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ABSTRACT

To a given holomorphic section of a Hermitian vector bundle, one can associate a residue current by means of Cauchy-Fantappiè-Leray type formulas. In this thesis we define products of such residue currents. We prove that, in the case of a complete intersection, the product of the residue currents of a tuple of sections coincides with the residue current of the direct sum of the sections.

Keywords: Residue current, Coleff-Herrera current, Bochner-Martinelli formula, Hermitian vector bundle, Cauchy-Fantappiè-Leray formula, complete intersection.

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1. Introduction and Background

Let $f$ be a holomorphic function at 0 in $\mathbb{C}$. One can prove that there exists a distribution $U^f$ such that $f U^f = 1$, as was first done by Schwartz [13]. One way of defining such a $U^f$, sometimes denoted by $[1/f]$, is as the principal value distribution of $1/f$, given by

$$
\varphi \mapsto \lim_{\varepsilon \to 0} \int_{|f| > \varepsilon} \frac{\varphi}{f},
$$

Applying the $\partial$ operator to $U^f$, we obtain a $(0, 1)$-current, which we call the residue current of $f$ and which we denote by $\partial[1/f]$ or $R^f$. Clearly, $R^f$ has support on $Y = f^{-1}(0)$ and it is easy to see that a function $\varphi$ that is holomorphic in a neighborhood of $Y$ belongs to the ideal generated by $f$ precisely when it annihilates $R^f$, that is when $\varphi R^f = 0$. Observe that these last properties do not depend on the exact definition of $U^f$, but only on the fact that $f U^f = 1$. By Stokes’ theorem, the action of $R^f$ is given by

$$
(1.1) \quad \partial[1/f](\varphi) = \lim_{\varepsilon \to 0} \int_{|f| > \varepsilon} \frac{1}{f} \partial \varphi = \lim_{\varepsilon \to 0} \int_{|f| > \varepsilon} \partial \left( \frac{\varphi}{f} \right) = \lim_{\varepsilon \to 0} \int_{|f| = \varepsilon} \frac{\varphi}{f},
$$

where the limit is taken over the regular values of $|f|$.

Given a tuple $f$ of holomorphic functions $f_1, \ldots, f_m$ defined in some domain $\Omega$ in $\mathbb{C}^n$, it is natural to ask whether there are analogues to the currents $U^f$ and $R^f$, that can be used to characterize the ideal $f$. In general, definitions of such residue currents lead to integration over singular varieties, and thus the theory of multidimensional residue currents relies heavily on Hironaka’s theorem of resolution of singularities from 1964, see [2].

In 1978, Coleff and Herrera [6] used the desingularization to construct a residue current of the tuple $f$ as a product of residue currents of the functions $f_i$. For a test form $\phi \in \mathcal{D}_{n,n-m}(\Omega)$, consider the residue integral

$$
I^\phi_f(\varepsilon) = \frac{1}{(2\pi i)^m} \int_{T^m} \frac{\phi}{f_1 \cdots f_m},
$$

where $T^m = \{|f_1| = \varepsilon_1, \ldots, |f_m| = \varepsilon_m\}$ is oriented as the distinguished boundary of the corresponding polyhedron. To see how $\lim_{\varepsilon \to 0} I^\phi_f(\varepsilon)$, if it exists, can be interpreted as the action of a current $\partial[1/f_1] \wedge \cdots \wedge \partial[1/f_m]$, let us consider the case of two functions $f_1$ and $f_2$. Assuming $f$ to be regular, that is $df_1 \wedge df_2 \neq 0$, on $Y = f^{-1}(0)$, we can locally choose $f_1$ and $f_2$ as coordinates, and for $\phi \in \mathcal{D}_{(n,n-2)}(\Omega)$ we get by Stokes’ theorem (compare
to (1.1) that
\[
\int_{|f_1|=\varepsilon_1,|f_2|=\varepsilon_2} \phi f_1 f_2 = \int_{|f_1|=\varepsilon_1,|f_2|>|\varepsilon_2|} d_f \phi (f_1 f_2) = \int_{|f_1|=\varepsilon_1,|f_2|>|\varepsilon_2|} \frac{1}{f_1} \partial f_2 (\frac{\phi}{f_2})
\]
\[
= \int_{|f_1|>|\varepsilon_1,|f_2|>|\varepsilon_2|} d_f \phi (\frac{1}{f_1} \partial f_2 (\frac{\phi}{f_2})) = \int_{|f_1|>|\varepsilon_1,|f_2|>|\varepsilon_2|} \partial f_1 (\frac{1}{f_1} \partial f_2 (\frac{\phi}{f_2})).
\]
In general \( I_f^\phi (\varepsilon) \) does not have a limit as \( \varepsilon \) tends to 0 and it is easy to find examples that illustrate this, see for instance Example 7.1.3 in [8]. However, Coleff and Herrera showed that \( \lim_{\varepsilon \to 0} I_f^\phi (\varepsilon) \) does exist if one lets \( \varepsilon \) tend to zero along a certain path, a so called admissible trajectory, for which each \( \varepsilon_j \) tends to zero faster than any power of \( \varepsilon_{j+1} \). In the case of a complete intersection, that is, when the variety \( Y \) has codimension \( m \), the limit is independent of the ordering of the functions and therefore, in this case, it is reasonable to expect the limit of \( I_f^\phi (\varepsilon) \) to exist unconditionally. There are however counterexamples by Passare and Tsikh [11] and Björk [5] that show that this is not the case.

One way to circumvent the problem of such “bad” trajectories is to consider the residue current as certain averages of \( I_f^\phi (\varepsilon) \), as first done by Passare in [9]. In order to avoid integration over the possibly singular varieties \( \{|f| = \varepsilon\} \) he modifies the definition of residue currents of Coleff and Herrera [6] somewhat. Regarding the currents as limits of smooth forms he defines \( [1/f] \) and \( \partial [1/f] \) as
\[
\lim_{\varepsilon \to 0} \frac{1}{f} \chi(|f|/\varepsilon) \quad \text{and} \quad \lim_{\varepsilon \to 0} \frac{1}{f} \partial \chi(|f|/\varepsilon),
\]
respectively, where \( \chi(x) \) is a smooth nondecreasing function, equal to 0 in a neighborhood of the origin and equal to 1 for \( x > c \) for some \( c > 0 \). One can show that the currents are independent of the particular choice of \( \chi \). Further, products of principal value and residue currents are constructed, based on those definitions. Let
\[
R_{m'}^m \rho_{(m-m')} \left[ \frac{1}{f} \right] (s) = \lim_{\varepsilon \to 0} \frac{1}{f_1} \cdots \frac{1}{f_m} \partial \chi_1 \wedge \cdots \wedge \partial \chi_{m'} \chi_{m'+1} \cdots \chi_m,
\]
where \( \chi_j = \chi(|f_j|/\varepsilon s) \). Here, \( R_{m'}^m \rho_{(m-m')} \) should be interpreted as the product of the residue currents of \( f_1, \ldots, f_m \) and the principal value currents of \( f_{m'+1}, \ldots, f_m \). In particular, \( R_{m'}^m [1/f] (s) \) corresponds to \( I_f^\phi (\varepsilon^n, \ldots, \varepsilon^m) \).

Finally, the product \( R_{m'}^m \rho_{(m-m')} [1/f] \) is defined as the mean value of \( R_{m'}^m \rho_{(m-m')} [1/f] (s) \) over the simplex
\[
\Sigma_m = \{ s \in \mathbb{R}^m, s_j > 0, \sum s_j = 1 \},
\]
that is,
\[ R^{m'} \mathcal{P}^{(m-m')}[1/f] = \int_{\Sigma_m} R^{m'} \mathcal{P}^{(m-m')}[1/f](s) dS_m(s), \]
where \( S_m \) is the Lebesgue measure, normalized with respect to \( \Sigma_m \). In case \( f \) defines a complete intersection, the current \( R^m[1/f](s) \) is shown to agree with the Coleff-Herrera residue current, as expected since \( \lim_{\varepsilon \to 0} I(\varepsilon) \) is then independent of the path along which \( \varepsilon \) tends to 0 as long as it is an admissible trajectory.

Another way of averaging \( I_f^\phi(\varepsilon) \), is to take the Mellin transform of \( I_f^\phi(\varepsilon) \),
\[ \Gamma_f^\phi(\lambda) = \int_{\mathbb{R}^n} I_f^\phi(\varepsilon) \varepsilon^{\lambda-1} d\varepsilon, \]
where \( \lambda = (\lambda_1, \ldots, \lambda_m) \in \mathbb{C}^n \) and \( \varepsilon^{\lambda-1} d\varepsilon = \varepsilon_1^{\lambda_1-1} \cdots \varepsilon_m^{\lambda_m-1} d\varepsilon_1 \wedge \cdots \wedge d\varepsilon_m \). A computation shows that
\[ \Gamma_f^\phi(\lambda) = \frac{1}{\lambda_1 \cdots \lambda_m} \int_{\mathbb{R}^n} \nabla f_1 |f_1|^{2\lambda_1} \wedge \cdots \wedge \nabla f_m |f_m|^{2\lambda_m} \wedge \phi. \]
Now, \( \Gamma_f^\phi(\lambda) \) is meromorphic in \( \mathbb{C}^n \), and the polar structure at the origin is related to the limit behaviour of the residue integral, see [10]. In fact, if \( f \) is a complete intersection, it holds that
\[ \lambda^m \Gamma_f^\phi(\lambda_1, \ldots, \lambda_m)|_{\lambda=0} = \lim_{\varepsilon \to 0} I_f^\phi(\varepsilon), \]
where the limit is taken along an admissible trajectory. Thus, this yields an alternative definition of the Coleff-Herrera current. In the particular case when \( f \) consists of only two functions, the mapping \( \lambda \mapsto \lambda_1 \lambda_2 \Gamma_f^\phi(\lambda) \) is holomorphic at the origin as first shown by Berenstein and Yger, see the proof of Theorem 3.18 in [3]. Although not yet satisfactorily proven, this result is believed to extend to any finite number of functions.

The residue currents we have studied so far are based on the Cauchy kernel \( K = 1/(2\pi i) \, d\zeta/\zeta \). From the definition of the one-dimensional residue current, one sees that \( R^c \) times \( d\zeta \) can be expressed as \( \partial K \), and so \( R^c \wedge df = \partial f^* K \). It follows that the Coleff-Herrera current can be regarded as the pull-back under \( f \) of the multiple Cauchy kernel
\[ 1/(2\pi i)^m d\zeta_1/\zeta_1 \wedge \cdots d\zeta_m/\zeta_m, \]
and by that as the formal product of one-dimensional residue currents. Another approach to the multidimensional residue current would be to use other generalizations of the one-dimensional Cauchy kernel. In [12], Passare, Tsikh
and Yger define a residue current based on the Bochner-Martinelli kernel. Let
\[ b(\zeta) = \frac{\sum \zeta_j d\zeta_j}{2\pi i|\zeta|^2} \]
The Bochner-Martinelli kernel in \( \mathbb{C}^n \setminus \{0\} \) is given by
\[ B(\zeta) = b(\zeta) \wedge \left( \partial b(\zeta) \right)^{m-1} = \frac{\sum (-1)^{j-1} \zeta_j d\zeta_j}{(2\pi i)^m |\zeta|^{2m}} \],
where \( d\zeta_j \) denotes the form \( d\zeta_1 \wedge \ldots \wedge d\zeta_{j-1} \wedge d\zeta_{j+1} \wedge \ldots \wedge d\zeta_m \). Clearly, \( B(\zeta) \) is in \( L^1_{loc} \) and it is easy to show that
\[ \partial B(\zeta) = [0] = \frac{1}{(2\pi i)^m} \frac{1}{\zeta_m} \wedge \ldots \wedge \frac{1}{\zeta_1} \wedge d\zeta_1 \wedge \ldots \wedge d\zeta_m, \]
where the right hand expression should be recognized as the Coleff-Herrera current times \( d\zeta_1 \wedge \ldots \wedge d\zeta_m \). If we let \( B(f) = f^* B \) then \( B(f) \) is clearly smooth outside \( Y \). If \( f \) is regular on \( Y \), it is easy to see that \( B(f) \) is in \( L^1_{loc} \) and that \( \partial B(f) \) coincides with the Coleff-Herrera current \( R_{CH}^f \). Indeed, after a coordinate change we are back to the case (1.2). Thus, it is natural to expect the current \( \partial B(f) \) to have meaning also for a more general \( f \). Let
\[ R_{BM}^f(\varepsilon) = \int_{|f|^2 = \varepsilon} B(f) \wedge \phi \]
where \( |f|^2 = \sum |f_j|^2 \). In [12] it is shown that the limit of \( R_{BM}^f(\varepsilon) \) as \( \varepsilon \) tends to zero, always exists. This is done by proving the existence of the meromorphic continuation of the corresponding Mellin transform. We denote the limit by \( R_{BM}^f \). Moreover we have

**Theorem 1.1** (Passare, Tsikh, Yger [12]). Assume that \( f \) is a complete intersection. Then
\[ R_{BM}^f = R_{CH}^f. \]

The Bochner-Martinelli residue current has been used for investigations in the noncomplete intersection case. For example, in [4], Berenstein and Yger use the Bochner-Martinelli current to construct Green currents.

In [1], Andersson introduces an alternative approach to \( R_{BM}^f \), based on the Koszul complex. Considering sections of holomorphic vector bundles rather than tuples of functions yields globally defined currents. For further reference, we give a presentation of the construction. As in the one-dimensional case we start out by looking for holomorphic solutions to \( fU = 1 \), now read as \( \sum_{j=1}^m f_j U_j = 1 \). We adopt an invariant point of view and assume that \( f \) is a holomorphic section of the dual bundle \( E^* \) of a holomorphic \( m \)-bundle \( E \rightarrow X \) over a complex manifold \( X \). If \( e_1, \ldots, e_m \) is a
local holomorphic frame for $E$ and $e^*_i, \ldots, e^*_m$ is the dual frame, we can write $f = \sum f_j e^*_j$. On the exterior algebra over $E$ we have mappings

$$\delta_f : \Lambda^{\ell+1}E \to \Lambda^\ell E,$$

where $\delta_f$ is the interior multiplication by $f$. In particular, $\delta_f$ acts on a section $\psi = \sum \psi_j e_j$ of $E = \Lambda^1 E$ as $\delta_f \psi = \sum f_j \psi_j$, and thus we can formulate the original problem as finding holomorphic solutions to

(1.3) \quad \delta_f \psi = 1.

This kind of division problem can be solved by means of the Koszul complex. We start by looking for smooth solutions to (1.3). Outside $Y = f^{-1}(0)$ we can easily find one; $u_1 = \sum \tilde{f}_j / |f|^2$. However, in general $u_1$ is not holomorphic, so we need to compensate for that. Now, let us introduce the spaces $\mathcal{E}_{0,k}(X, \Lambda^\ell E)$ and $\mathcal{D}'_{0,k}(X, \Lambda^\ell E)$ of $(0,k)$ forms and currents, respectively, that take values in $\Lambda^\ell E$. Note that $\delta_f$ and $\bar{\partial}$ extend to the exterior algebra over $T^*_0 \oplus E$, where they anticommute. Thus, $\delta_f \bar{\partial} u_1 = -\bar{\partial} \delta_f u_1 = 0$, and since the Koszul complex of $f$ is exact outside $Y$ we can find a $u_2 \in \mathcal{E}_{0,1}(X, \Lambda^2 E)$ such that $\delta_f u_2 = \bar{\partial} u_1$. We proceed by successively solving

(1.4) \quad \delta_f u_k = \bar{\partial} u_{k-1},

where $u_k \in \mathcal{E}_{0,k-1}(X, \Lambda^k E)$. This procedure will terminate after a finite number of steps. In fact, $\bar{\partial} u_m = 0$, and so, if $X$ is Stein, by successively solving equations

(1.5) \quad \bar{\partial} v_{k-1} = u_{k-1} + \delta_f v_k

for $k \leq m$, we finally arrive at the desired holomorphic solution

$$\psi = u_1 + \delta_f u_2$$

to (1.3). Now, letting

$$\mathcal{L}^r(X, E) = \bigoplus_{k+\ell = r} \mathcal{D}'_{0,k}(X, \Lambda^{-\ell} E),$$

we introduce

$$\nabla_f = \delta_f - \bar{\partial} : \mathcal{L}^r \to \mathcal{L}^{r+1}.$$ 

Note that $\nabla_f$ is an antiderivation on the exterior algebra over $T^*_0 \oplus E$. With this notation, the system of equations (1.3) and (1.4) can now be written as

(1.6) \quad \nabla_f u = 1,

where $u \in \mathcal{L}^{-1}(X, E)$. To find a solution to (1.6) in $X \setminus Y$, let us assume that $E$ is equipped with some Hermitian metric and let $s$ be the section of $E$. 

with pointwise minimal norm such that \( \delta_f s = |f|^2 \). Outside \( Y = f^{-1}(0) \), the Cauchy-Fantappiè-Leray form

\[
    u = \frac{s}{\nabla_f s} = \frac{s}{\delta_f s - \partial_s} = \sum_\ell \frac{s \wedge (\bar{\partial} s)_{\ell-1}}{(\delta_f s)_{\ell}} = \sum_\ell \frac{s \wedge (\bar{\partial} s)_{\ell-1}}{|f|^{2\ell}}
\]

is well-defined (observe that \( \bar{\partial} s \) is of even degree), and \( \nabla u = 1 \). In the trivial metric the term of top degree corresponds to the Bochner-Martinelli form. Indeed, \( s = \sum_i \bar{f}_i e_j \), so that

\[
    u = \sum_\ell \frac{\sum_i \bar{f}_i e_j \wedge (\sum_i \bar{\partial} \bar{f}_i e_j)_{\ell-1}}{|f|^{2\ell}}.
\]

We wish to extend \( u \) to the entire \( X \). If we could find an extension such that (1.6) still holds, by solving equations (1.5), this would imply that 1 belongs to the ideal generated by \( f \), which is clearly not possible in general. Thus, there will appear residues. In [1], the existence of an analytic continuation of \( \lambda \to |f|^{2\lambda} u \) to \( \text{Re } \lambda > -\epsilon \) is proven. The value at \( \lambda = 0 \), denoted by \( U \), yields the desired extension of \( u \). (By analogy with the one-dimensional case, we will sometimes refer to \( U \) as the principal value current.) Moreover, \( \nabla U = 1 - R^f \), where \( R^f = \bar{\partial}|f|^{2\lambda} \wedge u|_{\lambda=0} \) now defines the residue current of \( f \). If \( f \) is a complete intersection, then \( R^f = R^f_{BM} \wedge e_1 \wedge \ldots \wedge e_m \), where \( R^f_{BM} \) is the Bochner-Martinelli current. Furthermore, from the construction it is easily verified that a holomorphic function that annihilates \( R^f \) belongs to the ideal generated by \( f \) locally.

To sum up, given \( f \) a function of functions \( f_1, \ldots, f_r \), we have discussed essentially two different ways of associating a residue current to \( f \). Either we can consider the product of the residue currents corresponding to each function \( f_i \) of the tuple, that is the Coleff-Herrera current, or we can define \( R^f \) by means of the Cauchy-Fantappiè-Leray (or Bochner-Martinelli) form of \( f \). Considering this, we are led to ask whether it is possible to define products not only of one function currents but also of the multifunction currents. That is, given two tuples of functions \( f \) and \( g \), is there a way to give meaning to the expression \( R^f \wedge R^g \)? Let us recall that, by the Mellin transform, the Coleff-Herrera current of \( f \) is equal to the value at \( \lambda = 0 \) of

\[
    \bar{\partial}|f_i|^{2\lambda} \frac{1}{f_1} \wedge \ldots \wedge \bar{\partial}|f_r|^{2\lambda} \frac{1}{f_r}.
\]

Now, if we assume that each \( f_i \) is a section of a line bundle \( L_i^* \) with frame \( e_i^* \), the Cauchy-Fantappiè-Leray form is just \( u^h = e_i/f_i \), so in fact this product times the frame elements \( e_1 \wedge \ldots \wedge e_r \) can be expressed as

\[
    \bar{\partial}|f_i|^{2\lambda} \wedge u^{f_i} \wedge \ldots \wedge \bar{\partial}|f_r|^{2\lambda} \wedge u^{f_r}.
\]
In light of this, it is most tempting to extend this definition to include not only sections of line bundles but sections $f_i$ of bundles of arbitrary rank. To be more accurate, we assume that $f_i$ is a section of the dual bundle of a hermitian $m_i$-bundle $E_i \to X$. Further, assuming that each $E_i$ is equipped with a hermitian metric, let $s_i$ be the section of $E_i$ of minimal norm such that $\delta_{f_i} s_i = |f_i|^2$, and let $u^{f_i}_j$ be the Cauchy-Fantappiè-Leray form defined as above. Since a section $f$ of a bundle $E$ has a natural interpretation as a section of a bundle $E \oplus E'$, that is as $f \oplus 0$, we have given meaning to the expression (1.7) as a form taking values in the exterior algebra over $E = E_1 \oplus \cdots \oplus E_r$. Thus, in accordance with the line bundle case, we can take the value at $\lambda = 0$ of (1.7) as a definition of $R^{f_1} \wedge \cdots \wedge R^{f_r}$, provided that the analytic continuation exists. However, this is assured by Proposition 1.2, where products are defined also of principal value currents.

**Proposition 1.2.** Let $f_i$ be a holomorphic section of the hermitian $m_i$-bundle $E_i^* \to X$. Let $u^{f_i}_j$ be the corresponding Cauchy-Fantappiè-Leray form and let $Y_i = f_i^{-1}(0)$. Then

$$\lambda \mapsto |f_i|^{2\lambda} u^{f_i} \wedge \ldots \wedge |f_{s+1}|^{2\lambda} u^{f_{s+1}} \wedge \bar{\partial}|f_s|^{2\lambda} u^{f_s} \wedge \ldots \wedge \bar{\partial}|f_1|^{2\lambda} u^{f_1}$$

has an analytic continuation as a current to $\lambda > -\epsilon$.

We define $T = U^{f_1} \wedge \ldots \wedge U^{f_{s+1}} \wedge R^{f_s} \wedge \ldots \wedge R^{f_1}$ as the value at $\lambda = 0$. Then $T$ has support on $\bigcap_{i=1}^s Y_i$ and it is alternating with respect to the principal value factors $U$ and commutative with respect to the residue factors $R$.

Of course there is nothing special about the ordering that we have chosen, that is first the principal value factors and then the residue factors. We can just as well mix $U$’s and $R$’s.

Since the term of top degree of $u^{f_i}$ corresponds to the classical Bochner-Martinelli kernel, the term of top degree of $R^{f_1} \wedge \ldots \wedge R^{f_1}$ can be interpreted as a product of Bochner-Martinelli currents in the sense of [12]. In general, however, there will also occur terms of lower degree. More precisely we have

**Proposition 1.3.** Let

$$T = U^{f_1} \wedge \ldots \wedge U^{f_{s+1}} \wedge R^{f_s} \wedge \ldots \wedge R^{f_1}$$

be defined as above. Let $m = m_1 + \ldots + m_r$. Then $T = T_p + \ldots + T_q$, where $T_i \in P_{0,l}(\lambda^* E)$, $p = \text{codim } Y_1 \cap \ldots \cap Y_s$ and $q = \min(m, n)$. In particular, if $f$ is a complete intersection, then $R^{f_1} \wedge \ldots \wedge R^{f_1}$ consists of only one term of top degree $m$.

Notice, that in the particular case when the bundles $E_i$ are all line bundles and thus the sections $f_i$ correspond to single functions, the current

$$R^{f_1} \wedge \ldots \wedge R^{f_r}$$
is just the Coleff-Herrera current of $f$ times $e_1 \wedge \ldots \wedge e_r$. Hence, we can formulate the equality in Theorem 1.1 as
\begin{equation}
R^{f_1 \oplus \cdots \oplus f_r} = R^{f_1} \wedge \ldots \wedge R^{f_r}.
\end{equation}

Now, the obvious question is, does this equality extend to hold for sections of vector bundles of arbitrary rank. Our main result states that this is indeed the case.

**Theorem 1.4.** Let $f_i$ be a holomorphic section of the Hermitian $m_i$-bundle $E_i^*$ and let $f$ denote the section $f_1 \oplus \cdots \oplus f_r$ of $E^\ast = E_1^\ast \oplus \cdots \oplus E_r^\ast$. If $f$ is a complete intersection, that is codim $f^{-1}(0) = m_1 + \ldots + m_r$, then
\begin{equation}
R^f = R^{f_1} \wedge \ldots \wedge R^{f_r}.
\end{equation}

That is, in a local perspective, given a tuple of functions split into sub-tuples, the product of the Bochner-Martinelli currents of each subtuple is equal to the Bochner-Martinelli current of the whole tuple of functions. We give an explicit proof of Theorem 1.4, involving finding a potential to the current $1 - R^{f_1} \wedge \ldots \wedge R^{f_r}$. Indeed we prove

**Theorem 1.5.** Let $f = f_1 \oplus \cdots \oplus f_r$ be a section of $E^* = E_1^* \oplus \cdots \oplus E_r^*$. Assume that $f$ is a complete intersection. Then there exists a current $V$ such that
\begin{equation}
\nabla_f V = 1 - R^{f_1} \wedge \ldots \wedge R^{f_r},
\end{equation}
and furthermore a current $U^f \wedge V$ such that
\begin{equation}
\nabla_f(U^f \wedge V) = V - U^f.
\end{equation}

At first it might seem a bit peculiar to denote the second potential by $U^f \wedge V$. However, notice that on a formal level, if we were allowed to multiply currents so that $\nabla_f$ acted as an antiderivation on the products, then
\begin{equation}
\nabla_f(U^f \wedge V) = (1 - R^f) \wedge V - U^f(1 - R^{f_1} \wedge \ldots \wedge R^{f_r}),
\end{equation}
since $U^f$ is of odd degree. From Proposition 1.3 we know that $R^f$ and $R^{f_1} \wedge \ldots \wedge R^{f_r}$ are of top degree, $m$, in $d\bar{z}_j$, since $f$ is a complete intersection, and thus it is reasonable to expect also the products $R^f \wedge V$ and $U^f \wedge R^{f_1} \wedge \ldots \wedge R^{f_r}$ to be of degree $m$ in $d\bar{z}_j$. However, since $U^f$ and $V$ are both in $L^{-1}(X, E)$, this is not possible unless the products vanish. Thus we are left with $V - U^f$, and the notation is motivated.

**Proof of Theorem 1.4.** Recall that $\nabla_f U^f = 1 - R^f$. Hence, applying $\nabla_f$ twice to $U^f \wedge V$ yields
\begin{equation}
0 = \nabla^2_f(U^f \wedge V) = \nabla_f(U^f - V) = R^{f_1} \wedge \ldots \wedge R^{f_r} - R^f,
\end{equation}
and thus we are done. $\blacksquare$
The disposition of this paper is as follows. In Section 2 we give proofs of Proposition 1.2 and Proposition 1.3. In Section 3 we prove Theorem 1.5. Finally, in Section 4 we give some examples of products of Cauchy-Fantappiè-Leray currents.

2. Products of residue currents of Cauchy-Fantappiè-Leray type

We start with the proof of Proposition 1.2. For further use a slightly more general formulation is appropriate. Namely, the proof of Theorem 1.5 requires a broader definition of products of currents. We need to allow also products of currents of sections of the bundle $E$, that are not necessarily orthogonal, at least in certain cases. Thus we give a new, somewhat unwieldy, version of Proposition 1.2 that however covers all the currents that we will be concerned with.

**Proposition 2.1.** Let $f = f_1 \oplus \ldots \oplus f_r$ be a holomorphic section of the bundle $E^* = E^*_1 \oplus \ldots \oplus E^*_r$, where $E^*_i$ is a Hermitian $m_i$-bundle. For a subset $I = \{I_1, \ldots, I_p\}$ of $\{1, \ldots, r\}$, let $f_I$ denote the section $f_{I_1} \oplus \ldots \oplus f_{I_p}$ of $E^*_I = E^*_{I_1} \oplus \ldots \oplus E^*_{I_p}$; let $u_I^I$ be the corresponding Cauchy-Fantappiè-Leray form, let $Y_I = f_I^{-1}(0)$, and let $m_I = m_{I_1} + \ldots + m_{I_p}$. If $I^1, \ldots, I^q$ are subsets of $\{1, \ldots, r\}$, then

$$
\lambda \mapsto |f_{I^1}|^{2\lambda} u_{I^1}^{I^1} \wedge \ldots \wedge |f_{I^1+1}|^{2\lambda} u_{I^1+1}^{I^1+1} \wedge \bar{\nabla} |f_{I^2}|^{2\lambda} \wedge u_{I^2}^{I^2} \wedge \ldots \wedge \bar{\nabla} |f_I|^{2\lambda} \wedge u_I^I
$$

has an analytic continuation to $\lambda > -\epsilon$.

We define $T = U_I^{J^1} \wedge \ldots \wedge U_I^{J^q} \wedge R^{I^{J^1}} \wedge \ldots \wedge R^{I^{J^q}}$ as the value at $\lambda = 0$. Then $T$ has support on $\bigcap_{i=1}^s Y_I^i$ and it is alternating with respect to the principal value factors $U$ and commutative with respect to the residue factors $R$.

Note that Proposition 1.2 corresponds to the particular case when each $I^i$ is just a singleton. The proof of Proposition 2.1 is very much inspired by the proof of Lemma 2.2 in [12] and Theorem 1.1 in [1]. It is based on the possibility to resolve singularities by Hironaka's theorem and the following lemma, which is proven essentially by integration by parts.

**Lemma 2.2.** Let $v$ be a strictly positive smooth function in $C$, $\varphi$ a test function in $C$, and $p$ a positive integer. Then

$$
\lambda \mapsto \int v^\lambda |s|^{2\lambda} \varphi(s) \frac{ds \wedge d\bar{s}}{sp}
$$

and

$$
\lambda \mapsto \int \bar{\nabla}(v^\lambda |s|^{2\lambda}) \wedge \varphi(s) \frac{ds}{sp}
$$
both have meromorphic continuations to the entire plane with poles at rational points on the negative real axis. At \( \lambda = 0 \) they are both independent of \( v \), and the second one is a distribution of \( \varphi \) supported at the origin and they only depend on powers of \( \partial / \partial s \) of the test function \( \varphi \). Moreover, if \( \varphi(s) = \exists \psi(s) \) or \( \varphi = d \xi \wedge \psi \), then the value of the second integral at \( \lambda = 0 \) is zero.

Proof of Proposition 2.1. We may assume that the bundle \( E = E_1 \oplus \cdots \oplus E_p \) is trivial since the statement is clearly local. Note that \( f_i = \sum f_{i,j} e_{i,j} \), where \( e_{i,j} \) is the trivial frame. The proof is based on the possibility to resolve singularities locally using Hironaka's theorem. Given a small enough neighborhood \( \mathcal{U} \) of a given point in \( X \) there exist a \( n \)-dimensional manifold \( \tilde{\mathcal{U}} \) and a proper analytic map \( \Pi_h : \tilde{\mathcal{U}} \to \mathcal{U} \) such that if \( Z = \{ \Pi_{i,j} f_{i,j} = 0 \} \) and \( Z = \Pi_h^{-1}(Z) \), then \( \Pi : \mathcal{U} \setminus Z \to \mathcal{U} \setminus Z \) is biholomorphic and such that moreover \( Z \) has normal crossings in \( \tilde{\mathcal{U}} \). This implies that locally in \( \mathcal{U} \) we have that \( \Pi_{i,j} f_{i,j} = a_{i,j} \mu_{i,j} \), where \( a_{i,j} \) are non-vanishing and \( \mu_{i,j} \) are monomials in some local coordinates \( \tau_k \). Further, given a finite number of monomials \( \mu_1, \ldots, \mu_m \) in some coordinates \( \tau_k \) defined in an \( n \)-dimensional manifold \( \mathcal{U}_t \), there exists a toric manifold \( \mathcal{U}_t \) and a proper analytic map \( \Pi_t : \mathcal{U}_t \to \mathcal{U}_t \) such that \( \Pi_t \) is biholomorphic outside the coordinate axes and moreover, locally it holds that, for some \( i, \Pi_t^* \mu_i \) divides all \( \Pi_t^* \mu_j \), see [3] and [7]. Clearly, if \( \mu_i \) divides \( \mu_j \) in \( \mathcal{U}_t \) then \( \Pi_t^* \mu_i \) divides \( \Pi_t^* \mu_j \) in \( \mathcal{U}_t \). Thus after a number, say \( q \), of such toric resolutions \( \Pi_{i_1} \) we can locally consider each section \( f_{i_1} \) as a monomial times a non-vanishing section. More precisely we have that \( \Pi^* f_{i_1} = \mu_i f_{i_1} \), where \( \Pi = \Pi_{i_q} \circ \cdots \circ \Pi_{i_1} \circ \Pi_h, \mu_i \) is a monomial and \( f_{i_1} \) is a non-vanishing section of \( E_{i_1} \).

Let \( \phi \) be a test form with compact support. After a partition of unity we may assume that it has support in a neighborhood \( \mathcal{U} \) as above. Then, since \( \Pi_h \) is proper, the support of \( \Pi_h^* \phi \) can be covered by a finite number of neighborhoods in which it holds that \( \Pi_h^* \phi = a_{i,j} \mu_{i,j} \). If \( \psi \) is a test form with support in such a neighborhood, then the support of \( \Pi_h^* \psi \) can be covered by finitely many neighborhoods in which we have the desired property that the pull-back of one monomial divides some of the other ones, and so on. Thus, for \( \text{Re} \lambda > 2 \max_i m_{i_1}, (2.1) \) is in \( L^1_{\text{loc}} \). Since \( \Pi \) is biholomorphic outside a zero-set we have that

\[
\int |f_{i_1}|^{2\lambda} u_f^{f_i} \wedge \ldots \wedge |f_{i_{s+1}}|^{2\lambda} u_f^{f_{i_{s+1}}} \wedge \bar{\partial}|f_{i_1}|^{2\lambda} u_f^{f_i} \wedge \ldots \wedge \bar{\partial}|f_{i_{s+1}}|^{2\lambda} u_f^{f_{i_{s+1}}} \wedge \phi
\]

is equal to a finite number of integrals of the form

\[
(2.2)
\]

\[
\Pi^*(|f_{i_1}|^{2\lambda} u_f^{f_i} \wedge \ldots \wedge |f_{i_{s+1}}|^{2\lambda} u_f^{f_{i_{s+1}}} \wedge \bar{\partial}|f_{i_1}|^{2\lambda} u_f^{f_i} \wedge \ldots \wedge \bar{\partial}|f_{i_{s+1}}|^{2\lambda} u_f^{f_{i_{s+1}}} \wedge \phi.
\]
Here
\[ \phi = \rho_1 \Pi_1^*(\ldots \rho_t \Pi_t^*(\rho_\eta \Pi_\eta^*(\phi)) ), \]
where the \( \rho_i \)'s are functions from some partitions of unity, so that the test form \( \phi \) has support in a neighborhood where it holds that \( \Pi^* f_{j_i} = \mu_i f^*_{j_i} \).
In such a coordinate neighborhood the pullback of \( s_{j_i} \) is \( \tilde{\mu}_i \) times a smooth form, so that \( \Pi^* (s_{j_i} \wedge (\partial s_{j_i})^{\ell-1}) \) is \( \tilde{\mu}_i^{\ell} \) times a smooth form. Moreover
\[ \Pi^* |f_{j_i}|^2 = |\mu_i|^2 a_i, \]
where \( a_i \) is a strictly positive smooth function. Thus
\[ \Pi^* u^{f_{j_i}} = \sum \tilde{\mu}_i |\alpha_i|^2 = \sum \alpha_i \mu_i, \]
where \( \alpha_i \) are smooth forms (taking values in \( \Lambda^2 E \)) and so (2.2) is equal to a finite sum of integrals
\[ (2.3) \int |\mu_1|^{2\lambda} a_{11}^{\lambda} \wedge \ldots \wedge |\mu_1|^{2\lambda} a_{11}^{\lambda} \wedge \alpha_{1,1,1,1} \wedge \wedge \alpha_{1,1,1,1} \wedge \wedge \phi. \]
Expanding each factor \( \widetilde{\partial}(|\mu_1|^{2\lambda} a_{11}^{\lambda}) \) by Leibniz’ rule results in a finite sum of terms. Letting \( \tilde{\partial} \) fall only on the monomials \( \mu_i \) yields integrals of the form
\[ (2.4) \int a_{1,1,1}^{\lambda} |\mu_{1,1,1}^{2\lambda} \frac{\alpha_{1,1,1}}{\mu_{1,1,1}} \wedge \wedge \partial|\sigma_{1,1,1}^{\lambda} |^{2\lambda} \wedge \wedge \partial|\sigma_{1,1,1}^{\lambda} |^{2\lambda} \wedge \phi, \]
where \( \sigma_1 \) is one of the coordinate functions \( \tau_j \) that divide \( \mu_i, a = a_1 \cdot a_1 \) is a strictly positive smooth function, \( \mu' \) and \( \mu'_{1,1,1} \) are monomials in \( \tau_j \) not divisible by any \( \sigma_i \) and \( \alpha_{1,1,1} = C \alpha_{1,1,1} \wedge \ldots \wedge \alpha_{1,1,1} \) is a smooth form, where \( C \) is just a constant that depends on the relation between \( q_{1,1,1} \) and the number of \( \sigma_i \)'s in \( \mu_i \). The remaining integrals, that arise when \( \tilde{\partial} \) falls on any of the \( a_i \), vanish in accordance with Lemma 2.2. Indeed, consider one of the integrals obtained when \( \tilde{\partial} \) falls on \( a_1 \),
\[ \lambda \int a_{1,1,1}^{\lambda} |\mu_{1,1,1}^{2\lambda} \frac{\alpha_{1,1,1}}{\mu_{1,1,1}} \wedge \wedge \partial|\sigma_{1,1,1}^{\lambda} |^{2\lambda} \wedge \wedge \partial|\sigma_{1,1,1}^{\lambda} |^{2\lambda} \wedge \phi. \]
This is just \( \lambda \) times an integral of the form (2.4), so provided that we can prove the existence of an analytic continuation of (2.4), it must clearly vanish at \( \lambda = 0 \).

Now an application of Lemma 2.2 for each \( \tau_k \) that divides any of the \( \mu_j \)'s gives the desired analytic continuation of (2.4) to \( \lambda > -\epsilon \). Note that for \( \sigma_1, \ldots, \sigma_s \) we get integrals of the second type, for the remaining \( \tau_i \) integrals of the first type, so that the value at \( \lambda = 0 \) is a current with support on \( \{ \sigma_s = 0 \} \cap \ldots \cap \{ \sigma_1 = 0 \} \). Thus the value of (2.3) at \( \lambda = 0 \) has support on
\[ \{ \mu_s = 0 \} \cap \ldots \cap \{ \mu_1 = 0 \} = \bar{Y}_{fs} \cap \ldots \cap \bar{Y}_{f1}, \]
where \( \bar{Y}_s = \Pi^{-1} Y_s \), and accordingly \( U^{f_1} \land \ldots \land U^{f_{s+1}} \land R^{f_{s+1}} \land \ldots \land R^{f_1} \) is a current with support on \( Y_1 \cap \ldots \cap Y_s \).

Since the form (2.1) is alternating with respect to the factors \( |f_1|^{2 \lambda} u^{f_1} \) and commutative with respect to the factors \( \bar{\partial}|f_1|^{2 \lambda} \land u^{f_1} \), it follows that \( U^{f_1} \land \ldots \land U^{f_{s+1}} \land R^{f_{s+1}} \land \ldots \land R^{f_1} \) is alternating with respect to the principal value factors and commutative with respect to the residue factors. \( \square \)

We continue with the proof of Proposition 1.3.

**Proof of Proposition 1.3.** Notice that \( T_\ell \) is the analytic continuation to \( \lambda = 0 \) of the terms

\[
(2.5) \quad |f_\ell|^{2 \lambda} u^{f_\ell} \land \ldots \land |f_{s+1}|^{2 \lambda} u^{f_{s+1}} \land \bar{\partial}|f_s|^{2 \lambda} \land u^{f_s} \land \ldots \land \bar{\partial}|f_1|^{2 \lambda} \land u^{f_1},
\]

where

\[
u^{f_i} = \frac{a_i \land (\bar{\partial} s_j)^{\ell_i-1}}{|f_i|^{2 \ell_i}}
\]

and the total degree in \( d\bar{z}_j \) (that is \( \ell_1 + \ldots + \ell_r - r + s \)) is \( \ell \). By the notion that a form (or current) is of degree \( k \) in \( d\bar{z}_j \), we just mean that it is a \((r,k)\)-form. In the same manner, we will say that a form is of degree \( \ell \) in \( e_j \) when it takes values in \( \Lambda^\ell E \).

Following the proof of Proposition 2.1, a term of the form (2.5), integrated against a test form \( \phi \), is equal to a sum of terms like

\[
(2.6) \quad \int |\mu_\ell|^{2 \lambda} a_\ell^\lambda \frac{\alpha_{\ell_1,\ell}}{\mu_\ell} \land \ldots \land |\mu_{s+1}|^{2 \lambda} a_{s+1}^\lambda \frac{\alpha_{s+1,\ell_{s+1}}}{\mu_{s+1}} \land \bar{\partial}(|\mu_s|^{2 \lambda} a_s^\lambda) \land \ldots \land \bar{\partial}(|\mu_1|^{2 \lambda} a_1^\lambda) \land \frac{\alpha_{1,\ell_1}}{\mu_1} \land \bar{\partial} \phi,
\]

where the \( \alpha_{i,\ell_i} \)'s are smooth forms of degree \( \ell_i \) in \( e_j \), the \( a_i \)'s are non-vanishing functions, the \( \mu_i \)'s are monomials in some local coordinates \( \tau_j \) and \( \phi \) is as in the previous proof. We can find a toric resolution such that locally one of \( \mu_1, \ldots, \mu_s \) divides the other ones, so without loss of generality we may assume that \( \mu_1 \) divides \( \mu_2, \ldots, \mu_s \).

We expand \( \bar{\partial}(|\mu_1|^{2 \lambda} a_1^\lambda) \) by Leibniz' rule. Observe that when \( \bar{\partial} \) falls on \( a_1^\lambda \) the integral vanishes as in the proof of Proposition 2.1, and thus it suffices to consider the case when \( \bar{\partial} \) falls on one of the \( \tau_j \) that divide \( \mu_1 \), say on \( |\sigma|^{2 \lambda} \). If \( \ell < p \), we claim that this part of (2.6) vanishes when integrating with respect to \( \sigma \). In fact, we may assume that \( \phi = \phi_1 \land d\bar{z}_I \), where \( \phi_1 \) is an \((n,0)\)-form and \( d\bar{z}_I = d\bar{z}_{I_1} \land \ldots \land d\bar{z}_{I_m} \). Now \( d\bar{z}_I \) vanishes on the variety \( Y_1 \cap \ldots \cap Y_s \) of codimension \( p \) for degree reasons. Consequently \( \Pi^*(d\bar{z}_I) \) vanishes on \( \bar{Y}_1 \cap \ldots \cap \bar{Y}_s \), and in particular on \( \{\sigma = 0\} \). However, this is a form in \( d\bar{r}_k \) with antiholomorphic coefficients since \( \Pi \) is holomorphic, and
therefore each of its terms contains a factor $d\sigma$ or a factor $\sigma$. Indeed, if $\Psi(\tau)$ is a form in $d\bar{\sigma}$ with antiholomorphic coefficients we can write

$$\Psi(\tau) = \Psi(\tau) \wedge d\bar{\sigma} + \Psi''(\tau),$$

where $\Psi''(\tau)$ does not contain $d\bar{\sigma}$. The first term clearly vanishes on $\{\sigma = 0\}$ since $d\sigma$ does. If $\Psi(\tau)$ vanishes on $\{\sigma = 0\}$, then $\Psi''(\tau)$ does, and hence it contains a factor $\bar{\sigma}$ due to antiholomorphicity. In both cases the $\sigma$-integral, and thereby (2.6), vanishes according to Lemma 2.2.

3. The complete intersection case

Recall from Section 1 that if $f = f_1, f_2$ defines a complete intersection, then

$$\lambda_1 \lambda_2 \Gamma(\lambda_1, \lambda_2) = \int \frac{\partial |f_1|^{2\lambda_1}}{f_1} \wedge \frac{\partial |f_2|^{2\lambda_2}}{f_2} \wedge \phi$$

is holomorphic in $\lambda$, and that the result is believed to extend to any finite number of functions $f_1$.

Thus, taking a closer look at the definition of the currents in Proposition 1.2 (and Proposition 2.1) a natural question is whether

$$(3.1) t(\lambda) := \frac{\partial |f_1|^{2\lambda_1} \wedge u^k \wedge \ldots \wedge \partial |f_1|^{2\lambda_1} \wedge u^f}{z_1^f z_2^f}$$

is holomorphic in $\lambda$.

In general, letting $\lambda$ tend to 0 along different paths yields different currents as shown by the following example. Let $f_1 = z_1$ and $f_2 = z_1 z_2$ in $\mathbb{C}^2$ and consider $t(\lambda)$ acting on a test form $\phi = \varphi(z_1, z_2) d\bar{z}_1 \wedge d\bar{z}_2$

$$\int \frac{\partial |f_1|^{2\lambda_1} \wedge \partial |f_2|^{2\lambda_2}}{f_1 f_2} \wedge \phi = \frac{\partial |z_1|^{2\lambda_1} \wedge \partial |z_1 z_2|^{2\lambda_2}}{z_1^f z_2^f} \wedge \phi.$$

As $\lambda$ tends to zero the integral approaches the value $\lambda_1/(\lambda_1 + \lambda_2) \varphi_{z_1}(0,0)$.

Thus the value at 0 depends on the ratio between $\lambda_1$ and $\lambda_2$.

In case we have two functions defining a complete intersection though, this phenomenon does not occur. By an integration by parts, we can write $t(\lambda)$ integrated against a test form $\phi$ as

$$\int |f_2|^{2\lambda_2} u^f \wedge \partial |f_1|^{2\lambda_1} \wedge u^f \wedge \bar{\phi}.$$

After a resolution of singularities this is equal to a sum of integrals of the form

$$\int |\mu_2|^{2\lambda_2} a_2^{\lambda_2} \frac{\Omega_2}{\mu_2^f} \wedge \bar{\partial} (|\mu_1|^{2\lambda_1} a_1^{\lambda_1}) \wedge \frac{\Omega_1}{\mu_1^f} \wedge \bar{\partial} \phi,$$

(see the proof of Proposition 2.1). Let $\sigma$ be one of the coordinate functions that divides $\mu_1$. If $\mu_2$ does not contain $\sigma$, then the $\sigma$-integral is clearly independent of $\lambda$. If $\mu_2$ does contain $\sigma$, at first we seem to end up in a
situation similar to the one above, where the result at $\lambda = 0$ depends on the relation between $\lambda_1$ and $\lambda_2$. However, the $\sigma$-integral vanishes for the same reason as $T_\ell$ vanishes when $\ell < p$ in the proof of Proposition 1.3. Thus, in this case, the definition of the current $T = t(\lambda)|_{\lambda=0}$ is robust in the sense that it does not depend on the particular path along which $\lambda$ tends to zero.

Provided that the Mellin transform of the residue integral is holomorphic in $\lambda$ in a neighborhood of $0 \in \mathbb{C}^\sigma$, it is reasonable to believe that also $t(\lambda)$ is. Presuming this to be true, we can give a soft proof of Theorem 1.4 based on Theorem 1.1. Indeed, if $f = \sum_{j} f_j e_j$ is a section of a bundle $E^\ast$ we let

$$t^f_{CFL}(\lambda) = \bar{\partial} |f|^{2\lambda} \wedge u^f,$$

and

$$t^f_{CH}(\lambda) = \bar{\partial} |f_1|^{2\lambda} \frac{1}{f_1} \wedge \ldots \wedge \bar{\partial} |f_m|^{2\lambda} \frac{1}{f_m},$$

where $CFL$ and $CH$ of course stand for Cauchy-Fantappiè-Leray and Coleff-Herrera, respectively. With this notation the equality in Theorem 1.1 can be expressed as

$$(3.2)$$

$$t^f_{CFL}(\lambda)|_{\lambda=0} = t^f_{CH}(\lambda)|_{\lambda=0}.$$

Now let $f$ and $g$ be sections of the bundles $E_1^\ast$ and $E_2^\ast$, respectively, and assume that $f \oplus g$ is a complete intersection. By definition,

$$R^f \wedge R^g = t^f_{CFL}(\lambda) \wedge t^g_{CFL}(\lambda)|_{\lambda=0},$$

and

$$R^{f \oplus g} = t^{f \oplus g}_{CFL}(\lambda)|_{\lambda=0},$$

so we want to prove that

$$t^f_{CFL}(\lambda) \wedge t^g_{CFL}(\lambda)|_{\lambda=0} = t^{f \oplus g}_{CFL}(\lambda)|_{\lambda=0}.$$

If Re $\lambda_2$ is large enough, $t^g_{CFL}(\lambda_2)$ is in $\mathcal{L}^1_{\text{loc}}$, and so by (3.2)

$$t^f_{CFL}(\lambda_1) \wedge t^g_{CFL}(\lambda_2)|_{\lambda_1=0} = t^f_{CH}(\lambda_1) \wedge t^g_{CFL}(\lambda_2)|_{\lambda_1=0},$$

and analogously, if Re $\lambda_1$ is large enough

$$t^f_{CH}(\lambda_1) \wedge t^g_{CFL}(\lambda_2)|_{\lambda_2=0} = t^f_{CH}(\lambda_1) \wedge t^g_{CH}(\lambda_2)|_{\lambda_2=0}.$$

Now, by assumption

$$(\lambda_1, \lambda_2) \mapsto t^f_{\bullet}(\lambda_1) \wedge t^g_{\bullet}(\lambda_2),$$

where $\bullet$ stands for either $CFL$ or $CH$, is holomorphic at the origin, and thus it follows that

$$t^f_{CFL}(\lambda) \wedge t^g_{CFL}(\lambda)|_{\lambda=0} = t^f_{CH}(\lambda) \wedge t^g_{CH}(\lambda)|_{\lambda=0},$$

but the right hand side is, by (3.2), equal to $t^{f \oplus g}_{CFL}(\lambda)|_{\lambda=0}$, and so we obtain Theorem 1.4 for $r = 2$. However, the argument easily extends to arbitrary $r$. 

The way we actually prove Theorem 1.4, that is, as already announced, by Theorem 1.5, is more direct and relies neither on the holomorphicity of the Mellin transform nor on Theorem 1.1. The proof is inspired by Proposition 4.2 in [1], in which potentials were used to prove Theorem 1.1. Our hope is that this construction of potentials will be of use for further investigations in the case of a noncomplete intersection.

Proof of Theorem 1.5. We let
\[ V = U^{f_1} + U^{f_2} \wedge R^{f_1} + U^{f_3} \wedge R^{f_2} \wedge R^{f_1} + \ldots + U^{f_r} \wedge R^{f_{r-1}} \wedge \ldots \wedge R^{f_1}. \]
To motivate this choice of \( V \), note that on a formal level
\[ \nabla(U^{f_i} \wedge R^{f_{i-1}} \wedge \ldots \wedge R^{f_1}) = R^{f_{i-1}} \wedge \ldots \wedge R^{f_1} - R^{f_i} \wedge \ldots \wedge R^{f_1}, \]
(observe that \( \nabla = \nabla_f \) acts on \( U^{f_i} \) just as \( \nabla_{f_i} \), so that \( \nabla U^{f_i} = 1 - R^{f_i} \)) so that
\[ \nabla V = 1 - R^{f_r} \wedge \ldots \wedge R^{f_1}. \]
Thus, to prove the first claim of the theorem we have to make this computation legitimate.

First, notice that if a form \( A(\lambda) \), depending on a parameter \( \lambda \), has an analytic continuation as a current to \( \lambda = 0 \), then clearly \( \nabla A(\lambda) \) has one. The action on a test form \( \phi \) is given by
\[ \pm \int A(\lambda) \wedge \nabla \phi. \]
However, by integration by parts with respect to \( \nabla \) and due to the uniqueness of analytic continuations, this is equal to
\[ \int \nabla A(\lambda) \wedge \phi. \]
To be able to perform the integration by parts in a stringent way we have to regard the currents \( T \in \mathcal{D}'_{\alpha}(\Lambda^\ell E) \) as functionals on \( \mathcal{D}_{n,n-k}(\Lambda^{n-\ell} E \wedge \Lambda^n E^*) \). So far we have been a little sloppy about this.

Thus, to compute \( \nabla V \) we consider the form
\[ v^\lambda = |f_1|^{2\lambda} u^{f_1} + |f_2|^{2\lambda} u^{f_2} \wedge \bar{\partial}|f_1|^{2\lambda} \wedge u^{f_1} + \ldots + |f_r|^{2\lambda} u^{f_r} \wedge \bar{\partial}|f_{r-1}|^{2\lambda} \wedge u^{f_r} \wedge \ldots \wedge \bar{\partial}|f_1|^{2\lambda} \wedge u^{f_1}, \]
\[ \ldots + |f_r|^{2\lambda} u^{f_r} \wedge \bar{\partial}|f_{r-1}|^{2\lambda} \wedge u^{f_r} \wedge \ldots \wedge \bar{\partial}|f_r|^{2\lambda} \wedge u^{f_1}, \]
since, by definition, \( v^\lambda|_{\lambda=0} = V \), and accordingly \( \nabla V = (\nabla v^\lambda)|_{\lambda=0} \). More precisely, to verify (3.3), let us consider (recall that \( \nabla u^{f_i} = 1 \))
\[ \nabla(|f_1|^{2\lambda} u^{f_1} \wedge \bar{\partial}|f_{i-1}|^{2\lambda} \wedge \ldots \wedge \bar{\partial}|f_1|^{2\lambda} \wedge u^{f_1}) = \]
\[ - \bar{\partial}|f_1|^{2\lambda} \wedge u^{f_1} \wedge \bar{\partial}|f_{i-1}|^{2\lambda} \wedge u^{f_i} \wedge \ldots \wedge \bar{\partial}|f_1|^{2\lambda} \wedge u^{f_1} + \]
\[ |f_1|^{2\lambda} \bar{\partial}|f_{i-1}|^{2\lambda} \wedge u^{f_i} \wedge \ldots \wedge \bar{\partial}|f_1|^{2\lambda} \wedge u^{f_1} + R_1, \]
where $\mathcal{R}$ is a sum of terms of the form

$$|f_i|^{2\lambda} u^{f_i} \wedge \bar{\partial}|f_{i-1}|^{2\lambda} \wedge u^{f_{i-1}} \wedge \ldots \wedge \bar{\partial}|f_j|^{2\lambda} \wedge \bar{\partial}|f_{j-1}|^{2\lambda} \wedge \ldots \wedge \bar{\partial}|f_1|^{2\lambda} \wedge u^{f_1}$$

that arise when $\nabla$ falls on any $u^{f_j}$, $j < i$. The value at $\lambda = 0$ of the first term is just $-R^{f_i} \wedge R^{f_{i-1}} \wedge \ldots \wedge R^{f_1}$, and it follows from Lemma 3.1 that the second term has an analytic continuation to $\lambda = 0$ equal to $R^{f_{i-1}} \wedge \ldots \wedge R^{f_1}$. The remaining terms, $\mathcal{R}$, vanish according to Lemma 3.2. Thus (1.10) is proved, and thereby the first part of the theorem.

Furthermore, let

$$U^f \wedge V = U^f \wedge U^{f_1} + U^f \wedge U^{f_2} \wedge R^{f_1} + \ldots + U^f \wedge U^{f_r} \wedge R^{f_{r-1}} \wedge \ldots \wedge R^{f_1}.$$

We compute $\nabla$ of each term. To do this we use a form as above whose analytic continuation to $\lambda = 0$ is equal to this particular current. Now, we actually need the extended version of Proposition 1.2, that is Proposition 2.1. Indeed, consider

$$\nabla(|f|^2 u^f \wedge |f_i|^{2\lambda} u^{f_i} \wedge \bar{\partial}|f_{i-1}|^{2\lambda} \wedge u^{f_{i-1}} \wedge \ldots \wedge \bar{\partial}|f_1|^{2\lambda} \wedge u^{f_1}) =$$

$$= -\bar{\partial}|f|^2 u^f \wedge |f_i|^{2\lambda} u^{f_i} \wedge \bar{\partial}|f_{i-1}|^{2\lambda} \wedge u^{f_{i-1}} \wedge \ldots \wedge \bar{\partial}|f_1|^{2\lambda} \wedge u^{f_1} +$$

$$|f|^{2\lambda} |f_i|^{2\lambda} u^{f_i} \wedge \bar{\partial}|f_{i-1}|^{2\lambda} \wedge u^{f_{i-1}} \wedge \ldots \wedge \bar{\partial}|f_1|^{2\lambda} \wedge u^{f_1} +$$

$$|f|^{2\lambda} u^f \wedge \bar{\partial}|f_i|^{2\lambda} u^{f_i} \wedge \bar{\partial}|f_{i-1}|^{2\lambda} \wedge u^{f_{i-1}} \wedge \ldots \wedge \bar{\partial}|f_1|^{2\lambda} \wedge u^{f_1}$$

$$= -|f|^{2\lambda} u^f \wedge |f_i|^{2\lambda} \wedge \bar{\partial}|f_{i-1}|^{2\lambda} \wedge u^{f_{i-1}} \wedge \ldots \wedge \bar{\partial}|f_1|^{2\lambda} \wedge u^{f_1} +$$

$$|f|^2 u^f \wedge \mathcal{R}.$$

The first term corresponds to $-R^f \wedge U^{f_i} \wedge R^{f_{i-1}} \wedge \ldots \wedge R^{f_1}$. Since $f$ is a complete intersection and $R^f$ therefore is of top degree in $d\bar{\partial}z_j$ according to Proposition 1.3, it is most reasonable to expect also this product to be of top degree in $d\bar{\partial}z_i$, but because of the factor $U^{f_i} \in L^{-1}(E_i)$ that is apparently not possible unless the product vanishes. This is indeed the case, as follows from Lemma 3.3. The second, third and fourth terms have analytic continuations as $U^{f_i} \wedge R^{f_{i-1}} \wedge \ldots \wedge R^{f_1}$, $U^f \wedge R^{f_i} \wedge \ldots \wedge R^{f_1}$ and $-U^f \wedge R^{f_{i-1}} \wedge \ldots \wedge R^{f_1}$, respectively, by Lemma 3.1. The remaining terms vanish according to Lemma 3.2. Hence

$$\nabla_f(U \wedge V) = \sum U^{f_i} \wedge R^{f_{i-1}} \wedge \ldots \wedge R^{f_1}$$

$$= \nabla_f(U \wedge V) - \sum(U^f \wedge R^{f_{i-1}} \wedge \ldots \wedge R^{f_1} - U^f \wedge R^{f_i} \wedge \ldots \wedge R^{f_1}) =$$

$$= V - U^f + U^f \wedge R^{f_i} \wedge \ldots \wedge R^{f_1}.$$
Finally, the term $U^f \wedge R^f \wedge \ldots \wedge R^f$ vanishes by Lemma 3.4, and thus taking the lemmas 3.1 to 3.4 for granted, the theorem is proved. □

What remains is the technical part; to prove the lemmas. We have tried to put them as simply as possible. Still the formulations may seem a bit strained though. Hopefully, the remarks will shed some light on what matters.

**Lemma 3.1.** Let $f = f_1 \oplus \cdots \oplus f_r$ be a section of $E^* = E^*_1 \oplus \cdots \oplus E^*_r$. Assume that $f$ is a complete intersection, let $s < r$ and $s' \leq r$. If $h = f$, or if $h = f_i$ for some $i > s$, then

(3.4) $|h|^{2\lambda} |f_r|^{2\lambda} u^f \wedge \ldots \wedge |f_{s+1}|^{2\lambda} u^{f_{s+1}} \wedge \bar{\partial}|f_s|^{2\lambda} u^s \wedge \ldots \wedge \bar{\partial}|f_1|^{2\lambda} u^1$ has an analytic continuation to Re $\lambda > -\epsilon$, which for $\lambda = 0$ is equal to the current $U^f \wedge \ldots \wedge U^{f_{s+1}} \wedge R^s \wedge \ldots \wedge R^1$.

Moreover,

(3.5) $|h|^{2\lambda} |f_r|^{2\lambda} u^f \wedge |f_{s+1}|^{2\lambda} u^{f_{s+1}} \wedge \ldots \wedge |f_1|^{2\lambda} u^1$ has an analytic continuation to Re $\lambda > -\epsilon$, which for $\lambda = 0$ is equal to the current $U^f \wedge U^{f_{s+1}} \wedge \ldots \wedge U^{f_1} \wedge R^s \wedge \ldots \wedge R^1$.

**Remark 1.** The crucial point is that inserting a factor $|h|^{2\lambda}$ has no effect on the value at $\lambda = 0$, as long as

$$\text{codim } \{h = 0\} \cap Y_s \cap \ldots \cap Y_1 > \text{codim } Y_s \cap \ldots \cap Y_1,$$

since then all possibly "dangerous" contributions to the current will vanish for degree reasons as in the proof of Proposition 1.3. There might be a more general formulation of the lemma that catches this behaviour better. Nevertheless, for the proof of Theorem 1.5 it suffices with the cases above. □

**Proof.** We give a proof of the first claim of the lemma. The second one, concerning (3.5), can be proved along the same lines.

For a compactly supported test form $\phi$, we consider

$$\int |h|^{2\lambda} |f_r|^{2\lambda} u^f \wedge \ldots \wedge |f_{s+1}|^{2\lambda} u^{f_{s+1}} \wedge \bar{\partial}|f_s|^{2\lambda} u^s \wedge \ldots \wedge \bar{\partial}|f_1|^{2\lambda} u^1 \wedge \phi.$$ 

After a resolution of singularities as described in the proof of Proposition 2.1, for Re $\lambda$ large enough, this integral is equal to a sum of

(3.6) $\int \left|\mu_1^{2\lambda} |\mu_r^{2\lambda} a^1 r^{\lambda} \frac{\alpha_1 r^{\lambda}}{\mu_1^{\lambda}} \wedge \ldots \wedge |\mu_{s+1}^{2\lambda} a_{s+1}^{\lambda} \frac{\alpha_{s+1} r^{\lambda}}{\mu_{s+1}^{\lambda}} \wedge \bar{\partial}(\mu_s^{2\lambda} a_s^{\lambda}) \wedge \ldots \wedge \bar{\partial}(\mu_1^{2\lambda} a_1^{\lambda}) \wedge \frac{\alpha_1 r^{\lambda}}{\mu_1^{\lambda}} \wedge \phi_1\right|$. 


where the $a_j$'s are strictly positive functions, the $\mu_j$'s are polynomials in some local coordinates $\tau_j$, the $\alpha_j t_j$'s are smooth forms and $\phi$ is as in the proof of Proposition 2.1. The existence of the analytic continuation to $\Re \lambda > -\varepsilon$ follows from Lemma 2.2 as before.

Our aim is to prove that the factor $|h|^{2\lambda}$ does not affect the value at $\lambda = 0$. Let $\sigma$ be one of the coordinate functions $\tau_k$ that divides $\mu_k$. When expanding each factor $\partial \overline{\partial}(|\mu_j|^{2\lambda} \alpha_j^k)$ by Leibniz' rule we get two different types of terms, integrals with one occurrence of a factor $\partial \overline{\partial} |\sigma^\alpha|^2 \lambda$ for some $\alpha$, and integrals with no such factors. In the second case the extra factor $|\sigma|^{2\lambda}$ does no harm, since, in fact, the value at $\lambda = 0$ is independent of the number of $|\sigma|^{2\lambda}$'s in the numerator as long as there is no $\overline{\sigma}$ in the denominator. Furthermore, we claim that each integral of the first kind actually vanishes at $\lambda = 0$. The argument is analogous to the one in the proof of Proposition 1.3. Let us first consider the case when $h = f$. Observe that the terms in (3.4) are of degree at most $m_1 + \ldots + m_r - r' + s \leq m - 1$ in $d\overline{z}_j$, where $m = m_1 + \ldots + m_r$. The crucial term $-1$ appears because of the (at least for the proof) necessary condition that $r < s$, that is that we have at least one factor $U$. Thus, it is enough to consider test forms of codimension in $d\overline{z}$ at most $m - 1$; by codimension we mean the difference between the dimension of $X$ and the degree. We assume that $\phi = \phi_j \wedge d\overline{z}_j$, where $\phi_j$ is a smooth $(n,0)$-form and $d\overline{z}_j = d\overline{z}_{j_1} \wedge \ldots \wedge d\overline{z}_{j_p}$ where $p \geq n - (m_1 + \ldots + m_r) + 1$. Now, $d\overline{z}_j$ vanishes on the variety $Y = f^{-1}(0)$, since it has codimension $m$, and accordingly $\Pi^*(d\overline{z}_j)$ vanishes on $Y = \Pi^{-1}Y$, and in particular on $\{\sigma = 0\}$. Since it is a form in $d\overline{z}_j$ with antiholomorphic coefficients, each of its terms contains a factor $\overline{\sigma}$ or $d\overline{\sigma}$ (see the proof of Proposition 1.3), and so in both cases the $\sigma$-integrals vanish according to Lemma 2.2.

In the second case, when $h = f$, the proof becomes slightly more complicated. We want to prove that the $\sigma$-integral vanishes due to the occurrence of a factor $\overline{\sigma}$ or $d\overline{\sigma}$ as above, but now the desired factors $\sigma$ and $d\sigma$ do not necessarily divide the test form $\phi$. We need to look at a “larger” form than $\phi$, in fact at the “largest” possible “$\sigma$-free” form. Without loss of generality we may assume that, for some numbers $s'$ and $r'$, $1 \leq s' \leq s \leq r' \leq r$, $\sigma$ divides $\mu_{s'+1}, \ldots, \mu_s$ and $\mu_{r'+1}, \ldots, \mu_{r}$ but neither $\mu_{s+1}, \ldots, \mu_s$ nor $\mu_{s+1}, \ldots, \mu_{s'}$. Recall that $u^h = \sum_{\ell} v_{s_{1 \ell}}^h / |f_i|^2$, where $v_{s_{1 \ell}}^h = s_i \wedge (\overline{\partial} s_{1 \ell})^{1/2}$. Let the smooth form

$$v_{s_{1 \ell}}^h \wedge \ldots \wedge v_{s_{1 \ell}^{s+1}}^h \wedge \overline{\partial} |f_{s'}|^2 \wedge v_{s'}^h \wedge \ldots \wedge \overline{\partial} |f_r|^2 \wedge v_{s_{1 \ell}}^h$$

be denoted by $F_{s_{1 \ell}}$, and let

$$Y' = \{f_{s'+1} = \ldots = f_s = f_{r'+1} + \ldots f_r = h = 0\}.$$
As above we may assume that $\phi$ consists of only one term $\phi_1 \wedge d\bar{z}_1$. Then, by inspection, the form $F_\ell \wedge d\bar{z}_1$ is of codegree at most
\[ m_{s'+1} + \ldots + m_s + m_{r'+1} + \ldots + m_r - r + r' \]
in $d\bar{z}_j$, which is strictly less than
\[ \text{codim} Y' = m_{s'+1} + \ldots + m_s + m_{r'+1} + \ldots + m_r + m_h, \]
because of the assumptions of complete intersection. Consequently $F_\ell \wedge d\bar{z}_1$ vanishes on $Y'$, and thus $\Pi^*(F_\ell \wedge d\bar{z}_1)$ vanishes on $\Pi^{-1}Y'$, and in particular on $\{\sigma = 0\}$. Since it is a form in $d\bar{z}_j$ with antiholomorphic coefficients, each of its terms contains a factor $\bar{\sigma}$ or a factor $d\bar{\sigma}$. Using that $\bar{\partial}|f|^{2\lambda} = \lambda |f|^{2(\lambda-1)} \bar{\partial}|f|^2$, we can write (3.6) as

\[
\pm \int \left( \mu_1 |f|^2 \lambda a_1^{\lambda} \frac{\partial_r \partial_\ell}{\partial_\ell} \wedge \ldots \wedge \mu_{r'+1} |f|^{2\lambda} a_{r'+1}^{\lambda} \frac{\partial_\ell' \partial_{r'+1}}{\partial_{r'+1}} \wedge \right.
\]
\[
\left. \bar{\sigma}(|\mu_1|^{2\lambda} a_1^{\lambda}) \wedge \frac{\partial_\ell}{\partial_{r'+1}} \wedge \ldots \wedge \bar{\sigma}(|\mu_{r'+1}|^{2\lambda} a_{r'+1}^{\lambda}) \wedge \frac{\partial_{r'+1}}{\partial_{r'+1}} \wedge \right.
\]
\[
\left. |\mu_r|^{2\lambda} a_r^{\lambda} \ldots |\mu_{s'+1}|^{2\lambda} a_{s'+1}^{\lambda} \lambda \mu_r^{\lambda} \mu_{r'+1}^{\lambda} \ldots |\mu_{s'+1}^{\lambda} a_{s'+1}^{\lambda} \lambda |\mu_1|^{2(t_1+1)} \mu_{r'+1}^{\lambda-1} \ldots |\mu_{s'+1}^{\lambda-1} a_{s'+1}^{\lambda-1} \right),
\]

where the sign depends on the relation between $r, r', s$ and $s'$. Now the only way a factor $\bar{\sigma}$ in the numerator (more precisely in $\Pi^*(F_\ell) \wedge \bar{\phi}$) could be canceled out when $\lambda$ is small, is by the occurrence of a factor $\bar{\sigma}$ in one of $\mu_1, \ldots, \mu_{r'}$, but that would obviously contradict the assumption made above. Hence each term in the integral must contain a factor $\bar{\sigma}$ or $d\bar{\sigma}$ independently of the value of $\lambda$ and thus the $\sigma$-integral vanishes according to Lemma 2.2.

\[ \square \]

**Lemma 3.2.** Let $f = f_1 \oplus \cdots \oplus f_r$ be a section of $E^* = E^*_1 \oplus \cdots \oplus E^*_r$. Assume that $f$ is a complete intersection. Then

\[ |f_1|^{2\lambda} u_1^{l_1} \wedge \ldots \wedge |f_{s'+1}|^{2\lambda} u_{s'+1}^{l_{s'+1}} \wedge \bar{\partial}|f_1|^{2\lambda} \wedge u_1^{l_1} \wedge \ldots \bar{\partial}|f_{s'+1}|^{2\lambda} \wedge u_{s'+1}^{l_{s'+1}} \]

has an analytic continuation to $\Re \lambda > -\epsilon$ that vanishes at $\lambda = 0$.

**Remark 2.** Morally, what this lemma says is that when applying Leibniz’ rule to $\nabla$ acting on a product of principal value and residue currents, there will be no contributions from $\nabla$ falling on a residue factor. Of course this is expected, since the residue currents are $\nabla$-closed.

**Proof.** The result follows from Lemma 3.1 after an integration by parts with respect to $\nabla$. (Recall that $\phi$ is a form taking values in $\Lambda^{n-\ell} E \wedge \Lambda^n E^*$.) Note
that $\partial |f_1|^{2\lambda} = -\nabla |f_1|^{2\lambda}$. By Stokes’ theorem,

$$
\int |f_r|^{2\lambda} u^{f_r} \wedge \ldots \wedge |f_{s+1}|^{2\lambda} u^{f_{s+1}} \wedge \partial |f_s|^{2\lambda} \wedge u^{f_s} \wedge \ldots \wedge \nabla |f_1|^{2\lambda} \wedge \partial |f_1|^{2\lambda} \wedge u^{f_1} \wedge \phi =
$$

$$
\pm \int (|f_1|^{2\lambda} - 1) \nabla (|f_r|^{2\lambda} u^{f_r} \wedge \ldots \wedge |f_{s+1}|^{2\lambda} u^{f_{s+1}} \wedge \partial |f_s|^{2\lambda} \wedge u^{f_s} \wedge \ldots \wedge \partial |f_1|^{2\lambda} \wedge u^{f_1} \wedge \phi),
$$

so it is enough to prove that this expression vanishes at $\lambda = 0$. However, the form

$$
\nabla (|f_r|^{2\lambda} u^{f_r} \wedge \ldots \wedge |f_{s+1}|^{2\lambda} u^{f_{s+1}} \wedge \partial |f_s|^{2\lambda} \wedge u^{f_s} \wedge \ldots \wedge \partial |f_1|^{2\lambda} \wedge u^{f_1})
$$

is precisely as in the hypothesis of Lemma 3.1 and $f_t$ is an $h$ of the second kind, so according to the lemma the factor $|f_1|^{2\lambda}$ does not have any effect on the value at $\lambda = 0$. Thus we are done. $\square$

**Lemma 3.3.** Let $f = f_1 \oplus \cdots \oplus f_r$ be a section of $E^* = E^*_1 \oplus \cdots \oplus E^*_r$ of rank $m$ and let $h = f \oplus f'$, where $f'$ is a section of the dual bundle of a holomorphic $m'$-bundle $E'$. Assume that $h$ is a complete intersection. If $r > s$, then

(3.8) \[ \partial |h|^{2\lambda} \wedge u^{h} \wedge |f_r|^{2\lambda} u^{f_r} \wedge \ldots \wedge |f_{s+1}|^{2\lambda} u^{f_{s+1}} \wedge \partial |f_s|^{2\lambda} \wedge u^{f_s} \wedge \ldots \wedge \partial |f_1|^{2\lambda} \wedge u^{f_1} \]

has an analytic continuation to $\text{Re} \lambda > -\epsilon$ that vanishes at $\lambda = 0$.

**Remark 3.** Notice that the value at $\lambda = 0$ corresponds to the current $R^h \wedge U^{f_r} \wedge \ldots \wedge U^{f_{s+1}} \wedge R^{f_s} \wedge \ldots \wedge R^{f_1}$. Since $h$ is a complete intersection, $R^h$ is of degree $m + m'$ in $d\bar{\omega}_j$ according to Proposition 1.3, and therefore it is reasonable to expect also the product to be of degree $m + m'$ in $d\bar{\omega}_j$. However, since the product contains at least one principal value factor, the degree in $e_j$ must be strictly larger than the degree in $d\bar{\omega}_j$ and so, the product must vanish. We will see that the assumption that $r > s$ is crucial also for the proof. \( \square \)

**Proof.** After a resolution of singularities as described in the proof of Proposition 2.1, we can write (3.8) integrated against a test form $\phi$ as a sum of terms of the type

(3.9) \[ \int \bar{\partial}(|\mu_h|^{2\lambda} a_h^{\lambda}) \frac{\alpha_h \bar{\ell}_h}{\mu_h} \wedge |\mu_r|^{2\lambda} a_r^{\lambda} \frac{\alpha_r \bar{\ell}_r}{\mu_r} \wedge \ldots \wedge |\mu_{s+1}|^{2\lambda} a_{s+1}^{\lambda} \frac{\alpha_{s+1} \bar{\ell}_{s+1}}{\mu_{s+1}} \wedge \tilde{\partial}(|\mu_s|^{2\lambda} a_s^{\lambda}) \wedge \alpha_s \bar{\ell}_s \wedge \ldots \wedge \tilde{\partial}(|\mu_1|^{2\lambda} a_1^{\lambda}) \wedge \alpha_1 \bar{\ell}_1 \wedge \phi, \]

\[ \text{where } a_h^{\lambda}, a_r^{\lambda}, \ldots, a_1^{\lambda} \text{ are the local coordinates associated with } h_r, \ldots, h_1 \text{ respectively.} \]
where the $\alpha_{i,\ell_i}'s$ are smooth forms of degree $\ell_i$ in $e_{j_i}$, the $a_i'$s are non-vanishing functions and the $\mu_i$'s are monomials in some local coordinates $\tau_i$ and $\phi$ is as in the previous proofs.

We expand the factor $\partial(|\mu_h|^{2\lambda} a_h)$ by Leibniz’ rule and consider the term obtained when $\partial$ falls on $|\sigma|^{2\lambda}$, where $\sigma$ is one of the $\tau_i$'s that divide $\mu_h$. We prove that this term vanishes when integrating with respect to $\sigma$. The term that arises when $\partial$ falls on $a_h^{\lambda}$ clearly vanishes as before (see the proof of Proposition 2.1). Since the rank of $E \oplus E'$ is $m + m'$, the terms in (3.8) are of degree at most $m + m' - 1$ in $d\bar{z}$, since we have at least one $U$-factor. Thus it is enough to consider test forms of codegree in $d\bar{z}$ at most $m + m' - 1$. As in the previous proofs we may assume that $\phi = \phi_I \wedge d\bar{z}_I$. It follows that $d\bar{z}_I$ vanishes on $Y = h^{-1}(0)$ for degree reasons, and thus $\Pi^*(d\bar{z}_I)$ vanishes on $\Pi^{-1}Y$. Since this is a form in $d\bar{f}_j$ with antiholomorphic coefficients, each of its terms contains a factor $\bar{\sigma}$ or $d\bar{\sigma}$ and consequently the $\sigma$-integral vanishes according to Lemma 2.2.

**Lemma 3.4.** Let $f = f_1 \oplus \cdots \oplus f_r$ be a section of $E^* = E^*_{\nu} \oplus \cdots \oplus E^*_{\nu_r}$. Assume that $f$ is a complete intersection. Let $h = f_{I_1} \oplus \cdots \oplus f_{I_p}$, where $I = \{I_1, \ldots, I_p\} \subset \{1, \ldots, r\}$. Then

$$|h|^{2\lambda} u^h \wedge \partial|f_{I_1}|^{2\lambda} \wedge u^{I_1} \wedge \cdots \wedge \partial|f_r|^{2\lambda} \wedge u^1$$

has an analytic continuation to $\Re \lambda > -\epsilon$ that vanishes at $\lambda = 0$.

**Remark 4.** The value at $\lambda = 0$ corresponds to the current $U^h \wedge R^1 \wedge \ldots \wedge R^I$. Since the $R$-part is of top degree according to Proposition 1.3 this product should formally vanish by arguments similar to those in Remark 3.

**Proof.** As in the proofs of the previous lemmas we start by a resolution of singularities. Thus, the form (3.10) integrated against a test form $\phi$ is equal to a sum of terms of the type

$$\int |\mu_h|^{2\lambda} a_h^{\lambda} \frac{\partial}{\partial \bar{h}} \wedge \bar{\partial}(|\mu_r|^{2\lambda} a_r^\lambda) \wedge \frac{\partial}{\partial \bar{h}} \wedge \cdots \wedge \bar{\partial}(|\mu_1|^{2\lambda} a_1^\lambda) \wedge \frac{\partial}{\partial \bar{h}} \wedge \bar{\phi},$$

where $\alpha_i, \ell_i$, $\mu_i$, and $\bar{\phi}$ are as above. Further, we can find a resolution to a certain toric variety so that locally one of the monomials $\mu_1, \ldots, \mu_r$ divides the other ones. Without loss of generality we may assume that $\mu_1$ divides all $\mu_j$’s. We expand $\partial(|\mu_1|^{2\lambda} a_1^\lambda)$ by Leibniz’ rule. The term obtained when $\partial$ falls on $a_1^\lambda$ vanishes as in the proof of Proposition 2.1, so it is enough to consider the terms that arise when $\partial$ falls on $|\sigma|^{2\lambda}$, where $\sigma$ is one of the coordinates in $\mu_1$.

We claim that the $\sigma$-integral vanishes at $\lambda = 0$. As usual, we observe that the terms of (3.10) are of degree at most $m - 1$ in $d\bar{z}_J$, where $\cdot - 1$ in this
case is due to the factor $U^h$, so it suffices to consider test forms of codegree at most $m - 1$. We assume that $\phi = \phi_l \land d\bar{z}_l$, where $\phi_l$ is an $(n, 0)$-form and $d\bar{z}_l = d\bar{z}_{l_1} \land \ldots \land d\bar{z}_{l_p}$, where $p \leq n - m + 1$. Then $d\bar{z}_l$ vanishes on the variety $Y = f^{-1}(0)$ for degree reasons, and accordingly $\Pi^*(d\bar{z})$ vanishes on $\Pi^{-1}Y$, and in particular on $\{\sigma_1 = 0\}$. By arguments as in the proof of Proposition 1.3 it follows that $\Pi^*(d\bar{z})$ must contain a factor $\bar{\sigma}$ or $d\bar{\sigma}$ since it is a form in $d\bar{r}_k$ with antiholomorphic coefficients, and hence the $\sigma$-integral vanishes as before. 

\[ \square \]

4. SOME EXAMPLES

We conclude this thesis by giving some examples of our residue currents. Although the sections we have chosen to work with are relatively simple, in fact we have just considered monomials, the computations become quite intricate, including repeated blow-ups. Yet we give a fairly detailed account of the calculations, since they make a nice complement to the proofs above. We are shown the toric resolutions in action and provided an illustration of the rather nontrivial argument about vanishing for degree reasons due to a factor $\bar{\sigma}$ or $d\bar{\sigma}$ (for some $\sigma$), which was first used in the proof of Proposition 1.3. Also, we see how combinatorics comes into play in a not always obvious way.

**Example** 1. We start by illustrating Theorem 1.4 in a very simple case of complete intersection. Let $f_1 = z_1^p$, $f_2 = z_2^p$ and $f_3 = z_3^p$, where $p$ is some integer ($> 1$ to make the example somewhat interesting), in a neighborhood $U$ of the origin in $\mathbb{C}^3$. Then $Y = f^{-1}(0)$ is simply the origin so $f$ is a complete intersection.

We want to compute and compare the currents $R^{f_1 \otimes f_2 \land f_3}$ and $R^{f_1 \otimes f_2 \otimes f_3}$. However, to warm up and to make the investigation complete we start by computing $R^{f_1} \land R^{f_2} \land R^{f_3}$. Since each $f_i$ depends only on $z_i$, the $f_i$'s are separated and thus the current can be regarded as the tensor product $R^{f_1} \otimes R^{f_2} \otimes R^{f_3}$, where

\[ R^{f_i} = \partial \left[ \frac{1}{f_i} \right] \land e_i = \partial \left[ \frac{1}{z_i^p} \right] \land e_i. \]

Here $\partial[1/f]$ is merely a short-hand notation for the value at $\lambda = 0$ of $\partial[f^{2\lambda}/f]$, which will be used in the sequel. By $[1/f]$ we will mean $|f^{2\lambda}|_{\lambda=0}$, that is just the principal value of $1/f$. Let $\phi = \varphi(z)dz$. By iterated integration by parts we have that

\begin{equation}
\int \partial \left[ \frac{1}{z^p} \right] \land \phi = \frac{2\pi i}{(p-1)!} \partial z^{p-1} \varphi(0),
\end{equation}

(4.1)
Thus, if \( \phi = \varphi(z_1, z_2, z_3) \, dz_1 \wedge dz_2 \wedge dz_3 \), it follows that
\[
R^{f_1} \otimes R^{f_2} \otimes R^{f_3}, \phi = \left( \frac{-2\pi i}{(p-1)!} \right)^3 \frac{\partial^{p-1} \partial^{p-1} \partial^{p-1}}{\partial z_1^{p-1} \partial z_2^{p-1} \partial z_3^{p-1}} \varphi(0, 0) \wedge e_1 \wedge e_2 \wedge e_3.
\]

We continue with the current \( R = R^{f_1 \oplus f_2} \otimes R^{f_3} \) as above. Thus, we need to compute \( R^{f_1 \oplus f_2} \). In the trivial metric
\[
s = \bar{f}_1 e_1 + \bar{f}_2 e_2 = \bar{z}_1^p e_1 + \bar{z}_2^p e_2,
\]
so that
\[
u = \frac{s}{|f|^2} + \frac{s \wedge \bar{\partial} s}{|f|^4} = \frac{\bar{z}_1^p e_1 + \bar{z}_2^p e_2}{|z|^2} + \frac{\bar{z}_1^p \bar{\partial}(\bar{z}_2^p) - \bar{\partial}(\bar{z}_1^p) \bar{z}_2^p}{|z|^4} \wedge e_1 \wedge e_2,
\]
where \(|z|^2 = |z_1|^2 + |z_2|^2\). To find the extension of \( u_2 \) across \( Y \) we consider the proper mapping \( \Pi: \tilde{U} \to U \), where \( \tilde{U} \) is the blow-up at the origin of \( U \).

We can cover \( U \) by two coordinate neighborhoods
\( \Omega_1 = \{ \tau; (\tau_1, \tau_2, \tau_3) = z \in U \} \) and \( \Omega_2 = \{ \sigma; (\sigma_1, \sigma_2, \sigma_3) = z \in U \} \).

Let us start by computing \( \bar{u} = \Pi^\ast \bar{u} \) in \( \Omega_1 \). Notice that
\[
\bar{s} = \Pi^\ast s = \bar{\tau}_1^p e_1 + \bar{\tau}_2^p \bar{\tau}_2^p e_2 = \bar{\tau}_1^p (e_1 + \bar{\tau}_2^p e_2).
\]
Recall that due to homogeneity, if \( s = \mu s' \), where \( \mu \) is a smooth function, then
\[
s \wedge (\bar{\partial} s)^{\ell-1} = \mu^\ell s'(\bar{\partial} s')^{\ell-1},
\]
so it follows that
\[
\bar{\tilde{u}} = \frac{\bar{e}_1 + \bar{\tau}_2^p e_2}{\bar{\tau}_1^p (1 + |\tau_2|^2 \bar{\tau}_2^p)} + \frac{\bar{\tau}^{p-1}_1 \bar{\partial} e_1 \wedge \bar{\partial} \bar{\tau}_2 \wedge e_2}{\bar{\tau}_1^p (1 + |\tau_2|^2 \bar{\tau}_2^p)^2}.
\]
The extension of \( \bar{u} \) across \( Y \) is simply
\[
\frac{1}{\bar{\tau}_1^p (1 + |\tau_2|^2 \bar{\tau}_2^p)} + \frac{1}{\bar{\tau}_1^p (1 + |\tau_2|^2 \bar{\tau}_2^p)^2},
\]
and therefore
\[
R^{f_1 \oplus f_2} = \bar{\partial} \left[ \frac{1}{\bar{\tau}_1^p} \wedge \bar{e}_1 + \bar{\tau}_2^p e_2 \right] + \bar{\partial} \left[ \frac{1}{\bar{\tau}_1^p} \wedge \frac{\bar{\tau}^{p-1}_1 \bar{\partial} e_1 \wedge \bar{\partial} \bar{\tau}_2 \wedge e_2}{(1 + |\tau_2|^2 \bar{\tau}_2^p)^2} \right],
\]
in \( \Omega_1 \). To compute the action of \( R^{f_1 \oplus f_2} \) let \( \phi \) be a test form in \( U \). Then
\[
(4.2) \quad \int_U R^{f_1 \oplus f_2} \wedge \phi = \int_{\tilde{U}} R^{\Pi(f_1 \oplus f_2)} \wedge \Pi^\ast \phi.
\]
Recall that, since \( f \) is a complete intersection, the part of degree 1 should vanish according to Proposition 1.3. To see how this happens, assume that
\( \phi = \phi_1 \, dz_1 \wedge dz_2 \), where \( \phi_1 = \varphi^1(z_1, z_2) \, dz_1 + \varphi^2(z_1, z_2) \, dz_2 \). Now, \( \Pi^\ast \phi_1 = \)
\( \varphi^1(\tau_1, \tau_2) d\tau_1 + \varphi^2(\tau_1, \tau_2) d(\tau_1, \tau_2) \), so each term of \( \Pi^* \phi \) contains a factor \( \tau_1 \) or \( d\tau_1 \) as predicted in the proof of Proposition 1.3, and consequently the \( R_1^{\Pi^* f_1 \circ f_2} \)-part of the integral vanishes.

Further, for the action of \( R_2^{\Pi^* f_1 \circ f_2} \), it is enough to consider a \((2,0)\)-form \( \phi = \varphi(x_1, x_2) dx_1 \wedge dx_2 \). Then

\[
\Pi^* \phi = \varphi(\tau_1, \tau_2) d\tau_1 \wedge d(\tau_1, \tau_2) = \tau_1 \varphi(\tau_1, \tau_2) d\tau_1 \wedge d\tau_2
\]
in \( \Omega_1 \), so the right hand side integral of (4.2) is equal to

\[
\int_{\tau_2} \frac{\partial}{\partial \tau_1} \left[ \frac{1}{\tau_1^{2p-1}} \right] \wedge \frac{p\tau_2^{p-1} e_1 \wedge d\tau_2 \wedge e_2}{(1 + |\tau_2|^{2p})^2} \wedge \varphi(\tau_1, \tau_2) d\tau_1 \wedge d\tau_2.
\]

From (4.1) we know that this is equal to

\[-\frac{2\pi i}{(2p-2)!} \int_{\tau_2} \frac{p\tau_2^{p-1} e_1 \wedge d\tau_2 \wedge e_2}{(1 + |\tau_2|^{2p})^2} \wedge \frac{\partial^{2p-2}}{\partial \tau_1^{2p-2}} \varphi(\tau_1, \tau_2)|_{\tau_1 = 0} d\tau_2.
\]

Observe that

\[
\frac{\partial}{\partial \tau_1} \varphi(\tau_1, \tau_2) = \sum_k \binom{r}{k} \tau_2^k \varphi_{r-k2k}(\tau_1, \tau_2),
\]

where \( \varphi_{r-k2k} \) means \( j \) derivatives with respect to the first variable and \( k \) derivatives with respect to the second one. Now, only the term with a factor \( \tau_2^{p-1} \) will contribute. The other terms vanish due to antisymmetry, as is easily verified. Thus we get

\[-\frac{2\pi i}{(2p-2)!} \frac{((2p-2)!}{(p-1)!} \int_{\tau_2} \frac{p|\tau_2|^{p-1} e_1 \wedge d\tau_2 \wedge e_2}{(1 + |\tau_2|^{2p})^2} \wedge d\tau_2 =
\]

\[-\frac{2\pi i}{(p-1)!} \int_{\tau_2} \frac{-2\pi i}{(2p-2)!} \frac{((2p-2)!}{(p-1)!} \varphi_{p-12p-1}(0,0)(-2\pi i) e_1 \wedge e_2 =
\]

\[-\frac{2\pi i}{(p-1)!} \frac{2}{\tau_2^{p-1}} \frac{\partial}{\partial \tau_2^{p-1}} \varphi(0,0,0) e_1 \wedge e_2.
\]

Hence

\[ R^{f_1 \circ f_2} \otimes R^{f_3} \phi = \left( \frac{-2\pi i}{(p-1)!} \right)^3 \frac{\partial^{p-1} \varphi}{\partial z_1^{p-1}} \frac{\partial^{p-1} \varphi}{\partial z_2^{p-1}} \frac{\partial^{p-1} \varphi}{\partial z_3^{p-1}} \varphi(0,0,0) e_1 \wedge e_2 \wedge e_3
\]
in \( \Omega_1 \). Since the current \( R^{\Pi^* f_1 \circ f_2} \) has support only when \( \tau_1 = 0 \) we can compute the integral over the entire \( \mathcal{U} \) by extending the integration to \( \tau_2 \in \mathbb{R}^1 \), so this is indeed the total action of the current.

It remains to calculate the current \( R^{f_1 \circ f_2 \circ f_3} \). In the trivial metric we have

\[ s = \tilde{f}_1 e_1 + \tilde{f}_2 e_2 + \tilde{f}_3 e_3 = \tilde{z}_1 e_1 + \tilde{z}_2 e_2 + \tilde{z}_3 e_3. \]
To find the extension across $Y$ we need a resolution of the singularity at the origin. Consider the proper mapping $\Pi: \bar{U} \to U$, where $\bar{U}$ is the blow-up of the origin. We can cover $\bar{U}$ by three coordinate neighborhoods

$$\Omega_1 = \{ \tau; \ (\tau_1, \tau_2, \tau_3) = z \in \bar{U} \} \ , \ \Omega_2 = \{ \sigma; \ (\sigma_1, \sigma_2, \sigma_3) = z \in U \} \quad \text{and} \end{array} \end{array}$$

$$\Omega_3 = \{ \rho; \ (\rho_1, \rho_2, \rho_3, \rho_4) = z \in U \} .$$

Let us compute $\bar{u}_3 = \Pi^* u_3$, that is the part of $u$ of top degree, in one of the neighborhoods; $\Omega_1$. We leave it to the reader to verify that the terms of lower degree vanish. First, notice that

$$\bar{s} = \Pi^* s = \bar{p}_1^p (e_1 + \bar{p}_2^p e_2 + \bar{p}_3^p e_3) ,$$

so that

$$\bar{u}_3 = \frac{p^2 \bar{p}_2^{p-1} \bar{p}_3^{p-1} e_1 \wedge d\bar{p}_2 \wedge e_2 \wedge d\bar{p}_3 \wedge e_3}{\tau_1^{3p} (1 + |\tau_2|^{2p} + |\tau_3|^{2p})^3} ,$$

and thus

$$R_3^{\Pi^* (f_1 \oplus f_2 \oplus f_3)} = \partial \left[ \frac{1}{\tau_1^{3p}} \right] \wedge \frac{p^2 \bar{p}_2^{p-1} \bar{p}_3^{p-1} e_1 \wedge d\bar{p}_2 \wedge e_2 \wedge d\bar{p}_3 \wedge e_3}{(1 + |\tau_2|^{2p} + |\tau_3|^{2p})^3}$$

in $\Omega_1$. Now, if $\varphi = \varphi(x_1, x_2, x_3) dz_1 \wedge dz_2 \wedge dz_3$ is a test form in $U$ we get

$$\int_U R_3^{\Pi^* (f_1 \oplus f_2 \oplus f_3)} \wedge \varphi = \int_U R_3^{\Pi^* (f_1 \oplus f_2 \oplus f_3)} \wedge \Pi^* \varphi ,$$

and since

$$\Pi^* \varphi = \varphi(\tau_1, \tau_1 \tau_2) d\tau_1 \wedge d(\tau_1 \tau_2) \wedge d(\tau_1 \tau_3) = \tau_1^2 \varphi(\tau_1, \tau_1 \tau_2) d\tau_1 \wedge d\tau_2 \wedge d\tau_3$$

in $\Omega_1$, the right hand side integral of (4.3) is equal to

$$\int_\tau \left[ \frac{1}{\tau_1^{3p-2}} \right] \wedge \frac{p^2 \bar{p}_2^{p-1} \bar{p}_3^{p-1} e_1 \wedge d\bar{p}_2 \wedge e_2 \wedge d\bar{p}_3 \wedge e_3}{(1 + |\tau_2|^{2p} + |\tau_3|^{2p})^3} \wedge \varphi(\tau_1, \tau_1 \tau_2, \tau_1 \tau_3) d\tau_1 \wedge d\tau_2 \wedge d\tau_3 =$$

$$- \frac{2\pi i}{(3p-3)!} \int_{\tau_2, \tau_3} \frac{p^2 \bar{p}_2^{p-1} \bar{p}_3^{p-1} e_1 \wedge d\bar{p}_2 \wedge e_2 \wedge d\bar{p}_3 \wedge e_3}{(1 + |\tau_2|^{2p} + |\tau_3|^{2p})^3} \wedge \frac{\partial^{3p-3}}{\partial \tau_1^{3p-3}} \varphi(\tau_1, \tau_1 \tau_2, \tau_1 \tau_3) \big|_{\tau_1 = 0} d\tau_2 \wedge d\tau_3 .$$

Moreover, we have that

$$\frac{\partial^r}{\partial \tau_1^j} \varphi(\tau_1, \tau_1 \tau_2, \tau_1 \tau_3) = \sum_{k,j} \binom{r}{k} \binom{r-k}{j} \tau_2^{-r} \tau_3^{-j} \varphi(\tau_1, \tau_1 \tau_2, \tau_1 \tau_3) .$$
The only term of $\frac{\partial^{3p-3}}{\partial \tau_1^{3p-3}} \varphi(\tau_1, \tau_2, \tau_1 \tau_3)|_{\tau_1=0}$ in (4.4) that does not vanish due to antisymmetry is the one with the factor $\tau_2^{p-1} \tau_3^{p-1}$ so the last integral is equal to

$$- \frac{2\pi i}{(3p-3)!} \left( \frac{(3p-3)}{(p-1)} \right) \left( \frac{(2p-2)}{(p-1)} \right) \varphi_{1^{p-1}2^{p-1}3^{p-1}}(0,0,0)$$

$$\int_{\tau_2 \tau_3} \frac{p^2 |\tau_2|^{2(p-1)} |\tau_3|^{2(p-1)} e_1 \wedge d\bar{\tau}_2 \wedge e_2 \wedge d\bar{\tau}_3 \wedge e_3}{(1 + |\tau_2|^{2p} + |\tau_3|^{2p})^3} \wedge d\tau_2 \wedge d\tau_3 =$$

$$- \frac{2\pi i}{(p-1)!} \varphi_{1^{p-1}2^{p-1}3^{p-1}}(0,0,0) (-2\pi i)^2 e_1 \wedge e_2 \wedge e_3 =$$

$$\left( \frac{-2\pi i}{(p-1)!} \right)^3 \frac{\partial^{p-1}}{\partial z_1^{p-1}} \frac{\partial^{p-1}}{\partial z_2^{p-1}} \frac{\partial^{p-1}}{\partial z_3^{p-1}} \varphi(0,0,0) e_1 \wedge e_2 \wedge e_3.$$ 

Since the current $R^{f_1 \oplus f_2 \oplus f_3}$ has support at the origin, and thus $R^{\Omega (f_1 \oplus f_2 \oplus f_3)}$ has support where $\tau_1 = 0$, we can compute the integral over the entire $\mathbb{U}$ simply by extending the integration to $\tau_2, \tau_3 \in \mathbb{C}^2$, so we have actually computed the total action of the current.

Hence we have shown that the three currents $R^{f_1 \wedge R^{\Phi} \wedge R^{\Phi}}$, $R^{f_1 \wedge f_2 \wedge R^{\Phi}}$ and $R^{f_1 \wedge f_2 \wedge f_3}$, in this case, are the same in accordance with Theorem 1.4. 

When generalizing Theorem 1.1, or rather its line bundle formulation (1.9), to sections of bundles of arbitrary rank, it is not obvious how one should interpret the assumption of $f$ being a complete intersection. In the formulation of Theorem 1.4 we require the codimension of $f^{-1}(0)$ to be equal to the rank of the bundle $E$. A less strong hypothesis would be to just demand the $f_i$’s to intersect properly, that is that codim $f^{-1}(0) = p_1 + \ldots + p_r$ if $p_i = \text{codim } f_i$. However, the following example shows that Theorem 1.4 does not extend to this case.

**Example 2.** Let $f_1 = z_1^2$, $f_2 = z_1 z_2$ and $g = z_2 z_3$ be defined in a neighborhood of the origin in $\mathbb{C}^3$. Then

$$Y_f = \{ f_1 = f_2 = 0 \} = \{ z_1 = 0 \},$$

$$Y_g = \{ g = 0 \} = \{ z_2 = 0 \} \cup \{ z_3 = 0 \},$$

and

$$Y = Y_f \cap Y_g = \{ z_1 = z_2 = 0 \} \cup \{ z_1 = z_3 = 0 \},$$

that is the union of the $z_3$-axis and the $z_2$-axis. Evidently, $Y_f$ and $Y_g$ have codimension 1, and $Y$ has codimension 2, so $f$ and $g$ intersect properly, although they do not define a complete intersection.
We want to compute and compare the currents $R^f \wedge R^g$ and $R^{f,g}$. However, we confine ourselves to computing the parts of the currents of degree 2, since we find them the most interesting. The top degree parts do not differ very much from the ones in Example 1. Therefore it suffices to consider test forms of bidegree $(3,1)$. Throughout this example we let $\phi = \phi_1 \wedge d\overline{z}$, where $d\overline{z} = d\overline{z}_1 \wedge d\overline{z}_2 \wedge d\overline{z}_3$ and $\phi_f = \varphi^1(z)d\overline{z}_1 + \varphi^2(z)d\overline{z}_2 + \varphi^3(z)d\overline{z}_3$.

We start with the current $R^f \wedge R^g$. Adopting the trivial metric we get
\[ s^f = \bar{f}_1 e_1 + \bar{f}_2 e_2 = \overline{z}_1 (\overline{z}_1 e_1 + \overline{z}_2 e_2) \]

and
\[ |f|^2 = |\overline{z}_1|^2 (|\overline{z}_1|^2 + |\overline{z}_2|^2), \]
so that
\[ u^f = \frac{\overline{z}_1 e_1 + \overline{z}_2 e_2}{|\overline{z}_1|^2 + |\overline{z}_2|^2} \]
(for the part of $R^f \wedge R^g$ of degree 2 we only need to care about $u^f$). If we restrict ourselves to test forms with support outside \{ $\overline{z}_2 = 0$ \}, where $|\overline{z}_1|^2 + |\overline{z}_2|^2$ is non-vanishing, the action on $\phi$ is given by
\[
\int \bar{\partial}(\overline{z}_1 |^2 (|\overline{z}_2|^2 + |\overline{z}_3|^2) \lambda) \frac{\overline{z}_1 e_1 + \overline{z}_2 e_2}{|\overline{z}_1|^2 + |\overline{z}_2|^2} \wedge \bar{\partial}|^{\overline{z}_2 \overline{z}_3}|^{2 \lambda} \frac{e_3}{\overline{z}_2 \overline{z}_3} \wedge \phi|_{\lambda=0} = \\
- \int \bar{\partial}\left[ \frac{1}{\overline{z}_1} \wedge \bar{\partial}\left[ \frac{1}{\overline{z}_2} \wedge \overline{\partial}\left[ \frac{1}{\overline{z}_3} \wedge e_2 \wedge e_3 \wedge \phi \right] \right] \right].
\]

In particular, this is a current with support on the $\overline{z}_2$-axis, which is to be expected, since $Y$ is the union of the $\overline{z}_e$- and $\overline{z}_3$-axes and $\phi$ now has support outside the $\overline{z}_3$-axis. More explicitly we can write this expression as
\[
\int \frac{1}{\overline{z}_2} \varphi^2(0, \overline{z}_2, 0) e_2 \wedge e_3 \wedge d\overline{z}_2 \wedge d\overline{z}_2.
\]

To include also test forms with support over \{ $\overline{z}_2 = 0$ \} we need to resolve the singularity of $f$ at the $\overline{z}_3$-axis. By the blow-up of the origin in the $\overline{z}_1 \overline{z}_2$-plane, using the notation of Example 1, we get
\[
\tilde{s}^f = \Pi^* s^f = \tau_2^2(e_1 + \tau_2 e_2)
\]
in $\Omega_1$, so that
\[
\tilde{u}^f = \frac{e_1 + \tau_2 e_2}{\tau_1 (1 + |\tau_2|^2)}.
\]
The action of $R^{II,f} \wedge R^{II,g}$ on $\tilde{\phi}$ is given by
\[
(4.5) \quad \int \bar{\partial}\left[ \tau_1 |^{4 \lambda} \wedge \frac{e_1 + \tau_2 e_2}{\tau_1 (1 + |\tau_2|^2)} \wedge \bar{\partial}\left[ \tau_1 \tau_2 \overline{z}_3 |^{2 \lambda} \right] \wedge \frac{e_3}{\tau_1 \tau_2 \overline{z}_3} \wedge \phi|_{\lambda=0} = \\
\frac{2}{3} \int \bar{\partial}\left[ \tau_1 |^{6 \lambda} \wedge \frac{e_1 + \tau_2 e_2}{\tau_1 (1 + |\tau_2|^2)} \wedge \bar{\partial}\left[ \tau_2 \overline{z}_3 |^{2 \lambda} \right] \wedge \frac{e_3}{\tau_2 \overline{z}_3} \wedge \phi|_{\lambda=0}.\right.
\]
Note that we have omitted the non-vanishing factor \((1 + |\tau_2|^2)^\lambda\) from the numerator. This is allowed since it has no effect on the value at \(\lambda = 0\) according to Lemma 2.2. Before proceeding, let us take a closer look at \(\phi = \Pi^* \phi_I \wedge \Pi^* dz\). Observing that

\[
\Pi^* dz = d\tau_1 \wedge d(\tau_1 \tau_2) \wedge dx_3 = \tau_1 d\tau_1 \wedge d\tau_2 \wedge dx_3,
\]

and that

\[
\Pi^* \phi_I = \varphi^1(\tau_1, \tau_1 \tau_2, z_3) \, d\tau_1 + \varphi^2(\tau_1, \tau_1 \tau_2, z_3) \, d(\tau_1 \tau_2) + \varphi^3(\tau_1, \tau_1 \tau_2, z_3) \, d\mathbb{S},
\]

we notice that if the second \(\bar{\partial}\) in the right hand side integral of (4.5) falls on \(|z_3|^{2\lambda}\), the only part of \(\phi\) that survives is the \(\varphi^2\)-part. However, this contains a factor \(d(\tau_1 \tau_2)\), which implies that the \(\tau_1\) integral vanishes according to Lemma 2.2. Thus, this \(\bar{\partial}\) must fall on \(|\tau_2|^{2\lambda}\) and therefore the integral is equal to

\[
\frac{2}{3} \left(2\pi i\right)^2 \int \frac{1}{\tau_1} \wedge e_1 \wedge \frac{1}{\tau_2} \wedge e_3 \wedge \phi = \\
\frac{2}{3} \left(2\pi i\right)^2 \int \frac{1}{\tau_1} \wedge e_1 \wedge \frac{1}{\tau_2} \wedge e_3 \wedge e_3 \wedge d\mathbb{S} \wedge dx_3 = \\
\left(2\pi i\right)^2 \int \varphi^3(0, 0, z_3) \, \frac{1}{z_3} e_1 \wedge e_3 \wedge d\mathbb{S} \wedge dx_3 = \\
- \frac{2}{3} \int \frac{1}{\tau_1} \wedge \frac{1}{\tau_2} \wedge e_1 \wedge e_3 \wedge \phi.
\]

For the last two equalities we have used (4.1) from Example 1.

Next, we compute the action of \(R^I f \wedge R^I g\) in §2. Now,

\[
\ddot{s}^f = \ddot{s}_1^2 \ddot{s}_2 (e_2 e_1 + e_2),
\]

and so

\[
\ddot{u}^f_1 = \ddot{s}_1 e_1 + e_2 \quad \frac{1}{\sigma_1^2 \sigma_2 (1 + |\sigma_2|^2)}.
\]

Thus, the action on \(\ddot{\phi}\) is given by

\[
\int \ddot{\phi} |\sigma_1|^2 |\sigma_2|^{2\lambda} \wedge \left( \ddot{s}_1 e_1 + e_2 \right) \wedge \ddot{s}_2 \sigma_1 \sigma_3 |\sigma_3|^{2\lambda} \wedge \ddot{e}_3 \wedge \ddot{\phi}_{\lambda=0}.
\]

If the first \(\ddot{\partial}\) falls on \(|\sigma_1|^{2\lambda}\), the second \(\ddot{\partial}\) must fall on \(|\sigma_3|^{2\lambda}\), and then the only part of \(\ddot{\phi}\) that comes into play is \(\ddot{s}_1 \varphi^2 \ddot{\sigma}_2\), so the \(\sigma_1\)-integral will vanish due to the factor \(\ddot{s}_1\). Thus the first \(\ddot{\partial}\) must fall on \(|\sigma_2|^{2\lambda}\). The term that appears when the second \(\ddot{\partial}\) falls on \(|\sigma_3|^{2\lambda}\) is exactly the current we obtained when we assumed that \(z_2 \neq 0\), as can be easily checked. Therefore, what


remains is
\[
\frac{1}{3} \int \bar{\partial} |\sigma_2|^{2\lambda} \wedge \frac{\sigma_2 e_1 + e_2}{\sigma_2 (1 + |\sigma_2|^2)} \wedge \bar{\partial} |\sigma_1|^{6\lambda} \wedge \frac{e_3}{\sigma_1 z_3} \wedge \bar{\phi}|_{\lambda=0} =
\]
\[
\frac{1}{3} \int \bar{\partial} \left[ \frac{1}{\sigma_2} \right] \wedge e_2 \wedge \bar{\partial} \left[ \frac{1}{\sigma_1^2} \right] \wedge \frac{1}{1 + z_3} \wedge e_3 \wedge \bar{\phi} =
\]
\[
(2\pi i)^2 \frac{1}{3} \int \varphi_2^3(0, 0, z_3) \frac{1}{z_3} \wedge e_2 \wedge e_3 \wedge d\bar{z}_3 \wedge dz_3 =
\]
\[
\frac{1}{3} \int \bar{\partial} \left[ \frac{1}{z_1} \right] \wedge \bar{\partial} \left[ \frac{1}{z_2} \right] \wedge \frac{1}{z_3} \wedge e_2 \wedge e_3 \wedge \bar{\phi},
\]
To sum up so far, the action of \( R_1^f \wedge R^g \) consists of two parts, one of which
is an integration of the test form along the \( z_2 \)-axis, and another of which is
an integration along the \( z_3 \)-axis of derivatives of the test form.

We continue with \( R_2^f \wedge g \). Now
\[
s = s^0 = z_1^2 e_1 + \bar{z}_1 \bar{z}_2 e_2 + \bar{z}_2 \bar{z}_3 e_3.
\]
To be able to compute the current we need a resolution of singularities. Recall the blow-up of the origin in \( \mathbb{C}^3 \) from Example 1. In the coordinate
neighborhood \( \Omega_1 \) we get
\[
\tilde{s} = \Pi^* s = \tilde{\tau}_1^2 s', \text{ where } s' = e_1 + \tilde{\tau}_2 e_2 + \tilde{\tau}_3 e_3,
\]
and thus the action of \( R_2^f \Pi^* (f \wedge g) \) on \( \tilde{\phi} \) there is given by
\[
(4.6) \quad \int \bar{\partial} |\tau_1|^{14} \frac{s' \wedge \bar{\partial} s'}{\tau_1 (1 + |\tau_2|^2 + |\tau_2 \tau_3|^2)^2} \wedge \bar{\phi}|_{\lambda=0}.
\]
However, all terms of
\[
\Pi^* \phi = \phi^1 d\tilde{\tau}_1 + \phi^2 d(\tilde{\tau}_2 \tilde{\tau}_3) + \phi^3 d(\tilde{\tau}_1 \tilde{\tau}_3)
\]
contain a factor \( \tilde{\tau}_1 \) or \( d\tilde{\tau}_1 \) and thus (4.6) vanishes at \( \lambda = 0 \) according to
Lemma 2.2.

Morally, this can be explained by the fact that the image of \( \Omega_1 \) under \( \Pi \)
is \( \mathcal{U} \) minus the plane \( \{ z_1 = 0 \} \) (plus the origin). However we know that
\( R_1^f \wedge R^g \) should have support in this plane (more precisely on the axes), and
thus it is not very surprising that we get no contribution from \( \Omega_1 \).

We go on to the second coordinate neighborhood \( \Omega_2 \). According to
the intuitive argument above, since the image of \( \Omega_2 \) is \( \mathcal{U} \) minus the plane
\( \{ z_2 = 0 \} \), we expect to obtain something with support on the \( z_2 \)-axis. First,
note that
\[
\tilde{s} = \tilde{\sigma}_1^2 (\tilde{\sigma}_2 e_1 + \tilde{\sigma}_2 e_2 + \tilde{\sigma}_3 e_3).
\]
To calculate the current close to the \( z_2 \)-axis, that is, where \( \sigma_2 \) and \( \sigma_3 \) are
both zero, we need to refine the resolution by a blow-up along the axis. For
a neighborhood $\mathcal{U}_\sigma$ of the origin in the $\sigma_2\sigma_3$-plane we consider the proper mapping $\Pi_\sigma : \mathcal{U}_\sigma \to \mathcal{U}_\sigma$, where $\mathcal{U}_\sigma$ is the blow-up of the origin in the $\sigma_2\sigma_3$-plane. We cover $\mathcal{U}_\sigma$ by two coordinate neighborhoods

$$\{x; (x_1,x_2) = (\sigma_2, \sigma_3) \in \mathcal{U}_\sigma \} \text{ and } \{y; (y_1,y_2,y_1) = (\sigma_2, \sigma_3) \in \mathcal{U}_\sigma \}.$$  

In the second one we have

$$\tilde{s} = \tilde{\sigma}_1^2 \tilde{y}_1 s', \text{ where } s' = \tilde{y}_1^2 e_1 + \tilde{y}_2 e_2 + e_3,$$

so that the action on $\phi$ here is given by

$$\int \tilde{\sigma}_1^2 y_1 |2\lambda \wedge \frac{s' \wedge \tilde{\sigma}_1^2 \wedge \tilde{y}_2 e_1 + \tilde{y}_2 e_2 + e_3}{\sigma_1^2 y_1^2 (1 + |\tilde{y}_2 y_1|^2 + |\tilde{y}_2|^2)^2} \wedge \tilde{\phi} = 0.$$

Now $\tilde{\sigma}_1$ must fall on the factor $y_1$, since otherwise the integral will vanish by the usual arguments, so we get

$$\int \tilde{\sigma}_1^2 y_1 |2\lambda \wedge \frac{e_3 \wedge \tilde{y}_2 e_1 + \tilde{y}_2 e_2 + e_3}{\sigma_1^2 y_1^2 (1 + |\tilde{y}_2 y_1|^2 + |\tilde{y}_2|^2)^2} \wedge \tilde{\phi}.$$

Now let us take a closer look at $\tilde{\phi}$. The pull-back of $dz$ is equal to $\sigma_1^2 y_1 dy_2 \wedge d\tilde{\sigma}_1 \wedge y_1$ and the only part of $\phi_1$ that matters is $\varphi^2(\sigma_1 y_1, \sigma_1, \sigma_1 y_1) d\tilde{\sigma}_1$. By an integration in $y_2$ we get

$$-(2\pi i)^2 \int \frac{1}{\sigma_1} \varphi^2 (0, \sigma_1, 0) e_2 \wedge e_3 \wedge d\tilde{\sigma}_1 \wedge d\sigma_1 =$$

$$-\int \tilde{\sigma}_1^2 y_1 |2\lambda \wedge \frac{1}{\tilde{z}_1} \wedge \frac{1}{\tilde{z}_2} \wedge \frac{1}{\tilde{z}_3} \wedge \phi \wedge e_2 \wedge e_3.$$

Some remarks are in order. First note that the integration in $y_2$ might be extended to $\mathbb{P}^1$ so that we have actually calculated the action in $\mathcal{U}_\sigma$. Also, let us just notice that this part of $R^{y2}$ coincides with the corresponding part of $R^{y1}$.

Finally we compute the contribution from $\Omega_3$. One realizes that a further resolution along the $z_1$-axis is necessary. For a neighborhood $\mathcal{U}_3$ of the origin in the $\rho_2\rho_3$-plane, we consider $\Pi_3 : \mathcal{U}_\rho \to \mathcal{U}_\rho$, where $\mathcal{U}_\rho$ is the blow-up of the origin. We cover this by two coordinate neighborhoods

$$\{x; (x_1,x_2) = (\rho_2, \rho_3) \in \mathcal{U}_\rho \} \text{ and } \{y; (y_1,y_2,y_1) = (\rho_2, \rho_3) \in \mathcal{U}_\rho \}.$$  

It is easily verified that there will be no contribution from the second neighborhood. As usual, the pull-back of the test form will contain factors that kill the current. So what remains is the first neighborhood. It turns out that we actually need a third resolution, now of the origin in the $x_1,x_2$-plane. The blow-up $\mathcal{U}_e$ of a neighborhood $\mathcal{U}_e$ of this point is covered by two coordinate neighborhoods

$$\{\xi; (\xi_1,\xi_2) = (x_1,x_2) \in \mathcal{U}_e \} \text{ and } \{\eta; (\eta_1,\eta_2,\eta_1) = (x_1,x_2) \in \mathcal{U}_e \}.$$
In the first one,

\[ s = \rho_1^2 \xi_1^2 s', \]

where \( s' = e_1 + \xi_1 \xi_2 e_2 + \xi_2 e_3, \)

so that the action on \( \tilde{\phi} \) is given by

\[
\int \frac{e_1 \wedge d\xi_2 \wedge e_3}{(1 + |\xi_1|^2 + |\xi_2|^2)^2} \wedge \tilde{\phi} \big|_{\lambda = 0}. 
\]

By the usual degree arguments, the \( \tilde{\phi} \) must fall on \( |\xi_1|^{2\lambda} \), and since \( e_1 \wedge d\xi_2 \wedge e_3 \) is the only part of \( s' \wedge \tilde{\partial} s' \) that does not contain \( \xi_1 \) or \( d\xi_1 \), the integral is equal to

\[
\int \tilde{\phi} \left[ \frac{1}{\xi_1^2} \right] \left[ \frac{1}{z_1^2} \right] \wedge \frac{e_1 \wedge d\xi_2 \wedge e_3}{(1 + |\xi_1|^2 + |\xi_2|^2)^2} \wedge \tilde{\phi}.
\]

The pull-back of \( dz \) is equal to \( \rho_1^2 \xi_1^2 \, d\xi_1 \wedge d\xi_2 \wedge d\rho_1 \), and the only surviving part of \( \psi_1 \) is \( \varphi^2(\rho_1 \xi_1, \rho_1 \xi_2, \rho_1) \, d\rho_1 \), so by an integration in \( \xi_2 \), we get

\[
(2\pi i)^3 \int \frac{1}{\rho_1} \psi_1^3(0, 0, \rho_1) \, e_1 \wedge e_3 \wedge d\rho_1 \wedge d\rho_1 =
\]

\[
\int \tilde{\phi} \left[ \frac{1}{z_1^2} \right] \wedge \tilde{\phi} \left[ \frac{1}{z_2^2} \right] \left[ \frac{1}{z_3} \right] \wedge e_1 \wedge e_3 \wedge \phi.
\]

Here we have used that the \( \xi_1 \)-derivative must fall on the first variable, since otherwise we get something that vanishes due to antisymmetry. By extending the integration to \( \xi_2 \in \mathbb{P}^1 \), we can compute the integral over the entire \( U_\varepsilon \), and therefore we have actually computed the action in \( \Omega_3 \).

Knowing the result of these computations, we could have skipped the first blow-up of the origin in \( \mathbb{C}^3 \). It would have been sufficient to resolve the singularities along the axes. However to be sure about what happens at the origin, the resolutions we made were necessary.

To conclude, the parts of the currents with support on the \( z_2 \)-axis coincide, while the parts with support on the \( z_3 \)-axis differ in the way they differentiate the test form. Hence Theorem 1.4 does not generalize to the case of proper intersections. \( \Box \)

References


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