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On the Dimensionality of Multilevel Coded Modulation in the High SNR Regime

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***Abstract*—In this paper, the dimensionality of the multilevel coded modulation (MLCM) scheme is addressed. This study is done for an MLCM scheme with a N-dimensional (ND) constellation constructed from the Cartesian product of N identical 1D constellations in the high SNR regime. It is demonstrated that multidimensional MLCM with Reed–Solomon code components has better trade-off between coding gain and complexity than a 1D scheme. Specifically, a 4D MLCM system gains 1.4 dB over a 1D MLCM system with lower complexity at a block error of 10^{-6}. The gain increases to 2.5 dB asymptotically.**

I. Introduction

Multidimensional multilevel coded modulation (MLCM) schemes have received relatively little attention, in contrast to, e.g., multidimensional trellis coded modulation (TCM) [1, Ch.8] [2]. Recently, dense multidimensional lattices such as the Schläfli, Gosset, Barnes-Wall, and Leech lattices were introduced as the constituent signal constellations for an MLCM scheme [3]. Although a 1-dimensional (1D) MLCM scheme can perform within 1 dB of capacity [4] [5], its complexity resulting from the large block length of the component codes, diminishes its practical interest. The better trade-off between complexity and performance provided by a multidimensional MLCM scheme may therefore be more practical, e.g., in optical communications, the high data rates (10-40 Gbs) make low-complexity solutions very important [6].

In this paper, we exploit the Cartesian product constellation which introduces a simpler set partitioning approach than dense lattices [3]. While there is no complexity comparison between two MLCM schemes with different dimensions in [3], we show for the first time, to our knowledge, that a multidimensional MLCM scheme has a better performance than the 1D one with the same complexity, which partly contradicts the result in [4]. Moreover, building on [3] and [7] a novel simple set partitioning algorithm is introduced. Furthermore, an analytical expression for the asymptotic coding gain (ACG) of an MLCM scheme is derived, in which the ACG of MLCM with affine component codes (see section III.C) is related to the minimum Euclidean distances (MED) of its layers. Then, the theoretic performance improvement due to the increase in the dimension of the constituent constellation is computed. In addition, the performances of 1, 2, and 4-dimensional (1D, 2D, and 4D) MLCM schemes for some specific constellations are compared at practical SNR (block error rate (BLER) around 10^{-6}) through simulation. The results, both analytic and numeric, show a high potential advantage of MLCM schemes with higher dimension in providing a better trade-off between complexity and coding gain.

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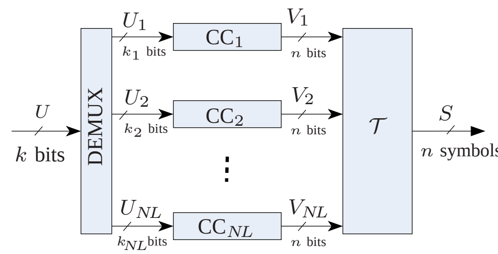

Fig. 1. An ND MLCM with NL component codes or layers.

II. System Model

We consider an ND constellation $\mathcal{C}$ as a Cartesian product of N 1D constellations with cardinality 2^L. The MLCM system consists of NL layers or component codes with the same block length n but different code rates R_i, Hamming distances δ_i, and correcting capabilities t_i for layer i. An ND set partitioning algorithm ($\mathcal{T}$ according to Fig. 1) maps NL encoded bits at each time instant to an ND symbol. In the system model shown in Fig. 1, the DEMUX unit splits the input bit vector U of length k bits into NL different vectors $U_1,\ldots,U_{NL}$ of lengths $k_1,\ldots,k_{NL}$, respectively, where $\sum_{l=1}^{NL} k_l = k$. The component codes $\mathrm{CC}_1, \ldots, \mathrm{CC}_{NL}$ encode these vectors into NL row code vectors $V_1, \ldots, V_{NL}$ of length n. We denote the normalized MED of the layer i by d_i (normalizing with $\sqrt{2\eta E_b}$, where E_b is the average bit energy and η is the spectral efficiency of the system). The channel model is a discrete-time memoryless additive white Gaussian noise channel with noise variance $N_0/2$. A multistage decoder (MSD) with soft or hard decision is applied in the MLCM receiver.

III. ACG of MLCM systems

We define the ACG as the ratio between the required SNR of two systems that achieve the same, asymptotically low, BLER. System $\mathcal{F}$ exploits a serially concatenated forward error correction (FEC) and modulation units which operate independently and system $\mathcal{M}$ is based on the MLCM approach. The two compared systems have the same information bit rate, pulse shape, bandwidth and delay. The component codes belong to the same family of codes and have the same block length n.

III.A-Soft decision decoding: The derivation of the upper bound on the BLER of a coded system follows the approach used in [8] for a QAM signal set. The BLER of a system with normalized MED d between the different code vectors of length n with multiplicity A exploiting soft decision decoding is obtained by $P_e \approx AQ\left(\sqrt{\eta d^2\rho_b}\right)$, where ρ_b is the signal to noise ratio per bit (E_b/N_0) of the system and $Q(x) \triangleq \frac{1}{\sqrt{2\pi}}\int_x^\infty \exp(-\frac{1}{2}x^2)\mathrm{d}x$. By the bound $Q(x) \leq \frac{1}{x\sqrt{2\pi}}\exp\left(-\frac{1}{2}x^2\right)$, which is quite tight asymptotically, we have $P_e \approx \frac{A}{d\sqrt{2\pi\eta\rho_b}}\exp\left(-\frac{1}{2}d^2\eta\rho_b\right)$. At asymptotically high SNR, we compute the reduction in SNR by considering equal P_e for the two different systems, with different ρ_b, d and A but the same η, and then taking the natural logarithm of both sides. Noting that $\ln\frac{A}{d\sqrt{2\pi\eta\rho_b}}$ is negligible for large SNRs, the ACG between the two proposed systems $\mathcal{M}$ and $\mathcal{F}$ is $d^2_{\mathcal{M}}/d^2_{\mathcal{F}}$, where $d^2_{\mathcal{F}} = \delta_{\mathcal{F}}d^2_{\min}$ is the squared normalized MED between different code vectors of system $\mathcal{F}$ [4], $d^2_{\mathcal{M}} = \min_i\{\delta_i d_i^2\}$ is the overall squared normalized MED between code vectors of system $\mathcal{M}$, $d_{\min}$ is the normalized MED between symbols in the signal constellation of system $\mathcal{F}$, and $\delta_{\mathcal{F}}$ is the minimum Hamming distance between code vectors of system $\mathcal{F}$.

III.B-Hard decision decoding: The asymptotic BLER of a binary coded system consisting of a block code with the correcting capability t, codeword length n, and a constellation with normalized MED d using hard decision decoding can be approximated very well [1, Eq. (7.5-7)] for high SNR ($1-p \approx 1$) by

$$P_e \approx \frac{1}{(d\sqrt{2\eta\pi\rho_b})^{t+1}}\binom{n}{t+1}\exp\left(-\frac{1}{2}(t+1)d^2\eta\rho_b\right). \quad (1)$$

Exploiting (1) and following the same approach as in the derivation of ACG for soft decision decoding, the ACG between systems $\mathcal{M}$ and $\mathcal{F}$ is $d^2_{h_\mathcal{M}}/d^2_{h_\mathcal{F}}$ where $d^2_{h_\mathcal{F}} = (t_\mathcal{F}+1)d^2_{\min}$ and $d^2_{h_\mathcal{M}} = \min_{i=1,\ldots,NL}\{(t_i+1)d_i^2\}$, where $t_\mathcal{F}$ is the correcting capability of system $\mathcal{F}$.

The optimal rate allocation, in terms of ACG, is given by maximizing $d_\mathcal{M}$ for a soft decision decoding MSD. The maximum is obtained when the balanced distance rule (BDR) [4] is satisfied $d^2_\mathcal{M} = \delta_i d_i^2, \quad 0 < i \leq NL$. Similarly, for a hard decision decoding MSD, according to (1), the minimization of the BLER is equivalent to maximizing $d^2_{h_\mathcal{M}} = \min_{i=1,\ldots,NL}\{(t_i+1)d_i^2\}$. Therefore using the same approach as with the BDR, we obtain

$$d^2_{h_\mathcal{M}} = (t_i+1)d_i^2,\ 0 < i \leq NL. \quad (2)$$

III.C-ACG with affine component codes: Affine codes are a family of block codes having a linear relation between their Hamming distances and code rates. RS and BCH codes [1, Ch. 7] are special cases of affine codes. In general, for an *affine code* with the length n, the minimum Hamming distance δ and code rate $R = k/n$ satisfy $\delta = \alpha + \beta R$, where α and β depend on n but not on R or k. For example, for the (255,k) RS codes, $\delta = 256 - 255R$.

Theorem 1: The ACG of an MLCM system with affine component codes is given by (for soft as well as hard decision MSD) ACG $= \frac{\bar{d}^2}{d^2_{\min}}$, where $\bar{d}^{-2} = \frac{1}{NL}\sum_{i=1}^{NL} d_i^{-2}$,

Proof: The two coded systems $\mathcal{F}$ and $\mathcal{M}$ have the same code rate $R_\mathcal{F} = R_\mathcal{M} = \frac{1}{NL}\sum_{i=1}^{NL} R_i$. Moreover, using the affine code definition for system $\mathcal{F}$, $\delta_\mathcal{F} = \alpha + \beta R_\mathcal{F}$, we obtain $\delta_\mathcal{F} = \alpha + \beta\left(\frac{1}{NL}\sum_{i=1}^{NL} R_i\right)$. Similarly, for each layer of the MLCM system, exploiting the property of affine codes, one can write $\delta_\mathcal{F} = \frac{1}{NL}\sum_{i=1}^{NL}\delta_i$. For soft decision decoding, assuming large enough block length codes, we ignore the fact that both δ and k are integers. It follows from the BDR that $\delta_\mathcal{F} = \frac{1}{NL}\sum_{i=1}^{NL}\frac{d^2_\mathcal{M}}{d_i^2}$, which substitute into ACG $= \frac{d^2_\mathcal{M}}{\delta_\mathcal{F}d^2_{\min}}$ completes the proof for the soft decision case. One may use an analogous approach for hard decision decoding by using the approximation $t = \left\lfloor\frac{\delta-1}{2}\right\rfloor \approx \frac{\delta}{2} - 1$ for large enough block length codes and (2) to obtain

$$\delta_\mathcal{F} \approx \frac{1}{NL}\sum_{i=1}^{NL}\frac{2d^2_{h_\mathcal{M}}}{d_i^2}. \quad (3)$$

On the other hand using the definition of $d_{h_\mathcal{F}}$, we have

$$\text{ACG} = \frac{d^2_{h_\mathcal{M}}}{(t_\mathcal{F}+1)d^2_{\min}} \approx \frac{2d^2_{h_\mathcal{M}}}{\delta_\mathcal{F}d^2_{\min}}. \quad (4)$$

Substituting (3) into (4), the proof is complete. ∎

The theorem is general in the sense that it is independent of α and β, so it holds for both RS and BCH codes (or any other affine code). It is also independent of n and R (within certain limits; in particular, $0 \leq R_i \leq 1$ for all component codes). Furthermore, this theorem holds for the MSD with soft and hard decision decoding.

IV. MULTIDIMENSIONAL SET PARTITIONING

In general, an arbitrary labeling of the constellation symbols can define the mapping function $\mathcal{T}$ of the MLCM system in Fig. 1, but Ungerboeck or block set partitioning [4] provides simpler implementations for the MSD. The 1D constellation $\mathcal{A}$ with a normalized MED of d_0 and cardinality of 2^L can be set partitioned into two subsets $\mathcal{A}_0$ and $\mathcal{A}_1$ with normalized MEDs of $2d_0$. Each of the subsets $\mathcal{A}_0$ and $\mathcal{A}_1$ can be further set partitioned into subsets $\mathcal{A}_{00}$, $\mathcal{A}_{01}$, $\mathcal{A}_{10}$, and $\mathcal{A}_{11}$ and so on, up to L steps with subsets $\mathcal{A}_{x_1,\ldots,x_L}$, $x_i \in \{0,1\}, 0 < i \leq L$ (the same notation as [4] and [3]). The set partitioning of an ND constellation $\mathcal{C} = \mathcal{A}^N$, based on the subsets of the 1D constellation $\mathcal{A}$ and the $(N-1)$D constellation $\mathcal{C}' = \mathcal{A}^{N-1}$, can be written as

$$\begin{aligned}
\mathcal{C}_0 &= \mathcal{A}_0 \times \mathcal{C}'_0 \cup \mathcal{A}_1 \times \mathcal{C}'_1\\
\mathcal{C}_1 &= \mathcal{A}_0 \times \mathcal{C}'_1 \cup \mathcal{A}_1 \times \mathcal{C}'_0\\
\mathcal{C}_{00} &= \mathcal{A}_0 \times \mathcal{C}'_0 \quad , \quad \mathcal{C}_{10} = \mathcal{A}_0 \times \mathcal{C}'_1\\
\mathcal{C}_{01} &= \mathcal{A}_1 \times \mathcal{C}'_1 \quad , \quad \mathcal{C}_{11} = \mathcal{A}_1 \times \mathcal{C}'_0\\
\mathcal{C}_{000} &= \mathcal{A}_{00} \times \mathcal{C}'_{00} \cup \mathcal{A}_{01} \times \mathcal{C}'_{01}\\
\mathcal{C}_{001} &= \mathcal{A}_{00} \times \mathcal{C}'_{01} \cup \mathcal{A}_{01} \times \mathcal{C}'_{00}\\
&\vdots
\end{aligned}$$

assuming that a set partitioning of $\mathcal{C}'$ into $\mathcal{C}'_0$, $\mathcal{C}'_1$, $\mathcal{C}'_{00}$,... is available. For $N = 4$, provided that $\mathcal{A}$ is an ASK constellation labeled by the natural binary code, this method generates Wei's set partitioning [2] approach for 4D QAM. Applying the above recursive approach in NL steps, we can do set partitioning of any ND constellation ($\mathcal{A}^N$).

Example: For 64-ASK^3, which is the Cartesian product of three 4-ASK constellations, the set partitioning is done in 6 steps. The neighboring coefficients (see [9]) are $\frac{9}{2}$, $\frac{9}{2}$, $\frac{81}{16}$, 3, 2, 1 and $\frac{2}{15}$, $\frac{4}{15}$, $\frac{4}{15}$, $\frac{8}{15}$, $\frac{8}{15}$, $\frac{8}{15}$ are the squared normalized MEDs of layers 1, …, 6.

V. Complexity and performance comparison

In this section we show that even though the 1D turbo coded MLCM designed based on the capacity rule can achieve within about 1dB of the Shannon limit regardless of dimensionality [4], multidimensional MLCM schemes introduce a better trade-off between complexity and coding gain than 1D schemes.

Wachsmann $et\ al.$ investigated the dimensionality of MLCM [4] for convolutional and turbo block component codes by using examples with 4-ASK and 16-QAM constellations. It is stated in [4] that for a fixed spectral efficiency (bits/sec/Hz), the MLCM scheme with a 1D constellation (4-ASK) has 0.25 dB higher power efficiency than the system with a 2D constellation (16-QAM), and also a lower complexity exploiting turbo component codes. Here, we show that this gain is obtained at the cost of higher complexity.

Finding the closest ND symbol to the received vector among the 2^{NL} symbols in $\mathcal{C} = \mathcal{A}^N$ requires approximately N times the computational complexity of finding the closest 1D symbol in the constituent 2^L-point constellation $\mathcal{A}$, neglecting the $N-1$ additions which one may need to compute the ND MED from N 1D MEDs [2]. This complexity analysis implies that one may compare the complexity of the receivers for two MLCM schemes with different dimensions by taking into account solely the complexity of the component code decoders per dimension. The evaluated 1D system in [4] uses two turbo codes (R_1/R_2 = 0.52/0.98) of length $2N$ whereas the 4D system uses three turbo codes ($R_1/R_2/R_3/R_4$ = 0.29/0.75/0.96/1) not four ($R_4 = 1$) of length N, in order to transmit at the same bit rate. For a turbo code using convolutional component (CC) codes with block length N, the decoder complexity C_{TC} is mainly determined by the complexity of its CCs decoders [10]. For a trellis decoding structure, the complexity of the decoder depends on the block length linearly [1, Ch. 8]. Therefore, the complexity of the 1D MLCM scheme ($2C_{\text{TC}}$ per dimension) is higher than the 2D one ($\frac{3}{2}C_{\text{TC}}$ per dimension). The conclusion in [4] about the MLCM scheme with CC codes (not turbo) shows the benefit of using a 2D constellation instead of a 1D one, which is consistent with our results for affine block codes.

The ACG of three MLCM schemes with 256-ASK^4 (4D), 16-ASK^2 (2D) and 4-ASK (1D) constellations, computed using Theorem 1, is seen in Table I. Surprisedly the ACG can be improved by 1.25 dB by increasing the dimension by a factor 2, while for a fixed data delay, due to the same number of component code decoders in each dimension, the complexity is almost the same. This gain, which by Theorem 1 can be proved to be exactly 4/3 (1.25 dB), is independent of the rate R, code vector length n, and even of the code type, as long as it is affine. In practical SNRs, the minimization of the BLER for the 4D MLCM system (see [9]) introduces only five component codes, while the 1D scheme needs component code in all its layers. This leads to 2, $\frac{3}{2}$, and $\frac{5}{4}$ decoders per dimension for 1D, 2D, and 4D MLCM schemes, respectively. As seen in Fig. 2, the performance of the system is improved by 1.4 dB by exploiting 4D MLCM, with less complexity. However, this gain is limited to the high-SNR regime, and it does not apply to capacity-achieving codes.

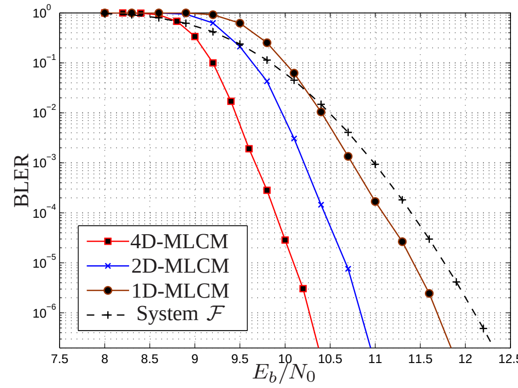

Fig. 2. Performance comparison of three MLCM systems with RS component codes over GF(2^7), $n = 889$ bits, hard decision MSD-All the systems have the same average code rate $R = 0.929$, but different symbol rates to support the same spectral efficiency $\eta = 3.72$ bits/sec/Hz.

TABLE I
ACG of three MLCM schemes with affine component codes

Constellation	$\{d_1, d_2, \ldots, d_{NL}\}$	ACG (dB)
1D	$\{1, 2\}$	2.04
2D	$\{1, \sqrt{2}, 2, 2\sqrt{2}\}$	3.29
4D	$\{1, \sqrt{2}, \sqrt{2}, 2, 2, 2\sqrt{2}, 2\sqrt{2}, 4\}$	4.54

VI. Conclusion

A potential advantage of multidimensional MLCM in providing better trade-off between complexity and coding was shown for affine component codes. The results illustrate that for practical SNRs, we can design 4D MLCM schemes with lower complexity and higher power efficiency than with 1D constellations.

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