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Bearing-only target localization with uncertainties in observer position

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***Abstract*—In this paper, the bearing-only target localization problem when the observer positions are subject to error is investigated. In this problem, the angle of arrival of the transmitted signal between target and observer are used to estimate the target position. It is assumed that not only the bearing measurements are corrupted by noise but also the exact position of observer is not available to the estimator. The accuracy of estimated location of target depends on the reliability of information from the observer position. Therefore, the previously published algorithms considering only the bearing measurement noise do not meet the expected performance when the observer positions are subject to error. The maximum likelihood, the least squares and total least square algorithms and a new method of localization based on weighted total least squares approach are developed for this problem. The corresponding Cramér-Rao lower bound (CRLB) is derived for this problem. Computer simulations are performed to evaluate the performance of the proposed algorithms. Simulation results show that the new method can attain the CRLB for sufficiently high SNR.**

***Index Terms*—bearing-only, localization, maximum likelihood, weighted total least squares.**

I. INTRODUCTION

In bearing-only localization problem, a moving observer is used to find the location of a fixed target or to track a moving target. In this work, we concentrate on fixed target localization. Bearing measurements are obtained from different points along the trajectory line of moving observer. The location of target is estimated from the intersection point of bearing lines among different positions of observer and target [1].

A various works have been done on the bearing-only localization. In [1], the performance of the Maximum Likelihood (ML) estimator, least squares (LS) estimator, and weighted least squares (WLS) estimator, also called Stansfield algorithm, were examined. The Stansfield and ML estimators for different observer trajectories were also indicated in [2] where the performances of the estimators are enhanced by finding the optimal observer trajectories.

Most of the works done in literatures are based on the assumption that the exact position of the observer is available. However, this assumption is not realistic in practice. Recently, some works have been carried out into bearing-only localization problem with uncertainties about observer position. In [3] total least squares (TLS) and ML estimators were developed for this kind of problem. The doppler-bearing tracking problem in the presence of observer position error in the case of one and two observers was also investigated in [4].

In the this paper, we assume that the exact position of the observer is not available. ML estimator is investigated in two cases; when the estimator does not know about the observer position error and also when it does. In ML algorithm, it is required to solve a nonlinear problem which is computationally intensive. In addition, having a good initial guess is inevitable to guarantee that the algorithm converges to the global minimum of the cost function. Therefore, we also focus on some linear algorithms. First, LS, WLS, and TLS, an extension to LS, are applied for the problem. Then, we apply the novel technique based on weighted TLS (WTLS) estimator for our problem in order to improve the performance of TLS. The Cramér-Rao lower bound (CRLB) of bearing-only localization with uncertainties in observer position is obtained and comparison among the proposed algorithms and CRLB is made.

The paper is organized as follows. In Section II, the model of bearing-only localization and the corresponding CRLB are described. In Section III, we derive different algorithms formulation for bearing-only localization. In section IV, computer simulations are presented to evaluate the performance of the proposed algorithms. Finally, Section V concludes the paper.

II. LOCALIZATION MODEL AND CRLB

Let $\mathbf{s}^o = [x_s^o, y_s^o]^T \in \mathbb{R}^2$ be the coordinate of the target to be estimated. The observer collects bearing measurements at M distinct points $\mathbf{x}_i^o = [x_i^o, y_i^o]^T \in \mathbb{R}^2$, $i = 1, 2, \ldots, M$. In the absence of the measurement noise, the relation between true bearing angle (in radians) and true location of the target is

$$\alpha_i^o = \tan^{-1} \frac{y_s^o - y_i^o}{x_s^o - x_i^o}. \tag{1}$$

where $\tan^{-1}\{\cdot\}$ is four-quadrant inverse tangent. Let $\boldsymbol{\alpha}$ be the bearing measurement vector consisting of the true bearing corrupted by additive noise,

$$\boldsymbol{\alpha} = [\alpha_1, \alpha_2, \ldots, \alpha_M]^T = \boldsymbol{\alpha}^o + \mathbf{n}, \tag{2}$$

where $\mathbf{n}$ is the bearing measurement error vector modeled as zero mean Gaussian random vector with covariance matrix $\boldsymbol{\Psi}_{\boldsymbol{\alpha}}$. In the current model, we assume that the exact position of the observer is not available. Let $\mathbf{x}_i = [x_i, y_i]^T$ be the nominal value of the observer position at the ith point and $\mathbf{x} = [\mathbf{x}_1^T, \mathbf{x}_2^T, \ldots, \mathbf{x}_M^T]^T$ be the vector of nominal observer positions available to the estimator, then,

$$\mathbf{x} = \mathbf{x}^o + \mathbf{v}, \tag{3}$$

where $\mathbf{v}$ is observer position error vector assumed to be zero mean Gaussian random vector with covariance matrix of $\mathbf{\Psi_x}$. Note that $\mathbf{\Psi_x} = \text{blkdiag}[\mathbf{\Psi_{x_1}}, \mathbf{\Psi_{x_2}}, \ldots, \mathbf{\Psi_{x_M}}]$, where $\text{blkdiag}\{\cdot\}$ denotes the block diagonal matrix, and $\mathbf{\Psi_{x_i}}$ is covariance matrix of noise over ith position of observer. We assume that the bearing measurement and observer position errors, i.e., $\mathbf{n}$ and $\mathbf{v}$, are statistically independent. This assumption has been previously considered for similar cases in [3], [4], but might not be valid for all bearing measurement systems.

To compute CRLB, we consider the same approach used in [4]. Let $\boldsymbol{\theta} = [\boldsymbol{\theta}_1^T, \boldsymbol{\theta}_2^T]^T = [\mathbf{s}^{oT}, \mathbf{x}^{oT}]^T$ be the unknown parameter vector to be estimated. Note that since the true position of the observer is not known for the estimator, it should also be estimated. Let $\boldsymbol{\beta} = [\boldsymbol{\alpha}^T, \mathbf{x}^T]^T$ be the data vector consisting of bearing measurements and nominal positions of the observer. The bearing measurements and nominal positions are statistically independent, therefore, the probability density function (PDF) of the data vector is the product of their individual PDFs. The CRLB of the unknown parameters is computed by the inverse of the Fisher information matrix [5]. Partitioning the Fisher matrix and taking the inverse of partitioned matrix, we have [4],

$$\text{CRLB}(\mathbf{s}^o) = \mathbf{X}^{-1} + \mathbf{X}^{-1}\mathbf{Y}\,\text{CRLB}(\mathbf{x}^o)\,\mathbf{Y}^T\mathbf{X}^{-1}, \quad (4a)$$

$$\text{CRLB}(\mathbf{x}^o) = (\mathbf{Z} - \mathbf{Y}^T\mathbf{X}^{-1}\mathbf{Y})^{-1} = \mathbf{\Psi_x}, \quad (4b)$$

where $\mathbf{X} = \mathbf{A}^T\mathbf{\Psi}_{\boldsymbol{\alpha}}^{-1}\mathbf{A}$, $\mathbf{Y} = \mathbf{A}^T\mathbf{\Psi}_{\boldsymbol{\alpha}}^{-1}\mathbf{B}$, $\mathbf{Z} = \mathbf{B}^T\mathbf{\Psi}_{\boldsymbol{\alpha}}^{-1}\mathbf{B} + \mathbf{\Psi}_{\mathbf{x}}^{-1}$, $\mathbf{A} = \partial\boldsymbol{\alpha}^o/\partial\mathbf{s}^o$, and $\mathbf{B} = \partial\boldsymbol{\alpha}^o/\partial\mathbf{x}^o$. Let $\mathbf{a}_i^T$ and $\mathbf{b}_i^T$ be the ith row of matrix $\mathbf{A}$ and matrix $\mathbf{B}$ respectively,

$$\mathbf{a}_i^T = \left[-\frac{y_s^o - y_i^o}{d_i} \quad \frac{x_s^o - x_i^o}{d_i}\right], \quad (5a)$$

$$\mathbf{b}_i^T = -\left[\mathbf{0}_{(2i-2)\times 1}^T \quad \mathbf{a}_i^T \quad \mathbf{0}_{(2M-2i)\times 1}^T\right], \quad (5b)$$

where $d_i = \|\mathbf{s}^o - \mathbf{x}_i^o\|$ is the Euclidian distance between target and observer at the ith point. It can be seen from (4b) that the CRLB of the observer position is equal to the covariance matrix of the observer position error. Moreover, (4a) shows that the CRLB of the target location depends on the covariance matrix of the observer position error $\mathbf{\Psi_x}$. By setting $\mathbf{\Psi_x} = \mathbf{0}$ (i.e., the exact position of the observer is known), the CRLB of the target location reduces to $\text{CRLB}(\mathbf{s}^o) = \mathbf{X}^{-1}$ which is the same as the CRLB derived in [1] when the exact position of the observer is available to the estimator.

III. Localization Algorithms

In this section, we develop different algorithms for solving the bearing-only localization problem defined in Section II. We start with ML algorithm then we continue with introducing the linear algorithms, i.e., LS, and TLS.

A. ML Algorithm

First, the ML estimator assumes that the exact positions of the observer are available. Since the bearing measurement has a Gaussian PDF, ML problem turns into the following nonlinear minimization problem [5],

$$\hat{\boldsymbol{\theta}}_{1,\text{ML}} = \arg\min_{\boldsymbol{\theta}_1} (\boldsymbol{\alpha} - \mathbf{g}_1(\boldsymbol{\theta}_1))^T\mathbf{C}_1(\boldsymbol{\alpha} - \mathbf{g}_1(\boldsymbol{\theta}_1)), \quad (6)$$

where $\mathbf{C}_1 = \mathbf{\Psi}_{\boldsymbol{\alpha}}^{-1}$, $\mathbf{g}_1(\boldsymbol{\theta}_1) = [g_{1,1}(\boldsymbol{\theta}_1), g_{1,2}(\boldsymbol{\theta}_1), \ldots, g_{1,M}(\boldsymbol{\theta}_1)]^T$, and $g_{1,i}(\boldsymbol{\theta}_1) = \tan^{-1}(y_s - y_i)/(x_s - x_i)$. Above minimization can be approximated by the Gauss-Newton (GN) method [5],

$$\boldsymbol{\theta}_1^{k+1} = \boldsymbol{\theta}_1^k + \left(\mathbf{H}_{1,k}^T\,\mathbf{C}_1\mathbf{H}_{1,k}\right)^{-1}\mathbf{H}_{1,k}^T\,\mathbf{C}_1(\boldsymbol{\alpha} - \mathbf{g}_1(\boldsymbol{\theta}_1^k)), \quad (7)$$

where $\mathbf{H}_{1,k} = \partial\mathbf{g}_1(\boldsymbol{\theta}_1)/\partial\boldsymbol{\theta}_1|_{\boldsymbol{\theta}_1 = \boldsymbol{\theta}_1^k}$. Note that $\mathbf{H}_{1,k}$ is equal to $\mathbf{A}$ when $\mathbf{s}^o = \boldsymbol{\theta}_1^k$ and $\mathbf{x}^o = \mathbf{x}$. Now assume that the ML estimator tries to estimate the observer position as well as the target location using the joint PDF of bearing measurement and observer position. Consequently, the ML estimate is

$$\hat{\boldsymbol{\theta}}_{\text{ML}} = \arg\min_{\boldsymbol{\theta}} (\boldsymbol{\beta} - \mathbf{g}(\boldsymbol{\theta}))^T\mathbf{C}\,(\boldsymbol{\beta} - \mathbf{g}(\boldsymbol{\theta})), \quad (8)$$

where $\mathbf{C} = \text{blkdiag}[\mathbf{C}_1, \mathbf{C}_2] = \text{blkdiag}[\mathbf{\Psi}_{\boldsymbol{\alpha}}^{-1}, \mathbf{\Psi}_{\mathbf{x}}^{-1}]$, and $\mathbf{g}(\boldsymbol{\theta}) = [\mathbf{g}_1(\boldsymbol{\theta})^T, \boldsymbol{\theta}_2]^T$. Similar to (6), the minimization of (8) can be approximated using GN method [5], therefore,

$$\boldsymbol{\theta}^{k+1} = \boldsymbol{\theta}^k + \left(\mathbf{H}_k^T\,\mathbf{C}\,\mathbf{H}_k\right)^{-1}\mathbf{H}_k^T\,\mathbf{C}\,(\boldsymbol{\beta} - \mathbf{g}(\boldsymbol{\theta}^k)), \quad (9)$$

where $\mathbf{H}_k = \partial\mathbf{g}(\boldsymbol{\theta})/\partial\boldsymbol{\theta}|_{\boldsymbol{\theta}=\boldsymbol{\theta}^k}$. Partitioning the second term of right hand side of (9) for $\boldsymbol{\theta}_1$ and $\boldsymbol{\theta}_2$ yields

$$\begin{bmatrix} \hat{\mathbf{X}}_k^{-1}\mathbf{H}_{1,k}^T\mathbf{C}_1(\boldsymbol{\alpha} - \mathbf{g}_1(\boldsymbol{\theta}^k)) + \hat{\mathbf{X}}_k^{-1}\mathbf{C}_2(\mathbf{x} - \boldsymbol{\theta}_2^k) \\ \mathbf{U}_k^{-1}\mathbf{C}_2(\mathbf{x} - \boldsymbol{\theta}_2^k) \end{bmatrix}, \quad (10)$$

where $\mathbf{U}_k = \hat{\mathbf{Z}}_k - \hat{\mathbf{Y}}_k^T\hat{\mathbf{X}}_k^{-1}\hat{\mathbf{Y}}_k$, and $\hat{\mathbf{X}}_k$, $\hat{\mathbf{Y}}_k$, and $\hat{\mathbf{Z}}_k$ are equal to $\mathbf{X}$, $\mathbf{Y}$, and $\mathbf{Z}$ respectively by setting $\mathbf{s}^o = \boldsymbol{\theta}_1^k$ and $\mathbf{x}^o = \boldsymbol{\theta}_2^k$. Based on our computer simulations, for any initialization of $\boldsymbol{\theta}_2$ sufficiently close to $\mathbf{x}$, $\boldsymbol{\theta}_2^k$ converges to $\mathbf{x}$ after some iterations, therefore, the term $\mathbf{x} - \boldsymbol{\theta}_2^k$ in (10) vanishes and the final solution of (9) for $\boldsymbol{\theta}_2$ would be identical to the nominal position of observer. In addition, hereafter, the updating terms for $\boldsymbol{\theta}_1$ in (10) would be the same as given in (7) and eventually after convergence, (9) reaches to the same estimate for the target location as (7). It should be mentioned that the minimization of (6) and (8) using MATLAB routine `fminsearch` (a derivative-free method) also yields the same estimate for target location. In conclusion, according to our simulations, we think that both ML estimators ignoring and considering observer position uncertainties achieve the same result for target location. Furthermore, by applying the joint PDF, we are unable to find a better estimate for the observer position than the nominal value.

B. LS Algorithm

The LS algorithm is based on assumption that the bearing measurement errors are sufficiently small [1]. Consider (1), it can be written as

$$\tan(\alpha_i^o) = \frac{\sin(\alpha_i^o)}{\cos(\alpha_i^o)} = \frac{y_s^o - y_i^o}{x_s^o - x_i^o}. \quad (11)$$

By cross multiplying

$$x_s^o\sin(\alpha_i^o) - y_s^o\cos(\alpha_i^o) = x_i^o\sin(\alpha_i^o) - y_i^o\cos(\alpha_i^o). \quad (12)$$

In the presence of noise, (12) can be expressed in matrix form

$$\mathbf{G}\boldsymbol{\theta}_1 = \mathbf{h}, \quad (13)$$

where $\boldsymbol{\theta}_1$ defined earlier is location of target and

$$\mathbf{G}=\begin{bmatrix} \sin\alpha_1 & -\cos\alpha_1 \\ \vdots & \vdots \\ \sin\alpha_M & -\cos\alpha_M \end{bmatrix}, \mathbf{h}=\begin{bmatrix} x_1\sin\alpha_1 - y_1\cos\alpha_1 \\ \vdots \\ x_M\sin\alpha_M - y_M\cos\alpha_M \end{bmatrix}. \quad (14)$$

The least squares solution of (13) is (if $\mathbf{G}$ is full rank) [5],

$$\hat{\boldsymbol{\theta}}_{1,\mathrm{LS}} = (\mathbf{G}^T\mathbf{G})^{-1}\mathbf{G}^T\mathbf{h}. \quad (15)$$

Unlike the ML estimator, the LS estimator has a closed-form solution and does not need iterative computation. The performance of the LS algorithm can be enhanced by defining a weighting matrix to minimization problem. The weighted least squares solution of (13) is [5],

$$\hat{\boldsymbol{\theta}}_{1,\mathrm{WLS}} = (\mathbf{G}^T\mathbf{W}\mathbf{G})^{-1}\mathbf{G}^T\mathbf{W}\mathbf{h}, \quad (16)$$

where $\mathbf{W}$ is the weighting matrix which is equal to the inverse of the covariance matrix of the residual error in (12). By replacing true values with noisy ones in (12), expanding trigonometry elements, and using the approximations $\sin(n_i) \approx n_i$ and $\cos(n_i) \approx 1$ which are valid if the bearing measurement noises are sufficiently small, the residual error becomes

$$\epsilon_i = n_i d_i + n_i \dot{\mathbf{g}}_i^T \mathbf{v}_i + \mathbf{g}_i^T \mathbf{v}_i, \quad (17)$$

where $\mathbf{g}_i = [\sin(\alpha_i^o), -\cos(\alpha_i^o)]^T$, $\dot{\mathbf{g}}_i = \left[\begin{smallmatrix} 0 & -1 \\ 1 & 0 \end{smallmatrix}\right]\mathbf{g}_i$, and $\mathbf{v}_i = [v_{x,i}, v_{y,i}]^T$ is the noise vector of the ith position of the observer. Therefore, the weighting matrix would be

$$\begin{aligned}\mathbf{W} &= \mathrm{E}[\boldsymbol{\epsilon}\boldsymbol{\epsilon}^T]^{-1} \\ &= (\mathbf{B}_1\boldsymbol{\Psi}_{\boldsymbol{\alpha}}\mathbf{B}_1^T + \boldsymbol{\Psi}_{\boldsymbol{\alpha}}\mathbf{D}_2\boldsymbol{\Psi}_{\mathbf{x}}\mathbf{D}_2^T + \mathbf{D}_1\boldsymbol{\Psi}_{\mathbf{x}}\mathbf{D}_1^T)^{-1}, \quad (18)\end{aligned}$$

where $\mathbf{B}_1 = \mathrm{diag}(d_1, d_2, \ldots, d_M)$ with diag $\{\cdot\}$ denoting the diagonal matrix, $\mathbf{D}_1 = \mathrm{blkdiag}[\mathbf{g}_1^T, \mathbf{g}_2^T, \ldots, \mathbf{g}_M^T]$, and $\mathbf{D}_2 = \mathrm{blkdiag}[\dot{\mathbf{g}}_1^T, \dot{\mathbf{g}}_2^T, \ldots, \dot{\mathbf{g}}_M^T]$. Note that if we assume that the true position of the observer is known (i.e., $\boldsymbol{\Psi}_{\mathbf{x}} = \mathbf{0}$), (18) reduces to the expression given in [1] for the weighting matrix of WLS when the exact position of the observer is available to the estimator (the so-called Stansfield estimator).

It should be noted that the weighting matrix $\mathbf{W}$ depends on the true position of the target $\mathbf{s}^o$ which is not available for estimator. Therefore, the WLS estimator can be approxiamted in two steps. In the first step, we use identity matrix for WLS algorithm, i.e., $\mathbf{W} = \mathbf{I}_{M\times M}$. Indeed, WLS estimator changes to LS estimator defined in (15). For next step, we use estimated target location for computing the weighing matrix (18) for WLS algorithm.

C. TLS Algorithm

The TLS is an extension to the classic least squares [6]. Consider (13), the disturbance of bearing measurement as well as observer position affect both matrix $\mathbf{G}$ and vector $\mathbf{h}$. The LS algorithm only respects disturbance in vector $\mathbf{h}$, while the TLS takes errors in both vector $\mathbf{h}$ and matrix $\mathbf{G}$ into account. The TLS solution of (13) is [6],

$$\hat{\boldsymbol{\theta}}_{1,\mathrm{TLS}} = (\mathbf{G}^T\mathbf{G} - \sigma_s^2\,\mathbf{I})^{-1}\mathbf{G}^T\mathbf{h}, \quad (19)$$

where σ_s is the smallest singular value of matrix $[\mathbf{G}\ \mathbf{h}]$. It has been stated that the TLS algorithm has better performance than LS algorithm if we have errors in both data matrix and observation vector [6]. The TLS was developed for bearing localization in [3] where simulation results were used to compare the TLS and LS algorithms and it has been showed that TLS has better performance than LS. In simulation section, we will see this conclusion is not true for every situation.

D. WTLS Algorithm

In TLS algorithm, we presume that the errors in both matrix $\mathbf{G}$ and vector $\mathbf{h}$ are independent and identically distributed (IID). This assumption is not valid in the bearing-only localization problem. Hence, we introduce WTLS estimator which considers correlated noises with different statistical properties for the matrix $\mathbf{G}$ and vector $\mathbf{h}$. The classification of the WTLS was mentioned in [6] based on the structure of the weighting matrix. In contrast to the classic TLS, the WTLS has no closed-form solution. Currently WTLS is formulated as an optimization problem and solved by iterative algorithms. In this section, for the first time we apply WTLS algorithm for bearing-only localization model. In the WTLS algorithm, we not only have to compute the covariance matrix of the residual error in the vector $\mathbf{h}$, but also we require the covariance matrix of residual error in the matrix $\mathbf{G}$, and the covariance between residual errors in $\mathbf{h}$ and $\mathbf{G}$. Consider the ith row of (13), substituting true parameters with noisy ones and extracting residual errors, we have

$$\boldsymbol{\epsilon}_{\mathbf{g},i} = [n_i\cos(\alpha_i^o),\, n_i\sin(\alpha_i^o)]^T = n_i\dot{\mathbf{g}}_i, \quad (20)$$

which is residual error of the ith row of matrix $\mathbf{G}$. The covariance matrix of (20) would be

$$\boldsymbol{\Psi}_{\mathbf{g},i} = \mathrm{E}[\boldsymbol{\epsilon}_{\mathbf{g},i}\boldsymbol{\epsilon}_{\mathbf{g},i}^T] = \dot{\mathbf{g}}_i\boldsymbol{\Psi}_{\boldsymbol{\alpha},i}\dot{\mathbf{g}}_i^T, \quad (21)$$

where $\boldsymbol{\Psi}_{\boldsymbol{\alpha},i} = [\boldsymbol{\Psi}_{\boldsymbol{\alpha}}]_{ii}$. The residual error of the ith element of vector $\mathbf{h}$ is

$$\epsilon_{h,i} = n_i\dot{\mathbf{g}}_i^T\mathbf{x}_i + n_i\dot{\mathbf{g}}_i^T\mathbf{v}_i + \mathbf{g}_i^T\mathbf{v}_i. \quad (22)$$

The covariance matrix of (22) becomes

$$\begin{aligned}\boldsymbol{\Psi}_{h,i} = \mathrm{E}[\epsilon_{h,i}^2] = \mathbf{x}_i^T\dot{\mathbf{g}}_i\boldsymbol{\Psi}_{\boldsymbol{\alpha},i}\dot{\mathbf{g}}_i^T\mathbf{x}_i + \\ \boldsymbol{\Psi}_{\boldsymbol{\alpha},i}\dot{\mathbf{g}}_i^T\boldsymbol{\Psi}_{\mathbf{x},i}\dot{\mathbf{g}}_i + \mathbf{g}_i^T\boldsymbol{\Psi}_{\mathbf{x},i}\mathbf{g}_i, \quad (23)\end{aligned}$$

where $\boldsymbol{\Psi}_{\mathbf{x},i} = \boldsymbol{\Psi}_{\mathbf{x}_i}$. Moreover, it is required to derive the covariance matrix between residual errors in $\mathbf{h}$ and $\mathbf{G}$,

$$\boldsymbol{\Psi}_{\mathbf{g}h,i} = \mathrm{E}[\boldsymbol{\epsilon}_{\mathbf{g},i}\epsilon_{h,i}] = \dot{\mathbf{g}}_i\boldsymbol{\Psi}_{\boldsymbol{\alpha},i}\dot{\mathbf{g}}_i^T\mathbf{x}_i. \quad (24)$$

It should be noted that in the above derivations we have used the approximations applied in (17). Now, we will define the WTLS solution based on the algorithm developed in [7]. First, we rewrite (13) as

$$\mathbf{F}\boldsymbol{\Theta} = \mathbf{0}, \quad (25)$$

where $\boldsymbol{\Theta} = [\boldsymbol{\theta}_1^T, -1]^T$, and $\mathbf{F} = [\mathbf{G}, \mathbf{h}]$. Let $\boldsymbol{\Psi}_{\mathbf{f},i}$ be the covariance matrix of the ith row of $\mathbf{F}$, then

$$\boldsymbol{\Psi}_{\mathbf{f},i} = \begin{bmatrix} \boldsymbol{\Psi}_{\mathbf{g},i} & \boldsymbol{\Psi}_{\mathbf{g}h,i} \\ \boldsymbol{\Psi}_{\mathbf{g}h,i}^T & \boldsymbol{\Psi}_{h,i} \end{bmatrix}. \quad (26)$$

Therefore, the WTLS problem is defined as [7],

$$\hat{\boldsymbol{\theta}}_{1,\mathrm{WTLS}} = \underset{\boldsymbol{\theta}_1,\Delta\mathbf{f}_i}{\arg\min} \sum_{i=1}^{M} \|\boldsymbol{\Psi}_{\mathbf{f},i}^{-1/2}\Delta\mathbf{f}_i\|_2^2 \tag{27a}$$

$$\text{subject to } (\mathbf{F}+\Delta\mathbf{F})\boldsymbol{\Theta} = \mathbf{0}, \tag{27b}$$

where $\Delta\mathbf{F}$ is a correction matrix trying to compensate errors in matrix $\mathbf{F}$, $\Delta\mathbf{f}_i$ is ith row of matrix $\Delta\mathbf{F}$, and $\|\cdot\|_2$ denotes 2-norm. We have assumed that residual errors in each row of $\mathbf{F}$ are statistically independent. This type of WTLS problem is classified as row-wise WTLS [6]. The problem in (27) is an optimization problem. The full details of minimization procedure is given in [7]. After some computations, (27) turns into the following minimization problem

$$\hat{\boldsymbol{\theta}}_{1,\mathrm{WTLS}} = \underset{\boldsymbol{\theta}_1}{\arg\min} f(\boldsymbol{\theta}_1) = \underset{\boldsymbol{\theta}_1}{\arg\min} \sum_{i=1}^{M} \frac{r_i^2(\boldsymbol{\theta}_1)}{q_i(\boldsymbol{\theta}_1)}, \tag{28}$$

where $[r_1(\boldsymbol{\theta}_1), r_2(\boldsymbol{\theta}_1), \ldots, r_M(\boldsymbol{\theta}_1)]^T = \mathbf{G}\boldsymbol{\theta}_1 - \mathbf{h}$, and $q_i(\boldsymbol{\theta}_1) = \boldsymbol{\Theta}^T\boldsymbol{\Psi}_{\mathbf{f},i}\boldsymbol{\Theta}$. Indeed, $f(\boldsymbol{\theta}_1)$ is the cost function of WTLS should be minimized. To find the minimum of the cost function, the derivative of $f(\boldsymbol{\theta}_1)$ is equated to zero, i.e., $f'(\boldsymbol{\theta}_1) = \partial f(\boldsymbol{\theta}_1)/\partial\boldsymbol{\theta}_1 = 0$, where

$$f'(\boldsymbol{\theta}_1) = 2\sum_{i=1}^{M}\left[\mathbf{g}_i\frac{r_i(\boldsymbol{\theta}_1)}{q_i(\boldsymbol{\theta}_1)} - (\boldsymbol{\Psi}_{\mathbf{g},i}\boldsymbol{\theta}_1 - \boldsymbol{\Psi}_{\mathbf{g}h,i})\frac{r_i^2(\boldsymbol{\theta}_1)}{q_i^2(\boldsymbol{\theta}_1)}\right]. \tag{29}$$

(29) has probably several roots but the root corresponding to the global minimum of (28) is the WTLS estimation of target location. In [7] an iterative linear approximation algorithm has been suggested for solving (29) which seems to be inappropriate in some conditions. Effective numerical methods for finding the roots of (29) can be found in [8]. In our computer simulations, we have employed MATLAB routine `fsolve` with default settings, which uses *Dogleg* algorithm. Like the ML estimator, WTLS also has convergence problem due to the nonlinearity behavior of the cost function [7]. Although it has been shown that for large sample size and sufficiently close initialization, the algorithm converges certainly to the global minimum of the cost function [7], it is still possible that the algorithm either converges to a local minimum or diverges.

IV. Simulation Results

To evaluate the performance of the proposed algorithms, computer simulations are conducted. We consider two scenarios for the simulations. In the first scenario which is the same as the configuration in [3], the target is located at $[55, 35]^T$, the observer trajectory is $y = -0.2x + 14$ for $5 < x < 45$, and observer obtains M bearing measurements in equal distant points. In the second scenario, the target location remains as the first scenario and the observer trajectory is $y = 3x + 30$ for $5 < x < 45$. The bearing measurements and the nominal observer position are generated by adding zero mean Gaussian random variables with covariance matrix $\boldsymbol{\Psi}_{\boldsymbol{\alpha}} = \sigma_\alpha^2\mathbf{I}_{M\times M}$ and $\boldsymbol{\Psi}_{\mathbf{x}} = \sigma_x^2\mathbf{I}_{2M\times 2M}$ respectively to true values. The values of σ_α^2 and σ_x^2 are indicated in each figure. The mean square error (MSE) of each algorithm is computed by averaging of 10000 independent realizations. The plotted CRLB is computed as $\mathrm{trace}[\mathrm{CRLB}(\mathbf{s}^o)]$ in (4a).

Fig. 1 shows the MSE of the proposed algorithms versus the standard deviation of bearing measurement noise when the number of observations is $M = 20$ and the standard deviation of observer position noise is $\sigma_x = 0.1$ m. The ML estimator is calculated using GN method [9]. We have used the true position of target as the initialization of ML and WTLS to increase the probability that the algorithms converge to the global minimum. It can be seen that the WLS algorithm performs better than LS. The TLS has remarkably better performance than LS and WLS. Furthermore, WTLS, and ML have very close performance and can attain the CRLB accuracy for bearing noise standard deviation under 5°.

The MSE of proposed algorithms as a function of the standard deviation of bearing measurement noise for the second scenario is shown in Fig. 2. The number of observations and observer position noise remain as Fig. 1. The WTLS and ML show similar performance and achieve the CRLB for bearing noise standard deviation under 5°. In this case, the TLS has not better MSE than the LS. The reason is that, for TLS we assume that the errors in matrix $\mathbf{G}$ and vector $\mathbf{h}$ are independent and equally sized, however, (20) and (22) show that the errors in $\mathbf{G}$ and $\mathbf{h}$ depend on the observer position and since the distance between the first and last observation in the second scenario is almost three times more than the first scenario, the errors in $\mathbf{G}$ and $\mathbf{h}$ of the latter will be unequally sized more severely than the former. Consequently, the assumption in TLS is not valid anymore and its performance will degrade.

In Fig. 3, we compare the MSE of the proposed algorithms in the first scenario versus standard deviation of observer position noise. The number of observations is $M = 20$ and the standard deviation of the bearing noise is $\sigma_\alpha = 2^\circ$. The MSE of all algorithms get worse as the noise on observer position increases. The ML and WTLS have the optimum performance for lower noise (less than 0.2 m). However, the MSE of ML intensifies as the noise on the observer position increases, which is consistent with the results in [3], while the WTLS performance stays close to the CRLB. The ML estimator is expected to be asymptotically efficient, but efficiency is not guaranteed for a finite number of observations [5]. Therefore, we expect the ML gets back to an efficient estimate for sufficiently large data records as indicated in Fig. 4.

Fig. 4 depicts the MSE of proposed algorithms for different number of observations in the first scenario when the standard deviation of bearing measurement and observer position noises are 4° and 1 m respectively. It can be seen that when the number of observations increases, the MSE of all algorithms diminishes. However, the MSE decline for LS and WLS is very slow and almost flat for the large number of observations (i.e., greater than 160) because they do not consider the disturbances in the matrix $\mathbf{G}$ in (13). On the other hand, the WTLS obtains the CRLB performance by increasing the number of observations presenting asymptotically efficient behavior. The MSE of the ML is also interesting. It has inferior performance

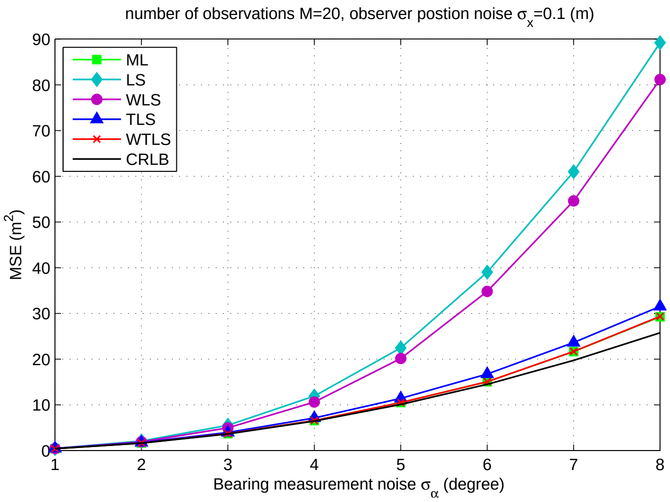

Fig. 1. The MSE performance of the proposed algorithms versus standard deviation of bearing measurement noise (the first scenario).

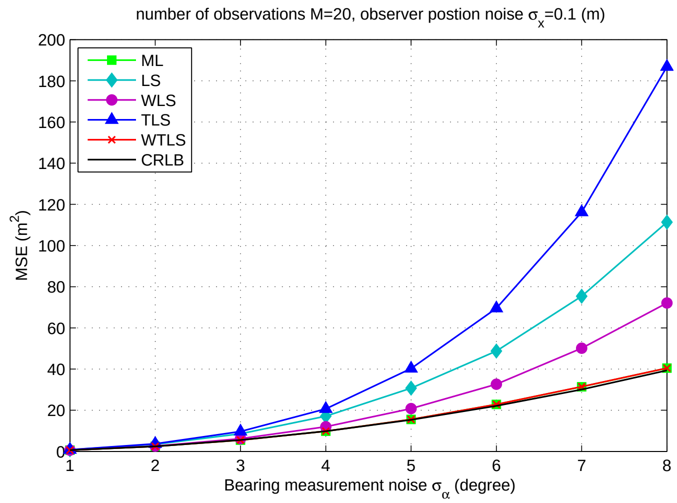

Fig. 2. The MSE performance of the proposed algorithms versus standard deviation of bearing measurement noise (the second scenario).

compared to TLS for the number of observations less than 20, but it surpasses TLS for greater number of observations. We can conclude that it might be optimal for large data records.

V. Conclusion

The bearing-only localization problem with uncertainties in observer position was surveyed in this paper. The Cramér-Rao lower bound (CRLB) of the proposed localization model was derived under this assumption that the bearing measurement noise and observer position noise are independent. The maximum likelihood, linear least squares, weighted least squares, and total least squares estimators were developed for this problem and additionally a novel method of positioning based on the weighted total least squares was introduced. Computer simulations were conducted to assess the performance of the proposed algorithms. Simulation results demonstrated that the novel method outperforms other methods and obtains the CRLB accuracy asymptotically.

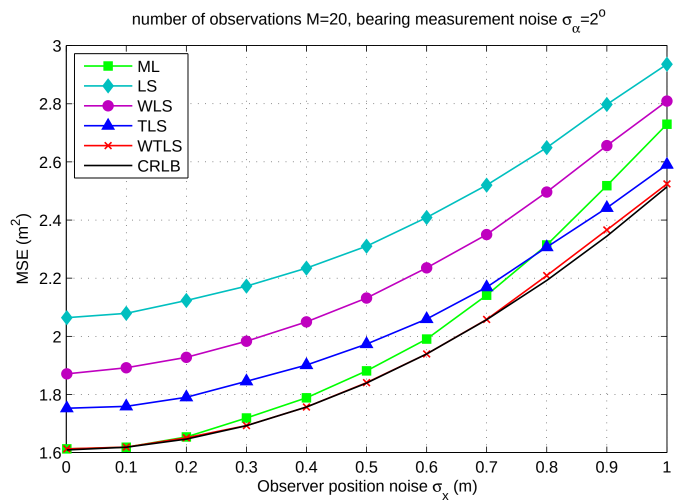

Fig. 3. The MSE performance of the proposed algorithms versus standard deviation of observer position noise (the first scenario).

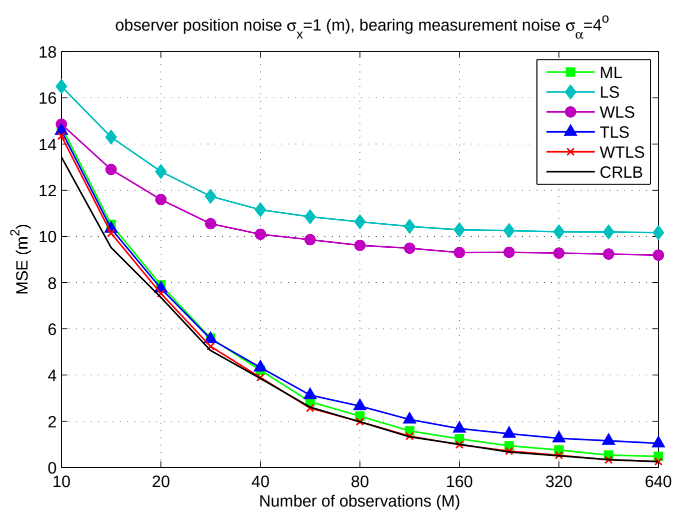

Fig. 4. The MSE performance of the proposed algorithms versus the number of observations (the first scenario).

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