On Quaternionic Shimura Surfaces

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Let k be a real quadratic field, and let A be a totally indefinite quaternion algebra which allows an involution of type 2, that is, an involution inducing the non-trivial automorphism on k. Let Λ be a maximal order in A. The elements of Λ with norm 1 act naturally on $\mathcal{H}\times\mathcal{H}$, where $\mathcal{H}$ is the complex upper half plane. Let Γ denote the image of Λ^1 in $\mathrm{Aut}(\mathcal{H}\times\mathcal{H})$, and X the quotient surface $\mathcal{H}\times\mathcal{H}/\Gamma$. We let Y be the minimal desingularisation of the compactification of X. If $A = M_2(k)$, then X is a so called Hilbert modular surface. Such surfaces are rather well investigated. We look at the case when A is a skew field. In this case, X is compact, so it only has quotient singularities. We also examine quotients by some extensions of Γ to larger discrete subgroups of $\mathrm{Aut}(\mathcal{H}\times\mathcal{H})$.

We construct a family of curves on Y, which corresponds to the so called modular curves in the case of Hilbert modular surfaces. The main part of the work consists of a study of various aspects of these curves. They are parametrised by the elements β of a quaternary lattice (L, q), which consists of what we call integral Λ-hermitian forms. There is a close connection between the quadratic space L and the order Λ via Clifford algebras.

To each curve F_β there is an associated quaternion order Λ_β over $\mathbb{Z}$ and a map $\mathcal{H}/\Lambda^1_\beta \to F_\beta$, which is generically 1 to 1 or 2 to 1. We determine the genus of the order Λ_β. To do this, we study, among other things, a certain one-to-one correspondence between primitive orders and hermitian planes in the local case.

For each positive integer N, we define a curve F_N in the same way as it is done in the case of Hilbert modular surfaces. We determine the number of irreducible components of F_N. To each intersection point of curves, we associate an integral binary quadratic form. We derive a formula for the number of points on X, which are associated to a given form. This gives a possibility to completely determine the configuration of curves.

Finally, we study the particular case when $k = \mathbb{Q}(\sqrt{13})$ and the discriminant of the algebra A is (3). We construct a natural tower $\Gamma \subset \Gamma_{\mathrm{I}} \subset \Gamma_{\mathrm{II}} \subset \Gamma_{\mathrm{III}}$ of discrete subgroups of $\mathrm{Aut}(\mathcal{H}\times\mathcal{H})$, where each group extension is of degree 2, and consider the minimal desingularisation of the corresponding quotients. We prove, using the modular curves, that Y is a minimal surface of general type, Y_{I} is a $K3$-surface blown up 4 times, Y_{II} is an Enriques surface blown up 2 times, and Y_{III} is a rational surface with Euler characteristic $e = 12$. We also construct an elliptic fibration on Y_{II}, which we use to conclude that Y_{II} is a so called special Enriques surface.

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