$.

4) It is important that all normal operators of finite multicyclicity are 2-semidiagonal. More generally, if $f_1, ..., f_n$ are Lipschitz functions on the spectrum of a normal operator A, then the family $(f_1(A), ..., f_n(A))$ is p-semidiagonal where p is the Hausdorff dimension of $\sigma(A)$.

Slightly more general, if the essential dimension of a family $\mathbb{A} = (A_1, ..., A_n)$ of commuting normal operators does not exceed $p \le 2$ then $\mathbb{A}$ is p-semidiagonal.

5) Any multicyclic almost normal operator A is 2-semidiagonal ([25]).

Recall that an operator A is *almost normal* if its self-commutator $[A^*, A]$ is nuclear.

Theorem 3.1. *If the left coefficient family* $\mathbb{A}$ *of an elementary operator* Δ *is* 1-*semidiagonal then the equality (*3.3*) holds.*

If $\mathbb{A}$ *is* $p/(p-2)$-*semidiagonal then (*3.4*) holds.*

Of course one could impose the same restriction (of 1-semidiagonality) on the right coefficient family $\mathbb{B}$ of Δ.

Problem 3. *Find a condition which involves both coefficient families so that* (3.3) *or* (3.4) *is valid. In particular, does* (3.3) *hold if both* $\mathbb{A}$ *and* $\mathbb{B}$ *are* 2*-semidiagonal?*

To outline the proof of Theorem 3.1, note that the operators Δ and Δ_p are intertwined by the injection $\Phi_p : \mathcal{S}_p \to \mathcal{B}(H)$. Similarly the injection $\Phi_{p_1,p_2} : \mathcal{S}_{p_1} \to \mathcal{S}_{p_2}$, intertwines Δ_{p_1} and Δ_{p_2} for $p_1 \leq p_2$. If $\mathbb{A}$ is p-semidiagonal then one can use the projections P_n from the definition of semidiagonality, to construct an AII for these intertwinings by setting $F_n(X) = P_n X$. We describe firstly the results that relate properties of Δ and Δ_2: this is important because the latter acts on the Hilbert space $\mathcal{H} = \mathcal{S}_2$ and its adjoint is $\widetilde{\Delta}_2$.

Theorem 3.2. *(i) If the left coefficient family* $\mathbb{A}$ *of* Δ *is 2-semidiagonal then there exists an AII for* $(\Phi_2, \Delta_2, \Delta)$.

(ii) If $\mathbb{A}$ *is 1-semidiagonal then*

*(a) there exists a *-AIS for* $(\Phi_2, \Delta_2, \Delta)$ *and* $(\Phi_2, \widetilde{\Delta}_2, \widetilde{\Delta})$*;*

(b) there exists an AII for $(\Phi_2, \Delta_2, \Delta_\infty)$*;*

*(c) there exists a *-AII for* $(\Phi_2, \Delta_2, \Delta_\infty)$ *and* $(\Phi_2, \widetilde{\Delta}_2, \widetilde{\Delta}_\infty)$.

(iii) If $\mathbb{A}$ *is* $p/(p-2)$*-semidiagonal,* $p > 2$*, then there exists a *-AII for* $(\Phi_{2,p}, \Delta_2, \Delta_p)$ *and* $(\Phi_{2,p}, \widetilde{\Delta}_2, \widetilde{\Delta}_p)$

Now to prove Theorem 3.1 it suffices to apply Theorem 3.2 (parts (ii-a) and (iii)) and Corollary 2.6.

3.2. Hyponormality

Let us say that an elementary operator Δ is *formally positive* if $tr(\Delta(X)X^*) \geq 0$ for each $X \in \mathcal{S}_2$. Furthermore Δ is *formally hyponormal* if $\widetilde{\Delta}\Delta - \Delta\widetilde{\Delta}$ is formally positive.

Theorem 3.3. *Let* Δ *be formally hyponormal and let its left coefficient family* $\mathbb{A}$ *be 2-semidiagonal.*

(i) If $\Delta(X) \in \mathcal{S}_2$*, for some* $X \in \mathcal{B}(H)$*, then* $\widetilde{\Delta}(X) \in \mathcal{S}_2$ *and*

$$\|\Delta(X)\|_2 \geq \|\widetilde{\Delta}(X)\|_2.$$

As a consequence we get

(ii) $\ker \Delta \subset \ker \widetilde{\Delta}$.

Indeed, by Theorem 3.2 (i), the assumption of 2-semidiagonality implies the existence of AII for $(\Phi_2, \Delta_2, \Delta)$. Since formal hyponormality of Δ means that Δ_2 is a hyponormal operator, the inequality $\|\widetilde{\Delta}(X)\|_2 \leq \|\Delta(X)\|_2$ holds for each operator $X \in \mathcal{S}_2$. It remains to apply Theorem 2.3 to the intertwinings $(\Phi_2, \Delta_2, \Delta)$ and $(\Phi_2, \widetilde{\Delta}_2, \widetilde{\Delta})$.

Considering elementary operators of the form $\Delta = L_A + R_B$ it is easy to see that $\widetilde{\Delta}\Delta - \Delta\widetilde{\Delta} = L_{[A^*,A]} + R_{[B,B^*]}$. Therefore such operator is formally hyponormal if A and B^* are hyponormal. One can also show that the converse is also true. Note that if a hyponormal operator A is multicyclic then its selfcommutator $[A^*, A]$ is nuclear ([4]) so A is almost normal. Taking into account that multicyclic normal operators are 2-semidiagonal we apply Theorem 3.3 and obtain

Corollary 3.4. *Let operators A and B^* be hyponormal and multicyclic. Then*

$$\|AX - XB\|_2 \geq \|A^*X - XB^*\|_2 \tag{3.9}$$

for each operator X such that $AX - XB \in \mathcal{S}_2$.

This result from [22] was proved earlier under more restrictive conditions on A, B and X (see [7] and numerous references therein).

Theorem 3.3 is related to the following result which is not restricted by hyponormality assumptions.

Theorem 3.5. *If $\mathbb{A}$ is 1-semidiagonal then*

$$\|\Delta(X)\|_2^2 = tr(X^*\widetilde{\Delta}\Delta(X)) \tag{3.10}$$

for each compact operator X such that $\widetilde{\Delta}\Delta(X) \in \mathcal{S}_1$.

This statement can be proved in the same way as the previous results by using Theorem 2.5 (ii) and Theorem 3.2 (ii-c).

To see its relation to Theorem 3.3, note that if Δ is formally hyponormal then clearly $tr(X^*\widetilde{\Delta}\Delta(X)) \geq tr(X^*\Delta\widetilde{\Delta}(X))$.

3.3. Ranges of derivations and traces of commutators

Now we will study which trace class operators can belong to the range of an elementary operator Δ. For this we need information about AII's for the intertwinings $(\Phi_1, \Delta_1, \Delta)$ and $(\Phi_{1,p}, \Delta_1, \Delta_p)$.

It will be convenient, for each $p \in (1;\infty)$, to denote by p' the number $\frac{p}{p-1}$, and set $p' = 1$ if $p = \infty$, $p' = \infty$ if $p = 1$.

Theorem 3.6. *(i) If the left coefficient family $\mathbb{A}$ of Δ is 1-semidiagonal then there exists an AIS for $(\Phi_1, \Delta_1, \Delta)$.*

(ii) If $\mathbb{A}$ is p'-semidiagonal then there exists an AII for $(\Phi_{1,p}, \Delta_1, \Delta_p)$.

Applying Theorem 2.1 we deduce from part (ii) of Theorem 3.6

Corollary 3.7. *If $\mathbb{A}$ is p'-semidiagonal then $\mathcal{S}_1 \cap \Delta(\mathcal{S}_p)$ is contained in the $\|\cdot\|_1$-closure of $\Delta(\mathcal{S}_1)$.*

In particular,

Corollary 3.8. *Let the coefficients of Δ satisfy the condition $\sum_{i=1}^n B_iA_i = 0$. If $\mathbb{A}$ is p'-semidiagonal then $tr(\Delta(X)) = 0$ for each operator $X \in \mathcal{S}_p$ for which $\Delta(X) \in \mathcal{S}_1$.*

Indeed if $Y \in \mathcal{S}_1$ then $tr(\Delta(Y)) = tr(\sum_{i=1}^n B_iA_iY) = 0$, so the result follows from Corollary 3.7 and the continuity of trace on $\mathcal{S}_1$.

The result can be applied to the problem "when the trace of a commutator is equal to 0?". It is well known that a commutator $[A, X]$ has zero trace if X

is nuclear or if both operators A and X are Hilbert-Schmidt. Weiss [27] proved that the same is true if X is Hilbert-Schmidt and A is normal. The following proposition, which is an easy consequence of Corollary 3.8, widely extends this result.

Corollary 3.9. *Let $p > 1$. If $\{A_k\}_{k=1}^n$ is p'-semidiagonal, $X_k \in \mathcal{S}_p$ and $\sum_{k=1}^n [A_k, X_k] \in \mathcal{S}_1$ then*

$$tr(\sum_{k=1}^n [A_k, X_k]) = 0.$$

For $p = \infty$, this gives, for instance, that the commutator of a Hermitian operator (or a weighted shift, or a Bishop's operator) with a compact operator has zero trace if nuclear. Taking $p = 2$, we obtain the same for the commutator of a normal (or almost normal) multicyclic operator with a Hilbert-Schmidt operator. The restriction on multicyclicity can be easily removed.

Choosing in Proposition 3.9 for A_k the multiplication operators M_{f_k} on $L_2([0,1])$ and for X_k the integral operators with kernels $F_k \in L_2([0,1]^2)$, one obtains the following result:

Corollary 3.10. *If $f_k \in Lip_{1/2}([0,1])$, $1 \le k \le n$, then no functions $F_k \in L_2([0,1]^2)$ satisfying the condition*

$$\sum_{k=1}^n (f_k(x) - f_k(y))F_k(x,y) = 1. \tag{3.11}$$

It was asked by Weiss if this is true for $f_k \in C[0,1]$; the answer is negative (see [22, Proposition 8.7] which shows that 1/2 in Corollary 3.10 cannot be even changed by 1/3)

The constant p' in Proposition 3.9 is sharp for $p = 2$, i.e. the condition $X_k \in \mathcal{S}_2$ cannot be weakened to $X_k \in \mathcal{S}_q$ for any $q > 2$ (see [22, Example 8.5]).

Problem 4. *Is the constant p' sharp for all $p \ge 1$?*

4. Normal elementary operators

In this section the coefficient families $\mathbb{A}, \mathbb{B}$ of an elementary operator Δ are assumed to be commutative and to consist of normal operators. Such elementary operators are called normal.

4.1. Spectral subspaces

It is well known (see for example a much more general result of [6]) that

$$\sigma(\Delta) = \{\sum_{k=1}^n \lambda_i \mu_i : \lambda \in \sigma(\mathbb{A}),\ \mu \in \sigma(\mathbb{B})\}. \tag{4.1}$$

We will study here some other spectral characteristics of elementary operators.

Recall that an operator T on a Banach space $\mathcal{X}$ is decomposable if to each compact set $\alpha \subset \sigma(T)$ there corresponds a closed subspace $E_T(\alpha)$ invariant for all operators commuting with T and satisfying the following conditions:

(a) $\sigma(T|E_T(\alpha)) \subset \alpha$;

(b) If U, V are open subsets of $\mathbb{C}$ and $U \cup V \supset \sigma(T)$ then there are compact sets $\alpha \subset U$, $\beta \subset V$ with $E_T(\alpha) + E_T(\beta) = \mathcal{X}$.

The subspaces $E_T(\alpha)$ are called *spectral subspaces* of T; the map $\alpha \to E_T(\alpha)$ is *the spectral capacity* of T.

To describe spectral subspaces $E_\Delta(\alpha)$ of a normal elementary operator Δ we need the following notion. Let $E_{\mathbb{A}}(\cdot)$ and $E_{\mathbb{B}}(\cdot)$ be the spectral measures of the coefficient families $\mathbb{A}$ and $\mathbb{B}$ of Δ on $\sigma(\mathbb{A})$ and $\sigma(\mathbb{B})$ respectively. An operator $X \in \mathcal{B}(H)$ is said to be supported by a set $M \subset \sigma(\mathbb{A}) \times \sigma(\mathbb{B})$ if $E_{\mathbb{A}}(U) X E_{\mathbb{B}}(V) = 0$ for every Borel rectangle $U \times V \subset \sigma(\mathbb{A}) \times \sigma(\mathbb{B})$ non-intersecting M. We will write in this case $\operatorname{supp} X \subset M$.

It is not difficult to check that for each subset $M \subset \sigma(\mathbb{A}) \times \sigma(\mathbb{B})$, the subspace H_M of all operators supported by M is invariant for Δ. Moreover it can be deduced in the same way as (4.1) that $\sigma(\Delta|H_M) \subset \{\sum_{k=1}^n \lambda_i \mu_i : (\lambda, \mu) \in M\}$.

For every compact $\alpha \subset \sigma(\Delta)$, we set

$$M_{\mathbb{A},\mathbb{B}}(\alpha) = \{(\lambda, \mu) \in \sigma(\mathbb{A}) \times \sigma(\mathbb{B}) : \sum_{k=1}^{n} \lambda_k \mu_k \in \alpha\}.$$

Set now

$$E_\Delta(\alpha) = \{X \in \mathcal{B}(H) : \operatorname{supp} X \subset M_{\mathbb{A},\mathbb{B}}(\alpha)\}.$$

In other words $E_\Delta(\alpha) = H_{M_{\mathbb{A},\mathbb{B}}(\alpha)}$.

Theorem 4.1. [18] *A normal elementary operator Δ is decomposable; the map $\alpha \to E_\Delta(\alpha)$ is its spectral capacity.*

In particular the space $E_\Delta(\{0\})$ of all operators supported by $M_{\mathbb{A},\mathbb{B}}(0)$ coincides with $\{X : \|\Delta^n(X)\|^{1/n} \to 0\}$; in other words it is *the root space* of Δ.

It is important that the set $M_{\mathbb{A},\mathbb{B}}(\{0\})$ has a comparatively simple structure: it is the union of a null set and a countable family of rectangles. More precisely, let $\mu_{\mathbb{A}}$, $\mu_{\mathbb{B}}$ be scalar spectral measures of A and B, and let $m = \mu_{\mathbb{A}} \times \mu_{\mathbb{B}}$ be the product measure on $\sigma(\mathbb{A}) \times \sigma(\mathbb{B})$.

Lemma 4.2. *There are measurable sets $A_i \subset \sigma(\mathbb{A})$, $B_i \subset \sigma(\mathbb{B})$, $1 \le i < \infty$, and an m-null subset C of $\sigma(\mathbb{A}) \times \sigma(\mathbb{B})$ such that $M_{\mathbb{A},\mathbb{B}}(0) = (\cup_{i=1}^\infty A_i \times B_i) \cup C$.*

It follows from Lemma 4.2 that the space of all Hilbert-Schmidt operators in $E_\Delta(\{0\})$ is generated by the union of the spaces of all Hilbert-Schmidt operators supported by rectangles $A_i \times B_i$. Using this one obtains easily the following result:

Corollary 4.3. *The space $\mathcal{S}_1 \cap E_\Delta(\{0\})$ is dense in $\mathcal{S}_2 \cap E_\Delta(\{0\})$ with respect to the norm of the ideal $\mathcal{S}_2$.*

Since for a normal operator on a Hilbert space the root space coincides with the kernel, the result can be reformulated as follows: $\ker \Delta_1$ is a dense subspace of $\ker \Delta_2$.

Now we can show that for normal elementary operators the equality (3.8) holds in a much more stronger version.

Let $\overline{M}^{w^*}$ denote the weak*-closure of a set $M \subset \mathcal{B}(H)$.

Theorem 4.4. *For every normal elementary operator* Δ,

$$\overline{\Delta(\mathcal{B}(H))}^{w^*} \cap \ker \widetilde{\Delta}_2 = 0. \tag{4.2}$$

Indeed it is trivial to check that $tr(\Delta(X)Y) = 0$ for each $X \in \mathcal{B}(H)$ and $Y \in \ker \widetilde{\Delta}_1$. Hence by continuity $tr(ZY) = 0$ for $Z \in \overline{\Delta(\mathcal{B}(H))}^{w^*}$.

If also $Z \in \mathcal{S}_2$ then, again by continuity, $tr(ZY) = 0$ for all Y from the closure of $\ker \widetilde{\Delta}_1$ in $\mathcal{S}_2$. But by Corollary 4.3, this closure coincides with $\ker \widetilde{\Delta}_2$, which in its turn equals $\ker \Delta_2$ because Δ_2 is a normal operator and $\widetilde{\Delta}_2$ is its adjoint. So if $Z \in \overline{\Delta(\mathcal{B}(H))}^{w^*} \cap \ker \widetilde{\Delta}_2$ then $Z = 0$.

Note that we proved a more general statement:

$$\overline{\Delta(\mathcal{B}(H))}^{w^*} \cap \mathcal{S}_2 \text{ is orthogonal to } \ker \widetilde{\Delta}_2. \tag{4.3}$$

From this one can easily deduce the following result of Turnšek [24]:

Proposition 4.5. *For every normal elementary operator* Δ *the intersection of its range with* $\mathcal{S}_2$ *is contained in the* $\|\cdot\|_2$*-closure of* $\Delta(\mathcal{S}_2)$.

Indeed the $\|\cdot\|_2$-closure of $\Delta(\mathcal{S}_2)$ is just the orthogonal complement of $\ker \widetilde{\Delta}_2$.

Problem 5. *For which* $p > 2$ *the equality does* (4.2) *hold with* $\tilde{\Delta}_p$ *instead of* $\tilde{\Delta}_2$*?*

Remark 4.6. We note that $\overline{\Delta(\mathcal{B}(H))}^{w^*} \cap \ker \widetilde{\Delta} = 0$ does not hold in general even for normal inner derivations. One can easily construct $\Delta = L_A - R_A$ such that $\overline{\Delta(\mathcal{B}(H))}^{w^*} = \mathcal{B}(H)$.

4.2. Ascent

The relation between the kernel of an operator T and its root space is a subject of the so called *theory of thin spectral structure*. Since the root space contains the kernels of all operators T^n, a natural intermediate question is the interrelations of these spaces. If the chain $\ker T \subset \ker T^2 \subset \ldots$ stabilizes on number m: $\ker T^m = \ker T^{m+1} = ..$ then we say that the ascent of T equals m.

We will see that the ascent m of a normal elementary operator can be estimated via the essential dimension of its left coefficient family, and that $E_\Delta(0) = \ker \Delta^m$.

Theorem 4.7. *If ess-dim* $\mathbb{A} \leq r$ *then* $E_\Delta(0) = \ker \Delta^{(r/2]}$ *where* $(r/2] = [(r+1)/2]$, *the minimal integer* $\geq r/2$

For operators of the form $X \to AX + XB$ this result was obtained by Anderson and Foias [2]; clearly in this case it states that $E_\Delta(0) = \ker \Delta$.

The first step of the proof is to show that $\|\Delta|E_\Delta(0)\| \le 2|\sigma(\mathbb{B})| \operatorname{diam} \sigma(\mathbb{A})$, where by $|\sigma(\mathbb{B})|$ we mean $\sup\{|\lambda| : \lambda \in \sigma(\mathbb{B})\}$. Hence

$$\|\Delta^k|E_\Delta(0)\| \le (2|\sigma(\mathbb{B})| \operatorname{diam} \sigma(\mathbb{A}))^k$$

for each k.

If γ is a compact subset of $\sigma(\mathbb{A})$ then changing Δ by $\Delta R_{E_{\mathbb{A}}(\gamma)}$ we obtain the inequality $\|\Delta^k R_{E_{\mathbb{A}}(\gamma)}|E_\Delta(0)\| \le (2|\sigma(\mathbb{B})| \operatorname{diam} \gamma)^k$. Hence taking $X \in E_\Delta(0)$ and setting $T = \Delta^k(X)$ we obtain:

$$\|TE_{\mathbb{A}}(\gamma)\| \le C(\operatorname{diam} \gamma)^k \tag{4.4}$$

for all $\gamma \subset \sigma(\mathbb{A})$.

For each finite covering of $\sigma(\mathbb{A})$ by disjoint Borel sets $\gamma_1, ..., \gamma_N$, one can consider an orthogonal system $\{e_1, ..., e_N\}$ of unit vectors e_j supported in $E_{\mathbb{A}}(\gamma_j)$; the condition (4.4) gives that $\|TP_N\|_2^2 \le C \sum_{j=1}^N (\operatorname{diam} \gamma_j)^{2k}$ where P_N is the projection onto the subspace generated by $\{e_1, ..., e_N\}$. If $\mathbb{A}$ is cyclic (this can be assumed without reducing generality) then such orthogonal systems can approximate a basis. It follows that if the Hausdorff dimension of $\sigma(\mathbb{A})$ does not exceed $2k$ then each operator $T \in \mathcal{B}(H)$ satisfying the condition (4.4) belongs to $\mathcal{S}_2$.

We get that $\Delta^k(X) \in \mathcal{S}_2$ for all $X \in E_\Delta(0)$. Since $E_\Delta(0)$ is invariant for Δ, $\Delta^k(X) \in \mathcal{S}_2 \cap E_\Delta(0) = \ker \Delta_2 = \ker \widetilde{\Delta}_2$. By Theorem 4.4, $\Delta^k(X) = 0$.

4.3. Fuglede type theorems

The famous Fuglede-Putnam theorem says that $\ker \Delta = \ker \tilde{\Delta}$ if Δ is an operator of the form $X \mapsto N_1 X - X N_2$, where N_1, N_2 are normal operators. A natural question arising in this context is whether for every normal elementary operator Δ the equations

$$\Delta(X) = 0 \tag{4.5}$$

and

$$\widetilde{\Delta}(X) = 0 \tag{4.6}$$

are equivalent. This question was answered negatively in [19]. The construction of the counterexample is based on a modification of the famous L.Schwartz example of a set of spectral non-synthesis for the Fourier algebra of the group $\mathbb{R}^3$.

It is clear that $E_\Delta(0) = H_{M_{\mathbb{A},\mathbb{B}}} = E_{\widetilde{\Delta}}(0)$. Applying Theorem 4.7 we obtain the following version of the Fuglede - Putnam theorem:

Theorem 4.8. *If ess-dim* $\mathbb{A} \le r$ *then* $\ker \Delta^{(r/2]} = \ker \widetilde{\Delta}^{(r/2]}$.

The following consequence of Theorem 4.8 can be also immediately deduced from Theorem 3.3 (ii).

Theorem 4.9. *If ess-dim* $\mathbb{A} \le 2$ *then equations* (4.5) *and* (4.6) *are equivalent.*

It is reasonable to regard the work around Fuglede Theorem in a more wide way - as the study of conditions for the coincidence or inclusion of kernels of elementary operators. In other words: which linear operator equations are equivalent, how the solution spaces of different equations are related? The following theorem gives information for the case of 2-dimensional coefficient families.

Theorem 4.10. *Let ess-dim $\mathbb{A} \leq 2$. Let $\{f_i\}_{i=1}^m$ be Lipschitz functions on $\sigma(\mathbb{A})$, $\{g_i\}_{i=1}^m$ be Borel functions on $\sigma(\mathbb{B})$. Assume that $\sum_{i=1}^m f_i(\lambda)g_i(\mu) = 0$ for each $(\lambda, \mu) \in \sigma(\mathbb{A}) \times \sigma(\mathbb{B})$ satisfying the condition $\sum_{k=1}^n \lambda_k \mu_k = 0$. Then each solution X of the linear operator equation*

$$\sum_{k=1}^{n} A_k X B_k = 0 \tag{4.7}$$

satisfies equation

$$\sum_{i=1}^{m} f_i(\mathbb{A}) X g_i(\mathbb{B}) = 0. \tag{4.8}$$

Clearly Theorem 4.9 is a special case of Theorem 4.10 – it corresponds to the functions $f_i(\lambda) = g_i(\lambda) = \overline{\lambda_i}$.

To deduce Theorem 4.10 from Theorem 4.7, denote by $f(\mathbb{A})$ and $g(\mathbb{B})$ respectively the families $\{f_i(\mathbb{A}) : 1 \leq i \leq m\}$ and $\{g_i(\mathbb{B}) : 1 \leq i \leq m\}$. Let $\Delta' = \sum_{i=1}^m L_{f_i(\mathbb{A})} R_{g_i(\mathbb{B})}$. In our assumptions $M_{\mathbb{A},\mathbb{B}}(0) \subset M_{f(\mathbb{A}),g(\mathbb{B})}(0)$ whence $E_\Delta(0) \subset E_{\Delta'}(0)$. Furthermore ess-dim $f(\mathbb{A}) \leq 2$, so $E_{\Delta'}(0) = \ker \Delta'$ by Theorem 4.7.

Clearly the “Fuglede theorem”, for arbitrary normal Δ, holds in $\mathcal{S}_2$ (that is the equations (4.5) and (4.6) are equivalent in $\mathcal{S}_2$) and hence in $\mathcal{S}_p$, $p < 2$.

Problem 6. *Let Δ be a normal elementary operator. Are equations (4.5) and (4.6) equivalent in $\mathcal{S}_p$, $2 < p \leq \infty$?*

One can ask about the validity of similar results in C^*-algebras different from $\mathcal{B}(H)$. Namely, let $\mathcal{A}$ be a C^*-algebra. Consider elementary operators Δ on $\mathcal{A}$, i.e. $\Delta(c) = \sum_{k=1}^n a_k c b_k$, where $a_k, b_k \in \mathcal{A}$. We say that Δ is normal if $\{a_k\}_{k=1}^n$ and $\{b_k\}_{k=1}^n$ are families of commuting normal elements in $\mathcal{A}$.

Let us call a C*-algebra $\mathcal{A}$ 1-Fuglede (or (F1), for short) if $\ker \tilde{\Delta} = \ker \Delta$ for any normal elementary operator Δ. If $\ker \Delta^2 = \ker \Delta$ for any normal Δ then $\mathcal{A}$ is said to be 2-Fuglede (F2). If $\ker \tilde{\Delta}\Delta = \ker \Delta$ for any Δ then $\mathcal{A}$ is said to be 3-Fuglede (F3). Clearly (F3) $\Longrightarrow$ (F2) $\Longrightarrow$ (F1).

It follows from the above discussion that the algebra $\mathcal{B}(H)$ is not (F1) (for any infinite dimensional H). It is not difficult to see that

(i) the algebra, $\mathcal{S}_\infty$, of compact operators is (F2);

(ii) as a consequence each CCR-algebra is (F2);

(iii) each C*-algebra with a faithful family of traces (for example a simple unital hyperfinite C*-algebra) is 3-Fuglede.

Problem 7. *Which C^*-algebras $\mathcal{A}$ are F1, F2 and F3? Is $\mathcal{S}_\infty$ 3-Fuglede? Is any GCR-algebra (F2)? What about the Calkin algebra $\mathcal{B}(H)/\mathcal{S}_\infty$?*

We will briefly list other extensions of the Fuglede Theorem.

E.A.Gorin [8] discovered that if in the classical Rosenblum's proof of the Fuglede Theorem one uses instead of the Liouville theorem the Phragmen-Lindelef Theorem, then the result extends to a more general class of operators than the normal ones.

Theorem 4.11. *Let an operator A, acting on a Banach space $\mathfrak{X}$, be decomposed into a sum $A = T + iS$, where $[S, T] = 0$, S has real spectrum and T satisfies the condition*

$$\|exp(itT)\| = o(|t|) \text{ for t} \to \infty. \tag{4.9}$$

Then $\ker A = \ker T \cap \ker S$.

Applying this theorem to multiplication operators we obtain

Corollary 4.12. *Let an operator N on a Banach space $\mathfrak{X}$ can be written as a sum $N = U + iV$ where $[U, V] = 0$, V has real spectrum and U satisfies condition*

$$\|\exp(itU)\| = o(|t|^{1/2}). \tag{4.10}$$

Set $N' = U - iV$. Then the equations $NX - XN = 0$ and $N'X - XN' = 0$ are equivalent.

An operator T on a Banach space is called *Hermitian* if $\|\exp(itT)\| = 1$ for all $t \in \mathbb{R}$.

The following result belongs to K.Boyadzhiev [5].

Theorem 4.13. *Let $P(t_1, ..., t_n)$ be a polynomial in n variables without zeros in $\mathbb{R}^n \setminus \{0\}$. If $T_1, ..., T_n$ are commuting Hermitian operators then* $\ker P(T_1, ..., T_n) \subset \cap_{k=1}^n \ker T_k$.

To deduce the Fuglede Theorem from Theorem 4.13, one should take $P(t_1, t_2) = t_1 + it_2$, $T_1 = L_A - R_A$, $T_2 = L_B - R_B$ where A, B are the real and imaginary parts of a normal operator $N \in \mathcal{B}(H)$.

E.A.Gorin [8] obtained also the following "non-commutative" version of the Fuglede Theorem.

Let $\mathcal{F}_2$ be the free algebra with generators a, b. Consider the formal power series $f(z) = \exp(za)\exp(zb)$ with coefficients in $\mathcal{F}_2$. Then there is a formal power series $g(z)$ with coefficients in $\mathcal{F}_2$ such that $f(z) = \exp(g(z))$. Let us denote by $c_k(a, b)$ the coefficients of this power series:

$$g(z) = c_0(a, b) + c_1(a, b)z + ...$$

It is easy to obtain explicit formulas for $c_k(a, b)$ for small k. For example $c_0(a, b) = 1$, $c_1(a, b) = a + b$, $c_2(a, b) = [a, b]/2$.

Since $c_k(a, b)$ are elements of $\mathcal{F}_2$ ("non-commutative polynomials"), one can calculate $c_k(A, B)$ for each pair A, B of elements of any algebra.

Theorem 4.14. *Let T, S be operators on a Banach space $\mathfrak{X}$ such that T satisfies (4.10), S has real spectrum, and let $X \in B(\mathfrak{X})$. If $[c_k(T, iS), X] = 0$ for all $k \in \mathbb{N}$ then $[T, X] = [S, X] = 0$.*

To see that this theorem extends Corollary 4.12 note that if $[a, b] = 0$ then $c_k(a, b) = 0$ for all $k \geq 2$. Hence in this case the assumption becomes just $[c_1(a, ib), x] = 0$, that is $a + ib$ commutes with x.

4.4. Norm inequalities

The following result is a special case of Theorem 3.3 (i).

Theorem 4.15. *Suppose that ess-dim $\mathbb{A} \leq 2$. If $\Delta(X) \in \mathcal{S}_2$ then $\widetilde{\Delta}(X) \in \mathcal{S}_2$ and*

$$\|\Delta(X)\|_2 = \|\widetilde{\Delta}(X)\|_2. \tag{4.11}$$

It extends the famous result of Gary Weiss [29] which states that (4.11) holds for $\Delta = L_{N_1} - R_{N_2}$ where N_i are normal operators.

The arguments similar to ones used in Theorem 3.3 allow to extend Theorem 4.15 in spirit of Theorem 4.10:

Theorem 4.16. *Let ess-dim $\mathbb{A} \leq 2$. Let $\{f_i\}_{i=1}^m$ be Lipschitz functions on $\sigma(\mathbb{A})$, $\{g_i\}_{i=1}^m$ be Borel functions on $\sigma(\mathbb{B})$. Assume that $|\sum_{i=1}^m f_i(\lambda)g_i(\mu)| \leq |\sum_{k=1}^n \lambda_k\mu_k|$ for each $(\lambda, \mu) \in \sigma(\mathbb{A}) \times \sigma(\mathbb{B})$.*

If $\sum_{k=1}^n A_kXB_k \in \mathcal{S}_2$, for some $X \in \mathcal{B}(H)$, then $\sum_{i=1}^m f_i(\mathbb{A})Xg_i(\mathbb{B}) \in \mathcal{S}_2$ and

$$\|\sum_{i=1}^m f_i(\mathbb{A})Xg_i(\mathbb{B})\|_2 \leq \|\sum_{k=1}^n A_kXB_k\|_2.$$

Kittaneh [14] established the following special case of this result: if A is a normal operator and f is a Lipschitz function on $\sigma(A)$ then

$$\|f(A)X - Xf(A)\|_2 \leq k(f)\|AX - XA\|_2 \tag{4.12}$$

where $k(f)$ is the Lipschitz constant of f.

Without restrictions on the dimension of spectra the equality (4.11) can fail because even the kernels of Δ and $\widetilde{\Delta}$ can differ. Nevertheless Weiss [28] proved the following remarkable result:

Theorem 4.17. *Let Δ be a normal elementary operator. If both $\Delta(X)$ and $\widetilde{\Delta}(X)$ are Hilbert-Schmidt operators then (4.11) holds.*

One can prove Theorem 4.17 applying Corollary 2.8. Take as usually $\mathcal{H} = \mathcal{S}_2$, $S = \Delta_2$, $\mathfrak{Y} = \mathcal{B}(H)$, $T_1 = \Delta$, $T_2 = \widetilde{\Delta}$, then the identity inclusion $\Phi_2 : \mathcal{S}_2 \to \mathcal{B}(H)$ intertwines the operators. The condition $\ker S \cap \Phi^{-1}(T_2\mathfrak{Y}) = 0$, which is just $\ker \Delta_2 \cap \widetilde{\Delta}(\mathcal{B}(H)) = 0$, holds by Theorem 4.4. So inequality (2.1) holds which means that $\|\widetilde{\Delta}(X)\|_2 \leq \|\Delta(X)\|_2$ if $\Delta(X) \in \mathcal{S}_2$ and $\widetilde{\Delta}(X) \in \mathcal{S}_2$. Interchanging $\widetilde{\Delta}$ and Δ we obtain (4.11).

Using Theorem 2.7 instead of Corollary 2.8 one can obtain the following more general result:

Theorem 4.18. *Let $\{f_i\}_{i=1}^m$ be Lipschitz functions on $\sigma(\mathbb{A})$, $\{g_i\}_{i=1}^m$ be Borel functions on $\sigma(\mathbb{B})$. Assume that $|\sum_{i=1}^m f_i(\lambda)g_i(\mu)| \le |\sum_{k=1}^n \lambda_k\mu_k|$ for each $(\lambda,\mu) \in \sigma(\mathbb{A}) \times \sigma(\mathbb{B})$.*

If $\sum_{k=1}^n A_kXB_k \in \mathcal{S}_2$ and $\sum_{i=1}^m f_i(\mathbb{A})Xg_i(\mathbb{B}) \in \mathcal{S}_2$, for some $X \in \mathcal{B}(H)$, then

$$\|\sum_{i=1}^m f_i(\mathbb{A})Xg_i(\mathbb{B})\|_2 \le \|\sum_{k=1}^n A_kXB_k\|_2.$$

The mentioned result of (4.11) extends to other Schatten ideals: for every $p \in (1,\infty)$ there is a constant $c = c_p > 0$ such that

$$\|A^*X - XA^*\|_p \le c\|AX - XA\|_p \tag{4.13}$$

for all normal operators A and all operators X (Abdessemed and Davies [1], for $p > 2$, Shulman [20], for $p < 2$). This is not the case for $p = 1$ and $p = \infty$ (Yu.B. Farforovskaya [10]). Kissin and Shulman [12] proved that the statement remains true for the operator norm and $\mathcal{S}_1$-norm, if one imposes a restriction on spectra of normal operators. Namely, for each C^2-smooth Jordan line L there is constant $c = c_L$ such that

$$\|A^*X - XA^*\| \le c\|AX - XA\| \tag{4.14}$$

if $\sigma(A) \subset L$. It was also proved in [12] that a kind of smoothness of spectrum is necessary for the validity of (4.14): if for some normal operator A the inequality (4.14) holds for all X, then given a sequence $\lambda_n \in \sigma(A)$ converging to $\lambda \in \sigma(A)$, there is a limit $\lim(\lambda_n - \lambda)/|\lambda_n - \lambda| = f(\lambda)$.

Potapov and Sukochev [15] using a powerful technique of Banach space geometry and harmonic analysis proved that (4.12) extends to all S_p ideals, if $A = A^*$. For all normal A this was proved in [13].

If we note that for each X, the map $\delta_X : A \mapsto AX - XA$ is a derivation of the algebra $\mathcal{B}(H)$ then the question arises if the above results can be extended to derivations of more general class (defined on subalgebras of $\mathcal{B}(H)$). The following theorem was proved in [20].

Theorem 4.19. *Let δ be a derivation from a *-subalgebra $D(\delta) \subset \mathcal{B}(H)$ to $\mathcal{B}(H)$, and let $A \in D(\delta)$ be a normal operator without eigenvalues. Then*

(i) If $\delta(A) \in \mathcal{S}_p$ and $\delta(A^*) \in K(H)$ then $\delta(A^*) \in \mathcal{S}_p$ and

$$\|\delta(A^*)\|_p \le 2c_p\|\delta(A)\|_p; \tag{4.15}$$

(ii) if moreover $p = 2$ and δ is closed then

$$\|\delta(f(A))\|_2 \le k(f)\|\delta(A)\|_2 \tag{4.16}$$

for each Lipschitz function f on $\sigma(A)$ with Lipschitz constant $k(f)$.

The extension of part (ii) to all $p \in (1,\infty)$ was obtained in [13].

Let $A \in \mathcal{B}(H)$ be a normal operator. As it was mentioned above the map $AX - XA \mapsto A^*X - XA^*$ is not bounded in general as a map on $\mathcal{B}(H)$. However this map and more general maps of the form $AX - XA \mapsto f(A)X - Xf(A)$ (f

is a measurable function) are norm closable. In fact, if $AX_n - X_nA \to 0$ and $f(A)X_n - X_nf(A) \to B$ as $n \to \infty$ then

$$[B, A] = \lim_{n\to\infty}[[f(A), X_n], A] = \lim_{n\to\infty}[f(A), [X_n, A]] = 0.$$

Thus B is in the commutant $\{A\}'$ of A. Hence B commutes with $f(A)$ and therefore belongs to $\overline{\delta_{f(A)}(\mathcal{B}(H))} \cap \ker \widetilde{\delta_{f(A)}}$ which is $\{0\}$ by [2].

It would be interesting to see what happen if we weaken the condition on the topology replacing it by the weak-*-topology on $\mathcal{B}(H)$. It was proved in [21] that the Fuglede map $AX - XA \mapsto A^*X - XA^*$ is not w^*-closable if $\sigma(A)$ has non-empty interior and the spectral measure of A is equivalent to the Lebesgue measure on the interior of $\sigma(A)$. The proof uses a characterization of so-called Toeplitz w^*-closable multipliers given in [21]. However the method does not work for general maps of the type $AX - XA \mapsto f(A)X - Xf(A)$. Some sufficient conditions are established in [21].

Problem 8. *For which continuous functions f and normal operators $A \in \mathcal{B}(H)$ is the map on $\mathcal{B}(H)$ given by $AX - XA \mapsto f(A)X - Xf(A)$ closable in weak-*-topology?*

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