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Wideband Sequential Spectrum Sensing with Varying Thresholds

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***Abstract*—In this contribution, time varying threshold sequential detectors are employed for energy detection-based spectrum sensing in low-SNR regimes. Sequential detection is proven to be faster (on average) than any other multi-sample detector for a set of given probabilities of detection and false-alarm. In this report, exact performance of a sequential detector for spectrum sensing is analyzed using the direct method. The theoretical results presented herein are verified with Monte-Carlo simulations. It is shown that for a SNR of -10 dB, among tests with Wald and triangular thresholds with similar probabilities of mis-detection and false-alarm, triangular performs 54% faster in terms of maximum detection time (90 percentile).**

***Index Terms*—Spectrum Sensing, Cognitive Radio, Sequential Analysis, Sequential Probability Ratio Test, UWB and DAA.**

I. Introduction

Early work by Abraham Wald [1] in statistics introduced a special class of sequential tests called sequential probability ratio test (SPRT), which addressed several different problems involving samples from two or more statistical distributions, e.g., abrupt change detection [2]. This test (and its variants) was deemed useful in several fields such as when addressing detection problems with low signal to noise ratio (SNR) or attempting signal detection with few samples. It was also shown that SPRT is optimum in the sense of probability of detection and false alarm, Bayesian risk and detection time, which is the average sample number (ASN) needed for detecting the target [3]. Further, it was shown that SPRT is optimum when the distribution of signal and noise is known a-priori. Also, in comparison to the fixed sample size (FSS) detector, it performs much faster in terms of the ASN, especially when only noise is present. However, when the target signal fluctuates from design distribution or has very low power in comparison to the noise power, the ASN will be very large and it will need to be truncated.

The above characteristics of SPRT are particularly appealing in the context of cognitive radio research, where a primary goal is to make use of under-utilized radio spectrum. Cognitive radios sense the spectrum to detect any activities of the legacy systems in the bands they have license to operate. In many cognitive radio applications, such as ultra wide-band (UWB) detect and avoid (DAA), the key issue is spectrum sensing. Spectrum sensing is performed by a normally non-coherent receiver in the designated band [4]. In spectrum sensing, there exist several problems dependent on the setup, which include but are not limited to

- Low-SNR or wide-bandwidth scenarios [5]
- No information about transmission type for the primary users (PUs) of the band
- Hidden or exposed terminal cases
- Bursty and hopping primary transmissions.

These challenges are very similar to the ones addressed by sequential tools, which thus became good candidates for employment in the context of spectrum sensing. Note that SPRT has attracted a lot of attention due to its optimality in ASN [6]-[14].

Even though Wald SPRT is widely used in spectrum sensing literature, when there exists a mismatch between design and actual parameters of the distributions or when there is a change of distribution in the middle of the test, the maximum number of samples needed by SPRT to reach a decision could be prohibitively high [15]-[19]. In this paper we are introducing a certain class of truncated SPRT in spectrum sensing to ensure that the test would be terminated at a certain number of samples.

Performance of different detectors is one of the most important issues for the design and optimization of the proper scheme for a certain environment. However, there have been few attempts to compute the exact probabilities of false-alarm P_{F} and mis-detection P_{M} and the distribution of the sample number. Knowledge of these measures can provide insight as to the selection, design and optimization of the best scheme for any detection problem. Aroian and Robison [20] introduced a method for calculating the exact P_{F} and P_{M} of a sequential detector based on the distribution of samples and thresholds.

In this contribution, sequential detectors with varying thresholds are used for energy-based spectrum sensing. The performance of this class of sequential spectrum sensors is evaluated in terms of the probabilities of false-alarm, mis-detection and sample number distributions. This performance is compared with the standard fixed thresholds introduced by Wald [1].

In the next section, a system model and the design of a simple sequential detector is outlined. Then, Section III

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demonstrates the mathematical derivation for the exact calculation of performance measures for any detector. In Section IV the results for the performances of different sequential detectors are presented. The last section concludes the paper and outlines, in brief, our main contributions.

II. System model and sequential detector

In spectrum sensing, the target is to detect the existence or absence of primary users (PUs) of a frequency band or another cognitive transmitter in the designated band. Here, the existence of a PU transmission is denoted by hypothesis H_1 and its absence is denoted by H_0.

A. Signal and noise models

In order to model the two hypotheses, an additive white noise model is assumed. In this model, H_0 is modeled as noise only and H_1 is modeled as signal plus noise. The receiver is an energy detector which generates $x_k = r_k^2$ for each received signal sample r_k. The following is a review of the assumptions that this model is based on.

1) Noise only: This model assumes that the noise $n(t)$ is Gaussian with variance σ_n^2 ($n(t) \sim N(0, \sigma_n^2)$), and the received signal will be $r_k = n(kt_s) \sim N(0, \sigma_n^2)$, where t_s is the sampling interval. The probability density function of $x_k is$

$$p_0(x) = \frac{1}{\sigma_n 2^{1/2}\Gamma(\frac{1}{2})} x^{-1/2} e^{-x/2\sigma_n^2}\ ,\ x \geq 0, \quad (1)$$

where p_0 is the Chi-square distribution with one degree of freedom.

2) Signal plus noise: This model assumes that the noise is Gaussian with variance σ_n^2, the signal is Gaussian with variance σ_s^2, and $r_k = s(kt_s) + n(kt_s)$, $r_k \sim N(0, \sigma_r^2)$, where $\sigma_r^2 = \sigma_s^2 + \sigma_n^2$. The PDF of x_k is

$$p_1(x) = \frac{1}{\sigma_r 2^{1/2}\Gamma(\frac{1}{2})} x^{-1/2} e^{-x/2\sigma_r^2}\ ,\ x \geq 0. \quad (2)$$

It should be noted that, assuming a Gaussian signal where there exists no information about PU transmission is a valid choice [7]. Additionally, for an AWGN channel, a Gaussian transmission is shown to be capacity achieving.

B. Log-Likelihood ratio calculation and sequential observations

According to Wald [1], in order to perform sequential analysis, one should calculate the cumulative sum of log likelihood ratios (LLRs). The LLR z_k for an energy sample will be

$$\begin{aligned} z_k &= \log\ \frac{p\{x_k|H_1\}}{p\{x_k|H_0\}} = \log\ \frac{p_1(x_k)}{p_0(x_k)} \\ &= \log\ \frac{\sigma_n e^{-x_k/2\sigma_r^2}}{\sigma_r e^{-x_k/2\sigma_n^2}} = \log\ \frac{\sigma_n}{\sigma_r} + \frac{x_k}{2}(\frac{1}{\sigma_n^2} - \frac{1}{\sigma_r^2}). \quad (3) \end{aligned}$$

It should be noted that σ_s^2 and σ_n^2 are decided based on the requirements of the sensing. However, the actual values will be different from the design values.

Normally, it is assumed that the process x_k is i.i.d. Based on the i.i.d. assumption, the cumulative LLR for sequential measurements may be rewritten as

$$\begin{aligned} Z_k &= \log\ \frac{p_1(x_1)p_1(x_2)...}{p_0(x_1)p_0(x_2)...} = \sum_{i=1}^{k} z_i \\ &= k\ \log\ \frac{\sigma_n}{\sigma_r} + \frac{1}{2}(\frac{1}{\sigma_n^2} - \frac{1}{\sigma_r^2}) \sum_{i=1}^{k} x_i. \quad (4) \end{aligned}$$

Thus, the sequential observation becomes a summation of measurements (x_k). The measurements from each sensing period are observed sequentially until the cumulative LLR reach one of the thresholds.

III. Direct method for performance evaluation

In this section, Aroian's method [20] is adapted for characterizing the exact performance of energy-based sequential spectrum sensing.

A. Distribution of test statistic

The single sample energy of a Gaussian signal (or noise) is Chi-square distributed with with one degree of freedom (and Gaussian variance of σ_r^2 or σ_n^2). We denote the PDF of x_k with $f(x)$ and cumulative density function (CDF) with $F(x)$, where f (and F) is the PDF (and CDF) of the Chi-square distribution with one degree of freedom.

The test statistic at the kth observation, $W_k = W_{k-1}^{\mathrm{T}} + x_k$, in a sequential test is compared with a lower (R_k) and an upper (A_k) threshold, where W_{k-1}^{T} is the truncated version of W_{k-1} after a comparison with R_{k-1} and A_{k-1}, and T denotes this truncation. This comparison will introduce three regions

- Accepting hypothesis if $W_k \geq A_k$ (with probability $P_k(A)$)
- Continuation of the test if $R_k < W_k < A_k$ (with probability $P_k(C)$)
- Rejecting hypothesis if $W_k \leq R_k$ (with probability $P_k(R)$).

The PDF (or CDF) of W_k and W_k^{T} are denoted by $\hat{f}_k$ ($\hat{F}_k$) and f_k (F_k) respectively. The truncated CDF of a random variable (W_k^{T}) can be calculated from the original $\hat{F}_k$ as

$$F_k(x_k) = \begin{cases} 0, & \text{if } x_k < R_k; \\ \frac{\hat{F}_k(x_k) - \hat{F}_k(R_k)}{\hat{F}_k(A_k) - \hat{F}_k(R_k)}, & \text{if } R_k \leq x_k \leq A_k; \\ 1, & \text{if } x_k > A_k. \end{cases} \quad (5)$$

Hence, x_k and W_{k-1}^{T} are independent of each other. Thus, the PDF of the sum $W_k = W_{k-1}^{\mathrm{T}} + x_k$ is the convolution of each one PDF as $\hat{f}_k(x) = f(x) * f_{k-1}(x)$. It is essential to have the truncated CDF of W_k (F_k) here. To determine this one may start from the PDF

$$f_k(x) = \begin{cases} \zeta_k \int_{R_{k-1}}^{A_{k-1}} f(x-t) f_{k-1}(t) dt, & \text{if } k \geq 2; \\ f(x)\ ,\ R_1 < x < A_1, & \text{if } k = 1. \end{cases} \quad (6)$$
$$\text{for } R_k < x < A_k$$

where $\zeta_k = \frac{1}{\hat{F}_k(A_{k-1}) - \hat{F}_k(R_{k-1})}$. Thus, CDF may be used for calculating P_{F} and P_{M}.

B. Expressions for P_{F} and P_{M}

To characterize the performance of a sequential detector, a measure called the operating characteristic function (OCF) is calculated for each hypothesis as

$$\mathrm{OCF}(\sigma^2) = \sum_{k=1}^{\infty} P_k(R) \prod_{i=1}^{k-1} P_i(C), \tag{7}$$

where $\sigma^2 = \sigma_n^2$ or σ_r^2. The probability of mis-detection and false alarm can be written as

$$P_{\mathrm{F}} = 1 - \mathrm{OCF}(\sigma_n^2) \tag{8}$$

$$P_{\mathrm{M}} = \mathrm{OCF}(\sigma_r^2) \tag{9}$$

The other performance measure is the detection time distribution

$$P_k(T) = \Pr\{T \leq k\} = 1 - \prod_{i=1}^{m-1} P_i(C), \tag{10}$$

where T is the detection time (in samples). It should be noted that equation (7) is different from the one presented by Aroian [20]. First, the upper bound in the summation could be extended to infinity to be suitable for all kinds of thresholding methods (including the non-truncating ones). Second, $P_k(A)$ has been changed to $P_k(R)$ because in this context the interest is in acceptance or rejection of hypothesis H_1 and not H_0. In order to calculate P_{F} and P_{M}, $P_k(R)$ and $P_k(C)$ are thus needed. They can be calculated from the distributions $f_k(x)$ and $F_k(x)$as

$$\begin{aligned} P_k(R) &= \zeta_k \int_{R_{k-1}}^{A_{k-1}} F(R_k - t) f_{k-1}(t) dt \quad (11)\\ &= \zeta_k \hat{F}_k(R_k) \\ P_k(A) &= 1 - \zeta_k \int_{R_{k-1}}^{A_{k-1}} F(A_k - t) f_{k-1}(t) dt \quad (12)\\ &= 1 - \zeta_k \hat{F}_k(A_k) \\ P_k(C) &= 1 - P_k(R) - P_k(A). \quad (13) \end{aligned}$$

Hence, the calculation of all properties of a sequential detector amounts to calculating the CDF of the test statistic $\hat{F}_k(x)$.

The next section introduces three different sets of thresholds for evaluating the above expressions.

IV. Thresholds for Sequential Spectrum Sensing

In this section, three sets of thresholds will be compared in the spectrum sensing setup. In the first set, Wald's fixed thresholds based on desired P_{F} and P_{M} will be examined. Wald's thresholds are lines parallel to x-axis in the plane of LLR vs. sample number. The next two sets of thresholds are time-varying and will finish the test at a finite number.

A. Wald's thresholds for simulation of the theory

In order to calculate $P_k(A)$ and $P_k(R)$, upper and lower thresholds A_k and R_k, respectively, are needed. Wald's thresholds comprise a set of the thresholds for LLRs, which are given by [1]

$$\gamma_{0_k} = \log \frac{\beta}{1-\alpha} \tag{14}$$

$$\gamma_{1_k} = \log \frac{1-\beta}{\alpha}, \tag{15}$$

where α and β represent the desired probabilities of false-alarm and mis-detection, respectively. However, there is a linear transformation between LLRs and energy samples presented in expression (4). Based on Wald's thresholds (14) and (15) and the transformation (4), the transformation presented in the following expressions may be used to find the thresholds for energy.

$$R_k = \frac{\gamma_{0_k} - k \log(\frac{\sigma_n}{\sigma_r})}{\frac{1}{2}(\frac{1}{\sigma_n^2} - \frac{1}{\sigma_r^2})} \tag{16}$$

$$A_k = \frac{\gamma_{1_k} - k \log(\frac{\sigma_n}{\sigma_r})}{\frac{1}{2}(\frac{1}{\sigma_n^2} - \frac{1}{\sigma_r^2})}. \tag{17}$$

B. Varying thresholds

Wald's thresholds have been proven to be optimum in the sense of ASN. However, in certain situations such as low-SNR scenarios or when the actual signal variance is smaller than the design variance, the maximum detection time for Wald's thresholds could be very large. Such a high value is not appropriate for spectrum sensing. This issue attracts some attention for a certain class of thresholds which ends the test (truncate). Sequential tests (thresholds) other than Wald's may not be optimum in ASN. However, they may be optimum in terms of minimax [15], [19]. In this paper, two of these thresholds are modified for the spectrum sensing problem. The first set, which is known as triangular (trapezoidal) thresholds, was originally proposed by Anderson [15]. In this work we use modified triangular thresholds as

$$\gamma_{0_k} = \alpha'(1 - \frac{k}{k_{\mathrm{t}}}) \tag{18}$$

$$\gamma_{1_k} = \beta'(1 - \frac{k}{k_{\mathrm{t}}}), \tag{19}$$

where α' and β' are two design parameters and k_{t} is the truncation point. It should be noted that these α' and β' do not correspond to the desired probabilities of false-alarm and mis-detection.

The second set of varying thresholds, known as reversed parabolic, was originally introduced by Ferebee [19]. The modified version is

$$\gamma_{0_k} = \frac{1}{k_{\mathrm{t}} - 1} \alpha' \sqrt{k_{\mathrm{t}} - k} \tag{20}$$

$$\gamma_{1_k} = \frac{1}{k_{\mathrm{t}} - 1} \beta' \sqrt{k_{\mathrm{t}} - k}. \tag{21}$$

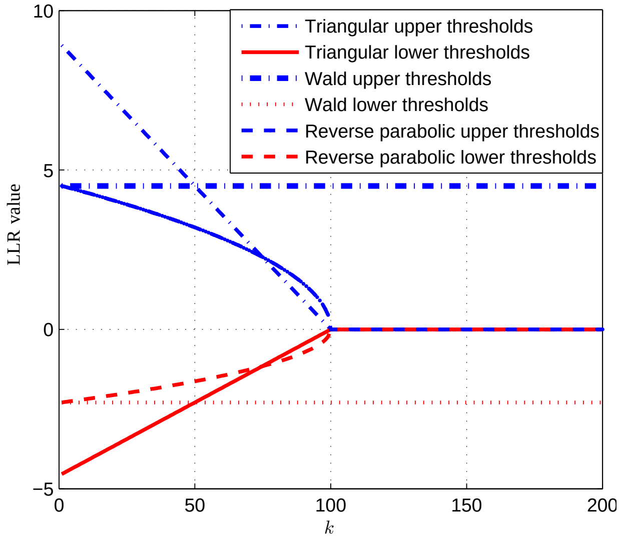

Figure 1. Wald and varying thresholds for $\alpha = 0.01$, $\beta = 0.1$, and $k_{\mathrm{t}} = 100$.

These thresholds are for LLR values and (16) and (17) can be used to make them suitable for energy samples.
Figure 1 presents a comparison between shapes of varying thresholds and Wald's thresholds. The performance of each of these thresholds is evaluated and compared with Wald's in the next section.

V. Performance evaluation and results

This section, first, discusses the evaluation setup by which the sequential detectors are assessed. Then, some results and a comparison are presented.

A. Evaluation setup

In simulating the performance of a spectrum sensing algorithm, the ratio of primary transmission signal to cognitive receiver noise is important. In this paper we are dealing with non-coherent detector and for such a detector detection is hard at negative SNR $= \sigma_s^2/\sigma_n^2$ (in dB). The results presented hereafter are obtained with the SNR ranging from -20 to 10 dB. Different thresholds are evaluated with different parameters under different SNRs. The results of each test are compared in terms of probabilities of false alarm and mis-detection, and detection times T_0 and T_1 under hypothesis H_0 and H_1 respectively. CDFs of T_0 and T_1 are computed due to the fact that they are random variables. To make the comparison between T_0 and T_1 easier, 50 percentile and 90 percentile values are found from their CDFs. Here, design parameters for triangular thresholds and reversed parabolic thresholds are optimized numerically by simulations.

B. Results

Figure 2 presents $\Psi_k = P_k(A)\prod_{i=1}^{k-1} P_i(C)$ estimated from simulations and calculated by the direct method described in secion III for Wald's thesholds when $\sigma^2 = 1.1$ and 1.5. These results show that the theory agrees well with the simulations. A small integration interval is involved in evaluating the convolution and CDF of test statistics in (12) and (13) numerically.

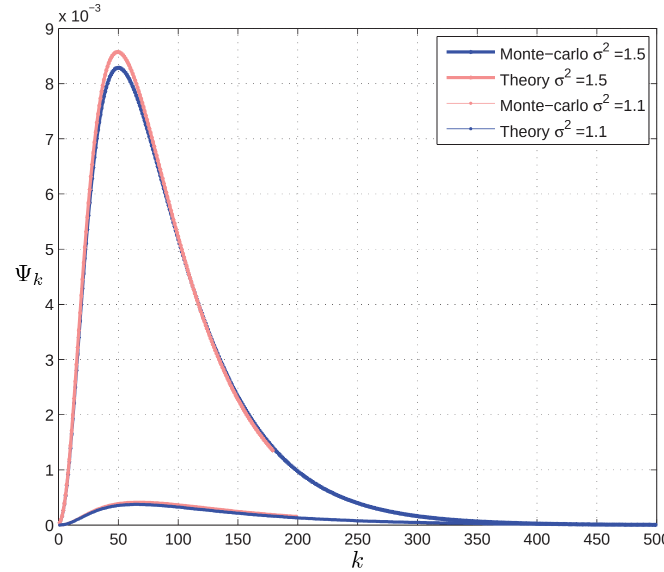

Figure 2. Theory vs. Monte-Carlo methods for $\Psi_k = P_k(A)\prod_{i=1}^{m-1} P_i(C)$ for $\sigma^2 = 1.1$, 1.5 and Wald's thresholds presented in Figure 1.

The smaller this interval, the higher the accuracy of integration will be.

For creating figure 3, first, tests having similar (with precision of 0.1%) P_{F}s or similar P_{M}s are selected and then minimum and maximum of 90 percentile of the total detection times $(T_0 + T_1)$ of those tests are presented vs. SNR. It is observed that, in the low-SNR regimes, the detector with the triangular thresholds performs faster than the one with Wald's thresholds for the same probabilities of mis-detection and false-alarm. At an SNR of -10 dB, the detector with triangular threshold performs 54% faster than the detector with Wald's thresholds on maximum total detection time. This smaller detection time translates directly to faster detection of PUs and hence less interference and more reutilization of vacant spectrum for the CR. This advantage of the detector with triangular threshold disappears at higher SNRs.

Figure 4 shows P_{F} vs. P_{M} for detectors with Wald and triangular thresholds with exactly the same 90 percentile detection times T_0 and T_1 when the SNR is -3 dB. In cognitive radio, where the amount of interference and hence P_{M} is constrained to lower values, triangular thresholds are preferable for P_{M}s lower than certain values. However, if the detector design is constrained on lower P_{F}s then Wald thresholds are beneficial.

Varying thresholds have slightly more complex structure. However, for one setup, the thresholds are calculated once. In fact, in comparison with considerable detection speed gain, this complexity is negligible.

VI. Conclusion

This paper evaluates the exact performance of sequential detectors for energy based spectrum sensing. Sequential detectors with both fixed and varying thresholds are investigated. It is shown that Wald's thresholds are not the best choice

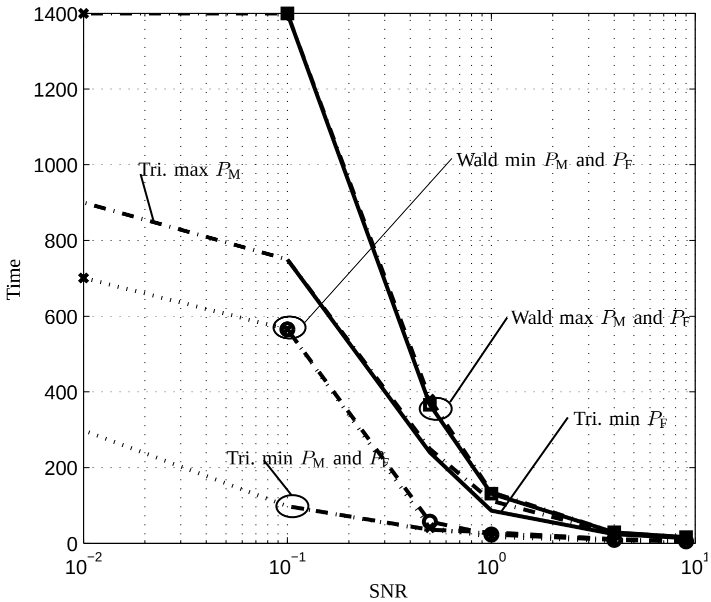

Figure 3. Minimum and maximum of the total detection time for tests with Wald and triangular thresholds when their P_M or P_F are similar (with precision of 0.1%).

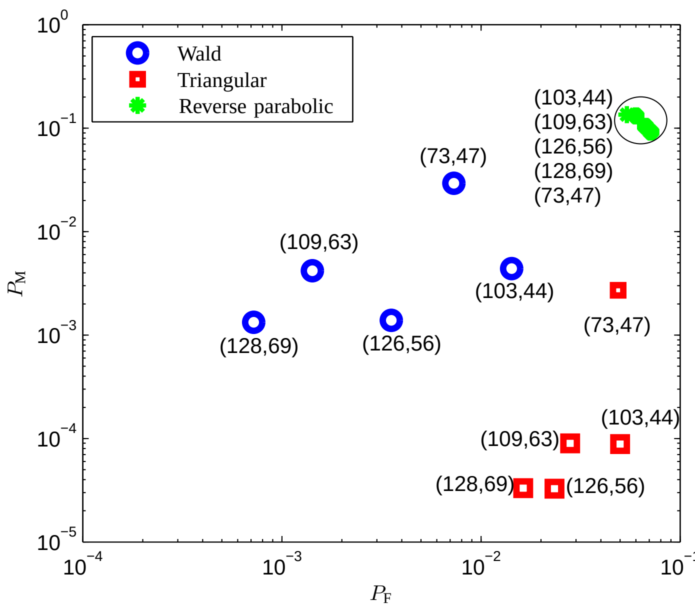

Figure 4. Probability of false alarm vs. probability of mis-detection for Wald and triangular and their corresponding detection times (T_0,T_1) for the SNR of -3 dB.

for all applications, e.g. spectrum sensing. In the low-SNR regime in spectrum sensing, we are not only interested in minimizing the average detection time (ASN) but also in minimizing the maximum detection time. It is shown that thresholds that limit the maximum detection time, like the triangular method, will perform better than Wald's in terms of detection time. Detectors with triangular thresholds have 54% and 17% smaller maximum and minimum total detection time, respectively, than detectors with Wald thresholds for an SNR of -10dB. Lower detection times in spectrum sensing applications such as UWB DAA achieve higher spectrum reutilization.

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