Abstract

Let p be an odd prime and let C_{p^n} be a cyclic group of order p^n and ζ_n be a primitive p^{n+1}-th root of unity. There exists an exact sequence

$$0 \to V_n^- \times V_n^+ \to \operatorname{Pic} \mathbb{Z}C_{p^{n+1}} \to \operatorname{Pic} \mathbb{Z}C_{p^n} \times \operatorname{Cl} \mathbb{Z}[\zeta_n] \to 0,$$

where V_n^- is known explicitly. In this thesis we deal with some problems regarding V_n^+. This group is a quotient of a group denoted $\mathcal{V}_n^+$ and a conjecture of Kervaire and Murthy from 1977 states the two groups are isomorphic. The conjecture also states that these groups are in fact isomorphic to the p-component of $\operatorname{Cl} \mathbb{Z}[\zeta_{n-1}]$.

In the first paper in this thesis we introduce a new technique and give a new proof of the known result that $\mathcal{V}_n^+$, and hence also V_n^+, is trivial when p is a regular prime. The proof is based on a generalization to $\mathbb{Z}[\zeta_n]$ of Kummer's famous result stating that a unit in $\mathbb{Z}[\zeta_0]$ congruent to 1 modulo p is a p-th power of another unit if p is a regular prime.

In the second paper we consider the structure of $\mathcal{V}_n^+$ and its relations with V_n^+ under three various assumptions on the prime p. All these assumptions are valid for all primes up to 4.000.000 and no primes for which the assumptions fail are known. Under two of these assumptions we prove that $V_n^+ \cong \mathcal{V}_n^+$.

In the third paper we give an exact formula for $\mathcal{V}_n^+$ under the most general assumption, namely that p is semi-regular, which by Vandiver's conjecture should be all primes. We also prove that $\mathcal{V}_n^+$ and $\operatorname{Cl}^{(p)} \mathbb{Z}[\zeta_{n-1}]$ have the same number of generators, which can be considered as a "weak version" of the Kervaire-Murthy conjecture that $\mathcal{V}_n^+ \cong \operatorname{Cl}^{(p)} \mathbb{Z}[\zeta_{n-1}]$.

In the final paper we discuss a family of rings $A_{k,l}$ which in some sense fit in between $\mathbb{Z}C_{p^n}$ and $\mathbb{Z}[\zeta_n]$. Under one of our assumptions above we give an exact sequence

$$0 \to V_{k,l}^- \oplus V_{k,l}^+ \to \operatorname{Pic} A_{k,l} \to \operatorname{Cl} \mathbb{Q}(\zeta_{k+l-1}) \oplus \operatorname{Pic} A_{k,l-1} \to 0$$

and calculate $V_{k,l}^-$ and $V_{k,l}^+$ explicitly.

Keywords: Picard group, Grothendieck group, integer group ring, cyclic group p-group, cyclotomic field, class group, Kummer's lemma, semi-regular prime.

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