THESIS FOR THE DEGREE OF LICENTIATE OF ENGINEERING

Determinantal two-block-factors and Dynamical Percolation for some Interacting Particle Systems

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Abstract

In this thesis we first analyze a class of one-dependent trigonometric determinantal processes and show that they are two-block-factors. We do this by constructing the two-block-factors explicitly. We hope that this description will enable one to use more standard probabilistic techniques when studying this class of processes.

Second we investigate the dynamic stability of percolation for the stochastic Ising model and the contact process. This is a natural extension of what previously has been done for non-interacting particle systems. The main question we ask is; if we have percolation at a fixed time in a time-dependent system, do we have percolation at all times? A key tool in the analysis is the concept of $\epsilon-$stability which we introduce here.

Keywords: Determinantal processes, k-dependence, k-block-factors, percolation, stochastic Ising models, contact process.

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I would like to thank my parents for always supporting my career decisions, even when I strayed from the more sensible path. Finally I would like to thank my sister Malin for all the encouragement and inspiration.

This thesis consists of the following papers:

Paper I: *One-dependent trigonometric determinantal processes are two-block-factors* Annals of Probability (To Appear)
Given a trigonometric polynomial $f : [0,1] \to [0,1]$ of degree m, one can define a corresponding stationary process $\{X_i\}_{i\in\mathbb{Z}}$ via determinants of the Toeplitz matrix for f. We show that for $m = 1$ this process, which is trivially one-dependent, is a two-block-factor.

Paper II: *Dynamical Stability of Percolation for Some Interacting Particle Systems and $\epsilon-$stability* (Submitted)
In this paper we will investigate dynamic stability of percolation for the stochastic Ising model and the contact process. We also introduce the notion of downwards and upwards ϵ–stability which will be a key tool for our analysis.

Paper I

One-dependent trigonometric determinantal processes are two-block-factors

Erik I. Broman*

September 29, 2003

Abstract

Given a trigonometric polynomial $f : [0,1] \to [0,1]$ of degree m, one can define a corresponding stationary process $\{X_i\}_{i\in\mathbb{Z}}$ via determinants of the Toeplitz matrix for f. We show that for $m = 1$ this process, which is trivially one-dependent, is a two-block-factor.

AMS subject classification: 60G10

Keywords and phrases: Determinantal processes, k-dependence,k-block-factors

Short title: Determinants and two-block-factors

1 Introduction

We will start by defining a family of probability measures $\mathbf{P}^f$ on the Borel sets of $\{0,1\}^{\mathbb{Z}}$ where $f : [0,1] \to [0,1]$ is a Lebesgue-measurable function (see [9]). For such an f, define the probability of the cylinder sets by

$$\begin{aligned}\mathbf{P}^f[\eta(e_1) = \cdots = \eta(e_k) = 1] &:= \mathbf{P}^f[\{\eta \in \{0,1\}^{\mathbb{Z}} : \eta(e_1) = \cdots = \eta(e_k) = 1\}] \\ &:= \det[\hat{f}(e_j - e_i)]_{1\leq,i,j\leq k},\end{aligned}$$

where $e_1, \ldots, e_k$ are distinct elements in $\mathbb{Z}$ and $k \geq 1$. Here $\hat{f}$ denotes the Fourier coefficients of f, defined by

$$\hat{f}(k) := \int_0^1 f(x)e^{-i2\pi kx}dx.$$

In [9] it is proven that $\mathbf{P}^f$ is indeed a probability measure. In fact they showed this for the more general case of $f : \mathbb{T}^d \to [0,1]$ where $\mathbb{T}^d := \mathbb{R}^d/\mathbb{Z}^d$;

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in this case the resulting process is indexed by $\mathbb{Z}^d$. This result rests very strongly on the results in [8]. Except for the two definitions below, $\{X_i\}_{i\in\mathbb{Z}}$ will always denote a process distributed according to some measure $\mathbf{P}^f$. Throughout this paper, equality in distribution will be denoted by $=_{\mathcal{D}}$. Let the function $f : [0,1] \to [0,1]$ be of the form

$$f(x) = \sum_{k=-m}^{m} a_k e^{-i2\pi kx}.$$

It is then easily checked that the process $\{X_i\}_{i\in\mathbb{Z}}$ corresponding to the probability measure $\mathbf{P}^f$ is m-dependent according to the definition below.

Definition 1.1 *A process $\{X_i\}_{i\in\mathbb{Z}}$ is called m-dependent if $\{X_i\}_{i<k}$ is independent of $\{X_i\}_{i\geq k+m}$ for all integers k.*

We will also need the definition of an m-block-factor.

Definition 1.2 *The process $\{X_i\}_{i\in\mathbb{Z}}$ is called an m-block-factor if there exists a function h of m variables and an i.i.d. process $\{Y_i\}_{i\in\mathbb{Z}}$ such that $\{X_i\}_{i\in\mathbb{Z}} =_{\mathcal{D}} \{h(Y_i, \ldots, Y_{i+m-1})\}_{i\in\mathbb{Z}}$.*

We will as usual not distinguish between the process $\{X_i\}_{i\in\mathbb{Z}}$ and the corresponding probability measure $\mathbf{P}^f$.

Observe that an (m+1)-block-factor is trivially m-dependent. For some time, it was an open question whether all m-dependent processes were in fact (m+1)-block-factors (see [4],[5],[6],[7]). However, in [2] the authors constructed a family of one-dependent processes which are not two-block-factors, and in [3] the authors constructed a one-dependent process which is not a k-block factor for any k. In [1] the authors construct a one-dependent stationary Markov process with five states which is not a two-block-factor, they also prove that this result is sharp in the sense that every one-dependent stationary Markov process with not more that four states is a two-block-factor. In view of the above it is a natural question to ask whether a certain m-dependent process is an (m+1)-block-factor or not.

$\mathbf{P}^f$ as defined above is an m-dependent "trigonometric determinantal probability measure". These probability measures are special cases of general determinantal probability measures, see [10] or [8] for definitions and results. Determinantal processes arise in numerous contexts e.g. mathematical physics, random matrix theory and representation theory to name a few. For a survey see [10], for further results see [8] and for results concerning the discrete stationary case, see [9]. In [9], they ask whether $\mathbf{P}^f$ above is an (m+1)-block-factor. In that paper they say that if one can find sufficiently explicit block factors for all trigonometric polynomials, then one can find explicit factors of i.i.d. processes giving $\mathbf{P}^f$, where f is any function such that $f : \mathbb{T} \to [0,1]$. This in turn would enable one to use more standard probabilistic techniques when studying such a $\mathbf{P}^f$. We answer their question positively for m=1 in Theorem (1.3), constructing an explicit two-block-factor.

Theorem 1.3 *If $f : [0,1] \to [0,1]$ is given by*

$$f(x) = b + ae^{-i2\pi x} + ce^{i2\pi x},$$

then the corresponding process $\{X_i\}_{i\in\mathbb{Z}}$ is a two-block-factor.

2 Proof of theorem 1.3

Proof of theorem 1.3.

With f as in the statement of the theorem, it follows that $\bar{a} = c$, $b \geq 0$ and hence if $a = a_1 + ia_2$

$$f(x) = b + 2a_1\cos(2\pi x) + 2a_2\sin(2\pi x) = b + 2|a|\cos(2\pi x - \phi), \tag{1}$$

for some suitable choice of ϕ. Let, as usual, $\mathbf{P}^f$ be the corresponding probability measure, and write

$$D_k := \det\, [\hat{f}(j-i)]_{1\leq,i,j\leq k+1}$$

where $k \geq 0$.

Note that the process $\{X_i\}_{i\in\mathbb{Z}}$ distributed according to $\mathbf{P}^f$ is obviously stationary. Since $\mathbf{P}^f$ is one-dependent, it is easily seen that it is uniquely determined among the one-dependent processes by the values of

$$\mathbf{P}^f[\eta(i) = \cdots = \eta(i+k) = 1] = \mathbf{P}^f[\eta(1) = \cdots = \eta(1+k) = 1]$$

as k varies over the nonnegative integers.

We have that for $k \geq 2$

$$D_k = \det\, [\hat{f}(j-i)]_{1\leq,i,j\leq k+1} = \begin{vmatrix} b & a & 0 & 0 & 0 & \cdots \\ \bar{a} & b & a & 0 & 0 & \cdots \\ 0 & \bar{a} & b & a & 0 & \cdots \\ 0 & 0 & \bar{a} & b & a & \ldots \\ \vdots & \vdots & \vdots & \ddots & \ddots & \ddots \end{vmatrix} \tag{2}$$

$$= b\begin{vmatrix} b & a & 0 & 0 & 0 & \cdots \\ \bar{a} & b & a & 0 & 0 & \cdots \\ 0 & \bar{a} & b & a & 0 & \cdots \\ 0 & 0 & \bar{a} & b & a & \ldots \\ \vdots & \vdots & \vdots & \ddots & \ddots & \ddots \end{vmatrix} - a\begin{vmatrix} \bar{a} & a & 0 & 0 & 0 & \cdots \\ 0 & b & a & 0 & 0 & \cdots \\ 0 & \bar{a} & b & a & 0 & \cdots \\ 0 & 0 & \bar{a} & b & a & \ldots \\ \vdots & \vdots & \vdots & \ddots & \ddots & \ddots \end{vmatrix}$$

$$= bD_{k-1} - |a|^2 D_{k-2},$$

where the determinant on the left-hand side of the third equality has size $(k+1) \times (k+1)$, and the two on the right-hand side have size $k \times k$. Furthermore

$$D_0 = |b| = b \tag{3}$$

$$D_1 = \begin{vmatrix} b & a \\ \bar{a} & b \end{vmatrix} = b^2 - |a|^2. \tag{4}$$

The characteristic equation corresponding to equation (2) is

$$r^2 - br + |a|^2 = 0, \tag{5}$$

which has two roots

$$r_1 = \frac{b}{2} + \sqrt{\frac{b^2}{4} - |a|^2}, \tag{6}$$

and

$$r_2 = \frac{b}{2} - \sqrt{\frac{b^2}{4} - |a|^2}. \tag{7}$$

Case 1: Assume that $r_1 = r_2 = r$ so that $r = \frac{b}{2}$ and

$$\frac{b^2}{4} = |a|^2$$

and so (since $b, |a| \geq 0$)

$$b = 2|a|.$$

We have by equation (1) that

$$\max_{x\in[0,1]} f(x) = \max_{x\in[0,1]} (b + 2|a|\cos(2\pi x - \phi)) = b + 2|a| = 2b$$

and since $f : [0,1] \to [0,1]$ we get $b \leq 1/2$ and so $|a| \leq 1/4$.

With $r_1 = r_2 = r$, it follows from the basic theory of difference equations that the solution to equation (2) is

$$D_k = (C_1 k + C_2)r^k \ \forall k \geq 0,$$

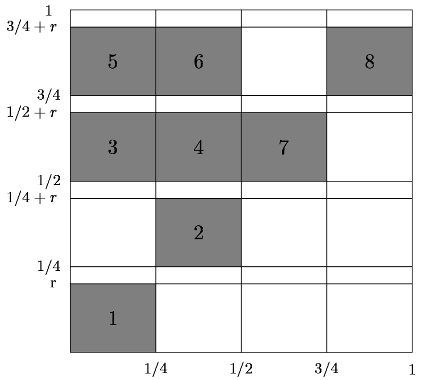

Figure 1: This figure shows A (the shaded area).

for some constants C_1, C_2 yet to be determined. Using (3) and (4), we get that $C_2 = D_0 = b = 2r$ and using this we get $(C_1 + 2r)r = D_1 = b^2 - |a|^2 = b^2 - b^2/4 = 3r^2$. Hence $C_1 = r$ and so

$$D_k = (kr + 2r)r^k \ \forall k \geq 0. \tag{8}$$

We will now construct a two-block-factor which we will show to be distributed according to $\mathbf{P}^f$. Let $\{Y_i\}_{i\in\mathbb{Z}}$ be i.i.d. uniform on $[0, 1]$. Define $h : [0, 1] \times [0, 1] \to [0, 1]$ by $h = I_A$ where

$$\begin{aligned} A &\\ &= [0, \frac{1}{4}] \times [0, r] \cup [0, \frac{1}{4}] \times [\frac{1}{2}, \frac{1}{2} + r] \cup [0, \frac{1}{4}] \times [\frac{3}{4}, \frac{3}{4} + r] \\ &\cup [\frac{1}{4}, \frac{1}{2}] \times [\frac{1}{4}, \frac{1}{4} + r] \cup [\frac{1}{4}, \frac{1}{2}] \times [\frac{1}{2}, \frac{1}{2} + r] \cup [\frac{1}{4}, \frac{1}{2}] \times [\frac{3}{4}, \frac{3}{4} + r] \\ &\cup [\frac{1}{2}, \frac{3}{4}] \times [\frac{1}{2}, \frac{1}{2} + r] \cup [\frac{3}{4}, 1] \times [\frac{3}{4}, \frac{3}{4} + r]. \end{aligned}$$

A is depicted as the grey area of figure (1). Observe that $r = |a| \leq 1/4$.

We will show that

$$\mathbf{P}[h(Y_i, Y_{i+1}) = \cdots = h(Y_{i+k}, Y_{i+k+1}) = 1] = D_k \ \forall k \geq 0.$$

Since $\{h(Y_i, Y_{i+1})\}_{i\in\mathbb{Z}}$ is one-dependent, this gives us $\{h(Y_i, Y_{i+1})\}_{i\in\mathbb{Z}} =_{\mathcal{D}} \mathbf{P}^f$ as desired. We first observe that the size of the shaded area of figure (1) is $8\frac{1}{4}r = 2r = b$, so that $\mathbf{P}[h(Y_i, Y_{i+1}) = 1] = D_0$.

If $h(Y_i, Y_{i+1}) = \cdots = h(Y_{i+k}, Y_{i+k+1}) = 1$, then (Y_{i+l}, Y_{i+l+1}) must be in one of the boxes marked 1 through 8 of figure (1) $\forall l \in \{0, \ldots, k\}$. If (Y_i, Y_{i+1}) is in the box marked 1, then $Y_{i+1} \in [0, r]$ and so (Y_{i+1}, Y_{i+2}) must be in one of the boxes marked 1, 3 or 5 because otherwise $(Y_{i+1}, Y_{i+2}) \notin A$. Similar "rules" apply if (Y_i, Y_{i+1}) is in one of the other seven boxes. We see that for any ω such that $h(Y_i(\omega), Y_{i+1}(\omega)) = \cdots = h(Y_{i+k}(\omega), Y_{i+k+1}(\omega)) = 1$ there is a natural sequence $j_0 j_1 \cdots j_k(\omega) \in \{1, \ldots, 8\}^{k+1}$ associated to it, where the value of j_l indicates that $(Y_{i+l}(\omega), Y_{i+l+1}(\omega))$ is in the box marked with that value. In any such sequence the number 1 can only be followed by either 1, 3 or 5, as described above, while the number 2 can only be followed by either 2, 4 or 6. Additionally any one of the numbers 3, 4 or 7 must be followed by a 7, while any one of 5, 6 or 8 must be followed by an 8.

We claim that the number of sequences $j_0 j_1 \cdots j_k$ described above is $(4k + 8)$. To see this, observe that every such sequence with $j_k \notin \{1, 2\}$ can be extended into a sequence $j_0 j_1 \cdots j_{k+1}$ in only one way, while if $j_k \in \{1, 2\}$ it can be extended in three ways. Observe also that there are only two sequences $j_0 j_1 \cdots j_k$ ending in 1 or 2.

The set of ω giving a specific sequence $j_0 j_1 \cdots j_k \in \{1, \ldots, 8\}^{k+1}$ has probability $(1/4)r^{k+1}$ since Y_i must be in an interval of length 1/4, while $Y_{i+1}, \ldots, Y_{i+k+1}$ all must be within intervals of length r. Hence the total probability of having $h(Y_i, Y_{i+1}) = \cdots = h(Y_{i+k}, Y_{i+k+1}) = 1$ is $(4k + 8)(1/4)r^{k+1} = (kr + 2r)r^k$. Comparing with equation (8) we see that

$$\mathbf{P}[h(Y_i, Y_{i+1}) = \cdots = h(Y_{i+k}, Y_{i+k+1}) = 1] = D_k$$

$\forall k \geq 0$ and we conclude that $\{h(Y_i, Y_{i+1})\}_{i\in\mathbb{Z}} =_{\mathcal{D}} \mathbf{P}^f$ and so this case is proved.

Case 2: It remains to consider $r_1 \neq r_2$. According to equations (6) and (7)

$$r_1 + r_2 = b,$$

and

$$r_1 r_2 = |a|^2.$$

In this case the solution to equation (2) is, again, from basic difference equation theory,

$$D_k = C_1 r_1^k + C_2 r_2^k \ \forall k \geq 0,$$

for some constants C_1, C_2 yet to be determined. Using this with equation (3) we get

$$C_1 + C_2 = D_0 = r_1 + r_2,$$

and using equation (4) we get

$$C_1 r_1 + C_2 r_2 = D_1 = b^2 - |a|^2 = (r_1 + r_2)^2 - r_1 r_2 = r_1^2 + r_1 r_2 + r_2^2.$$

A straightforward calculation yields

$$C_1 = \frac{r_1^2}{r_1 - r_2}$$

and

$$C_2 = -\frac{r_2^2}{r_1 - r_2}$$

and therefore for $k \geq 1$,

$$D_k = \frac{r_1^{k+2} - r_2^{k+2}}{r_1 - r_2} = \frac{r_1^{k+2} - r_1^{k+1}r_2 + r_2(r_1^{k+1} - r_2^{k+1})}{r_1 - r_2} = r_1^{k+1} + r_2 D_{k-1}. \tag{9}$$

Assume that $b \leq \frac{1}{2}$ so that $2(r_1+r_2) \leq 1$. We will now construct a two-block-factor which we will show to be distributed according to $\mathbf{P}^f$. Let $\{Y_i\}_{i\in\mathbb{Z}}$ be i.i.d. uniform on $[0,1]$ and again take $h : [0,1] \times [0,1] \to [0,1]$ to be the function $h = I_A$ where A is now

$$\begin{aligned} A &\\ &= [0, Cr_1] \times [0, r_1] \cup [0, Cr_1] \times [2Cr_1, 2Cr_1 + r_2] \\ &\cup [0, Cr_1] \times [2Cr_1 + Cr_2, 2Cr_1 + Cr_2 + r_2] \\ &\cup [Cr_1, 2Cr_1] \times [Cr_1, Cr_1 + r_1] \\ &\cup [Cr_1, 2Cr_1] \times [2Cr_1, 2Cr_1 + r_2] \\ &\cup [Cr_1, 2Cr_1] \times [2Cr_1 + Cr_2, 2Cr_1 + Cr_2 + r_2] \\ &\cup [2Cr_1, 2Cr_1 + Cr_2] \times [2Cr_1, 2Cr_1 + r_2] \\ &\cup [2Cr_1 + Cr_2, 1] \times [2Cr_1 + Cr_2, 2Cr_1 + Cr_2 + r_2], \end{aligned}$$

and $C = \frac{1}{2(r_1+r_2)} \geq 1$. A is the shaded area of figure (2).

Again we will show that

$$\mathbf{P}[h(Y_i, Y_{i+1}) = \cdots = h(Y_{i+k}, Y_{i+k+1}) = 1] = D_k \ \forall k \geq 0.$$

Since again $\{h(Y_i, Y_{i+1})\}_{i\in\mathbb{Z}}$ is one-dependent this gives us $\{h(Y_i, Y_{i+1})\}_{i\in\mathbb{Z}} =_{\mathcal{D}} \mathbf{P}^f$. We observe that the size of the shaded area of figure (2) equals

$$2Cr_1r_1 + 4Cr_1r_2 + 2Cr_2r_2 = 2C(r_1 + r_2)^2 = r_1 + r_2$$

by our choice of C, and so $\mathbf{P}[h(Y_i, Y_{i+1}) = 1] = D_0$.

For any ω such that $h(Y_i(\omega), Y_{i+1}(\omega)) = \cdots = h(Y_{i+k}(\omega), Y_{i+k+1}(\omega)) = 1$ there is a natural sequence $j_0 j_1 \cdots j_k(\omega) \in \{1, \ldots, 8\}^{k+1}$ associated to it as before. Let $\{\omega : j_0 j_1 \cdots j_k(\omega)\}$ denote the set of ω giving a specific sequence $j_0 j_1 \cdots j_k$, and for convenience we will write $\mathbf{P}[j_0 j_1 \cdots j_k]$ instead of $\mathbf{P}[\{\omega : j_0 j_1 \cdots j_k(\omega)\}]$. Assume that $j_{k-1} \in \{3, 4, 5, 6, 7, 8\}$, we get

$$\mathbf{P}[j_0 j_1 \cdots j_k] = r_2 \mathbf{P}[j_0 j_1 \cdots j_{k-1}]$$

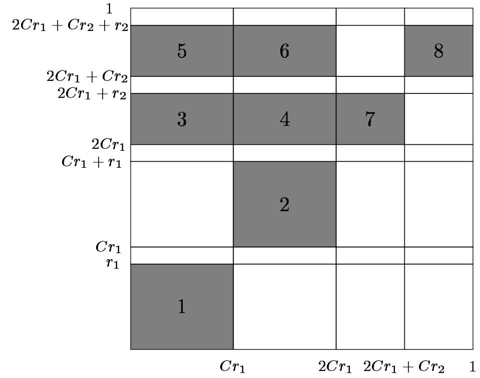

Figure 2: This figure shows A (the shaded area).

since j_k is either 7 or 8 (depending on the value of j_{k-1}). If instead $j_{k-1} = 1$ then j_k must be either 1, 3 or 5 and of course $j_l = 1$ for all $l \leq (k-1)$. Hence in this case

$$\mathbf{P}[j_0 j_1 \cdots j_k] = r_2 \mathbf{P}[j_0 j_1 \cdots j_{k-1}] = r_2 \mathbf{P}[\underbrace{11 \cdots 1}_{k}] = r_2 C r_1^{k+1}$$

if j_k is equal to 3 or 5 and

$$\mathbf{P}[j_0 j_1 \cdots j_k] = \mathbf{P}[\underbrace{11 \cdots 1}_{k+1}] = C r_1^{k+2}$$

if $j_k = 1$. Similarly if $j_{k-1} = 2$ then j_k must be either 2, 4 or 5 and of course $j_l = 2$ for all $l \leq (k-1)$. Hence

$$\mathbf{P}[j_0 j_1 \cdots j_k] = r_2 \mathbf{P}[j_0 j_1 \cdots j_{k-1}] = r_2 \mathbf{P}[\underbrace{22 \cdots 2}_{k}] = r_2 C r_1^{k+1}$$

if j_k is equal to 4 or 6 and

$$\mathbf{P}[j_0 j_1 \cdots j_k] = \mathbf{P}[\underbrace{22 \cdots 2}_{k+1}] = C r_1^{k+2}$$

if $j_k = 2$.

Let $\mathcal{A}_k$ be the set of all sequences $j_0 j_1 \cdots j_k$ corresponding to the event $h(Y_i, Y_{i+1}) = \cdots = h(Y_{i+k}, Y_{i+k+1}) = 1$. We have that

$$
\begin{aligned}
&\mathbf{P}[h(Y_i, Y_{i+1}) = \cdots = h(Y_{i+k}, Y_{i+k+1}) = 1] \\
&\quad = \sum_{\mathcal{A}_k} \mathbf{P}[j_0 j_1 \cdots j_k] \\
&\quad = \sum_{\substack{\mathcal{A}_k \\ j_{k-1} \notin \{1,2\}}} \mathbf{P}[j_0 j_1 \cdots j_k] + \sum_{\substack{\mathcal{A}_k \\ j_{k-1} \in \{1,2\}}} \mathbf{P}[j_0 j_1 \cdots j_k] \\
&\quad = r_2 \sum_{\substack{\mathcal{A}_{k-1} \\ j_{k-1} \notin \{1,2\}}} \mathbf{P}[j_0 j_1 \cdots j_{k-1}] + 4 r_2 C r_1^{k+1} + 2 C r_1^{k+2} \\
&\quad = r_2 \left(\sum_{\substack{\mathcal{A}_{k-1} \\ j_{k-1} \notin \{1,2\}}} \mathbf{P}[j_0 j_1 \cdots j_{k-1}] + \mathbf{P}[\underbrace{11 \cdots 1}_{k}] + \mathbf{P}[\underbrace{22 \cdots 2}_{k}] \right) \\
&\qquad + 2 r_2 C r_1^{k+1} + 2 C r_1^{k+2} \\
&\quad = r_2 \sum_{\mathcal{A}_{k-1}} \mathbf{P}[j_0 j_1 \cdots j_{k-1}] + 2 C r_1^{k+1} (r_1 + r_2) \\
&\quad = r_2 \mathbf{P}[h(Y_i, Y_{i+1}) = \cdots = h(Y_{i+k-1}, Y_{i+k}) = 1] + r_1^{k+1}.
\end{aligned}
$$

Comparing this to equation (9), and using $\mathbf{P}[h(Y_i, Y_{i+1}) = 1] = D_0$ we see that

$$\mathbf{P}[h(Y_i, Y_{i+1}) = \cdots = h(Y_{i+k}, Y_{i+k+1}) = 1] = D_k$$

for all $k \geq 0$, and so this case is also proved.

Finally the case $b > \frac{1}{2}$ remains. Take

$$g(x) = 1 - f(x) = 1 - b - 2|a| \cos(2\pi x - \phi) = 1 - b + 2|a| \cos(2\pi x - \phi'),$$

for some suitable choice of ϕ'. Since $1 - b \leq \frac{1}{2}$, it follows from above that we can construct a two-block-factor $\{h(Y_i, Y_{i+1})\}_{i \in \mathbb{Z}}$ such that

$$\{h(Y_i, Y_{i+1})\}_{i \in \mathbb{Z}} =_{\mathcal{D}} \mathbf{P}^g.$$

With $\tilde{h} = 1 - h$, we get a new two-block-factor $\{\tilde{h}(Y_i, Y_{i+1})\}_{i \in \mathbb{Z}}$ with ones and zeros flipped. Lemma 2.4 in [9] then shows that $\{\tilde{h}(Y_i, Y_{i+1})\}_{i \in \mathbb{Z}}$ has distribution $\mathbf{P}^{1-g}$, which in turn is $\mathbf{P}^f$.

QED

When trying to generalise theorem 1.3 to the case where f is a trigonometric polynomial of degree m, one must consider not only the values of

$$\mathbf{P}^f[\eta(1) = \cdots = \eta(1+k) = 1],$$

but also the values of

$$\mathbf{P}^f[\eta(e_1) = 1 \cdots = \eta(e_k) = 1]$$

where $e_i \in \mathbb{Z}\ \forall i \in \{1, \ldots, k\}$ but where e_i is not necessarily equal to $e_{i-1} + 1$. Analysing these new cylinder events adds to the complexity of the problem and therefore, in our opinion, the generalisation of theorem 1.3 (if indeed the generalisation is true) does not seem to be trivial.

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One-dependent trigonometric determinantal processes are two-block-factors

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Abstract

Given a trigonometric polynomial $f : [0,1] \to [0,1]$ of degree m, one can define a corresponding stationary process $\{X_i\}_{i\in\mathbb{Z}}$ via determinants of the Toeplitz matrix for f. We show that for $m = 1$ this process, which is trivially one-dependent, is a two-block-factor.

AMS subject classification: 60G10

Keywords and phrases: Determinantal processes, k-dependence,k-block-factors

Short title: Determinants and two-block-factors

1 Introduction

We will start by defining a family of probability measures $\mathbf{P}^f$ on the Borel sets of $\{0,1\}^{\mathbb{Z}}$ where $f : [0,1] \to [0,1]$ is a Lebesgue-measurable function (see [9]). For such an f, define the probability of the cylinder sets by

$$\begin{aligned}\mathbf{P}^f[\eta(e_1) = \cdots = \eta(e_k) = 1] &:= \mathbf{P}^f[\{\eta \in \{0,1\}^{\mathbb{Z}} : \eta(e_1) = \cdots = \eta(e_k) = 1\}] \\ &:= \det[\hat{f}(e_j - e_i)]_{1\leq,i,j\leq k},\end{aligned}$$

where $e_1, \ldots, e_k$ are distinct elements in $\mathbb{Z}$ and $k \geq 1$. Here $\hat{f}$ denotes the Fourier coefficients of f, defined by

$$\hat{f}(k) := \int_0^1 f(x)e^{-i2\pi kx}dx.$$

In [9] it is proven that $\mathbf{P}^f$ is indeed a probability measure. In fact they showed this for the more general case of $f : \mathbb{T}^d \to [0,1]$ where $\mathbb{T}^d := \mathbb{R}^d/\mathbb{Z}^d$;

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Paper II

Dynamical Stability of Percolation for Some Interacting Particle Systems and ϵ–Stability

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Abstract

In this paper we will investigate dynamic stability of percolation for the stochastic Ising model and the contact process. We also introduce the notion of downwards and upwards ϵ–stability which will be a key tool for our analysis.

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Keywords and phrases: percolation, stochastic Ising models, contact process,

Short title: Dynamical Stability for IPS

1 Introduction

Consider bond percolation on an infinite connected locally finite graph G, where for some $p \in [0,1]$ each edge (bond) of G is, independently of all others, open with probability p and closed with probability $1-p$. Write π_p for this product measure. The main questions in percolation theory (see [9]) deal with the possible existence of infinite connected components (clusters) in the random subgraph of G consisting of all sites and all open edges. Write $\mathcal{C}$ for the event that there exists such an infinite cluster. By Kolmogorov's 0-1 law, the probability of $\mathcal{C}$ is, for fixed G and p, either 0 or 1. Since $\pi_p(\mathcal{C})$ is nondecreasing in p, there exists a critical probability $p_c = p_c(G) \in [0,1]$ such that

$$\pi_p(\mathcal{C}) = \begin{cases} 0 & \text{for } p < p_c \\ 1 & \text{for } p > p_c. \end{cases}$$

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At $p = p_c$ we can have either $\pi_p(\mathcal{C}) = 0$ or $\pi_p(\mathcal{C}) = 1$, depending on G.

In [14], the authors initiated the study of dynamical percolation. In this model, with p fixed, the edges of G switch back and forth according to independent 2 state Markov chains where 0 switches to 1 at rate p and 1 switches to 0 at rate $1-p$. In this way, if we start with distribution π_p, the distribution of the system at all times is π_p. The general question studied in [14] was whether there could exist atypical times at which the percolation structure looks different than at a fixed time.

The point of the present paper is to initiate a study of dynamical percolation for *interacting* systems where the edges or sites flip at rates which depend on the neighbors. We point out that in a different direction such questions in continuous space but without interactions related to contiuum percolation have been studied in [2].

Ising model results. Precise definitions of the following Ising model measures and the stochastic Ising model will be given in Section 2. Fix a graph $G = (S, E)$. Let $\mu^{+,\beta,h}$ be the plus state for the Ising model with inverse temperature β and external field h on G; (this is a probability measure on $\{-1,1\}^S$). Let $\Psi^{+,\beta,h}$ denote the corresponding stochastic Ising model; (this is a stationary continuous time Markov chain on $\{-1,1\}^S$ with marginal distribution $\mu^{+,\beta,h}$). Let $\mathcal{C}^+$ ($\mathcal{C}^-$) denote the event that there exists an infinite cluster of sites with spin 1 (-1) and let $\mathcal{C}_t^+$ ($\mathcal{C}_t^-$) denote the event that there exists an infinite cluster of sites with spin 1 (-1) at time t. It is known that the family $\mu^{+,\beta,h}$, is, for fixed β, stochastically increasing (to be defined later) in h.

Theorem 1.1 *Consider a graph $G = (S, E)$ of bounded degree. Fix $\beta \geq 0$ and let $h_c = h_c(\beta)$ be defined by*

$$h_c := \inf\{h : \mu^{+,\beta,h}(\mathcal{C}^+) = 1\}.$$

Then for all $h > h_c$,

$$\Psi^{+,\beta,h}(\mathcal{C}_t^+ \text{ occurs for every } t) = 1$$

and for all $h < h_c$

$$\Psi^{+,\beta,h}(\exists t \geq 0 : \mathcal{C}_t^+ \text{ occurs }) = 0.$$

If we modify h_c to be instead

$$h'_c := \sup\{h : \mu^{+,\beta,h}(\mathcal{C}^-) = 1\},$$

the same two claims hold with $\mathcal{C}_t^+$ replaced by $\mathcal{C}_t^-$ and with $h < h'_c$ and $h > h'_c$ reversed.

This result tells us what happens in the subcritical and supercritical cases (with respect to h with β held fixed). It is the analogue of the easier Proposition 1.1 in [14] where it is proved that if $p < p_c$ ($p > p_c$), then, with probability 1, there is percolation at no time (at all times).

The following easy lemma gives us information about when h_c is nontrivial.

Lemma 1.2 *Assume the graph G has bounded degree and let β be arbitrary. Then $h_c > -\infty$. If $p_c(site) < 1$, then $h_c < \infty$. Similar results hold if h_c is replaced by h'_c.*

The following theorems, where we restrict to $\mathbb{Z}^d$, will only discuss the case $h = 0$. However, this will in many cases give us information about the "critical" case $(\beta, h_c(\beta))$ since in a number of situations, $h_c(\beta) = 0$. For example, this is true on all $\mathbb{Z}^d$ with $d \geq 2$ and β sufficiently large. We also mention that while the relationship between h_c and h'_c in Theorem 1.1 might in general be complicated, for $\mathbb{Z}^d$, one easily has that $h_c = -h'_c$; this follows from the known fact that the plus and minus states are the same when $h \neq 0$. When $h = 0$, we will abbreviate $\mu^{+,\beta,0}$ by $\mu^{+,\beta}$ and $\Psi^{+,\beta,0}$ by $\Psi^{+,\beta}$.

We first study percolation of -1's and then percolation of 1's. Let

$$\beta_p(2) := \inf\{\beta : \sum_{l=1}^{\infty} l3^{l-1}e^{-2\beta l} < \infty\} = \frac{\log 3}{2}.$$

We will refer to $\beta_p(2)$ as the critical inverse temperature of the Peierls regime for $\mathbb{Z}^2$. The choice of $\beta_p(2)$ might at first look quite arbitrary, but it is exactly what is needed to carry out a contour argument (known as Peierls argument) for $\mathbb{Z}^2$. For $d \geq 3$, there is a $\beta_p(d)$, such that for β larger than $\beta_p(d)$, a similar (although topologically more complicated) argument works for $\mathbb{Z}^d$. As a result of this "contour argument", it is well known and easy to show that for $\beta > \beta_p(d)$, we have that

$$\mu^{+,\beta}(\mathcal{C}^-) = 0. \tag{1}$$

Our next result is a dynamical version of equation (1) and we emphasize that this corresponds to the critical case as it is easy to check that for these β's, $h_c(\beta) = 0$.

Theorem 1.3 *For $\mathbb{Z}^d$ with $d \geq 2$ and $\beta > \beta_p(d)$*

$$\Psi^{+,\beta}(\exists t \geq 0 : \mathcal{C}_t^- \textit{ occurs}) = 0.$$

It is well known that $\beta_p(d) \geq \beta_c(d)$, the latter being the critical inverse temperature for the Ising model on $\mathbb{Z}^d$. For $d = 2$, Theorem 1.3 can be extended down to the critical inverse temperature $\beta_c(2)$. First, it is known (see [4]) that on $\mathbb{Z}^2$, for *all* β

$$\mu^{+,\beta}(\mathcal{C}^-) = 0. \tag{2}$$

Our dynamical analogue for $\beta > \beta_c$ is the following where we again point out that this is also a critical case as it is easy to check that for these β's, we also have $h_c(\beta) = 0$.

Theorem 1.4 *For the stochastic Ising model $\Psi^{+,\beta}$ on $\mathbb{Z}^2$ with parameter $\beta > \beta_c$,*

$$\Psi^{+,\beta}(\exists t \geq 0 : \mathcal{C}_t^- \textit{ occurs}) = 0.$$

Interestingly, equation (1) is not always true for $\beta > \beta_c(d)$ although, as stated, it is true for $\mathbb{Z}^2$ or β sufficiently large. In [1], it is shown that for $\mathbb{Z}^d$ with large d, there exists $\beta^+ > \beta_c(d)$ such that the probability in equation (1) is in fact 1 for all $\beta < \beta^+$. Moreover, they show that for these β, there exists $h > 0$ with

$$\mu^{+,\beta,h}(\mathcal{C}^-) = 1.$$

For such β's, this means that $h'_c > 0$ and hence it immediately follows from Theorem 1.1 that

$$\Psi^{+,\beta}(\mathcal{C}_t^- \text{ occurs for every } t) = 1.$$

Note that for these values of β, the case $h = 0$ is a *non*-critical case.

We next look at percolation of 1's under $\mu^{+,\beta}$. In the above results, we have not discussed the case of percolation of -1's when $\beta \leq \beta_c$. However, by symmetry, this is the same as studying percolation of 1's in this case and so we can now move over to the study of $\mathcal{C}^+$.

First, it is well known (see for example [3]) that for any graph of bounded degree when $\mu^{+,\beta,h} \neq \mu^{-,\beta,h}$, $\mu^{+,\beta,h}(\mathcal{C}^+) = 1$. In particular, for any graph G of bounded degree and for $\beta > \beta_c(G)$,

$$\mu^{+,\beta}(\mathcal{C}^+) = 1. \tag{3}$$

Our next result is a dynamical version of equation (3) for $\mathbb{Z}^d$. We mention that this result sometimes corresponds to a critical case and sometimes not. For $\beta > \beta_p(d)$ in $\mathbb{Z}^d$ or $\beta > \beta_c(2)$ in $\mathbb{Z}^2$, we have seen that $h_c = 0$ and so, in these cases, this next result covers the critical case. However, as pointed out, for d large and β just a little higher than β_c, the result in [1] gives us that $h_c < 0$ and hence in this case, this next theorem already follows from Theorem 1.1.

Theorem 1.5 *For the stochastic Ising model $\Psi^{+,\beta}$ on $\mathbb{Z}^d$ with parameter $\beta > \beta_c(d)$,*

$$\Psi^{+,\beta}(\mathcal{C}_t^+ \text{ occurs for every } t) = 1.$$

(The proof we give actually works for any graph of bounded degree). We mention that while $\beta > \beta_c$ is a sufficient condition for equation (3) to hold, it is certainly not necessary. For example, on $\mathbb{Z}^3$ we have that $\mu^{+,0}(\mathcal{C}^+) = 1$ since $\mu^{+,0} = \pi_{1/2}$ and the critical value for site percolation on $\mathbb{Z}^3$ is less than $1/2$. The reason β_c appears is the connection between the Ising model and the random cluster model; β_c corresponds to the critical value for percolation in the corresponding random cluster model.

We are now left with the case $\beta \leq \beta_c$. We will not be able to say too much since it is not known in all cases whether one has percolation at a fixed time. We first however have the following easy result for $d \geq 3$. We do not prove this result since it follows easily from the fact that the critical value for site percolation on $\mathbb{Z}^d$ is less than $1/2$ for $d \geq 3$ as this gives easily that $h_c(\beta) < 0$ for β sufficiently small and hence Theorem 1.1 is applicable.

Note that the case $\beta = 0$ follows from the result in [14] mentioned above.

Proposition 1.6 *For $d \geq 3$, there exists $\beta_1(d) > 0$ such that for all $\beta < \beta_1(d)$, we have that*

$$\Psi^{+,\beta}(\mathcal{C}_t^+ \text{ occurs for every } t) = 1.$$

Finally, due to work of Higuchi, we can determine what happens with $\beta < \beta_c$ for $\mathbb{Z}^2$. It is shown in [15] that for $\mathbb{Z}^2$, for all $\beta < \beta_c$, we have that $h_c(\beta) > 0$. The following result follows from this fact and Theorem 1.1.

Theorem 1.7 *For $d = 2$, for all $\beta < \beta_c$, we have that*

$$\Psi^{+,\beta}(\exists t \geq 0 : \mathcal{C}_t^+ \text{ occurs }) = 0.$$

We note that even though it is known that for $\mathbb{Z}^2$, $\mu^{+,\beta_c}(\mathcal{C}^+) = 0$, we cannot conclude as above that

$$\Psi^{+,\beta_c}(\exists t \geq 0 : \mathcal{C}_t^+ \text{ occurs }) = 0$$

since it is known (see [16]) that $h_c(\beta_c) = 0$. We finally mention that it is also interestingly known (see again [16]) that for $\beta < \beta_c$, $\mu^{+,\beta,h_c(\beta)}(\mathcal{C}^+) = 0$.

Contact process results. Precise definitions of the following items will be given in Section 2. Fix a graph $G = (S, E)$. Consider the contact process on a graph $G = (S, E)$ with parameter λ. Denote by μ_λ the stochastically largest invariant measure, the so-called "upper invariant measure"; (this is a probability measure on $\{0,1\}^S$). Let Ψ^λ denote the corresponding stationary contact process; (this is a stationary continuous time Markov chain on $\{0,1\}^S$ with marginal distribution μ_λ). If $0 < \lambda_1 < \lambda_2$, it is well known that μ_{λ_1} is stochastically smaller than μ_{λ_2}, denoted by

$$\mu_{\lambda_1} \preceq \mu_{\lambda_2}$$

(see Section 2 for this precise definition).

Theorem 1.8 *Consider the contact process Ψ^λ on a graph $G = (S, E)$, with initial and stationary distribution μ_λ. Let λ_p be defined by*

$$\lambda_p := \inf\{\lambda : \mu_\lambda(\mathcal{C}^+) = 1\}.$$

We have that for all $\lambda > \lambda_p$,

$$\Psi^\lambda(\mathcal{C}_t^+ \text{ occurs for every } t) = 1.$$

In order for this theorem to be nonvacuous, we need to know that $\lambda_p < \infty$ for at least some graph. While this seems to be an open question for $\mathbb{Z}^d$, the fact that there exists λ such that $\mu_\lambda(\mathcal{C}^+) > 0$ for $\mathbb{T}^d$ with $d \geq 2$ follows from [11]. Here $\mathbb{T}^d$ is the unique infinite connected graph without circuits and in which each site has exactly $d+1$ neighbours; $\mathbb{T}^d$ is commonly known as the homogenous tree of order d. Combined with a 0-1 law which we develop, Proposition 4.2, we obtain that $\lambda_p < \infty$.

When we prove Theorem 1.1, we will in fact, prove a more general theorem which holds for a large class of systems. However, this proof will only work for models satisfying the so-called FKG lattice condition (which we call "monotone" in this paper.) We now point out the important fact that for $\lambda > 2$, in 1 dimension, the upper invariant measure for the contact process, while having positive correlations, is *not* monotone (see [19]). These terms are defined in Section 2. One would also believe it is never monotone whenever the measure is not δ_0. Hence Theorem 1.8 does not follow from the generalization of Theorem 1.1 which will come later.

$\epsilon-$stability. We now introduce the concepts of upwards and downwards $\epsilon-$stability. While we mainly introduce these as a technical tool to be used in our main results, we believe that they are of independent interest. Let S be a countable set. Take any probability measure μ on $\{-1,1\}^S$ and let X be a $\{-1,1\}^S$ valued random variable with distribution μ. Let Z be a $\{-1,1\}^S$ valued random variable with distribution $\pi_{1-\epsilon}$ and be independent of X. Define $X^{(-,\epsilon)}$ by letting $X^{(-,\epsilon)}(s) = \min(X(s), Z(s))$ for every $s \in S$, and let $\mu^{(-,\epsilon)}$ denote the distribution of $X^{(-,\epsilon)}$. In a similar way, define $X^{(+,\epsilon)}$ by letting $X^{(+,\epsilon)}(s) = \max(X(s), Z(s))$ for every $s \in S$, where Z has distribution π_ϵ and is independent of X. Denote the distribution of $X^{(+,\epsilon)}$ by $\mu^{(+,\epsilon)}$.

Definition 1.9 *Let (μ_1, μ_2) be a pair of probability measures on $\{-1,1\}^S$, where S is a countable set. Assume that*

$$\mu_1 \preceq \mu_2.$$

If there exists an $\epsilon > 0$ such that

$$\mu_1 \preceq \mu_2^{(-,\epsilon)},$$

then we say that this pair of measures is downwards ϵ-stable. If the pair is downwards ϵ-stable for some $\epsilon > 0$, we say that the pair is downwards stable. Analogously, if there exists an $\epsilon > 0$ such that

$$\mu_1^{(+,\epsilon)} \preceq \mu_2,$$

then we say that the pair (μ_1, μ_2) is upwards ϵ-stable and that it is upwards stable if the pair is upwards ϵ-stable for some $\epsilon > 0$.

For probability measures on $\{0,1\}^S$, we have identical definitions.

The relevance of downward (or upward) $\epsilon-$stability to our dynamical percolation analysis will be explained in Section 5. In Section 3, we will prove $\epsilon-$stability for general monotone systems which will eventually lead to a proof of Theorem 1.1 (and its generalization). We now state a similar and key result for the contact process.

Theorem 1.10 *Let G be a graph of bounded degree, $0 < \lambda_1 < \lambda_2$ and μ_{λ_1}, μ_{λ_2} be the upper invariant measures for the contact process on $\{0,1\}^S$ with parameters λ_1 and λ_2 respectively. Then $(\mu_{\lambda_1}, \mu_{\lambda_2})$ is downwards stable.*

Remark: We do not know whether $(\mu_{\lambda_1}, \mu_{\lambda_2})$ is upwards stable.

We finally mention how the above questions that we study are related to classical Markov process theory. Let $(\Omega, \mathcal{F}, \mathbb{P})$ be the probability space where a stationary Markov process $\{X_t\}_{t\geq 0}$ taking values in some state space $\mathcal{S}$ is defined. Letting μ denote the distribution of X_t (for any t), consider an event $\mathcal{A} \subseteq \mathcal{S}$ with $\mu(\mathcal{A}) = 1$. Let $\mathcal{A}_t$ be the event that $\mathcal{A}$ occurs at time t.

We say that $\mathcal{A}$ is a *dynamicaly stable* event if $\mathbb{P}(\mathcal{A}_t\ \forall t \geq 0) = 1$. In Markov process terminology, this is equivalent to saying that $\mathcal{A}^c$ has capacity zero. All the questions in this paper deal with showing, for various models and parameters, that the event that there exists an infinite connected component of sites which are all open is dynamically stable.

The rest of this paper is divided into 9 sections. In Section 2, we will give all necessary preliminaries and precise definitions of our models. Sections 3 and 4 will deal with the concept of ϵ–stability. In Section 3, we develop what will be needed to prove Theorem 1.1 and its generalization. In Section 4, we will prove Theorem 1.10 (which is the key to Theorem 1.8) as well as give a proof that $\lambda_p < \infty$ for trees. In Section 5, we prove 2 elementary lemmas which relate the notion of ϵ–stability to dynamical questions. In the remaining sections, proofs of the remaining results are given. We note that the proof of Theorem 1.4 will use the proof of Theorem 1.5 and hence will come afterwards.

We end with one bit of notation. If μ is a probability measure on some set U, we write $X \sim \mu$ to mean that X is a random variable taking values in U with distribution μ.

2 Models and definitions

Before presenting the interacting particle systems discussed in this paper we will present some definitions and results related to stochastic domination. Let S be any countable set. For $\sigma, \sigma' \in \{-1,1\}^S$ we write $\sigma \preceq \sigma'$ if $\sigma(s) \leq \sigma'(s)$ for every $s \in S$. An increasing function f is a function $f : \{-1,1\}^S \to \mathbb{R}$ such that $f(\sigma) \leq f(\sigma')$ for all $\sigma \preceq \sigma'$. For two probability measures μ, μ' on $\{-1,1\}^S$ we write $\mu \preceq \mu'$ if for every continuous increasing function f we have that $\mu(f) \leq \mu'(f)$. When $\{-1,1\}^S$ is replaced by $\{0,1\}^S$, we have identical definitions.

A very useful result is the so called Holley's inequality, which appeared first in [17]. We will present a variant of the theorem by Holley; it is not the most general but is sufficient for our purposes.

Theorem 2.1 *Take S to be a finite set. Let μ, μ' be probability measures on $\{-1,1\}^S$ which assign positive probability to all configurations $\sigma \in \{-1,1\}^S$. Assume that*

$$\mu(\sigma(s) = 1 | \sigma(S \setminus s) = \xi) \leq \mu'(\sigma(s) = 1 | \sigma(S \setminus s) = \xi')$$

for every $s \in S$ and $\xi \preceq \xi'$ where $\xi, \xi' \in \{-1,1\}^{S\setminus s}$. Then $\mu \preceq \mu'$.

Proof. See [8] or [12] for a proof.

QED

Two properties of probability measures which are often encountered within the field of interacting particle systems are the monotonicity property and the property of positive correlations presented below.

Definition 2.2 *Take S to be a finite set. A measure μ on $\{-1,1\}^S$ which assigns positive probability to every configuration $\sigma \in \{-1,1\}^S$ is called monotone if for every $s \in S$ and $\xi \preceq \xi'$ where $\xi, \xi' \in \{-1,1\}^{S\setminus s}$,*

$$\mu(\sigma(s) = 1 | \sigma(S \setminus s) = \xi) \leq \mu(\sigma(s) = 1 | \sigma(S \setminus s) = \xi').$$

We point out immediately, that it is known that this is equivalent to the so-called FKG lattice conidition.

Definition 2.3 *A measure μ on $\{-1,1\}^S$ is said to have positive correlations if for all bounded increasing functions $f, g : \{-1,1\}^S \to \mathbb{R}$, we have*

$$\mu(fg) \geq \mu(f)\mu(g).$$

The following important result is sometimes known as the FKG inequality (see [6]).

Theorem 2.4 *Take S to be a finite set. Let μ be a monotone probability measure on $\{-1,1\}^S$ which assigns positive probability to every configuration. Then μ has positive correlations.*

Proof. See [8] for a proof.

QED

In this section and also later in this paper we will talk about convergence of probability measures. Convergence will always mean weak convergence, where $\{0,1\}^S$ is given the product topology.

2.1 The Ising model

Take $G = (S, E)$, where $|S| < \infty$. The Ising measure $\mu^{\beta,h}$ on $\{-1,1\}^S$ at inverse temperature $\beta \geq 0$, external field h and with free boundary conditions is defined as follows. For any configuration $\sigma \in \{-1,1\}^S$, let

$$H^{\beta,h}(\sigma) = -\beta \sum_{\substack{\{s,t\}\in E \\ s,t \in S}} \sigma(s)\sigma(t) - h \sum_{s \in S} \sigma(s). \tag{4}$$

$H^{\beta,h}$ is called the Hamiltonian. Define $\mu^{\beta,h}$ by assigning the probability

$$\mu^{\beta,h}(\sigma) = \frac{e^{-H^{\beta,h}(\sigma)}}{Z} \tag{5}$$

to any configuration $\sigma \in \{-1,1\}^S$ where Z is a normalization constant. Of course Z depends on the graph and the values β and h, but this will not be important for us and therefore not reflected in the notation.

Take $S_n := \Lambda_{n+1} = \{-n-1, \ldots, n+1\}^d$ and E_n to be the set of all nearest neighbor pairs of S_n. Given a configuration ξ on $\{-1,1\}^{\mathbb{Z}^d \setminus \Lambda_n}$, let, for $\sigma \in \{-1,1\}^S$,

$$H_n^{\xi,\beta,h}(\sigma) = -\beta \sum_{\substack{\{s,t\}\in E \\ s,t \in \Lambda_n}} \sigma(s)\sigma(t) - h \sum_{s \in \Lambda_n} \sigma(s) - \beta \sum_{\substack{\{s,t\}\in E \\ s \in \Lambda_n \\ t \in \Lambda_{n+1} \setminus \Lambda_n}} \sigma(s)\xi(t) \tag{6}$$

be our Hamiltonian. Here ξ is called a boundary condition. Again we define a probability measure using equation (5) but using the Hamiltonian of equation (6) instead. This Ising measure will be denoted by $\mu_n^{\xi,\beta,h}$. The cases $\xi \equiv 1$ and $\xi \equiv -1$ are especially important and the corresponding Ising measures are denoted by $\mu_n^{+,\beta,h}$ and $\mu_n^{-,\beta,h}$ respectively. We view $\mu_n^{+,\beta,h}$ ($\mu_n^{-,\beta,h}$) as a probability measure on $\{-1,1\}^{\mathbb{Z}^d}$ by letting, with probability 1, the configuration be identically 1 (-1) outside Λ_n. It is known (see [18], page 189) that the sequences $\{\mu_n^{+,\beta,h}\}$ and $\{\mu_n^{-,\beta,h}\}$ converge as n tends to infinity; these limits are denoted by $\mu^{+,\beta,h}$ and $\mu^{-,\beta,h}$.

The same kind of construction can be carried out on any infinite connected locally finite graph $G = (S, E)$. One defines a Hamiltonian analogous to the one in equation (6) but with Λ_n replaced by any $\Lambda \subseteq S$ where $|\Lambda| < \infty$. With $\xi \equiv 1$ or $\xi \equiv -1$, one then considers the corresponding limits of Ising measures as $\Lambda \uparrow S$, the limit being independent of the particular choice of sequence. See for instance [8] for how this is carried out in detail. Fix $h = 0$ and abbreviate $\mu^{+,\beta,0}$ and $\mu^{-,\beta,0}$ by $\mu^{+,\beta}$ and $\mu^{-,\beta}$. It is well known ([7], [8]) that for any graph, there exists $\beta_c \in [0, \infty]$ such that for $0 \leq \beta < \beta_c$, we have that $\mu^{-,\beta} = \mu^{+,\beta}$ (and there is then a unique so called Gibbs state) and for $\beta > \beta_c$, $\mu^{-,\beta} \neq \mu^{+,\beta}$. For $\mathbb{Z}^d$ with $d \geq 2$, and many other graphs, $\beta_c \in (0, \infty)$. β_c is sometimes referred to as the critical inverse temperature for phase transition in the Ising model. Furthermore in [13], the author shows that if G is of bounded degree, the condition $\beta_c < \infty$ is equivalent to the condition $p_c < 1$, where p_c is the critical parameter value for site percolation on G. It is easy to see that for any graph of bounded degree $p_c > 0$ (see the proof of theorem 1.10 of [9]). This in turn implies via the connection between the random cluster model and the Ising model, described below, that $\beta_c > 0$ for any graph of bounded degree.

2.2 Spin Systems.

A configuration $\sigma \in \{-1,1\}^S$ can be seen as particles on a discrete set S having one of two different "spins" represented by -1 and 1. To this we will add a stochastic dynamics, and assume that the system is described by "flip rate intensities" which we will denote by $\{C(s,\sigma)\}_{s\in S,\ \sigma\in\{-1,1\}^S}$. $C(s,\sigma)$ represents the rate at which site s changes its state when the present configuration is σ. Of course $C(s,\sigma)\geq 0\ \forall s\in S,\sigma\in\{-1,1\}^S$, and we assume that the interaction is nearest neighbour in the sense that the flip rate of a site $s\in S$ only depends on the configuration σ at sites t with $\{s,t\}\in E$. We will limit ourselves to only allow one site flip in every transition and we will only consider flip rate intensities such that

$$\sup_{s,\sigma} C(s,\sigma)<\infty.$$

In many cases we will consider translation invariant systems and then this last condition will hold trivially. Furthermore we will always assume the trivial condition that for every $s\in S$

$$\sup_{\sigma:\sigma(s)=0} C(s,\sigma(s))>0,\quad \sup_{\sigma:\sigma(s)=1} C(s,\sigma(s))>0.$$

We will call such an object a spin system (see [18] or [5] for results concerning general spin systems). Given such rates, one can obtain a Markov process Ψ on $\{-1,1\}^S$ governed by these flip rates; see [18]. Such a Markov process with a specified initial distribution μ on $\{-1,1\}^S$ will be denoted by Ψ^μ. Given a Markov process, μ will be called an invariant distribution for the process if the projections of Ψ^μ onto $\{-1,1\}^S$ at any fixed time $t\geq 0$ is μ. In this case, Ψ^μ will be a stationary Markov process on $\{-1,1\}^S$ all of whose marginal distributions are μ. Of course the state space $\{-1,1\}^S$ can be exchanged for either $\{0,1\}^S$ or $\{0,1\}^E$.

Sometimes we will work with two different sets of flip rates $\{C_1(s,\sigma)\}_{s\in S,\ \sigma\in\{-1,1\}^S}$ and $\{C_2(s,\sigma)\}_{s\in S,\ \sigma\in\{-1,1\}^S}$, governing two Markov processes Ψ_1 and Ψ_2 respectively. We will write $C_1\preceq C_2$ if the following conditions are satisfied;

$$C_2(s,\sigma_2)\geq C_1(s,\sigma_1)\ \forall s\in S,\ \forall \sigma_1\preceq\sigma_2 \text{ s.t. } \sigma_1(s)=\sigma_2(s)=0, \tag{7}$$

and

$$C_1(s,\sigma_1)\geq C_2(s,\sigma_2)\ \forall s\in S,\ \forall \sigma_1\preceq\sigma_2 \text{ s.t. } \sigma_1(s)=\sigma_2(s)=1. \tag{8}$$

The point of $C_1\preceq C_2$ is that a coupling of Ψ_1 and Ψ_2 will then exist for which $\{(\eta,\delta):\eta(s)\leq\delta(s)\forall s\in S\}$ is closed for the process; see [18].

2.3 Stochastic Ising models

We will now briefly discuss stochastic Ising models. We will omit most details; for an extensive discussion and analysis see again [18]. Consider

$G_n=(S_n,E_n)$ defined in the subsection 2.1. Given β and h, it is possible to construct flip rates C_n on $\{-1,1\}^{S_n}$ for which $\mu_n^{+,\beta,h}$ is reversible and invariant. We denote by $\Psi_n^{+,\beta,h}$ the corresponding stationary Markov process with initial distribution $\mu_n^{+,\beta,h}$. One possible choice of flip rate intensities are that for every $s\in\Lambda_n$ and $\sigma\in\{-1,1\}^S$,

$$C_n(s,\sigma)=\exp[-\beta(\sum_{\substack{\{t,s\}\in E\\ t\in\Lambda_n}}\sigma(s)\sigma(t)+\sum_{\substack{\{t,s\}\in E\\ s\in\Lambda_n\\ t\in\Lambda_{n+1}\setminus\Lambda_n}}\sigma(s))-h\sum_{s\in\Lambda_n}\sigma(s)].$$

s's in $\Lambda_{n+1}\setminus\Lambda_n$ are kept fixed at 1. Observe that if $s\in\Lambda_{n-1}$, the second sum is over an empty set. A straightforward calculation gives

$$C_n(s,\sigma)\mu_n^{+,\beta,h}(\sigma)=C_n(s,\sigma_s)\mu_n^{+,\beta,h}(\sigma_s), \tag{9}$$

where

$$\sigma_s(t)=\begin{cases}\sigma(t) & \text{if } t\neq s\\ -\sigma(t) & \text{if } t=s.\end{cases}$$

This shows that indeed $\mu_n^{+,\beta,h}$ is reversible and invariant for C_n. Any spin rates satisfying equation (9) is called a stochastic Ising model (on our finite set). One can show that there exists a limiting distribution $\Psi^{+,\beta,h}$ of $\Psi_n^{+,\beta,h}$ when n tends to infinity; see [18], Theorem 2.2, page 17 and Theorem 2.7, page 139. Furthermore $\Psi^{+,\beta,h}$ is a stationary Markov process on $\{-1,1\}^{\mathbb{Z}^d}$ with marginal distribution $\mu^{+,\beta,h}$ governed by flip rate intensities

$$C(s,\sigma)=\exp(-\beta\sum_{\{t,s\}\in E}\sigma(s)\sigma(t)-h\sum_{s\in\mathbb{Z}^d}\sigma(s)); \tag{10}$$

see [18] Theorem 2.7 page 139. It is also possible to construct $\Psi^{+,\beta}$ directly on $\{-1,1\}^{\mathbb{Z}^d}$ without going through the limiting procedure. Furthermore there are several possible choices of flip rate intensities that can be used to construct a stationary and reversible Markov process on $\{-1,1\}^{\mathbb{Z}^d}$ with marginal distribution $\mu^{+,\beta}$. In [18], a stochastic Ising model is defined to be any spin system with flip rate intensities $\{C(s,\sigma)\}_{s\in S,\sigma\in\{-1,1\}^S}$ satisfying that for each $s\in S$

$$C(s,\sigma)\exp(\beta\sum_{\{t,s\}\in E}\sigma(s)\sigma(t)+h\sum_{s\in S}\sigma(s)) \tag{11}$$

is independent of $\sigma(s)$. Therefore, when we refer to a stochastic Ising model $\Psi^{+,\beta}$ with marginal distribution $\mu^{+,\beta}$ we will have this definition in mind. It is particularly easy to see that equation (11) (or the condition of detailed balance as it is often referred to) is satisfied for the flip rate intensities of equation (10) but there are many other examples. It is known that the set of so called Gibbs states are exactly the same as the class of reversible measures with respect to the flip rates satisfying equation (11); see [18] page 190-196. Note also that for S finite the condition of equation (11) is equivalent to that of equation (9).

While we defined above stochastic Ising models on $\{-1,1\}^{\mathbb{Z}^d}$, this construction can be done on more general graphs (see [18]).

2.4 The random cluster model

Unlike all other models in this paper, the random cluster model deals with configurations on the edges E of a graph $G=(S,E)$. We will review the definition of the regular random cluster measure on general finite graphs and the "wired" random cluster measure on $\Lambda_n \subseteq \mathbb{Z}^d$. We will also recall the limiting measures and in the next subsection the connection between the random cluster model and the Ising model. In doing so we will follow the outlines of [8] and [12] closely.

Take a finite graph $G=(S,E)$. Define the random cluster measure $\nu_G^{p,q}$ on $\{0,1\}^E$ with parameters $p\in[0,1]$ and $q>0$ as the probability measure which assigns to the configuration $\eta\in\{0,1\}^E$ the probability

$$\nu_G^{p,q}(\eta)=\frac{q^{k(\eta)}}{Z}\prod_{e\in E}p^{\eta(e)}(1-p)^{1-\eta(e)}. \tag{12}$$

Here Z is again a normalization constant and $k(\eta)$ is the number of connected components of η. From now on we will always take $q=2$ and therefore we will suppress q in the notation.

Take $G_n=(S_n,E_n)$, where $S_n=\Lambda_{n+1}\subseteq\mathbb{Z}^d$ and E_n is the set of all nearest neighbour pairs of Λ_{n+1}. Write ν_n^p for $\nu_{G_n}^p$, and define

$$\tilde{\nu}_n^p(\cdot)=\nu_n^p(\cdot|\text{ all edges of } E_n \text{ with both end sites in } \Lambda_{n+1}\setminus\Lambda_n \text{ are present}). \tag{13}$$

This is the so called "wired" random cluster measure. It is called "wired" since all edges of the boundary are present. It is immediate from the defining equations (12) and (13) that for $e\in E_n$ and any $\xi\in\{0,1\}^{E_n\setminus e}$

$$\tilde{\nu}_n^p(\eta(e)=1|\eta(E_n\setminus e)=\xi)=\begin{cases} p, & \text{if the endpoints of } e \text{ are} \\ & \text{connected in } \xi, \\ \frac{p}{2-p} & \text{otherwise.}\end{cases} \tag{14}$$

One can show (see [8] or [12]) that when n tends to infinity, the measures $\{\tilde{\nu}_n^p\}_{n\in\mathbb{N}^+}$ converge to a measure $\tilde{\nu}^p$. Furthermore, the construction of $\tilde{\nu}_n^p$ on $\{0,1\}^{E_n}$ can be done on any finite graph by connecting all sites of the

boundary of the graph with each other. As a consequence, we can also define random cluster measures on more general graphs than $\mathbb{Z}^d$, see for example [10].

2.5 The random cluster model and the Ising model

Take $G_n=(S_n,E_n)$ as in Section 2.4. As in [12], let $\mathbf{P}_n^p$ be the probability measure on $\{-1,1,\}^{S_n}\times\{0,1\}^{E_n}$ defined in the following way.

1. Assign each site of $\Lambda_{n+1}\setminus\Lambda_n$ and every edge with both endpoints in $\Lambda_{n+1}\setminus\Lambda_n$ the value 1.

2. Assign each site of Λ_n the value 1 or -1 with equal probability, assign each edge with not more than one endpoint in $\Lambda_{n+1}\setminus\Lambda_n$ the value 0 or 1 with probabilities $1-p$ and p respectively. Do this independently for all sites and edges.

3. Condition on the event that no two sites with different spins have an open edge connecting them.

One can then check that $\mathbf{P}_n^p(\sigma,\{0,1\}^{E_n})=\mu_n^{+,\beta}(\sigma)$ with $\beta=-\log(1-p)/2$, and that $\mathbf{P}_n^p(\{-1,1\}^{S_n},\eta)=\tilde{\nu}_n^p(\eta)$. The same kind of construction can be carried out on any finite graph $G=(S,E)$.

2.6 The contact process

Consider a graph $G=(S,E)$ of bounded degree. In the contact process the state space is $\{0,1\}^S$. Let $\lambda>0$, and define the flip rate intensities to be

$$C(s,\sigma)=\begin{cases}1 & \text{if } \sigma(s)=1\\ \lambda\sum_{(s',s)\in E}\sigma(s') & \text{if } \sigma(s)=0.\end{cases}$$

If we let the initial distribution be $\sigma\equiv1$, the distribution of this process at time t which we will denote by $\delta_1 T_\lambda(t)$ is known to converge as t tends to infinity. This is simply because it is a so called "attractive" process and $\sigma\equiv1$ is the maximal state and $\{\delta_1T_\lambda(t)\}$ is stochastically decreasing; see [18] page 265. This limiting distribution will be referred to as the upper invariant measure for the contact process with parameter λ and will be denoted by μ_λ. We then let Ψ^λ denote the stationary Markov process on $\{0,1\}^S$ with initial (and invariant) distribution μ_λ.

3 ϵ-stability for monotone measures

In this section, we prove stability results for classes of monotone measures. The finite case is covered by Lemma 3.2, while the countable case is discussed in Proposition 3.3.

For any $|S| < \infty$, $s \in S$, $\xi \in \{0,1\}^{S\setminus s}$ and probability measure μ on $\{0,1\}^S$ write $\mu^{(*,\epsilon)}(i|\xi)$ for $\mu^{(*,\epsilon)}(\sigma(s) = i|\sigma(S \setminus s) = \xi)$, $\mu^{(*,\epsilon)}(i \cap \xi)$ for $\mu^{(*,\epsilon)}(\{\sigma(s) = i\} \cap \{\sigma(S \setminus s) = \xi\})$ and $\mu^{(*,\epsilon)}(\xi)$ for $\mu^{(*,\epsilon)}(\sigma(S\setminus s) = \xi)$. Here, $*$ can represent either $+$ or $-$ and $i \in \{0,1\}$. Note that s is suppressed in the notation and so should be understood from context.

We begin with an easy lemma whose proof is left to the reader. The idea is that if the configuration outside of s is ξ under $\mu^{(-,\epsilon)}$, it must have been at least as large under μ "before flipping some 1's to 0's"; then use monotonicity.

Lemma 3.1 *Assume that μ is a monotone probability measure on $\{0,1\}^S$ where $|S| < \infty$. Take $s \in S$ and let $\xi \in \{0,1\}^{S\setminus s}$. Then, for any $\epsilon > 0$, we have that*

$$\mu^{(-,\epsilon)}(1|\xi) \geq (1-\epsilon)\mu(1|\xi)$$

and that

$$\mu^{(+,\epsilon)}(0|\xi) \geq (1-\epsilon)\mu(0|\xi).$$

Lemma 3.2 *Let μ_1, μ_2 be monotone probability measures on $\{0,1\}^S$ where $|S| < \infty$. Assume that*

$$A := \inf_{\substack{s\in S \\ \xi\in\{0,1\}^{S\setminus s}}} [\mu_2(\sigma(s) = 1|\sigma(S\setminus s) \equiv \xi) - \mu_1(\sigma(s) = 1|\sigma(S\setminus s) \equiv \xi)] > 0.$$

Then for any choice of $\epsilon > 0$, such that

$$A > \frac{1}{1-\epsilon} - 1,$$

we have

$$\mu_1 \preceq \mu_2^{(-,\epsilon)},$$

and

$$\mu_1^{(+,\epsilon)} \preceq \mu_2.$$

Hence (μ_1, μ_2) is both downwards and upwards stable.

Proof. We prove only the first statement; the second is proved in the same way. Monotonicity of μ_2, Lemma 3.1, the definition of A and our choice of

ϵ give us that for any $s \in S$ and $\xi \in \{0,1\}^{S\setminus s}$

$$\begin{aligned}
&\mu_2^{(-,\epsilon)}(1|\xi) \\
&\quad \geq (1-\epsilon)\mu_2(1|\xi) \geq (1-\epsilon)(A + \mu_1(1|\xi)) \\
&\quad \geq (1-\epsilon)\frac{\mu_1(1|\xi)}{1-\epsilon} = \mu_1(1|\xi).
\end{aligned}$$

Since μ_1 is monotone, we get

$$\mu_1(1|\tilde{\xi}) \leq \mu_1(1|\xi) \ \forall \tilde{\xi} \preceq \xi,$$

and therefore that

$$\mu_1(1|\tilde{\xi}) \leq \mu_2^{(-,\epsilon)}(1|\xi) \ \forall \tilde{\xi} \preceq \xi.$$

The proof is completed by the use of Holley's inequality, Theorem 2.1.

QED

Remark: Observe that if we could show that $\mu_2^{(-,\epsilon)}$ is monotone, the assumption that μ_1 is monotone would not be needed for the result $\mu_1 \preceq \mu_2^{(-,\epsilon)}$.

Proposition 3.3 *Let S be any finite or countable set and consider $(S_n)_{n\in\mathbb{N}^+}$, a collection of sets such that $|S_n| < \infty \ \forall n \in \mathbb{N}^+$ and $S_n \uparrow S$. Let $(\mu_{1,n})_{n\in\mathbb{N}^+}$, $(\mu_{2,n})_{n\in\mathbb{N}^+}$, be two collections of measures, where $\mu_{1,n}, \mu_{2,n}$ are measures on $\{0,1\}^{S_n}$ for every $n \in \mathbb{N}^+$. Furthermore, assume that all these measures are monotone, that $\mu_{1,n} \to \mu_1$ and that $\mu_{2,n} \to \mu_2$. Set*

$$A_n := \inf_{\substack{s\in S_n \\ \xi\in\{0,1\}^{S_n\setminus s}}} [\mu_{2,n}(\sigma(s) = 1|\sigma(S\setminus s) \equiv \xi) - \mu_{1,n}(\sigma(s) = 1|\sigma(S\setminus s) \equiv \xi)].$$

If

$$\inf_{n\in\mathbb{N}^+} A_n > 0,$$

then (μ_1, μ_2) is both upwards and downwards stable.

Proof. Take $\epsilon > 0$ such that

$$\inf_{n\in\mathbb{N}^+} A_n > \frac{1}{1-\epsilon} - 1.$$

With this choice of ϵ, Lemma 3.2 says that $(\mu_{1,n}, \mu_{2,n})$ are both downwards and upwards $\epsilon-$stable. Since $\mu_{1,n} \to \mu_1$ and $\mu_{2,n} \to \mu_2$ we easily get that $\mu_{2,n}^{(-,\epsilon)} \to \mu_2^{(-,\epsilon)}$ and $\mu_{1,n}^{(+,\epsilon)} \to \mu_1^{(+,\epsilon)}$. Furthermore since the relations

$$\mu_{1,n} \preceq \mu_{2,n}^{(-,\epsilon)}$$

and

$$\mu_{1,n}^{(+,\epsilon)} \preceq \mu_{2,n}$$

are easily seen to be preserved under weak limits, we get that

$$\mu_1 \preceq \mu_2^{(-,\epsilon)} \text{ and } \mu_1^{(+,\epsilon)} \preceq \mu_2.$$

QED

4 ϵ–stability for the contact process and a 0-1 Law

The conditions in our next proposition might seem overly technical; however, these represent the essential features of the contact process (after a small suitable time rescaling) and therefore we feel it is instructive to highlight these features. In Proposition 4.1 and Lemmas 5.1, 5.2 and 8.1 we will use the so-called graphical representation to define our processes; see for instance [18] page 172.

Proposition 4.1 *Let μ_1 and μ_2 be two measures defined on $\{0,1\}^S$, where S is a countable set. Assume that $\mu_1 \preceq \mu_2$ and that there exists two stationary Markov processes Ψ_1 and Ψ_2, governed by flip rate intensities $\{C_1(s,\sigma_1)\}_{s\in S,\sigma_1\in\{0,1\}^S}$ and $\{C_2(s,\sigma_2)\}_{s\in S,\sigma_2\in\{0,1\}^S}$ respectively, and with marginal distributions μ_1 and μ_2. Assume that $C_1 \preceq C_2$ (conditions (7) and (8) of the introduction). Consider the following conditions;*

1. *There exists an $\epsilon_1 > 0$ such that*
$$C_2(s,\sigma_2) - C_1(s,\sigma_1) \geq \epsilon_1 \quad (15)$$
$$\forall s \in S,\ \forall \sigma_2 \succeq \sigma_1 \text{ s.t. } \sigma_2(s) = 0 \text{ and } C_1(s,\sigma_1) \neq 0.$$

2. *There exists an $\epsilon_2 > 0$ such that*
$$C_1(s,\sigma_1) - C_2(s,\sigma_2) \geq \epsilon_2 \quad (16)$$
$$\forall s \in S,\ \forall \sigma_2 \succeq \sigma_1 \text{ s.t. } \sigma_1(s) = 1 \text{ and } C_2(s,\sigma_2) \neq 0.$$

3. *There exists an $\epsilon_3 > 0$ such that*
$$C_1(s,\sigma_1) \geq \epsilon_3\ \forall s \in S,\ \forall \sigma_1 \text{ s.t. } \sigma_1(s) = 1, \quad (17)$$

4. *There exists an $\epsilon_4 > 0$ such that*
$$C_2(s,\sigma_2) \geq \epsilon_4\ \forall s \in S,\ \forall \sigma_2 \text{ s.t. } \sigma_2(s) = 0. \quad (18)$$

If conditions 1 2 and 3 are satisfied, then (μ_1,μ_2) is downwards stable.
If conditions 1 2 and 4 are satisfied, then (μ_1,μ_2) is upwards stable.

Proof. We will prove the first statement, the second follows by symmetry. Define

$$\lambda := \sup_{s,\sigma_2:\sigma_2(s)=0} C_2(s,\sigma_2) + \sup_{s,\sigma_1:\sigma_1(s)=1} C_1(s,\sigma_1).$$

Our aim is to construct a coupling of the processes $\{X_{1,t}\}_{t\geq 0} \sim \Psi_1$ and $\{X_{2,t}\}_{t\geq 0} \sim \Psi_2$ such that $\mu_1 \sim X_{1,t} \preceq X_{2,t} \sim \mu_2\ \forall t \geq 0$ in such a way that we prove the proposition. Before presenting the actual coupling we will discuss the idea behind it. For every site $s \in S$ associate an independent Poisson process with parameter λ. Next, let $\{U_{s,k}\}_{s\in S,k\geq 1}$ and $\{U'_{s,k}\}_{s\in S,k\geq 1}$ be independent uniform $[0,1]$ random variables also independent of the Poisson processes. If τ is an arrival time for the Poisson process at site s, we write $U_{s,\tau}$ for $U_{s,k}$ where k is such that τ is the kth arrival of the Poisson process at site s. Now, let τ be an arrival time for the Poisson process associated to a site s. For $i \in \{1,2\}$, let X_{i,τ^-} and X_{i,τ^+} denote the configurations before and after the arrival. We will let the outcome of $U_{s,\tau}$ decide what happens with the $\{X_{2,t}\}_{t\geq 0}$ process at time $t = \tau$, and then we will let $U'_{s,\tau}$ together with $U_{s,\tau}$ decide what happens with the $\{X_{1,t}\}_{t\geq 0}$ process at time $t = \tau$. As we will see, we will do this so that $X_{1,t} \preceq X_{2,t}$ for all $t \geq 0$. Furthermore, we will do this in such a way that there exists an $\epsilon \in (0,1)$ such that if $U'_{s,\tau} \geq 1-\epsilon$, then $X_{1,\tau^+}(s) = 0$ regardless of the outcome of $U_{s,\tau}$. Consider now the process $\{X_t^\epsilon\}_{t\geq 0}$ we get by taking $X_0^\epsilon(s) = 1$ for every $s \in S$ and letting $\{X_t^\epsilon(s)\}_{t\geq 0}$ be updated at every arrival time τ for the Poisson process associated to s, and updated in such a way that $X^\epsilon_{\tau^+}(s) = 0$ if $U'_{s,\tau} \geq 1-\epsilon$, and $X^\epsilon_{\tau^+}(s) = 1$ if $U'_{s,\tau} < 1-\epsilon$. Of course the distribution of X_t^ϵ will converge to $\pi_{1-\epsilon}$. Observe that whenever $X_t^\epsilon(s) = 0$ we have that $X_{1,t}(s) = 0$. Therefore we can conclude that

$$X_{1,t} \preceq \min(X_{2,t}, X_t^\epsilon)\ \forall t \geq 0. \quad (19)$$

Furthermore since the process $\{X_t^\epsilon\}_{t\geq 0}$ does not depend on any $U_{s,\tau}$ we have that $X_t^\epsilon(s)$ is conditionally independent of $X_{2,t}$ if there has been an arrival for the Poisson process associated to s before time t. Let s_i, $i \in \{1,\dots,n\}$ be distinct sites in S and let $\mathcal{A}_t$ be the event that all Poisson processes associated to s_1 through s_n have had an arrival by time t. Of course $\mathbb{P}(\mathcal{A}_t) = (1-e^{-\lambda t})^n$ and so we get that

$$\begin{aligned}&\mathbb{P}(X_{2,t}X_t^\epsilon(s_1) = \cdots = X_{2,t}X_t^\epsilon(s_n) = 1)\\ &\quad = \mathbb{P}(X_{2,t}X_t^\epsilon(s_1) = \cdots = X_{2,t}X_t^\epsilon(s_n) = 1|\mathcal{A}_t)\mathbb{P}(\mathcal{A}_t)\end{aligned}$$

$$
\begin{aligned}
&+\mathbb{P}(X_{2,t}X_t^\epsilon(s_1) = \cdots = X_{2,t}X_t^\epsilon(s_n) = 1|\mathcal{A}_t^c)\mathbb{P}(\mathcal{A}_t^c) \\
&= \mathbb{P}(\{X_{2,t}(s_1) = \cdots = X_{2,t}(s_n) = 1\} \cap \mathcal{A}_t)(1-\epsilon)^n \\
&\quad +\mathbb{P}(X_{2,t}X_t^\epsilon(s_1) = \cdots = X_{2,t}X_t^\epsilon(s_n) = 1|\mathcal{A}_t^c)\mathbb{P}(\mathcal{A}_t^c) \\
&\geq (\mathbb{P}(X_{2,t}(s_1) = \cdots = X_{2,t}(s_n) = 1) - \mathbb{P}(\mathcal{A}_t^c))(1-\epsilon)^n \\
&\quad +\mathbb{P}(X_{2,t}X_t^\epsilon(s_1) = \cdots = X_{2,t}X_t^\epsilon(s_n) = 1|\mathcal{A}_t^c)\mathbb{P}(\mathcal{A}_t^c) \\
&= \mu_2^{(-,\epsilon)}(\sigma(s_1) = \cdots = \sigma(s_n) = 1) \\
&\quad +\mathbb{P}(\mathcal{A}_t^c)(\mathbb{P}(X_{2,t}X_t^\epsilon(s_1) = \cdots = X_{2,t}X_t^\epsilon(s_n) = 1|\mathcal{A}_t^c) - (1-\epsilon)^n) \\
&\overset{t\to\infty}{\longrightarrow} \mu_2^{(-,\epsilon)}(\sigma(s_1) = \cdots = \sigma(s_n) = 1).
\end{aligned}
$$

In addition

$$
\begin{aligned}
&\mathbb{P}(X_{2,t}(s_1) = \cdots = X_{2,t}(s_n) = 1 \cap \mathcal{A}_t)(1-\epsilon)^n \\
&\leq \mathbb{P}(X_{2,t}(s_1) = \cdots = X_{2,t}(s_n) = 1)(1-\epsilon)^n \\
&= \mu_2^{(-,\epsilon)}(\sigma(s_1) = \cdots = \sigma(s_n) = 1).
\end{aligned}
$$

Hence, by inclusion exclusion, we have that the distribution of $\min(X_{2,t}, X_t^\epsilon)$ approaches $\mu_2^{(-,\epsilon)}$ as t tends to infinity. So by first taking the limit in equation (19), we get that $\mu_1 \preceq \mu_2^{(-,\epsilon)}$, as desired.

Now to the construction. Take $X_{1,0} \sim \mu_1$, $X_{2,0} \sim \mu_2$, such that $X_{1,0} \preceq X_{2,0}$. Let τ be an arrival time for the Poisson process associated to s. Take $U_{s,\tau}$ and $U'_{s,\tau}$. The following transition rules apply:

$$
\begin{array}{lll}
X_{2,\tau-} & X_{2,\tau+} & \text{if} \\
0 & 1 & U_{s,\tau} \leq \frac{C_2(s,X_{2,\tau-})}{\lambda} \\
1 & 0 & U_{s,\tau} \geq \frac{\lambda - C_2(s,X_{2,\tau-})}{\lambda}.
\end{array}
$$

It is easy to check that the process $\{X_{2,t}\}_{t\geq 0}$ thus constructed will have the right flip-rate intensities. The construction of $\{X_{1,t}\}_{t\geq 0}$ is slightly more complicated. If $C_2(s, X_{2,\tau-}) = 0$ and $X_{2,\tau-}(s) = 0$ then it follows from equation (7) that $C_1(s, X_{1,\tau-}) = 0$, and in that case we interpret $\frac{C_1(s,X_{1,\tau-})}{C_2(s,X_{2,\tau-})}$ as 0. Observe that $C_2(s, X_{2,\tau-})$ can be 0 when $X_{2,\tau-}(s) = 1$ but it will not cause any problems. With these observations in mind, these are the

transition rules we apply:

$$
\begin{array}{ccc}
(X_{1,\tau-}, X_{2,\tau-}) & (X_{1,\tau+}, X_{2,\tau+}) & \text{if} \\
(0,0) & (1,1) & U_{s,\tau} \leq \frac{C_2(s,X_{2,\tau-})}{\lambda} \text{ and } U'_{s,\tau} \leq \frac{C_1(s,X_{1,\tau-})}{C_2(s,X_{2,\tau-})} \\
(0,0) & (0,1) & U_{s,\tau} \leq \frac{C_2(s,X_{2,\tau-})}{\lambda} \text{ and } U'_{s,\tau} > \frac{C_1(s,X_{1,\tau-})}{C_2(s,X_{2,\tau-})} \\
(0,0) & (0,0) & \text{otherwise} \\
(0,1) & (0,0) & U_{s,\tau} \geq \frac{\lambda - C_2(s,X_{2,\tau-})}{\lambda} \\
(0,1) & (1,1) & U_{s,\tau} < \frac{\sup_{s,\sigma_2:\sigma_2(s)=0} C_2(s,\sigma_2)}{\lambda} \text{ and } \\
 & & U'_{s,\tau} \leq \frac{C_1(s,X_{1,\tau-})}{\sup_{s,\sigma_2:\sigma_2(s)=0} C_2(s,\sigma_2)} \\
(0,1) & (0,1) & \text{otherwise} \\
(1,1) & (0,0) & U_{s,\tau} \geq \frac{\lambda - C_2(s,X_{2,\tau-})}{\lambda} \\
(1,1) & (0,1) & U_{s,\tau} < \frac{\lambda - C_2(s,X_{2,\tau-})}{\lambda} \text{ and } \\
 & & U'_{s,\tau} \geq \frac{\lambda - C_1(s,X_{1,\tau-})}{\lambda - C_2(s,X_{2,\tau-})} \\
(1,1) & (1,1) & \text{otherwise}
\end{array}
$$

It is not difficult to check that all flip rate intensities are correct and that $X_{1,t} \preceq X_{2,t}$ for all $t \geq 0$. [Observe that by the definition of λ the events $\left\{U_{s,\tau} \geq \frac{\lambda - C_2(s,X_{2,\tau-})}{\lambda}\right\}$ and $\left\{U_{s,\tau} < \frac{\sup_{s,\sigma_2:\sigma_2(s)=0} C_2(s,\sigma_2)}{\lambda}\right\}$ are disjoint when $(X_{1,\tau-}, X_{2,\tau-}) = (0,1)$.]

We now want to show that there exists an $\epsilon > 0$ so that $U'_{s,\tau} \geq 1 - \epsilon$, implies that $X_{1,\tau+}(s) = 0$. Note that if $(X_{1,\tau-}, X_{2,\tau-}) = (0,0)$ and $C_1(s, X_{1,\tau-}) > 0$ ($\Rightarrow C_2(s, X_{2,\tau-}) > 0$) then

$$
\frac{C_1(s, X_{1,\tau-})}{C_2(s, X_{2,\tau-})} \leq \frac{C_2(s, X_{2,\tau-}) - \epsilon_1}{C_2(s, X_{2,\tau-})} \leq 1 - \frac{\epsilon_1}{\sup_{s,\sigma_2:\sigma_2(s)=0} C_2(s,\sigma_2)} < 1
$$

and if $(X_{1,\tau-}, X_{2,\tau-}) = (0,0)$ and $C_1(s, X_{1,\tau-}) = 0$ then

$$
\frac{C_1(s, X_{1,\tau-})}{C_2(s, X_{2,\tau-})} = 0.
$$

Furthermore if $(X_{1,\tau-}, X_{2,\tau-}) = (0,1)$ and $C_1(s, X_{1,\tau-}) > 0$, then

$$
\frac{C_1(s, X_{1,\tau-})}{\sup_{s,\sigma_2:\sigma_2(s)=0} C_2(s,\sigma_2)} \leq 1 - \frac{\epsilon_1}{\sup_{s,\sigma_2:\sigma_2(s)=0} C_2(s,\sigma_2)} < 1
$$

while again if $(X_{1,\tau^-}, X_{2,\tau^-}) = (0,1)$ and $C_1(s, X_{1,\tau^-}) = 0$, then the 0 never changes to a 1. Finally if $(X_{1,\tau^-}, X_{2,\tau^-}) = (1,1)$ and $C_2(s, X_{2,\tau^-}) > 0$ ($\Rightarrow C_1(s, X_{1,\tau^-}) > 0$), then

$$\frac{\lambda - C_1(s, X_{1,\tau^-})}{\lambda - C_2(s, X_{2,\tau^-})} \leq \frac{\lambda - C_2(s, X_{2,\tau^-}) - \epsilon_2}{\lambda - C_2(s, X_{2,\tau^-})} \leq 1 - \frac{\epsilon_2}{\lambda - C_2(s, X_{2,\tau^-})} \leq 1 - \frac{\epsilon_2}{\lambda},$$

and if $(X_{1,\tau^-}, X_{2,\tau^-}) = (1,1)$ and $C_2(s, X_{2,\tau^-}) = 0$,

$$\frac{\lambda - C_1(s, X_{1,\tau^-})}{\lambda - C_2(s, X_{2,\tau^-})} \leq \frac{\lambda - \epsilon_3}{\lambda} = 1 - \frac{\epsilon_3}{\lambda} < 1.$$

Therefore, whenever

$$U'_{s,\tau} \geq \max\left(1 - \frac{\epsilon_1}{\sup_{s,\sigma_2:\sigma_2(s)=0} C_2(s,\sigma_2)}, 1 - \frac{\epsilon_2}{\lambda}, 1 - \frac{\epsilon_3}{\lambda}\right),$$

we have that $X_{1,\tau^+}(s) = 0$ regardless of the outcome of $U_{s,\tau}$. Therefore (μ_1, μ_2) is downwards ϵ−stable where

$$\begin{aligned}\epsilon &:= 1 - \max\left(1 - \frac{\epsilon_1}{\sup_{s,\sigma_2:\sigma_2(s)=0} C_2(s,\sigma_2)}, 1 - \frac{\epsilon_2}{\lambda}, 1 - \frac{\epsilon_3}{\lambda}\right) \\ &= \min\left(\frac{\epsilon_1}{\sup_{s,\sigma_2:\sigma_2(s)=0} C_2(s,\sigma_2)}, \frac{\epsilon_2}{\lambda}, \frac{\epsilon_3}{\lambda}\right).\end{aligned}$$

QED

Proof of Theorem 1.10. Take $\delta > 0$ such that $\lambda_1(1+\delta) < \lambda_2$ and consider the process $\{X_t\}_{t\geq 0}$ constructed in the following way. Take $X_0 \equiv 1$ and let the process evolve with flip rate intensities

$$C_1(s,\sigma) = \begin{cases} 1+\delta & \text{if } \sigma(s) = 1 \\ \lambda_1(1+\delta)\sum_{s'\sim s}\sigma(s') & \text{if } \sigma(s) = 0. \end{cases} \tag{20}$$

Denote the limiting distribution of X_t as t tends to infinity by $\mu_{1+\delta,\lambda_1(1+\delta)}$. It is easy to see that this process is just a time-scaling of the contact process constructed in Section 2.6 with parameter λ_1. Recall that that process had limiting distribution μ_{λ_1}, the upper invariant measure for the contact process. Thus we have $\mu_{\lambda_1} = \mu_{1+\delta,\lambda_1(1+\delta)}$. By Proposition 4.1 with C_1 as above and C_2 as in Section 2.6 with parameter λ_2, there exists an $\epsilon > 0$ such that

$$\mu_{1+\delta,\lambda_1(1+\delta)} \preceq \mu_{\lambda_2}^{(-,\epsilon)}.$$

Hence $(\mu_{\lambda_1}, \mu_{\lambda_2})$ is downwards stable.

QED

The following is a 0-1 law for the upper invariant measure for the contact process.

Proposition 4.2 *Let $\mathcal{A} \subseteq \{0,1\}^{\mathbb{T}^d}$ where $d \geq 2$ be a set which is invariant under all graph automorphisms on $\mathbb{T}^d$. Then, for $\lambda > 0$, we have that*

$$\mu_\lambda(\mathcal{A}) \in \{0,1\}.$$

Proof. Let $\epsilon > 0$. By elementary measure theory, there exists a cylinder event $\mathcal{B}$ depending on finitely many coordinates such that

$$\mu_\lambda(\mathcal{A}\Delta\mathcal{B}) \leq \epsilon. \tag{21}$$

Let *supp*$\mathcal{B}$ denote the finite number of coordinates with respect to which $\mathcal{B}$ is measurable. Letting $\{T_\lambda(t)\}_{t\geq 0}$ denote the Markov semigroup for the contact process with parameter λ, we have that $\delta_1 T_\lambda(t) \to \mu_\lambda$ and also that $\mu_\lambda \preceq \delta_1 T_\lambda(t)$ for every $t \geq 0$. Choose t so that for all (equivalent some) sites s

$$\delta_1 T_\lambda(t)(\eta(s) = 1) \leq \mu_\lambda(\eta(s) = 1) + \frac{\epsilon}{2|supp\mathcal{B}|}.$$

It follows easily that if m is any coupling of $\delta_1 T_\lambda(t)$ and μ_λ which is concentrated on $\{(\eta,\delta) : \eta \preceq \delta\}$, then for any finite set S of sites

$$m(\{(\eta,\delta) : \eta(s) \neq \delta(s) \text{ occurs for some } s \in S) \leq \frac{|S|\epsilon}{2|supp\mathcal{B}|}.$$

In particular, if E is any event depending on at most $2|supp\mathcal{B}|$ sites, then

$$|\delta_1 T_\lambda(t)(E) - \mu_\lambda(E)| \leq \epsilon. \tag{22}$$

For this fixed t, Theorem 4.6 page 35 of [18] shows that there exists an automorphism $\gamma \in AUT(\mathbb{T}^d)$ such that

$$|\delta_1 T_\lambda(t)(\mathcal{B}\cap\gamma\mathcal{B}) - \delta_1 T_\lambda(t)(\mathcal{B})\delta_1 T_\lambda(t)(\gamma\mathcal{B})| \leq \epsilon. \tag{23}$$

Furthermore, since μ_λ is invariant under automorphisms (21) implies that

$$\mu_\lambda(\gamma\mathcal{A}\Delta\gamma\mathcal{B}) \leq \epsilon,$$

and since $\mathcal{A} = \gamma\mathcal{A}$, we have

$$\mu_\lambda(\mathcal{A}\Delta\gamma\mathcal{B}) \leq \epsilon.$$

It follows that

$$\mu_\lambda(\mathcal{B}\Delta\gamma\mathcal{B}) \le \mu_\lambda(\mathcal{A}\Delta\gamma\mathcal{B}) + \mu_\lambda(\mathcal{A}\Delta\mathcal{B}) \le 2\epsilon.$$

Next, Equation (22) implies that

$$|\delta_1 T_\lambda(t)(\mathcal{B}\Delta\gamma\mathcal{B}) - \mu_\lambda(\mathcal{B}\Delta\gamma\mathcal{B})| \le \epsilon,$$

and so

$$\delta_1 T_\lambda(t)(\mathcal{B}\Delta\gamma\mathcal{B}) \le 3\epsilon. \tag{24}$$

We get that

$$\begin{aligned} |\mu_\lambda(\mathcal{A}) - \mu_\lambda(\mathcal{A})^2| &= |\mu_\lambda(\mathcal{A}) - \mu_\lambda(\mathcal{A})\mu_\lambda(\gamma\mathcal{A})| \\ &\le |\mu_\lambda(\mathcal{B}) - \mu_\lambda(\mathcal{B})\mu_\lambda(\gamma\mathcal{B})| + 4\epsilon \\ &\le |\delta_1 T_\lambda(t)(\mathcal{B}) - \delta_1 T_\lambda(t)(\mathcal{B})\delta_1 T_\lambda(t)(\gamma\mathcal{B})| + 8\epsilon \\ &\le |\delta_1 T_\lambda(t)(\mathcal{B}) - \delta_1 T_\lambda(t)(\mathcal{B}\cap\gamma\mathcal{B})| + 9\epsilon \\ &\le \delta_1 T_\lambda(t)(\mathcal{B}\Delta\gamma\mathcal{B}) + 9\epsilon \le 12\epsilon. \end{aligned}$$

Where we used (21), (22) and (23) for the three first inequalities and (24) in the last. Since $\epsilon > 0$, was choosen arbitrarily we get that

$$\mu_\lambda(\mathcal{A}) = \mu_\lambda(\mathcal{A})^2$$

and so $\mu_\lambda(\mathcal{A}) \in \{0,1\}$.

QED

Remarks: The above proof works for any transitive and even quasi-transitive graph. For the case of $\mathbb{Z}^d$, this was proved in Proposition 2.16 page 143 of [18]. It is mentioned there that while $\delta_1 T_\lambda(t)$ is ergodic for each t, one cannot conclude immediately the ergodicity of μ_λ because the class of ergodic processes is not weakly closed. We point out however that there is another important notion of convergence given by the $\bar{d}$−metric (see [22] page 89 for definition) on stationary processes. Convergence in this metric is stronger than weak convergence and weaker than convergence in the total variation norm. It is also known that the ergodic processes are $\bar{d}$−closed and that weak convergence together with stochastic ordering implies $\bar{d}$−convergence. In this way, one can conclude ergodicity of μ_λ using the $\bar{d}$−metric giving an alternative proof of Proposition 2.16 of [18]. In fact, the proof of Proposition 4.2 is essentially based on this idea. However, because of the open question listed below, it is not so easy to formulate the $\bar{d}$−metric for tree indexed processes and so we choose a more hands on approach. Observe that the crucial property of $\bar{d}$−convergence which is

essentially used in the above proof is that for each fixed k, one has uniform convergence of the measures (in say the total variation norm) over all sets which depend on at most k points. (The point is that the k points can lie anywhere and hence this is much stronger than weak convergence).

Open Question related to defining the $\bar{d}$−metric for tree indexed processes: Assume that μ and ν are two automorphism invariant measures on $\{0,1\}^{\mathbb{T}^d}$ such that $\mu \preceq \nu$. Does there exist a $\mathbb{T}^d$−invariant coupling (X,Y) with $X \sim \mu$, $Y \sim \nu$ and $X \preceq Y$?

Proposition 4.3 *On $\mathbb{T}^d$, $d \ge 2$ there exists a λ_p such that for all $\lambda > \lambda_p$*

$$\mu_\lambda(\mathcal{C}^+) = 1.$$

Proof. By Theorem 1.33(c), page 275 in [18], for sufficiently large λ, $\mu_\lambda(\eta(s) = 1) \ge 2/3$. By [11] we have that if $\mu_\lambda(\eta(s) = 1) \ge 2/3$, then

$$\mu_\lambda(\mathcal{C}^+) > 0.$$

Finally, Proposition 4.2 then implies that

$$\mu_\lambda(\mathcal{C}^+) = 1.$$

QED

5 Relationship between ϵ−stability and dynamics

In the general setup we have a family of stationary Markov processes parametrised by one or two parameters, e.g. the contact processes Ψ^λ (λ is here the only parameter) or a stochastic Ising model $\Psi^{+,\beta,h}$ (β and h being the parameters). Many of the proofs in this paper will involve comparing the marginal distributions of these Markov processes for two different values of one of the involved parameters. Let p be the parameter and let $p_1 < p_2$. Assume that the marginal distributions are μ_{p_1} and μ_{p_2} respectively and that $\mu_{p_1} \preceq \mu_{p_2}$. Lemmas 5.1 and 5.2 shows that there is a close connection between showing that (μ_{p_1}, μ_{p_2}) is downwards ϵ−stable and that the infimum of the second process over a short time interval is stochastically larger than the first process.

Let Ψ^μ be a stationary Markov process on $\{0,1\}^S$ with marginal distribution μ and let $\{X_t\}_{t\ge 0} \sim \Psi^\mu$. For $\delta > 0$ and $s \in S$ define

$$X_{\inf,\delta}(s) := \inf_{t\in[0,\delta]} X_t(s),$$

and denote the distribution of $X_{\inf,\delta}$ by $\mu_{\inf,\delta}$. Similarly define

$$X_{\sup,\delta}(s) := \sup_{t\in[0,\delta]} X_t(s),$$

and denote the distribution of $X_{\sup,\delta}$ by $\mu_{\sup,\delta}$.

Lemma 5.1 *Take S to be the sites of a bounded degree graph. Let $\{C(s,\sigma)\}_{s\in S,\ \sigma\in\{-1,1\}^S}$ be the flip rate intensities for a stationary Markov process Ψ^μ on $\{-1,1\}^S$ with marginal distribution μ. Let*

$$\lambda := \sup_{(s,\sigma)} C(s,\sigma).$$

For any $\tau > 0$, if we set $\epsilon := 1 - e^{-\lambda\tau}$, we have that

$$\mu^{(-,\epsilon)} \preceq \mu_{\inf,\tau}.$$

Similarly, we get that

$$\mu_{\sup,\tau} \preceq \mu^{(+,\epsilon)}.$$

Proof. We will prove the first statement, the second statement follows by symmetry. Take $\tau > 0$. For every $s \in S$ associate an independent Poisson process with parameter λ. Define $\{(X_t^1, X_t^2)\}_{t\geq 0}$ in the following way. Let $X_0^1 \equiv X_0^2 \sim \mu$, and take t' to be an arrival time for the Poisson process of a site s. For $i \in \{1,2\}$, let $X^i_{t',-}$ and $X^i_{t',+}$ denote the configurations before and after the arrival. We let $X^1_{t',+}(s) \neq X^1_{t',-}(s)$ with probability $C(s, X^1_{t',-})/\lambda$ and we let $X^2_{t',+}(s) = 0$ and finally we let $X^1_{t',+}(S\setminus s) \equiv X^1_{t',-}(S\setminus s)$, $X^2_{t',+}(S\setminus s) \equiv X^2_{t',-}(S\setminus s)$. Do this independently for all arrival times for all Poisson processes of all sites. Observe that once $X_t^2(s)$ is 0, it remains so. Note also that $X_\tau^1 \sim \mu$, $X_\tau^2 \sim \mu^{(-,\epsilon)}$. Furthermore if $X_t^1(s) = 0$ for some $t \in [0,\tau]$ the construction guarantees that $X_\tau^2(s) = 0$ and therefore $X_\tau^2 \preceq X^1_{\inf,\tau} \sim \mu_{\inf,\tau}$.

QED

Lemma 5.2 *Take S to be the sites of any bounded degree graph. Let $\{C(s,\sigma)\}_{s\in S,\ \sigma\in\{-1,1\}^S}$ be the flip rate intensities of a stationary Markov process Ψ^μ on $\{-1,1\}^S$ with marginal distribution μ. Define*

$$\lambda_1 := \inf_{s,\sigma:\sigma(s)=1} C(s,\sigma).$$

If $\lambda_1 > 0$ then for any $0<\epsilon<1$, if we set $\tau := -\frac{\log(1-\epsilon)}{\lambda_1}$, we have that

$$\mu_{\inf,\tau} \preceq \mu^{(-,\epsilon)}.$$

Similarly, defining $\lambda_2 := \inf_{s,\sigma:\sigma(s)=0} C(s,\sigma)$, if $\lambda_2 > 0$, then for any $0<\epsilon<1$, if we set $\tau := -\frac{\log(1-\epsilon)}{\lambda_2}$, we have that

$$\mu^{(+,\epsilon)} \preceq \mu_{\sup,\tau}.$$

Proof. We will prove the first statement, the second statement follows by symmetry. For every $s \in S$ associate an independent Poisson process with parameter $\lambda := \sup_{(s,\sigma)} C(s,\sigma)$. Next, let $\{U_{s,k}\}_{s\in S, k\geq 1}$ be independent uniform $[0,1]$ random variables also independent of the Poisson processes. If t' is an arrival time for the Poisson process at site s, we write $U_{s,t'}$ for $U_{s,k}$ where k is such that t' is the kth arrival of the Poisson process at site s. Define $\{(X_t^1, X_t^2)\}_{t\geq 0}$ in the following way. Let $X_0^1 \equiv X_0^2 \sim \mu$, and take t' to be an arrival time for the Poisson process of a site s. We let $X^1_{t',+}(s) \neq X^1_{t',-}(s)$ if $U_{s,t'} \leq C(s, X^1_{t',-})/\lambda$. Furthermore we let $X^2_{t',+}(s) = 0$ if $U_{s,t'} \leq \lambda_1/\lambda$ or $X^2_{t',-}(s) = 0$, and finally we let $X^1_{t',+}(S\setminus s) \equiv X^1_{t',-}(S\setminus s)$, $X^2_{t',+}(S\setminus s) \equiv X^2_{t',-}(S\setminus s)$. Do this independently for all arrival times for all Poisson processes of all sites. Clearly $X_\tau^1 \sim \mu$ and $X_\tau^2 \sim \mu^{(-,\epsilon)}$. Furthermore, if $X_\tau^2(s) = 0$, then either $X_0^1(s) = X_0^2(s) = 0$ or there exists a $t \in [0,\tau]$ such that t is an arrival time for the Poisson process associated to s and $U_{s,t} \leq \lambda_1/\lambda$. Since $\lambda_1 \leq C(s, X^1_{t-})$ if $X^1_{t-}(s) = 1$, we get that either $X^1_{t+}(s)$ or $X^1_{t-}(s)$ is 0 and therefore $X^1_{\inf,\tau} \preceq X^2_\tau$.

QED

To illustrate why the condition $\lambda_1 > 0$ of Lemma 5.2 is needed, consider the case $\mu = \pi_p$ for some $p > 0$. With $\epsilon > 0$, if we assume the trivial dynamics $C(s,\sigma) = 0$ for all s, σ, we will of course not have that $\mu_{\inf,\tau} \preceq \mu^{(-,\epsilon)}$ for any $\tau > 0$.

6 Proof of Theorem 1.8

Proof of Theorem 1.8. Take $\lambda > \lambda_p$ and let $\lambda' = (\lambda + \lambda_p)/2$. By Theorem 1.10 there exists an $\epsilon > 0$ such that $(\mu_{\lambda'}, \mu_\lambda)$ is downwards $\epsilon-$stable. Lemma 5.1 gives us that there exists a $\tau > 0$ such that $\mu_\lambda^{(-,\epsilon)} \preceq \mu_{\lambda,\inf,\tau}$ and hence that $\mu_{\lambda'} \preceq \mu_{\lambda,\inf,\tau}$. Therefore, since $\mathcal{C}^+$ is an increasing event and $\lambda' > \lambda_p$, we have that

$$1 = \mu_{\lambda'}(\mathcal{C}^+) \leq \mu_{\lambda,\inf,\tau}(\mathcal{C}^+)$$

and so

$$\Psi^\lambda(\mathcal{C}_t^+ \ \forall t \in [0,\tau]) = 1.$$

The theorem now follows from countable additivity.

QED

7 Proof of Theorem 1.1

In this section we will deal with stationary distributions for interacting particle systems which are monotone in the sense of Definition 2.2.

Let $G = (S, E)$ be a countable connected locally finite graph and let $\Lambda \subseteq S$ be connected and $|\Lambda| < \infty$. Let $\{\mu_\Lambda^p\}_{p \in I}$, where $I \subseteq \mathbb{R}$ be a family of measures on $\{-1,1\}^\Lambda$ such that

$$\mu_\Lambda^{p_1} \preceq \mu_\Lambda^{p_2} \; \forall p_1 \leq p_2.$$

Assume that there exist stationary Markov processes Ψ_Λ^p governed by flip rate intensities $\{C_{p,\Lambda}(s,\sigma)\}_{s\in\Lambda,\sigma\in\{-1,1\}^\Lambda}$ and with marginal distributions μ_Λ^p. Furthermore assume that there exists limiting distributions Ψ^p of Ψ_Λ^p and μ^p of μ_Λ^p as $\Lambda \uparrow S$. Assume that μ_Λ^p are monotone for every p and Λ. For $p_1 < p_2$, let

$$A_{\Lambda,p_1,p_2} := \inf_{\substack{s\in\Lambda \\ \xi\in\{-1,1\}^{\Lambda\setminus s}}} [\mu_\Lambda^{p_2}(\sigma(s) = 1|\sigma(\Lambda\setminus s) \equiv \xi) - \mu_\Lambda^{p_1}(\sigma(s) = 1|\sigma(\Lambda\setminus s) \equiv \xi)]$$

and assume that for all $p_1 < p_2$

$$\inf_{\Lambda\subseteq S} A_{\Lambda,p_1,p_2} > 0.$$

For fixed $p_1 < p_2$ there exists by Proposition 3.3 an $\epsilon > 0$ such that (μ^{p_1}, μ^{p_2}) is both upwards and downwards ϵ−stable. Next, by Lemma 5.1 there exists a $\tau > 0$ such that

$$\mu^{p_2,(-,\epsilon)} \preceq \mu_{\text{inf},\tau}^{p_2},$$

and therefore

$$\mu^{p_1} \preceq \mu_{\text{inf},\tau}^{p_2}. \tag{25}$$

Theorem 7.1 *Consider the setup just described. Let $\mathcal{A}$ be an increasing event on $\{-1,1\}^S$ and let $\mathcal{A}_t$ be the event that $\mathcal{A}$ occurs at time t.*
(1) Let $a \in \mathbb{R}$. If

$$\mu^p(\mathcal{A}) = 1$$

for all $p \in I$ with $p > a$, then

$$\Psi^p(\mathcal{A}_t \textit{ occurs for every } t) = 1$$

for all $p \in I$ with $p > a$.
(2) Let $a \in \mathbb{R}$. If

$$\mu^p(\mathcal{A}) = 0$$

for all $p \in I$ with $p < a$, then

$$\Psi^p(\mathcal{A}_t \textit{ occurs for some } t) = 0$$

for all $p \in I$ with $p < a$.

Proof. We prove only (1) as (2) is proved in an identical way. Take $p > a$ and let $p_2 = (p + a)/2$. By the argument leading towards equation (25), there exists $\tau > 0$ such that

$$\mu^{p_2}(\mathcal{A}) \leq \mu_{\text{inf},\tau}^p(\mathcal{A}).$$

By using $\mu^{p_2}(\mathcal{A}) = 1$ and

$$\mu_{\text{inf},\tau}^p(\mathcal{A}) \leq \Psi^p(\mathcal{A}_t \text{ occurs for every } t \in [0,\tau]),$$

we get by countable additivity that

$$\Psi^p(\mathcal{A}_t \text{ occurs for every } t) = 1.$$

QED

We will now be able to prove Theorem 1.1 easily.

Proof of Theorem 1.1. We prove only the very first statement; all the other statements are proved in a similar manner. We fix $\beta \geq 0$ and then h will correspond to our parameter p in the above set up. For any $\Lambda \subseteq S$, any $s \in \Lambda$ and any $\xi \in \{-1,1\}^{\Lambda\setminus s}$, we have that

$$\mu_\Lambda^{+,\beta,h}(\sigma(s) = 1|\sigma(\Lambda \setminus s) = \xi) = \frac{1}{1 + e^{-2\beta(\sum_{t:t\sim s}\xi(t))-2h}}, \tag{26}$$

where we let $\xi(t) = 1$ if $t \in \Lambda^c$ in order to take the boundary condition into account. It is obvious from equation (26) and the definition of monotonicity that $\mu_\Lambda^{+,\beta,h}$ is monotone for any h and Λ. Letting $h_1 < h_2$, it is immediate that

$$A_{\Lambda,h_1,h_2} = \inf_{\substack{s\in\Lambda \\ \xi\in\{-1,1\}^{\Lambda\setminus s}}} [\frac{1}{1 + e^{-2\beta(\sum_{t:t\sim s}\xi(t))-2h_2}} - \frac{1}{1 + e^{-2\beta(\sum_{t:t\sim s}\xi(t))-2h_1}}] > 0,$$

where again $\xi(t) = 1$ for all $t \in \Lambda^c$. It is not hard to see that this strict inequality must hold uniformly in Λ; i.e.,

$$\inf_{\Lambda\subseteq S} A_{\Lambda,h_1,h_2} > 0.$$

It follows that all of the assumptions of Theorem 7.1 hold and part (1) of that result gives us what we want.

QED

Proof of Lemma 1.2. Fix $\beta \geq 0$. Given any $p \in (0,1)$, it is easy to see that there exists a real number h_2 such that for all $h \geq h_2$, for $s \in S$ and for all $\xi \in \{-1,1\}^{S \setminus s}$

$$\mu^{+,\beta,h}(\sigma(s) = 1 | \sigma(S \setminus s) = \xi) \geq p$$

and hence $\pi_p \preceq \mu^{+,\beta,h}$. It is also easy to see that there exists a real number h_1 such that for all $h < h_1$, for $s \in S$ and for all $\xi \in \{-1,1\}^{S \setminus s}$

$$\mu^{+,\beta,h}(\sigma(s) = 1 | \sigma(S \setminus s) = \xi) \leq p$$

and hence $\mu^{+,\beta,h} \preceq \pi_p$. The statements of the lemma easily follow from these facts.

QED

8 Proof of Theorem 1.3

In this section we will use a variant of the so called Peierls argument to prove Theorem 1.3. We prove this only for $\mathbb{Z}^2$; the proof (with more complicated topological details) can be carried out for $\mathbb{Z}^d$ with $d \geq 3$.

We will write $0 \overset{-,t}{\longleftrightarrow} \partial\Lambda_L$ for the event that there exists a path of sites in state -1 connecting the origin to $\partial\Lambda_L := \Lambda_{L+1} \setminus \Lambda_L$ at time t and we will write $0 \overset{-,t}{\longleftrightarrow} \infty$ for the event that there exists an infinite path of sites in state -1 containing the origin at time t. We will also write $0 \overset{+,t}{\longleftrightarrow} \partial\Lambda_L$ and $0 \overset{+,t}{\longleftrightarrow} \infty$ for the obvious analogous events. We will first need Lemma 8.1 and the concept of a dual graph. The dual graph $G_n^{dual} = (S_n^{dual}, E_n^{dual})$ of $G_n = (S_n, E_n)$ consists of the set of sites $S_n^{dual} := \{-n-\frac{1}{2}, \ldots, n+\frac{1}{2}\}^2$ and E_n^{dual} which is the set of nearest neighbor pairs of S_n^{dual}. In this paper we will only work with the edges of the dual graph. An edge $e \in E_n^{dual}$ crosses one (and only one) edge $f \in E_n$ and the end sites of this edge f will be called the sites (of G_n) associated to e. For a random spin configuration X on $\{-1,1\}^{S_n}$ define a random edge configuration Y on $\{0,1\}^{E_n^{dual}}$ in the following way:

$$Y(e) = \begin{cases} 0 & \text{if} \quad X(t) = X(s) \\ 1 & \text{if} \quad X(t) \neq X(s), \end{cases} \tag{27}$$

where s, t are the sites associated to edge $e \in E_n^{dual}$. In figure (1) we have drawn a configuration $\sigma \in \{-1,1\}^{S_1}$ and the induced edge configuration on $\{0,1\}^{E_1^{dual}}$.

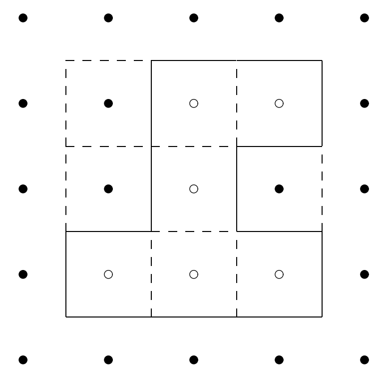

Figure 1: S_1 and the edges of it's dual graph. A solid circle marks a site with spin 1, while an empty circle has spin -1. A solid line is a present edge of the dual graph, and a dashed line is an absent edge of the dual graph.

Assume that the sites evolve according to the flip rate intensities $\{C_n(s,\sigma)\}_{s \in S_n,\ \sigma \in \{-1,1\}^{S_n}}$. Consider γ, a (finite) path of edges in the dual graph. Take γ' to be a subset of γ. Assume that all edges of γ' are absent and all edges of $\gamma \setminus \gamma'$ are present at $t = 0$. We want to estimate the probability of the event that all edges of γ' are present at some point (not necessarily all at the same time) during some time interval $[0, \tau]$. In other words we want to estimate the probability of the event $\{Y_{\sup,\tau}(\gamma') \equiv 1 | Y_0(\gamma') \equiv 0, Y_0(\gamma \setminus \gamma') \equiv 1\}$.

Lemma 8.1 *Let $\{C_n(s,\sigma)\}_{s \in S_n,\ \sigma \in \{-1,1\}^{S_n}}$ be the flip rate intensities for a stationary Markov process on $\{-1,1\}^{S_n}$ and let Y_t be defined as above. Let*

$$\lambda := \sup_{(s,\sigma)} C_n(s,\sigma) \ (< \infty).$$

For any $\tau > 0$ and any $\gamma' \subseteq E_n^{dual}$,

$$\mathbf{P}(Y_{\sup,\tau}(\gamma') \equiv 1 | Y_0(\gamma') \equiv 0, Y_0(E_n^{dual} \setminus \gamma')) \leq (4(1 - e^{-\lambda\tau})^{1/4})^{|\gamma'|}.$$

Proof.

Take $\tau > 0$. For every $s \in S_n$ associate an independent Poisson process with parameter λ. Define $\{X_t\}_{t \geq 0}$ in the following way. Let $X_0 \sim \mu$ and take t' to be an arrival time for the Poisson process of a site s. We let $X_{t',+}(s) \neq X_{t',-}(s)$ with probability $C(s, X_{t',-})/\lambda$. Do this independently for all arrival times for all Poisson processes associated to the different sites. It is immediate that $X_\tau \sim \mu$. Let s_i, $i \in \{1, \dots, l\}$ be distinct sites of S_n. The event $\{X_{\inf,\tau}(s_i) \neq X_{\sup,\tau}(s_i)\ \forall i \in \{1, \dots, l\}\}$ is contained in the event that every Poisson process associated to the sites $s_i, i \in \{1, \dots, l\}$ have had at least one arrival by time τ. The probability that a particular site has had an arrival by time τ is $1 - e^{-\lambda\tau}$. Furthermore this event is independent of the Poisson processes for all other sites. Therefore

$$\mathbf{P}(X_{\inf,\tau}(s_i) \neq X_{\sup,\tau}(s_i)\ \forall i \in \{1, \dots, l\}) \leq (1 - e^{-\lambda\tau})^l. \tag{28}$$

Given γ', consider the set of all sites associated to some edge of γ' and let $n_{\gamma'}$ be the cardinality of that set. Observe that $n_{\gamma'} \leq 2|\gamma'|$ and that in order for the event $\{Y_{\sup,\tau}(\gamma') \equiv 1 | Y_0(\gamma') \equiv 0, Y_0(E_n^{dual} \setminus \gamma')\}$ to occur, at least $|\gamma'|/4$ of the sites associated to γ' must flip during $[0, \tau]$. This is because one site is associated to at most 4 edges. Denote the event that at least $|\gamma'|/4$ of the sites associated to γ' flips during $[0, \tau]$ by $\mathcal{A}_{\tau,\gamma'}$. Take $\tilde{S}$ to be a subset of the sites associated to γ' such that $|\tilde{S}| \geq |\gamma'|/4$. By equation (28), the probability that all of these sites flips during $[0, \tau]$ is less than $(1 - e^{-\lambda\tau})^{|\tilde{S}|} \leq (1 - e^{-\lambda\tau})^{|\gamma'|/4}$. To conclude, observe that the number of subsets of the sites associated to γ' is $2^{2|\gamma'|}$. Hence, the probability of the event $\mathcal{A}_{\tau,\gamma'}$ must be less than $(1 - e^{-\lambda\tau})^{|\gamma'|/4} 2^{2|\gamma'|}$, and so

$$\begin{aligned} &\mathbf{P}(Y_{\sup,\tau}(\gamma') \equiv 1 | Y_0(\gamma') \equiv 0, Y_0(E_n^{dual} \setminus \gamma')) \\ &\quad \leq \mathbf{P}(\mathcal{A}_{\tau,\gamma'}) \leq ((1 - e^{-\lambda\tau})^{1/4} 4)^{|\gamma'|}. \end{aligned}$$

QED

Proof of Theorem 1.3. We will prove the theorem for $d = 2$. For $\beta > \beta_p$, choose $\delta_1 > 0$ so that $\beta' := \beta \frac{2-\delta_1}{2} > \beta_p$ and hence

$$\sum_{l=1}^{\infty} l 3^{l-1} e^{-2\beta' l} < \infty.$$

Next, choose N and $\epsilon < 1/2$ such that $\frac{4}{N} \leq \delta_1$, and $\epsilon^{\frac{1}{N}} \leq e^{-\beta(2-\delta_1)}$ and let τ be such that $\epsilon = 4(1 - e^{-\lambda\tau})^{1/4}$. Let $\delta > 0$ be arbitrary and choose L so that

$$3 \sum_{l=L}^{\infty} l 3^{l-1} e^{-2\beta' l} < \delta.$$

Let $\mathcal{E}_{L,\tau}$ be the event that $0 \overset{-,t}{\longleftrightarrow} \partial\Lambda_L$, for some $t \in [0, \tau]$. Let $\Psi_n^{+,\beta}$ be defined as in Section 2.3. We will show that

$$\Psi_n^{+,\beta}(\mathcal{E}_{L,\tau}) < \delta\ \forall n > L.$$

Since $\Psi_n^{+,\beta}(\mathcal{E}_{L,\tau}) \to \Psi^{+,\beta}(\mathcal{E}_{L,\tau})$, (see Section 2.3) we get that $\Psi^{+,\beta}(\mathcal{E}_{L,\tau}) \leq \delta$. Letting $L \to \infty$ and $\delta \to 0$, we get that

$$\Psi^{+,\beta}(\exists t \in [0, \tau] : 0 \overset{-,t}{\longleftrightarrow} \infty) = 0,$$

and then by countable additivity

$$\Psi^{+,\beta}(\exists t \geq 0 : 0 \overset{-,t}{\longleftrightarrow} \infty) = 0.$$

It is well known (see [7]) that if all sites in $\Lambda_{n+1} \setminus \Lambda_n$ takes the value $+1$,

$$\begin{aligned} &\mathcal{E}_{L,\tau} \\ &\quad \subseteq \{\exists \gamma \subseteq E_n^{dual}, t \in [0, \tau] : |\gamma| \geq L,\ \gamma \text{ surrounds the origin, } Y_t(\gamma) \equiv 1\} \\ &\quad \subseteq \{\exists \gamma \subseteq E_n^{dual} : |\gamma| \geq L,\ \gamma \text{ surrounds the origin, } Y_{\sup,\tau}(\gamma) \equiv 1\}. \end{aligned} \tag{29}$$

To prove $\Psi_n^{+,\beta}(\mathcal{E}_{L,\tau}) < \delta$, consider γ with $|\gamma| = l$ a contour in E_n^{dual} surrounding the origin. By Lemma 8.1, $\mathbf{P}(Y_{\sup,\tau}(\gamma') \equiv 1 | Y_0(\gamma') \equiv 0, Y_0(\gamma \setminus \gamma') \equiv 1) \leq \epsilon^{|\gamma'|}$ whenever $\gamma' \subseteq \gamma$. We get

$$\begin{aligned} &\mathbf{P}(Y_{\sup,\tau}(\gamma) \equiv 1) \\ &\quad = \sum_{k=0}^{l} \sum_{\substack{\gamma' \subseteq \gamma \\ |\gamma'| = k}} \mathbf{P}(Y_0(\gamma') \equiv 0, Y_0(\gamma \setminus \gamma') \equiv 1) \\ &\qquad\qquad \times \mathbf{P}(Y_{\sup,\tau}(\gamma') \equiv 1 | Y_0(\gamma') \equiv 0, Y_0(\gamma \setminus \gamma') \equiv 1) \\ &\quad \leq \sum_{k=0}^{l} \sum_{\substack{\gamma' \subseteq \gamma \\ |\gamma'| = k}} \mathbf{P}(Y_0(\gamma') \equiv 0, Y_0(\gamma \setminus \gamma') \equiv 1) \epsilon^k \\ &\quad = \sum_{k=0}^{l} \mathbf{P}(\{\text{all edges except k of } \gamma \text{ are present at } t = 0\}) \epsilon^k \\ &\quad = \sum_{k=0}^{l/N} \mathbf{P}(\{\text{all edges except k of } \gamma \text{ are present at } t = 0\}) \epsilon^k \\ &\qquad + \sum_{k=l/N+1}^{l} \mathbf{P}(\{\text{all edges except k of } \gamma \text{ are present at } t = 0\}) \epsilon^k. \end{aligned} \tag{30}$$

Obviously, l/N need not be an integer, but correcting for this is trivial and is left for the reader.

We need to estimate $\mathbf{P}(\{\text{all edges except k of } \gamma \text{ are present at } t = 0\})$. For this purpose, define T: $\{-1, 1\}^{S_n} \to \{-1, 1\}^{S_n}$, by

$$(T\sigma)(s) = \begin{cases} \sigma(s) & \text{if } \text{ s not inside } \gamma \\ -\sigma(s) & \text{if } \text{ s inside } \gamma \end{cases}$$

for all $\sigma \in \{-1,1\}^{S_n}$. Let $E_k = \{\sigma :$ all edges except k of γ are present$\}$. Since $H_n^{+,\beta}$ of equation (6) gives a contribution of $-\beta$ for adjacent pairs of equal spin and $+\beta$ for adjacent pairs of unequal spin, we have that for $\sigma \in E_k$, $H_n^{+,\beta}(T\sigma) = H_n^{+,\beta}(\sigma) - 2\beta(|\gamma| - k) + 2\beta k = H_n^{+,\beta}(\sigma) - 2\beta|\gamma| + 4\beta k$.

Hence, for $\sigma \in E_k$

$$\mu_n^{+,\beta}(\sigma) = \frac{e^{-H_n^{+,\beta}(\sigma)}}{Z} = \frac{e^{-H_n^{+,\beta}(T\sigma) - 2\beta|\gamma| + 4\beta k}}{Z},$$

and so

$$\begin{aligned} &\mu_n^{+,\beta}(E_k) \\ &= \sum_{\sigma \in E_k} \mu_n^{+,\beta}(\sigma) = e^{-2\beta l + 4\beta k} \sum_{\sigma \in E_k} \frac{e^{-H_n^{+,\beta}(T\sigma)}}{Z} \\ &\leq e^{-2\beta l + 4\beta k} \sum_{\sigma \in \{-1,1\}^{S_n}} \frac{e^{-H_n^{+,\beta}(T\sigma)}}{Z} = e^{-2\beta l + 4\beta k}, \end{aligned}$$

where the last equality follows from T being bijective. We then get that

$$\begin{aligned} &\sum_{k=0}^{l/N} \mathbf{P}(\{\text{all edges except k of } \gamma \text{ are present at } t = 0\})\epsilon^k \qquad (31) \\ &\leq \sum_{k=0}^{l/N} e^{-2\beta l + 4\beta k}\epsilon^k \leq e^{-2\beta l + \frac{4\beta l}{N}} \sum_{k=0}^{l/N} \epsilon^k \leq 2e^{-2\beta l + \frac{4\beta l}{N}} \\ &\leq 2e^{-\beta(2-\delta_1)l} = 2e^{-2\beta' l}. \end{aligned}$$

Furthermore

$$\begin{aligned} &\sum_{k=l/N+1}^{l} \mathbf{P}(\{\text{all edges except k of } \gamma \text{ are present at } t = 0\})\epsilon^k \qquad (32) \\ &\leq \epsilon^{l/N} \sum_{k=l/N+1}^{l} \mathbf{P}(\{\text{all edges except k of } \gamma \text{ are present at } t = 0\}) \\ &\leq \epsilon^{l/N} \leq e^{-\beta(2-\delta_1)l} = e^{-2\beta' l}, \end{aligned}$$

where we use that $\{$all edges except k of γ are present at $t = 0\}$ are disjoint events for different k. Hence equations (30), (31) and (32) combined gives us

$$\mathbf{P}(Y_{\sup,\tau}(\gamma) \equiv 1) \leq 3e^{-2\beta' l}$$

and so by equation (29), for all $n > L$,

$$\begin{aligned} &\Psi_n^{+,\beta}(\mathcal{E}_{L,\tau}) \\ &\quad \leq \Psi_n^{+,\beta}(\exists \gamma \subseteq E_n^{dual} : |\gamma| \geq L, \ \gamma \text{ surrounds the origin, } Y_{\sup,\tau}(\gamma) \equiv 1) \\ &\quad \leq \sum_{l=L}^{\infty} l3^{l-1} 3e^{-2\beta' l} < \delta, \end{aligned}$$

where the second to last inequality follows from the fact that the number of contours around the origin of length l is at most $l3^{l-1}$, (see [7]).

QED

Remark; For $\mathbb{Z}^d$, the proof is generalized by noting that the number of connected surfaces of size l surrounding the origin is at most $C(d)^l$, for some constant $C(d)$. The arguments are the same but the "topological details" are messier.

9 Proof of Theorem 1.5

We will start this subsection by presenting a theorem by T.M. Liggett, R.H. Schonmann and A.M. Stacey ([20]).

Theorem 9.1 *Let G=(S,E) be a graph with a countable set of sites in which every site has degree at most $\Delta \geq 1$, and in which every finite connected component of G contains a site of degree strictly less than Δ. Let $p, \alpha, r \in [0,1], q = 1 - p$, and suppose that*

$$\begin{aligned} (1-\alpha)(1-r)^{\Delta-1} &\geq q, \\ (1-\alpha)\alpha^{\Delta-1} &\geq q. \end{aligned}$$

If $\mu \in G(p)$, then $\pi_{\alpha r} \preceq \mu$. In particular, if $q \leq (\Delta-1)^{\Delta-1}/\Delta^{\Delta}$, then $\pi_\rho \preceq \mu$, where

$$\rho = \left(1 - \frac{q^{1/\Delta}}{(\Delta-1)^{(\Delta-1)/\Delta}}\right)(1 - (q(\Delta-1))^{1/\Delta}).$$

Here $G(p)$ denotes the set of Borel-measures on $\{-1,1\}^S$ such that if $\mu \in G(p)$, $X \sim \mu$ then for any site $s \in S$

$$\mathbb{P}[X(s) = 1 | \sigma(\{X(t) : \{s,t\} \notin E\})] \geq p \ a.s.$$

Observe that when $p \to 1 \Rightarrow q \to 0$ and so $\rho \to 1$. The above theorem is stated as the original in [20]. However, by considering the line-graph of $G = (S, E)$ it can be restated in the following way;

Corollary 9.2 *Let $\tilde{G} = (\tilde{S}, \tilde{E})$ be any countable graph of degree at most Δ. For each $0 < \rho < 1$ there exists a $0 < p < 1$ where $p = p(\Delta, \rho)$ such that if $Y \sim \nu$ where ν is a measure on the edges of $\tilde{G}$ such that for every edge $e \in \tilde{E}$*

$$\mathbb{P}[Y(e) = 1 | \sigma(\{Y(f) : e \not\sim f\})] \geq p \text{ a.s.}$$

we have that $\pi_\rho^{\tilde{E}} \preceq \nu$.

By $e \not\sim f$ we of course mean that the edges e and f does not have any endpoints in common. Here, $\pi_\rho^{\tilde{E}}$ is the product measure with density ρ on the edges of $\tilde{G}$.

Consider a graph $G = (S, E)$ and a subgraph $G' = (S', E')$ where $S' = S$ and $E' \subset E$. Let $X \sim \pi_p$ on S. We declare an edge $e \in E'$ to be closed if any of the endpoints takes the value 0 under X. Corollary 9.2 gives us that for any $\rho < 1$ there is a $p < 1$ such that this method of closing edges dominates independent bond percolation with density ρ on E'. Observe that we can choose p independent of E' since the maximal degree of E' is bounded above by the maximal degree of E.

Let $(X, Y) \sim \mathbf{P}_n^p$, defined in Section 2.5. Close every $e \in E_n$ such that $Y(e) = 1$ independently with probability ϵ thus creating $(X, Y^{(-,\epsilon)})$. Compare this to closing every site in S_n independently with parameter ϵ' (creating $X^{(-,\epsilon')}$) and defining

$$Y^{\epsilon'}(e) = \begin{cases} 1 & \text{if } Y(e) = 1 \text{ and neither one of the endpoints of } e \text{ flips} \\ 0 & \text{otherwise.} \end{cases}$$

By the arguments of the last paragraph we see that for a fixed ϵ there exists an ϵ' (that we can choose independent of (X, Y) and n) such that the first way (i.e. independent bond percolation) of removing edges is stochastically dominated by the latter. Hence

$$\begin{aligned} &\mathbf{P}_n^p((X, Y^{(-,\epsilon)}) \in (\{-1,1\}^{S_n}, \cdot) | (X, Y)) \\ &\quad \preceq \mathbf{P}_n^p((X^{(-,\epsilon')}, Y^{\epsilon'}) \in (\{-1,1\}^{S_n}, \cdot) | (X, Y)). \end{aligned}$$

By averaging over all possible (X, Y), the next lemma follows.

Lemma 9.3 *With notation as above, for any $\epsilon > 0$ there exists $\epsilon' > 0$ independent of n such that*

$$\mathbf{P}_n^p((X, Y^{(-,\epsilon)}) \in (\{-1,1\}^{S_n}, \cdot)) \preceq \mathbf{P}_n^p((X^{(-,\epsilon')}, Y^{\epsilon'}) \in (\{-1,1\}^{S_n}, \cdot)).$$

Observe that

$$\mathbf{P}_n^p((X, Y^{(-,\epsilon)}) \in (\{-1,1\}^{S_n}, \cdot)) =_{\mathcal{D}} \tilde{\nu}_n^{p,(-,\epsilon)}(\cdot) \tag{33}$$

and that

$$\mathbf{P}_n^p((X^{(-,\epsilon')}, Y^{\epsilon'}) \in (\cdot, \{-1,1\}^{E_n})) =_{\mathcal{D}} \mu_n^{+,\beta,(-,\epsilon')}(\cdot). \tag{34}$$

We are now ready to prove Theorem 1.5.

Proof of Theorem 1.5. For any choice of $\beta > \beta_c$ take $p = 1 - e^{-2\beta}$ and let $\delta \in (0, p - p_c)$. Equation (14) and Holley's inequality implies that

$$\tilde{\nu}_n^{p-\delta} \preceq \tilde{\nu}_n^p \; \forall n \in \mathbb{N}^+.$$

Since by equation (14) both $\tilde{\nu}_n^{p-\delta}$ and $\tilde{\nu}_n^p$ are monotone, there exists by Lemma 3.2 (it is easy to check that all other conditions of that lemma are satisfied) an $\epsilon > 0$ such that

$$\tilde{\nu}_n^{p-\delta} \preceq \tilde{\nu}_n^{p,(-,\epsilon)} \; \forall n \in \mathbb{N}^+. \tag{35}$$

In [12] they show that the limit $\lim_n \tilde{\nu}_n^{p-\delta}(0 \longleftrightarrow \partial\Lambda_n)$ exists and that

$$\lim_n \tilde{\nu}_n^{p-\delta}(0 \longleftrightarrow \partial\Lambda_n) > 0. \tag{36}$$

Here $\{0 \longleftrightarrow \partial\Lambda_n\}$ denotes the event that there exists a path of present edges connecting the origin to $\partial\Lambda_n := \Lambda_{n+1} \setminus \Lambda_n$. Since $\{0 \longleftrightarrow \partial\Lambda_n\}$ is an increasing event on the edges, Lemma 9.3 guarantees the existence of an $\epsilon' > 0$ such that

$$\begin{aligned} &\tilde{\nu}_n^{p,(-,\epsilon)}(0 \longleftrightarrow \partial\Lambda_n) \\ &\quad = \mathbf{P}_n^p((X, Y^{(-,\epsilon)}) \in (\{-1,1\}^{S_n}, 0, \longleftrightarrow \partial\Lambda_n)) \\ &\quad \leq \mathbf{P}_n^p((X^{(-,\epsilon')}, Y^{\epsilon'}) \in (\{-1,1\}^{S_n}, 0 \longleftrightarrow \partial\Lambda_n)) \; \forall n \in \mathbb{N}^+. \end{aligned}$$

If there exists a path of present edges connecting the origin to the boundary $\partial\Lambda_n$ under Y, all the sites of this path must have the value 1 under X. Similarly for $(X^{(-,\epsilon')}, Y^{\epsilon'})$, if there exists a path of present edges connecting the origin to the boundary $\partial\Lambda_n$ under $Y^{\epsilon'}$, all the sites of this path must have the value 1 under $X^{(-,\epsilon')}$. Hence

$$\begin{aligned} &\mathbf{P}_n^p((X^{(-,\epsilon')}, Y^{\epsilon'}) \in (\{-1,1\}^{S_n}, 0 \longleftrightarrow \partial\Lambda_n)) \\ &\quad = \mathbf{P}_n^p((X^{(-,\epsilon')}, Y^{\epsilon'}) \in (0 \overset{+}{\longleftrightarrow} \partial\Lambda_n, 0 \longleftrightarrow \partial\Lambda_n)) \\ &\quad \leq \mathbf{P}_n^p((X^{(-,\epsilon')}, Y^{\epsilon'}) \in (0 \overset{+}{\longleftrightarrow} \partial\Lambda_n, \{0,1\}^{E_n})) \\ &\quad = \mu_n^{+,\beta,(-,\epsilon')}(0 \overset{+}{\longleftrightarrow} \partial\Lambda_n). \end{aligned}$$

Of course

$$\mu_n^{+,\beta,(-,\epsilon')}(0 \stackrel{+}{\longleftrightarrow} \partial\Lambda_n) \leq \mu_n^{+,\beta,(-,\epsilon')}(0 \stackrel{+}{\longleftrightarrow} \partial\Lambda_L)\ \forall L < n.$$

Therefore, for any L we have that

$$\begin{aligned} 0 &< \lim_n \tilde{\nu}_n^{p-\delta}(0 \longleftrightarrow \partial\Lambda_n) \\ &\leq \lim_n \mu_n^{+,\beta,(-,\epsilon')}(0 \stackrel{+}{\longleftrightarrow} \partial\Lambda_L) = \mu^{+,\beta,(-,\epsilon')}(0 \stackrel{+}{\longleftrightarrow} \partial\Lambda_L), \end{aligned}$$

and so

$$0 < \lim_L \mu^{+,\beta,(-,\epsilon')}(0 \stackrel{+}{\longleftrightarrow} \partial\Lambda_L) = \mu^{+,\beta,(-,\epsilon')}(0 \stackrel{+}{\longleftrightarrow} \infty).$$

The limit in L exists since $\{0 \stackrel{+}{\longleftrightarrow} \partial\Lambda_{L_2}\} \subseteq \{0 \stackrel{+}{\longleftrightarrow} \partial\Lambda_{L_1}\}$ for $L_1 \leq L_2$. Since $\mu^{+,\beta}$ is ergodic (see [18] page 143 and 195) it follows that $\mu^{+,\beta,(-,\epsilon')}$ must also be ergodic. We conclude that

$$\mu^{+,\beta,(-,\epsilon')}(\mathcal{C}^+) = 1. \tag{37}$$

By Lemma 5.1, there exists a $\tau > 0$ such that

$$\mu^{+,\beta,(-,\epsilon')} \preceq \mu^{+,\beta}_{\text{inf},\tau}$$

and therefore

$$\mu^{+,\beta}_{\text{inf},\tau}(\mathcal{C}^+) = 1.$$

Therefore

$$\Psi^{+,\beta}(\mathcal{C}_t^+ \text{ occurs for every } t \in [0,\tau]) = 1.$$

Finally using countable additivity

$$\Psi^{+,\beta}(\mathcal{C}_t^+ \text{ occurs for every t}) = 1.$$

QED

10 Proof of Theorem 1.4

The aim of this section is to prove Theorem 1.4. For that we will use Theorem 1.5 and Lemma 10.1. We will not prove Lemma 10.1 since it follows immediately from the proof of Lemma 11.12 in [9] due to Y. Zhang.

A probability measure μ on $\{-1,1\}^S$ is said to have the finite energy property if all conditional probabilities on finite sets are strictly positive.

Lemma 10.1 *Take μ to be any probability measure on $\{-1,1\}^{\mathbb{Z}^2}$ which has positive correlations and the finite energy property. Assume further that μ is invariant under translations, rotations and reflections in the coordinate axes. If $\mu(\mathcal{C}^+) = 1$, then $\mu(\mathcal{C}^-) = 0$.*

Proof of Theorem 1.4. Fix $\beta > \beta_c$. By equation (37), there exists $\epsilon > 0$ such that

$$\mu^{+,\beta,(-,\epsilon)}(\mathcal{C}^+) = 1.$$

Since $\mu^{+,\beta}$ and $\pi_{1-\epsilon}$ both have positive correlations, it follows (see [18], page 78) that $\mu^{+,\beta,(-,\epsilon)}$ has positive correlations. Also, the finite energy property is easily seen to hold for $\mu^{+,\beta,(-,\epsilon)}$. Using this we can by Lemma 10.1 conclude that

$$\mu^{+,\beta,(-,\epsilon)}(\mathcal{C}^-) = 0.$$

By Lemma 5.1 there exists a $\tau > 0$ such that $\mu^{+,\beta,(-,\epsilon)} \preceq \mu^{+,\beta}_{\text{inf},\tau}$ and hence

$$\mu^{+,\beta}_{\text{inf},\tau}(\mathcal{C}^-) = 0.$$

It follows that

$$\Psi^{+,\beta}(\exists t \in [0,\tau] : \mathcal{C}_t^- \text{ occurs}) = 0,$$

and by countable additivity, we conclude

$$\Psi^{+,\beta}(\exists t \geq 0 : \mathcal{C}_t^- \text{ occurs}) = 0.$$

QED

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