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MULTIDIMENSIONAL CAYLEY TRANSFORMS AND PROJECTIVE OPERATORS

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Abstract

Let a be a closed operator on a Banach space X and let $\sigma(a)$ be the set of complex numbers z such that $z - a$ is not a bijection from its domain to X. If ϕ is a Möbius transformation of the extended complex plane $\widehat{\mathbb{C}} = \mathbb{C} \cup \{\infty\}$ such that $\phi^{-1}(\infty) \notin \sigma(a)$ then the operator $\phi(a)$, given by formal substitution of a in ϕ, is bounded and $\phi(\sigma(a)) = \sigma(\phi(a))$. The operator $\phi(a)$ is called a general one-variable Cayley transform of a. This thesis consists of the paper *Multidimensional Cayley Transforms and Projective Operators*. We give a generalisation of the one-variable Cayley transform to tuples of closed operators on X and characterise the set of tuples which can be transformed into tuples of bounded commuting operators via such a 'multidimensional Cayley transform'. Möbius transformations in several complex variables are naturally transformations of complex projective space. Our approach will be to define projective operators, which will have an invariant spectrum in projective space and which will be naturally transformed by these Möbius transformations. We will use the coordinate free approach to the analytic functional calculus described by Eschmeier and Putinar and we will construct an analytic functional calculus for the projective operators. We will also provide integral representations for this functional calculus.

Keywords: Cayley transform, closed operators, Taylor's functional calculus, n-dimensional complex projective space $\mathbb{CP}^n$, integral representations

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Abstract. We construct a multidimensional Cayley transform of tuples of closed operators into tuples of bounded commuting operators. To achieve this we define projective operators. Projective operators will have a natural spectrum in $\mathbb{CP}^n$ and they will admit an analytic functional calculus. We provide an integral representation for this functional calculus.

1. Introduction

The Cayley Transform is a one to one correspondence between the self-adjoint operators and the unitary operators such that 1 is not in the point spectrum. The correspondence is given by

$$a \mapsto (a+i)(a-i)^{-1}.$$

Slightly more generally, one can consider closed operators with nonempty resolvent sets on a Banach space X. Given such an operator a and assuming $\lambda \notin \sigma(a)$ one can form the bounded operator $(a+\mu)(a-\lambda)^{-1}$ where μ is any complex number. Transformations of the form

$$a \mapsto (a+\mu)(a-\lambda)^{-1} \tag{1}$$

from the closed operators on X, denoted $\mathscr{C}(X)$, to the bounded ones will be called general one-variable Cayley transforms. In Section 2 we review the basic theory of these transformations. See also e.g. [12] and [8]. One possible generalisation to higher dimensions, i.e. to several operators, is to Cayley transformation each of the operators. Given n closed operators $(a_1, \dots, a_n)$ with nonempty resolvent sets such that their resolvents commute, i.e. the operators are commuting in the strong sense, this construction makes it possible to define $\sigma(a_1, \dots, a_n)$; transform the tuple $(a_1, \dots, a_n)$ into a commuting bounded tuple, compute the Taylor spectrum for this tuple and apply the inverse transformation to this spectrum. This makes $\sigma(a_1, \dots, a_n)$ a subset of the product of n copies of $\mathbb{CP}^1$. This technique has been used for instance by Vasilescu in [12] to prove spectral theorems for unbounded self-adjoint operators. In a more general setting this has recently been studied by Andersson and Sjöstrand in [3]. In particular they provide integral formulas for the functional calculus for tuples of operators with real spectra.

This paper is concerned with another generalisation of the general one-variable Cayley transforms. We note that the one-variable Cayley transforms are Möbius transformations, or rational fractional transformations, of $\mathbb{C}$ in which we substitute an operator. Rational fractional transformations in several complex variables are transformations on $\mathbb{C}^n$ of the form

$$(z_1, \dots, z_n) \mapsto \Big(\frac{\lambda_{1,0} + \sum_1^n \lambda_{1,j} z_j}{\lambda_{0,0} + \sum_1^n \lambda_{0,j} z_j}, \dots, \frac{\lambda_{n,0} + \sum_1^n \lambda_{n,j} z_j}{\lambda_{0,0} + \sum_1^n \lambda_{0,j} z_j}\Big), \tag{2}$$

where $(\lambda_{i,j})$ is an invertible matrix. We characterise those tuples of closed operators that can be transformed to bounded commuting tuples via mappings of the form (2). Transformations of this type will be called *multidimensional Cayley transforms*. Note that (2) is only defined for z outside the affine hyperplane $\lambda_{0,0}+\sum_1^n \lambda_{0,j}z_j = 0$ and its image is not all of $\mathbb{C}^n$ but only $\mathbb{C}^n$ minus some other affine hyperplane. The appropriate way of looking at transformations like (2) is as transformations of n-dimensional complex projective space, $\mathbb{CP}^n$. In the homogeneous coordinates $[z_0, z_1, \ldots, z_n]$ the transformation takes the form

$$[z_0, \ldots, z_n] \mapsto [\sum_0^n \lambda_{0,j}z_j, \ldots, \sum_0^n \lambda_{n,j}z_j], \tag{3}$$

and it is a biholomorphic mapping from $\mathbb{CP}^n$ onto $\mathbb{CP}^n$. We will begin our study by defining equivalence classes of bounded commuting operators, called *projective operators*, which can be substituted into (3). This is not covered by Taylor's functional calculus, constructed in [10] and [9], because we do not have a mapping between domains in $\mathbb{C}^n$ and $\mathbb{C}^m$. However, our construction will rely on Taylor's construction. Let us therefore give a short review of Taylors analytic functional calculus.

Let $b = (b_1, \ldots, b_n)$ be commuting operators on a Banach space X and let $\Lambda^{p,0}X$ be the space of X-valued $(p,0)$-forms in $\mathbb{C}^n$. We denote by δ_{z-b} the operation on $\bigoplus_{p=0}^n \Lambda^{p,0}X$ of interior multiplication with the operator-valued vector field $\sum_1^n (z_j - b_j)\frac{\partial}{\partial z_j}$. Since b is commuting we have $\delta_{z-b} \circ \delta_{z-b} = 0$ and we get the Koszul-complex $K_\bullet(z-b, X)$:

$$0 \longleftarrow \Lambda^{0,0}X \xleftarrow{\delta_{z-b}} \Lambda^{1,0}X \xleftarrow{\delta_{z-b}} \cdots \xleftarrow{\delta_{z-b}} \Lambda^{n,0}X \longleftarrow 0 \tag{4}$$

Taylor defines in [10] the spectrum $\sigma(b)$ as the complement in $\mathbb{C}^n$ of the set of points z such that the Koszul-complex (4) is exact. We have a natural continuous algebra homomorphism $\mathscr{O}(\mathbb{C}^n) \to L(X)$ given by

$$\sum_0^\infty c_\alpha z^\alpha \mapsto \sum_0^\infty c_\alpha b^\alpha. \tag{5}$$

In [9] Taylor proves that this homomorphism can be extended to a homomorphism $\mathscr{O}(U) \to L(X)$ for any neighbourhood U of $\sigma(b)$.

Theorem 1.1 (Taylor, 1970)**.** *There is an extension of the homomorphism* (5) *to a continuous algebra homomorphism*

$$f \mapsto f(b) : \ \mathscr{O}(U) \to L(X)$$

for all open sets U such that $\sigma(b) \subseteq U$. If $f = (f_1, \ldots, f_m) \in \mathscr{O}(U, \mathbb{C}^m)$, then

$$f(\sigma(b)) = \sigma(f(b)),$$

where $f(b) = (f_1(b), \dots, f_n(b))$.

The statement $f(\sigma(b)) = \sigma(f(b))$ will be referred to as the Spectral Mapping Theorem. Taylor's first proof was based on Cauchy-Weil type integrals. Soon after he gave a proof by homological methods, see [11].

To handle functions between complex manifolds we need the coordinate-free approach to the functional calculus described by Eschmeier and Putinar in [5]. One key observation is that each continuous algebra homomorphism $\Psi : \mathscr{O}(U) \to L(X)$ corresponds to a unique continuous $\mathscr{O}(U)$-module structure $\Phi : \mathscr{O}(U) \times X \to X$ on X. In fact, given the algebra homomorphism Ψ we define the $\mathscr{O}(U)$-module structure Φ on X by letting $\Phi(f, x) = \Psi(f)x$. Conversely given Φ we get the algebra homomorphism by $\Psi(f)x = \Phi(f, x)$. We reformulate Taylor's Theorem in this terminology: The natural $\mathscr{O}(\mathbb{C}^n)$-module structure on X given by

$$(\sum_0^\infty c_\alpha z^\alpha, x) \mapsto \sum_0^\infty c_\alpha b^\alpha x$$

extends to a $\mathscr{O}(U)$-module structure for each neighbourhood U of $\sigma(b)$. In [5] Eschmeier and Putinar show more generally that given a Fréchet $\mathscr{O}(N)$-module $\mathscr{M}$ where N is a Stein space of finite dimension, there is a notion of spectrum $\sigma(N, \mathscr{M}) \subseteq N$ such that the $\mathscr{O}(N)$-module structure of $\mathscr{M}$ extends to a $\mathscr{O}(\sigma(N, \mathscr{M}))$ module structure. In the case that $\mathscr{M}$ is the natural $\mathscr{O}(\mathbb{C}^n)$-module obtained from an n-tuple of regular operators b on a Fréchet space then $\sigma(N, \mathscr{M}) = \sigma(b)$ and the abstract extended module structure is precisely the module structure corresponding to the $\mathscr{O}(\sigma(b))$-functional calculus. Moreover if N' also is a Stein space and $h : V \supseteq \sigma(N, \mathscr{M}) \to N'$ is holomorphic then

$$\overline{h(\sigma(N, \mathscr{M}))} = \sigma(N', (\mathscr{M}_{ext})^h)$$

where $\mathscr{M}_{ext}$ is the extended module and $(\mathscr{M}_{ext})^h$ is the $\mathscr{O}(N')$-module obtained by composing with h.

We will see that the projective operators have a natural spectrum in $\mathbb{CP}^n$ and that, for any f holomorphic in a neighbourhood of this spectrum, we get a pairing $(f, x) \mapsto \Phi(f)x$. In Section 7 we provide integral formulas representing this module structure by generalising the ideas in [1], where Andersson derives integral formulas for Taylor's functional calculus. If we denote a projective tuple by $[b]$ and its spectrum by $\sigma[b] \subseteq \mathbb{CP}^n$, and assume that $\sigma[b]$ does not intersect the hyperplane $[\lambda]$, we get that $P_\lambda = (p_1, \dots, p_n)$, a projection from the hyperplane $[\lambda]$ onto $\mathbb{C}^n$, is holomorphic in a neighbourhood of $\sigma[b]$. Then $P_\lambda([b]) = (c_1, \dots, c_n)$ will be a tuple of bounded commuting operators and $P_\lambda(\sigma[b]) = \sigma(c)$. Even if $[\lambda]$ is not disjoint with $\sigma[b]$, $P_\lambda([b])$ might still have meaning; not necessarily as a tuple of bounded commuting operators but as a tuple of closed operators. We say that a hyperplane is *admissible* for $[b]$ if $P_\lambda(\sigma[b])$ has meaning as a tuple of closed operators. In Section 4 we characterise those tuples of closed operators arising in this way. It turns out that being a projection of a projective operator corresponds to a certain commutation condition on the closed tuple. Tuples of closed operators satisfying this commutation condition will be called *affine operators*. It follows that the affine operators are precisely those which can be Cayley transformed to tuples of bounded commuting operators. In fact, given an affine operator $a = (a_1, \dots, a_n)$ there is a projective operator $[b]$ such that $P_\lambda([b]) = a$. Choosing a hyperplane $[\tilde{\lambda}]$ which does not intersect $\sigma[b]$ we get a multidimensional Cayley transform of a as the bounded commuting tuple $P_{\tilde{\lambda}}([b])$.

The disposition of the paper is as follows. In Section 2 we review the basic theory about general one-variable Cayley transforms. In Section 3 we define projective operators and prove that a projective operator has a well defined spectrum in $\mathbb{CP}^n$ and that it admits an analytic functional calculus. In Section 4 we study the behaviour of projective operators under various projections from $\mathbb{CP}^n$ to $\mathbb{C}^n$. We define affine operators as tuples of closed operators satisfying a certain commutation condition and show that these are precisely the closed tuples arising as such projections. Section 5 deals with the Taylor spectrum of affine operators and we show that a Spectral Mapping Theorem holds for the projections of projective operators to the affine ones. In Section 6 we apply our theory to Cayley transforms from $\mathbb{C}^n$ to $\mathbb{C}^n$ and show that a tuple of closed operators is affine if and only if it is a Cayley transform of a bounded commuting tuple, and that the Spectral Mapping Theorem holds. In Section 7 we provide an integral representation of the analytic functional calculus obtained in Section 3.

2. The one variable case

Let X be a Banach space and let $\mathscr{C}(X)$ be the set of closed, but not necessarily densely defined operators on X. For any linear operator a on X the spectrum of a $\sigma(a)$ is the complement in $\mathbb{C}$ of the set of points λ such that $\lambda - a$ is a bijection $\mathscr{D}(a) \to X$. The point spectrum $\sigma_p(a) \subseteq \sigma(a)$ is the set of $\lambda \in \mathbb{C}$ such that $\lambda - a$ is not injective. For $a \in \mathscr{C}(X)$ we have by the Closed Graph Theorem that $\lambda \notin \sigma(a)$ if and only if $\lambda - a$ has a bounded inverse. We let $\widehat{\mathbb{C}}$ denote the extended complex plane; $\mathbb{C} \cup \{\infty\}$ and we define the extended spectrum $\hat{\sigma}(a)$ as $\sigma(a)$ if a is bounded and $\sigma(a) \cup \{\infty\}$ if a is not bounded.

Let ϕ be a Möbius transformation of $\widehat{\mathbb{C}}$. We claim that $\phi(a)$ has meaning as an element in $\mathscr{C}(X)$ if $\phi^{-1}(\infty) \notin \sigma_p(a)$. Given the Möbius transformation ϕ we let $M_\phi \in \mathrm{GL}(2, \mathbb{C})$ be the corresponding 2×2-matrix. The matrix

M_ϕ acts naturally as a homeomorphism of $X \times X$. If $M_\phi = \{m_{j,k}\}_{1\leq j,k\leq 2}$ and $\phi^{-1}(\infty) \notin \sigma_p(a)$ we may put

$$\phi(a) = (m_{1,1}a + m_{1,2})(m_{2,1}a + m_{2,2})^{-1}. \tag{6}$$

We have to show that $\phi(a)$ is closed. From (6) we see that $\mathscr{D}(\phi(a)) = \mathscr{R}(m_{2,1}a + m_{2,2}) = \mathscr{R}(a - \phi^{-1}(\infty))$, where $\mathscr{D}$ and $\mathscr{R}$ denote the domain and range respectively, (excluding the case that $\phi(\infty) = \infty$ in which case a and $\phi(a)$ have the same domain). Now if $x \in \mathscr{D}(a)$ then $y = (m_{2,1}a + m_{2,2})x \in \mathscr{D}(\phi(a))$ and

$$M_\phi(x, ax) = (y, \phi(a)y). \tag{7}$$

Conversely if $y \in \mathscr{D}(\phi(a))$ then $x = (m_{2,1}a + m_{2,2})^{-1}y \in \mathscr{D}(a)$ and (7) holds. Hence we have that $M_\phi\mathrm{Graph}(a) = \mathrm{Graph}(\phi(a))$. Since $\mathrm{Graph}(a)$ is closed and M_ϕ is a homeomorphism we obtain that $\mathrm{Graph}(\phi(a))$ is closed as claimed. Moreover $\phi(a)$ is bounded if and only if $\phi^{-1}(\infty) \notin \hat{\sigma}(a)$. In fact, if $\phi^{-1}(\infty) \notin \hat{\sigma}(a)$ then either $\phi^{-1}(\infty) \neq \infty$ in which case $\mathscr{D}(\phi(a)) = \mathscr{R}(a - \phi^{-1}(\infty)) = X$ or $\phi^{-1}(\infty) = \infty$. In the latter case a has to be bounded and $\phi(a) = m_{1,1}a + m_{1,2}$. Therefore, in both cases $\phi(a)$ is bounded. Conversely, assume that $\phi(a)$ is bounded. If $\phi^{-1}(\infty) = \infty$ then $\mathscr{D}(a) = \mathscr{D}(\phi(a)) = X$ so a is bounded. If $\phi^{-1}(\infty) \neq \infty$ then $\mathscr{R}(a - \phi^{-1}(\infty)) = \mathscr{D}(\phi(a)) = X$ and hence $a - \phi^{-1}(\infty)$ is surjective. By assumption $\phi^{-1}(\infty) \notin \sigma_p(a)$ and so $a - \phi^{-1}(\infty)$ is injective as well. Thus $\phi^{-1}(\infty) \notin \hat{\sigma}(a)$. We conclude that the closed operators on X which can be Cayley transformed into bounded operators are precisely those which have nonempty resolvent sets. We finally prove a spectral mapping theorem for the general one-variable Cayley transforms.

Theorem 2.1. *Let $a \in \mathscr{C}(X)$ and let ϕ be a Möbius transformation such that $\phi^{-1}(\infty) \notin \sigma_p(a)$. Then*

$$\phi(\hat{\sigma}(a)) = \hat{\sigma}(\phi(a)).$$

Proof. Assume that $\lambda \notin \hat{\sigma}(a)$. We look at the possible cases. If $\lambda = \phi^{-1}(\infty)$ then we know from above that $\phi(a)$ is bounded, that is $\phi(\lambda) = \infty \notin \hat{\sigma}(\phi(a))$. If $\lambda \neq \phi^{-1}(\infty)$ then, in the case $\lambda = \infty$ we have

$$\phi(\lambda) - \phi(a) = \frac{\det M_\phi}{m_{2,1}^2}(\lambda - a)(a - \phi^{-1}(\infty))^{-1} \tag{8}$$

and if $\lambda \neq \infty$ then instead we have

$$\phi(\lambda) - \phi(a) = \frac{\det M_\phi}{m_{2,1}(m_{2,1}\lambda + m_{2,2})}(\lambda - a)(a - \phi^{-1}(\infty))^{-1}.$$

In either case $\phi(\lambda) - \phi(a)$ is an injective and surjective mapping $\mathscr{D}(\phi(a)) \to X$ and hence $\phi(\lambda) \notin \hat{\sigma}(\phi(a))$. Thus $\hat{\sigma}(\phi(a)) \subseteq \phi(\hat{\sigma}(a))$. But ϕ^{-1} is also a Möbius transformation and from (8) it has the property that $\phi(\infty) \notin \sigma_p(\phi(a))$. From what we have proved this far we conclude that

$$\hat{\sigma}(a) = \hat{\sigma}(\phi^{-1}\phi(a)) \subseteq \phi^{-1}(\hat{\sigma}(\phi(a))).$$

Hence $\phi(\hat{\sigma}(a)) \subseteq \hat{\sigma}(\phi(a))$. □

The preceding discussion suggests that the closed operator a defines some invariant object on $\mathbb{CP}^1 = \widehat{\mathbb{C}}$ if $\infty \notin \sigma_p(a)$. In the canonical affine part of $\widehat{\mathbb{C}}$ this object becomes the operator a and in some other affine part, corresponding to a Möbius transformation ϕ of the canonical one, it becomes $\phi(a)$ and has spectrum $\phi(\hat{\sigma}(a))$.

3. Projective operators and analytic functional calculus

In this section we define projective operators as the equivalence classes of an equivalence relation on a subset of the $n+1$-tuples of bounded commuting operators on a Banach space in such a way that the equivalence classes have well defined invariant Taylor spectrum in $\mathbb{CP}^n$. We will also show that the projective operators admit an analytic functional calculus.

Definition 3.1. Let $b = (b_0, \dots, b_n)$ and $\tilde{b} = (\tilde{b}_0, \dots, \tilde{b}_n)$ be tuples of bounded commuting operators on a Banach space X. We define $b \sim \tilde{b}$ if there are finitely many bounded commuting tuples b^j, $j = 1, \dots, m$ such that $b^1 = b$ and $b^m = \tilde{b}$ and for $j = 1, \dots, m-1$ we have $b^{j+1} = c_j b^j$ for some invertible $c_j \in (b^j)'$, the commutant of b^j.

Lemma 3.2. *The relation $\sim$ of definition 3.1 is an equivalence relation.*

Proof. We note that the relation R on bounded commuting $n+1$-tuples defined by $bR\tilde{b}$ if $\tilde{b} = cb$ for some invertible $c \in (b)'$ is reflexive and symmetric. Reflexivity is obvious since $e \in (b)'$. It is symmetric because if $\tilde{b} = cb$ for some invertible $c \in (b)'$ then $b = c^{-1}\tilde{b}$ and letting $\tilde{b} = (\tilde{b}_0, \dots, \tilde{b}_n)$ and $b = (b_0, \dots, b_n)$ we see

$$c^{-1}\tilde{b}_j = c^{-1}cb_j = b_j = b_j cc^{-1} = cb_j c^{-1} = \tilde{b}_j c^{-1}$$

so $c^{-1} \in (\tilde{b})'$. The relation $\sim$ is defined as the transitive closure of R so it is by definition transitive and it inherits reflexivity and symmetry from R. □

Remark 3.3. We will see later on, Remark 4.6, that for the tuples we will be interested in here there is a simpler description of the relation $\sim$. For these tuples it will also turn out, see Remark 4.5, that even though $\sim$ is defined as the transitive closure of R, any two representations for an equivalence class are not more than two steps from each other.

We denote the equivalence class containing b by $[b]$ and we let π denote the canonical mapping $\mathbb{C}^{n+1} \to \mathbb{CP}^n$.

Proposition 3.4. *Let $b = (b_0, \dots, b_n)$ be a commuting tuple of bounded operators on X and let $c \in (b)'$ be invertible. If $0 \notin \sigma(b)$ then $0 \notin \sigma(cb)$ and*

$$\pi\sigma(b_0, \dots, b_n) = \pi\sigma(cb_0, \dots, cb_n).$$

Proof. Define ψ and $\phi : \mathbb{C}^{n+2} \to \mathbb{C}^{n+1}$ by

$$\psi(z, z_0, \dots, z_n) = (zz_0, \dots, zz_n)$$

and

$$\phi(z, z_0, \dots, z_n) = (z_0, \dots, z_n)$$

respectively. We claim that $\sigma(c, b_0, \dots, b_n)$ avoids the hyperplane in $\mathbb{C}^{n+2}$ orthogonal to the vector $(1, 0, \dots, 0)$. In fact, we have

$$\sigma(c, b_0, \dots, b_n) \subseteq \sigma(c) \times \sigma(b_0, \dots, b_n) \tag{9}$$

see [10] and since c is invertible the claim follows. Moreover from (9) and the assumption that $0 \notin \sigma(b)$ we see that $\sigma(c, b_0, \dots, b_n)$ also avoids the coordinate axis $(z, 0, \dots, 0)$. Hence we may take a neighbourhood U of $\sigma(c, b_0, \dots, b_n)$ such that U does not intersect neither the hyperplane orthogonal to $(1, 0, \dots, 0)$ nor the coordinate axis $(z, 0, \dots, 0)$. Then the images V_1 and V_2 of U under ψ and ϕ respectively do not contain the origin and so the diagram

$$\begin{tikzcd} U \arrow[r, "\psi"] \arrow[d, "\phi"'] & V_1 \arrow[d, "\pi"] \\ V_2 \arrow[r, "\pi"'] & \mathbb{CP}^n \end{tikzcd} \tag{10}$$

must commute. By the Spectral Mapping Theorem

$$\sigma(b_0, \dots, b_n) = \sigma\, \phi(c, b_0, \dots b_n) = \phi\, \sigma(c, b_0, \dots, b_n)$$

$$\sigma(cb_0, \dots, cb_n) = \sigma\, \psi(c, b_0, \dots b_n) = \psi\, \sigma(c, b_0, \dots, b_n)$$

and since the diagram (10) commutes we conclude

$$\pi\, \sigma(cb_0, \dots, cb_n) = \pi\, \sigma(b_0, \dots, b_n).$$

□

Corollary 3.5. *Let $b \sim \tilde{b}$ and assume $0 \notin \sigma(b)$. Then $0 \notin \sigma(\tilde{b})$ and*

$$\pi\sigma(b) = \pi\sigma(\tilde{b}).$$

Proof. From the definition of $\sim$ we get commuting tuples $\{b^j\}_1^m$ such that $b^1 = b$ and $b^m = \tilde{b}$ and $b^{j+1} = c_j b^j$ where $c_j \in (b^j)'$ is invertible. From Proposition 3.4 we get $0 \notin \sigma(b^2)$ and

$$\pi\sigma(b) = \pi\sigma(b^2)$$

so inductively we obtain $0 \notin \sigma(b^j)$, $j = 1, \ldots, m$ and

$$\pi\sigma(b) = \pi\sigma(b^2) = \cdots = \pi\sigma(b^{m-1}) = \pi\sigma(\tilde{b}).$$

$\square$

It follows from the corollary that if $0 \notin \sigma(b)$ then $0 \notin \sigma(\tilde{b})$ for any $\tilde{b} \in [b]$ and $\pi\sigma(b) = \pi\sigma(\tilde{b})$. Hence we can make the following definitions.

Definition 3.6. Let b be a commuting tuple of bounded operators on a Banach space X such that $0 \notin \sigma(b)$. We define the *projective operator* $[b]$ as the equivalence class containing b.

Definition 3.7. Let $[b]$ be a projective operator. The *spectrum* $\sigma[b] \subseteq \mathbb{CP}^n$ for the projective operator $[b]$ is defined by

$$\sigma[b] = \pi\sigma(b).$$

We now use the coordinate free approach to functional calculus described in [5] to construct an analytic functional calculus for the projective tuples. The main theorem of this section is the following.

Theorem 3.8. *If $[b]$ is a projective operator, then there is a unique $\mathscr{O}(\sigma[b])$-module structure on X given by*

$$\mathscr{O}(\sigma[b]) \times X \to X \qquad\qquad (f, x) \mapsto f([b])x$$

and if $f = (f_1, \ldots, f_m) \in \mathscr{O}(\sigma[b], \mathbb{C}^m)$ then

$$\sigma(f([b]) = f(\sigma[b])$$

where $f([b]) = (f_1([b]), \ldots, f_m([b]))$.

Proof. We construct the module-structure as follows. Given some $f \in \mathscr{O}(\sigma[b])$ we consider the lift $\tilde{f}$ of f that makes the following diagram commute.

$$\begin{tikzcd} \mathbb{C}^{n+1} \arrow[d] \arrow[dr, "\tilde{f}"] & \\ \mathbb{CP}^n \arrow[r, "f"'] & \mathbb{C} \end{tikzcd}$$

Then $\tilde{f}$ is holomorphic in a neighbourhood of $\sigma(b)$ for any representative $b \in [b]$ and $\tilde{f}$ is constant on the complex lines through the origin (with the origin deleted). From Taylor's analytic functional calculus we get for each $b \in [b]$ an operator $\tilde{f}(b) \in L(X)$. We will see that in fact $\tilde{f}(b)$ is independent of representative b and our desired pairing $\mathscr{O}(\sigma[b]) \times X \to X$ will be $(f, x) \mapsto \tilde{f}(b)x$ where b is any representative of $[b]$.

Let b be a representative of $[b]$ and let $c \in (b)'$ be invertible. Put $\tilde{b} = cb$ and let ϕ and ψ be the mappings defined in Proposition 3.4. Let U_1 be a neighbourhood of $\sigma(b)$ in which $\tilde{f}$ is holomorphic and let V be a neighbourhood of $\sigma(c)$ such that $\overline{D(0,r)} \cap V = \emptyset$ for some $0 < r < 1$. Since c is invertible $0 \notin \sigma(c)$ and such a neighbourhood exists. Put

$$U = \bigcup_{\lambda \notin \overline{D(0,r)}} \lambda U_1.$$

Then $\tilde{f} \circ \phi$ and $\tilde{f} \circ \psi$ are holomorphic in $V \times U$. Moreover since $r < 1$ we have $\sigma(b) \subseteq U$ and so $\sigma(c, b) \subseteq \sigma(c) \times \sigma(b) \subseteq V \times U$. Now since $\tilde{f}$ is constant on the complex lines through the origin we have $\tilde{f} \circ \phi \mid_{V \times U} = \tilde{f} \circ \psi \mid_{V \times U}$ and we conclude from the composition rule

$$\tilde{f}(b) = \tilde{f} \circ \phi \mid_{V \times U} (c, b) = \tilde{f} \circ \psi \mid_{V \times U} (c, b) = \tilde{f}(cb) = \tilde{f}(\tilde{b}).$$

If b and $\tilde{b}$ are any two representatives of $[b]$ we have bounded commuting tuples $\{b^j\}_1^m$ with $b = b^1$, $\tilde{b} = b^m$ and $b^{j+1} = c_j b^j$, $j = 1, \ldots, m-1$ for invertible $c_j \in (b^j)'$. Inductively we obtain

$$\tilde{f}(b) = \tilde{f}(b^2) = \cdots = \tilde{f}(b^{m-1}) = \tilde{f}(\tilde{b}).$$

Thus f is well defined on $[b]$ and we write $f([b])$ for the operator $\tilde{f}(b)$.

To prove the Spectral Mapping property we proceed as follows. Since $\tilde{f}$ is constant on the complex lines through the origin $\tilde{f}(\sigma(b))$ is the same for all $b \in [b]$ and so from the Spectral Mapping Theorem (see e.g. [5]) we get

$$f(\sigma[b]) = \tilde{f}(\sigma(b)) = \sigma(\tilde{f}(b)) = \sigma(f([b])).$$

Uniqueness follows from the Spectral Mapping Property. See [5]. □

Let M be a complex manifold and assume $f : U \supseteq \sigma[b] \to M$ is holomorphic. We obtain a $\mathscr{O}(M)$-module structure $\mathscr{M}$ on X by

$$\mathscr{O}(M) \times X \to X \qquad (g, x) \mapsto g \circ f([b])x.$$

In [5] Eschmeier and Putinar defines the spectrum $\sigma(M, \mathscr{M}) \subseteq M$ of the module $\mathscr{M}$ and shows that the $\mathscr{O}(M)$-module structure extends uniquely to an $\mathscr{O}(\sigma(M, \mathscr{M}))$-module structure on X. Moreover they show a Spectral Mapping Theorem which in our case implies

$$\sigma(M, \mathscr{M}) = f(\sigma[b]).$$

It is shown that if $M = \mathbb{C}^m$ we can realise the extended module structure as the analytic functional calculus for an m-tuple of commuting bounded operators c on X by choosing coordinates on $\mathbb{C}^m$ and that the spectrum of the abstract module is precisely $\sigma(c)$. The Composition rule in Taylor's functional calculus is therefore built into the construction.

To stress the independence of coordinates in our study of projective tuples we adopt an invariant notation. For a subset M of $\mathbb{CP}^n$ we denote by M^* the *dual complement* of M. That is

$$M^* = \{[\lambda] \in \mathbb{CP}^{n*}; \langle z, \lambda\rangle \neq 0 \ \forall [z] \in M\}.$$

Geometrically M^* is the set of hyperplanes in $\mathbb{CP}^n$ which do not intersect M. The correspondence between hyperplanes in $\mathbb{CP}^n$ and points in $\mathbb{CP}^{n*}$ is the usual duality correspondence. To $[\lambda] \in \mathbb{CP}^{n*}$ we associate the hyperplane $\{[z]; \langle z, \lambda\rangle = 0\}$. We will not make any distinction between points in $\mathbb{CP}^{n*}$ and their corresponding hyperplanes and we will freely allow ourselves to speak about 'the hyperplane $[\lambda]$' if $[\lambda] \in \mathbb{CP}^{n*}$.

Lemma 3.9. *Let $[b]$ be a projective operator. Then*

$$\sigma[b]^* = \{[\lambda] \in \mathbb{CP}^{n*}; \langle b, \lambda\rangle \text{ is invertible}\}.$$

Proof. Since any two representatives of $[b]$ differ by an invertible operator we see that the statements in the lemma only depend on $[b]$. For the inclusion $\subseteq$ assume $[\mu] \in \sigma[b]^*$. Then from the definition we have $\langle z, \mu\rangle \neq 0$ for all $[z] \in \sigma[b]$. Thus the function $z \mapsto 1/\langle z, \mu\rangle$ from $\mathbb{C}^{n+1}$ to $\mathbb{C}$ is holomorphic in a neighbourhood of $\sigma(b)$. Hence from the functional calculus we see that $\langle b, \mu\rangle$ is invertible.

For the other inclusion assume $\langle b, \mu\rangle$ is invertible. We shall show that $\sigma(b)$ does not intersect the hyperplane μ. If $\mu = (1, 0, \dots, 0)$ we have to show that if b_0 is invertible then $\sigma(b)$ does not intersect the hyperplane orthogonal to $(1, 0, \dots, 0)$. But if b_0 is invertible then $0 \notin \sigma(b_0)$ and since

$$\sigma(b) \subseteq \sigma(b_0) \times \sigma(b_1, \dots, b_n)$$

(see [10]) $\sigma(b)$ can not intersect the hyperplane in question. For the general case let L be an invertible linear transformation sending μ to $(1, 0, \dots, 0)$. By the Spectral Mapping Theorem, to show that $\sigma(b)$ does not intersect μ is equivalent to show that $\sigma(L^{*-1}b)$ does not intersect $L\mu = (1, 0, \dots, 0)$. But the first component in $L^{*-1}b$ is

$$\langle L^{*-1}b, L\mu\rangle = \langle b, \mu\rangle$$

which is invertible by assumption and so the lemma follows. □

Remark 3.10. From now on we will always assume that $\sigma[b]$ avoids some hyperplane in $\mathbb{CP}^n$. We will do this because it will make it possible to realise the projective tuple as an ordinary n-tuple of bounded operators. Since the objective of this paper is to make multidimensional Cayley transforms of tuples of unbounded operators into tuples of bounded operators the assumption is natural.

If we fix some $[\tilde{\lambda}] \in \sigma[b]^*$ then the function

$$[z] \mapsto \frac{\langle z, \tilde{\lambda} \rangle}{\langle z, \lambda \rangle}$$

is holomorphic in a neighbourhood of $\sigma[b]$ if also $[\lambda] \in \sigma[b]^*$. Theorem 3.8 then implies that we get a well defined mapping in $\mathscr{O}(\sigma[b]^*, \cap_{b\in[b]}(b))$ given by

$$[\lambda] \mapsto \frac{\langle b, \tilde{\lambda} \rangle}{\langle b, \lambda \rangle}$$

This is the *Fantappie transform* of the $L(X)$-valued analytic functional $\mathscr{O}(\sigma[b]) \to L(X)$, $f \mapsto f([b])$ given by Theorem 3.8.

4. Affine operators

We extend the Fantappie transform to a larger set $\sigma[b]^*_{adm}$, called the set of admissible hyperplanes, and get a $\mathscr{C}(X)$-valued mapping instead. We will also define *affine operators* to be tuples of closed operators with certain commutation properties. We will show that affine tuples are precisely the tuples obtained by projecting a projective tuple from an admissible hyperplane. In order to keep track of the various domains of definition that turn up we start with some technical results.

In what follows we will often make implicit use of the following fact.

Proposition 4.1. *Let a be any closed operator on X and let b be bounded. Then, the operator ab with domain $\mathscr{D}(ab) = \{x \in X; bx \in \mathscr{D}(a)\}$ is closed.*

Proof. Assume that $\{x_j\}_0^\infty$ is a sequence in $\mathscr{D}(ab)$ such that $x_j \to x$ and $abx_j \to y$. Since b is bounded we have $bx_j \to bx$ and since a is closed we must have $bx \in \mathscr{D}(a)$, i.e. $x \in \mathscr{D}(ab)$, and $abx = y$. □

The following lemma generalises the fact that if b and c are bounded operators and b is invertible, then $bc = cb$ if and only if $b^{-1}c = cb^{-1}$.

Lemma 4.2. *Let b and c be bounded operators on X and assume b is injective. Then $bc = cb$ if and only if $cb^{-1} \subseteq b^{-1}c$. If this condition is fulfilled and in addition c is invertible then actually $cb^{-1} = b^{-1}c$.*

Proof. Assume $bc = cb$ and let $x \in \mathscr{D}(cb^{-1}) = \mathscr{D}(b^{-1})$. Then $x = by$ for some $y \in X$. Since b and c commute we get $cx = cby = bcy$ and so we must have $cx \in \mathscr{D}(b^{-1})$. Hence $\mathscr{D}(cb^{-1}) \subseteq \mathscr{D}(b^{-1}c)$ and

$$cb^{-1}x = cb^{-1}by = cy = b^{-1}bcy = b^{-1}cby = b^{-1}cx.$$

It follows that $cb^{-1} \subseteq b^{-1}c$. Conversely assume $cb^{-1} \subseteq b^{-1}c$. Note that if $x \in \mathscr{D}(b^{-1})$ then $cx \in \mathscr{D}(b^{-1})$ and so $b^{-1}cb \in L(X)$. By assumption

$$b^{-1}cb \supseteq cb^{-1}b = c$$

and because $c \in L(X)$ we must have equality. Multiplying by b from the left we obtain $cb = bc$.

For the last statement assume c is invertible and commutes with b. Then c^{-1} also commutes with b. To show $cb^{-1} = b^{-1}c$ it is enough to show $\mathscr{D}(b^{-1}c) \subseteq \mathscr{D}(cb^{-1})$ by the proof this far. Take $x \in \mathscr{D}(b^{-1}c)$, that is $cx \in \mathscr{D}(b^{-1})$. Then $cx = by$ for some $y \in X$. We get $x = c^{-1}by = bc^{-1}y$ so $x \in \mathscr{D}(b^{-1}) = \mathscr{D}(cb^{-1})$. $\square$

The next lemma and the remarks following it shed some light on the equivalence classes $[b]$.

Lemma 4.3. *Let $[b]$ be a projective tuple and assume that $[\lambda] \in \sigma[b]^*$. Then there is a representative b' for $[b]$ such that*

$$\langle b', \lambda \rangle = \sum_0^n \lambda_j b'_j = e.$$

Proof. If $\lambda \in \sigma[b]^*$ Lemma 3.9 says that $B = \langle b, \lambda \rangle$ is invertible. Then clearly $[B^{-1}b] = [b]$ and $b' = B^{-1}b$ is the desired representative. $\square$

Remark 4.4. There is no loss of generality in assuming that $\lambda_0 \neq 0$ because we may perturbate $[\lambda]$ a little and still belong to $\sigma[b]^*$.

Remark 4.5. We have defined the equivalence relation on commuting tuples as the transitive closure of a symmetric and reflexive relation R. The proof of Lemma 4.3 shows that given a class $[b]$ such that $\sigma[b]^*$ is nonempty, any representative is not more then one step from the representative b' with $\langle b', \lambda \rangle = e$. Hence if b and $\tilde{b}$ are any two representatives for $[b]$ then they are not more then two steps from each other.

Remark 4.6. Lemma 4.3 also enable us to to give an alternative description of the equivalence relation $\sim$ if we restrict ourselves to look at commuting $n+1$-tuples of operators with the additional property that their spectrum avoid some hyperplane through the origin in $\mathbb{C}^{n+1}$. In fact for such tuples, b and $\tilde{b}$, we have $b \sim \tilde{b}$ if and only if $\tilde{b} = cb$ for some invertible c. The only if part is clear. Conversely assume that $\tilde{b} = cb$ for some invertible c. The assumption on the spectrum for $\tilde{b}$ says precisely that $\sigma[\tilde{b}]^*$ is nonempty and so from Lemma 4.3 we see that we may assume that $\langle \tilde{b}, \lambda \rangle = e$ for some $[\lambda]$. Hence $\langle b, \lambda \rangle = c^{-1}$ so $c \in (b)'$ and therefore $[b] = [\tilde{b}]$.

Definition 4.7. Let $[b]$ be a projective tuple. We define $\sigma[b]^*_{adm}$, the set of *admissible* hyperplanes for $[b]$, by $[\alpha] \in \sigma[b]^*_{adm}$ if

$$\frac{\langle b, \alpha \rangle}{\langle b, \lambda \rangle}$$

is injective, where $[\lambda]$ is some hyperplane in $\sigma[b]^*$.

The definition clearly does not depend on the choice of $[\lambda]$ because if $[\tilde{\lambda}] \in \sigma[b]^*$ is some other choice, then

$$\frac{\langle b,\alpha\rangle}{\langle b,\tilde{\lambda}\rangle} = \frac{\langle b,\alpha\rangle}{\langle b,\lambda\rangle}\frac{\langle b,\lambda\rangle}{\langle b,\tilde{\lambda}\rangle}$$

and $\frac{\langle b,\lambda\rangle}{\langle b,\tilde{\lambda}\rangle}$ is invertible by the functional calculus.

Remark 4.8. Observe that $\sigma[b]^*_{adm}$ is not defined as the dual complement of something. It is defined directly as a subset of $\mathbb{CP}^{n*}$. However, in the one variable case $\sigma[b]^*_{adm}$ corresponds to the point spectrum in the following sense. If $[\lambda] \in \sigma[b]^* \subseteq \mathbb{CP}^1$ and P_λ a projection from the hyperplane $[\lambda]$ onto $\mathbb{C}$ then

$$\sigma[b]^*_{adm} = \left(P_\lambda^{-1}\sigma_p(P_\lambda([b]))\right)^*.$$

Proposition 4.9. *Let $[b]$ be a projective operator and let $[\lambda] \in \sigma[b]^*$ and $[\alpha] \in \sigma[b]^*_{adm}$. Then $\langle b,\alpha\rangle^{-1}\langle b,\lambda\rangle$ is a closed operator which does not depend on the particular representative $b \in [b]$. Moreover*

$$\langle b,\alpha\rangle^{-1}\langle b,\lambda\rangle = \langle b,\lambda\rangle\langle b,\alpha\rangle^{-1}.$$

and we denote this operator $\frac{\langle b,\lambda\rangle}{\langle b,\alpha\rangle}$. The domain of definition

$$\mathscr{D}(\frac{\langle b,\lambda\rangle}{\langle b,\alpha\rangle}) := \mathscr{D}_\alpha$$

does not depend on the choice of $[\lambda] \in \sigma[b]^*$. Finally if $[\beta_1], \ldots, [\beta_n]$ are any points such that $[\alpha], [\beta_1], \ldots, [\beta_n]$ are in general position then

$$\mathscr{D}_\alpha = \bigcap_{j=1}^{n} \mathscr{D}(\langle b,\alpha\rangle^{-1}\langle b,\beta_j\rangle).$$

Proof. It is clear that $\langle b,\alpha\rangle^{-1}\langle b,\lambda\rangle$ is a closed linear operator on X. Since $[\lambda] \in \sigma[b]^*$ we have that $\langle b,\lambda\rangle$ is invertible and so it follows from Lemma 4.2 that

$$\langle b,\alpha\rangle^{-1}\langle b,\lambda\rangle = (\langle b,\lambda\rangle\langle b,\alpha\rangle)^{-1}.$$

From this we immediately obtain that

$$\frac{\langle b,\lambda\rangle}{\langle b,\alpha\rangle} = \left(\frac{\langle b,\alpha\rangle}{\langle b,\lambda\rangle}\right)^{-1} \tag{11}$$

in the set theoretical sense and hence $\frac{\langle b,\lambda\rangle}{\langle b,\alpha\rangle}$ does not depend on the representative $b \in [b]$ since the right hand side of (11) does not. Moreover since $\mathscr{D}(\langle b,\lambda\rangle\langle b,\alpha\rangle^{-1}) = \mathscr{D}(\langle b,\tilde{\lambda}\rangle\langle b,\alpha\rangle^{-1})$ for any other $[\tilde{\lambda}] \in \sigma[b]^*$ the domain $\mathscr{D}_\alpha$ can not depend on the choice of $[\lambda] \in \sigma[b]^*$. For the last statement we first assume that $[\alpha] = [1, 0, \dots, 0]$ and $[\beta_j] = [0, \dots, 1, \dots, 0]$ where the 1 is in the j:s position. Then what we have to show is

$$\mathscr{D}(\frac{\langle b, \lambda\rangle}{b_0}) = \bigcap_0^n \mathscr{D}(b_0^{-1} b_j).$$

But from Lemma 4.2 we see

$$\mathscr{D}(b_0^{-1} b_j) \supseteq \mathscr{D}(b_j b_0^{-1}) = \mathscr{D}(\langle b, \lambda\rangle b_0^{-1})$$

and so $\mathscr{D}(\frac{\langle b,\lambda\rangle}{b_0}) \subseteq \bigcap_0^n \mathscr{D}(b_0^{-1} b_j)$. On the other hand

$$\bigcap_0^n \mathscr{D}(b_0^{-1} b_j) \subseteq \mathscr{D}(b_0^{-1} \langle b, \lambda\rangle)$$

so we are done. We reduce the general case to this one by considering the projective transformation P defined by

$$[z] \mapsto [\langle z, \alpha\rangle, \langle z, \beta_1\rangle, \dots, \langle z, \beta_n\rangle].$$

Then $P^{*-1}[\alpha] = [1, 0, \dots, 0]$ and $P^{*-1}[\beta_j] = [0, \dots, 1, \dots, 0]$. We want to show the equality

$$\mathscr{D}(\frac{\langle b, \lambda\rangle}{\langle b, \alpha\rangle}) = \bigcap_{j=1}^n \mathscr{D}(\langle b, \alpha\rangle^{-1} \langle b, \beta_j\rangle)$$

but this is equivalent to

$$\mathscr{D}(\frac{\langle Pb, P^{*-1}\lambda\rangle}{\langle Pb, P^{*-1}\alpha\rangle}) = \bigcap_{j=1}^n \mathscr{D}(\langle Pb, P^{*-1}\alpha\rangle^{-1} \langle Pb, P^{*-1}\beta_j\rangle).$$

Hence the proposition follows from the special case above. $\square$

Remark 4.10. We saw in the proof that there was no loss of generality in assuming that the hyperplanes were of a special kind because we could reduce to this case by a projective transformation of $\mathbb{CP}^n$. In order to simplify calculations in the proofs below we will often make such assumptions and it is supposed to be understood that there is no loss of generality in doing it.

Let us fix an $[\alpha] \in \sigma[b]^*_{adm}$ and $[\beta_1], \dots, [\beta_n] \in \mathbb{CP}^{n*}$ such that $[\alpha], [\beta_1], \dots, [\beta_n]$ are in general position. We denote the closed operator $\langle b, \alpha\rangle^{-1} \langle b, \beta_j\rangle$ by a_j.

Proposition 4.11. *With the hypothesis of the preceding proposition, if* $x \in \mathscr{D}(a_j) \cap \mathscr{D}(a_k)$ *then we have*

$$a_j x \in \mathscr{D}(a_k)$$

if and only if

$$a_k x \in \mathscr{D}(a_j).$$

If any of these conditions are satisfied then also

$$a_k a_j x = a_j a_k x.$$

Proof. We may assume $[\alpha] = [1, 0, \dots, 0]$ and $[\beta_j] = [0, \dots, 1, \dots, 0]$ and hence $a_j = b_0^{-1} b_j$. Suppose $x \in \mathscr{D}(b_0^{-1} b_j) \cap \mathscr{D}(b_0^{-1} b_k)$. Then from Lemma 4.2 we get

$$b_k b_0^{-1} b_j x = b_j b_0^{-1} b_k x.$$

Hence $b_0^{-1} b_j x \in \mathscr{D}(b_0^{-1} b_k)$ precisely when $b_0^{-1} b_k x \in \mathscr{D}(b_0^{-1} b_j)$ and

$$b_0^{-1} b_k b_0^{-1} b_j x = b_0^{-1} b_j b_0^{-1} b_k x.$$

□

Definition 4.12. A tuple $(a_1, \dots, a_n)$ of closed operators on X is said to be *affine* if

(i) it exists a $[\lambda] \in \mathbb{CP}^n$ such that the operator

$$a_0 := \lambda_0 + \sum_1^n \lambda_j a_j$$

with domain $\mathscr{D}(a_0) = \bigcap_1^n \mathscr{D}(a_j)$ is closed, injective and surjective,

(ii) the operators $a_0, a_1, \dots, a_n$ satisfy the following commutation conditions; if $x \in \mathscr{D}(a_j) \cap \mathscr{D}(a_j a_k)$ then $x \in \mathscr{D}(a_k a_j)$ and $a_j a_k x = a_k a_j x$ for $j, k = 0, 1, \dots, n$.

Remark 4.13. Definition 4.12 says that a tuple of closed operators is affine if some affine combination of them, called a_0, with domain $\bigcap_1^n \mathscr{D}(a_j)$, is closed, injective and surjective and that the tuple $a_0, a_1, \dots, a_n$ satisfies the commutation conditions (ii). In the one variable case this means precisely that $\sigma(a)$ is not all of $\mathbb{C}$ because $\lambda_0 + \lambda_1 a_1$ is injective and surjective iff $-\lambda_0/\lambda_1 \notin \sigma(a)$ and the commutation conditions are clearly satisfied. Hence from Section 2 we see that one closed operator is affine if and only if it can be Cayley transformed to a bounded operator.

Remark 4.14. Morally what condition (ii) should mean is that no matter how we define spectrum of a, the hyperplane $[\lambda]$ should avoid the closure in $\mathbb{CP}^n$ of it. For instance if $[\lambda] = [1, 0, \dots, 0]$, that is spectrum of a does not intersect the hyperplane at infinity, then one should expect that all the a_j are bounded. In fact if $[1, 0, \dots, 0]$ works as $[\lambda]$ in Definition 4.12 then condition (i) says that the domain of the identity is $\bigcap_1^n \mathscr{D}(a_j)$, that is $\mathscr{D}(a_j) = X$ for all j and so all the a_j are bounded by the Closed Graph Theorem.

Remark 4.15. Observe that we do not demand that the a_j have nonempty resolvent sets. We will see in Example 4.18 that there are affine tuples such that some of the components have all of $\mathbb{CP}^1$ as spectrum.

Remark 4.16. Condition (ii) of Definition 4.12 implies that the operators $a_1, \dots, a_n$ commute with the bounded operator a_0^{-1} in the sense that $a_0^{-1}a_j \subseteq a_j a_0^{-1}$. In fact, let $x \in \mathscr{D}(a_j)$. Then clearly $a_0^{-1}x \in \mathscr{D}(a_j) \cap \mathscr{D}(a_j a_0)$ and so condition (ii) implies that $a_0^{-1}x \in \mathscr{D}(a_0 a_j)$ and $a_0 a_j a_0^{-1} x = a_j a_0 a_0^{-1} x = a_j x$. Hence $a_j a_0^{-1} x = a_0^{-1} a_j x$ for all $x \in \mathscr{D}(a_j)$.

The operators we get when we project a projective operator from an admissible hyperplane are affine, in fact these are the only affine operators as we now show.

Theorem 4.17. *A tuple $a = (a_1, \dots, a_n)$ of closed operators on X is affine if and only if there is a projective tuple $[b]$ with $\sigma[b]^*$ nonempty, an $[\alpha] \in \sigma[b]^*_{adm}$ and $[\beta_1], \dots, [\beta_n] \in \mathbb{CP}^{n*}$ in general position together with $[\alpha]$, such that*

$$a_j = \langle b, \alpha \rangle^{-1} \langle b, \beta_j \rangle, \quad j = 1, \dots, n.$$

Proof. We may assume that $\alpha = [1, 0, \dots, 0]$, $\beta_j = [0, \dots, 0, 1, 0, \dots 0]$ where 1 is in the j:s place. First assume that $a_j = \langle b, \alpha \rangle^{-1} \langle b, \beta_j \rangle$, $j = 1, \dots, n$ for some projective tuple $[b]$, that is $a_j = b_0^{-1} b_j$. Let $[\lambda] \in \sigma[b]^*$ so that $B = \langle b, \lambda \rangle$ is invertible. From Proposition 4.9 we get that $b_0^{-1}B = Bb_0^{-1}$ and so we see that

$$a_0 := b_0^{-1}B = b_0^{-1} \sum_0^n \lambda_j b_j = \sum_0^n \lambda_j b_0^{-1} b_j = \lambda_0 + \sum_1^n \lambda_j a_j$$

has domain $\mathscr{D}(a_0) = \mathscr{D}(b_0^{-1}) = \bigcap_1^n \mathscr{D}(a_j)$ by Proposition 4.9, is closed, injective and surjective. Hence a satisfies condition (i) in Definition 4.12. Moreover Proposition 4.11 implies that if $x \in \mathscr{D}(a_j) \cap \mathscr{D}(a_j a_k)$ then $x \in \mathscr{D}(a_k a_j)$ and $a_j a_k x = a_k a_j x$ for $j, k = 1, \dots, n$. To see that this is also satisfied for $j = 0$ and $k = 0$ respectively we first assume that $x \in \mathscr{D}(a_0) \cap \mathscr{D}(a_0 a_k)$. Then since $x \in \mathscr{D}(a_0 a_k)$ we have that $b_0^{-1} b_k x \in \mathscr{D}(b_0^{-1})$ and since also $x \in \mathscr{D}(b_0^{-1})$ Lemma 4.2 implies that $b_0^{-1} b_k x = b_k b_0^{-1} x$. Hence $b_k b_0^{-1} x \in \mathscr{D}(b_0^{-1})$, that is $x \in \mathscr{D}(a_k a_0)$, and $b_0^{-1} b_0^{-1} b_k x = b_0^{-1} b_k b_0^{-1} x$ that is $a_0 a_k x = a_k a_0 x$. Now assume that $x \in \mathscr{D}(a_j) \cap \mathscr{D}(a_j a_0)$ which just means that $x \in \mathscr{D}(b_0^{-1})$ and $b_j b_0^{-1} x \in \mathscr{D}(b_0^{-1})$. From Lemma 4.2 we see that $b_j b_0^{-1} x = b_0^{-1} b_j x$ so $b_0^{-1} b_j x \in \mathscr{D}(b_0^{-1})$ and $b_0^{-1} b_0^{-1} b_j x = b_0^{-1} b_j b_0^{-1} x$. Hence $x \in \mathscr{D}(a_0 a_j)$ and $a_0 a_j x = a_j a_0 x$ so a also satisfies condition (ii) and thus a is affine.

Conversely assume that a is affine and take $[\lambda] \in \mathbb{CP}^n$ such that the operator $a_0 = \lambda_0 + \sum_1^n \lambda_j a_j$ satisfies the requirements of condition (i) in Definition 4.12. Then

$$b_0 := (\lambda_0 + \sum_1^n \lambda_j a_j)^{-1}, \;\; b_j := a_j (\lambda_0 + \sum_1^n \lambda_j a_j)^{-1} \;\; j = 1, \dots, n$$

are bounded operators by the Closed Graph Theorem. We claim that $b = (b_0, \dots, b_n)$ is commutative, that $\langle b, \lambda \rangle$ is invertible and that $a_j = b_0^{-1} b_j$. We start by showing commutativity. In Remark 4.16 we saw that it followed from condition (ii) that $a_0^{-1} a_j \subseteq a_j a_0^{-1}$, that is $b_0 a_j \subseteq a_j b_0$ for $j = 1, \dots, n$. Hence for any $x \in X$ we have $a_k b_0^2 x = b_0 a_k b_0 x \in \bigcap_1^n \mathscr{D}(a_j)$. So we see from condition (ii) that for any $x \in X$ we have

$$a_l b_0 a_k b_0 x = a_l a_k b_0^2 x = a_k a_l b_0^2 x = a_k b_0 a_l b_0 x.$$

Thus b is commutative. To see that $a_k = b_0^{-1} b_k$ we assume $x \in \mathscr{D}(a_k)$, then condition (ii), via Remark 4.16, implies that $a_k b_0 x = b_0 a_k x \in \bigcap_1^n \mathscr{D}(a_j)$. Hence $a_l a_k b_0 x = a_k a_l b_0 x$ for all l by condition (ii), and we obtain $b_0^{-1} a_k b_0 x = a_k x$. Thus $a_k \subseteq b_0^{-1} b_k$. To show equality it suffices to show $\mathscr{D}(b_0^{-1} b_k) \subseteq \mathscr{D}(a_k)$. Therefore assume $x \in \mathscr{D}(b_0^{-1} b_k)$, that is $a_k b_0 x \in \mathscr{D}(b_0^{-1})$ and so, again by condition (ii), we have $a_l a_k b_0 x = a_k a_l b_0 x$. Hence $a_l b_0 x \in \mathscr{D}(a_k)$ for all l and this gives us $x = b_0^{-1} b_0 x \in \mathscr{D}(a_k)$. Finally we observe

$$\langle b, \lambda \rangle = \sum_0^n \lambda_j b_j = \lambda_0 b_0 + \sum_1^n \lambda_j a_j b_0 = (\lambda_0 + \sum_1^n \lambda_j a_j) b_0 = e.$$

Hence $\sigma(b)$ avoids the hyperplane λ through the origin in $\mathbb{C}^{n+1}$ and hence $[b]$ is a projective tuple with $\sigma[b]^*$ nonempty. $\square$

Example 4.18. Let K be the compact subset of $\mathbb{C}^3$ defined by

$$K = \{(1, z_1, 0); |z_1| \leq 1\} \cup \{(1/z_1, 1, 1/z_1); |z_1| \geq 1\} \cup \{(0, 1, 0)\}.$$

Let $X = C(K)$ be the Banach space of continuous functions on K and let b_j denote the operator on X of multiplication with the coordinate function z_j, $j = 0, 1, 2$. Then $b = (b_0, b_1, b_2)$ defines a projective operator $[b]$ and $\sigma[b] = \pi(K)$, the projection of K on $\mathbb{CP}^3$. Moreover, one checks that the hyperplane $[2, 1, -3/2]$ avoids $\sigma[b]$. Clearly b_0 is injective and so the hyperplane $[1, 0, 0]$ is admissible. We get the affine operator $(a_1, a_2) = (b_0^{-1} b_1, b_0^{-1} b_2)$. We claim that $\sigma(a_1) = \mathbb{C}$. Let $w \in \mathbb{C}$ be arbitrary and take a point $(z_0, z_1, z_2) \in K$ such that $z_1/z_0 = w$. If $f \in C(K)$ is such that $f(z_0, z_1, z_2) \neq 0$ then f is not in the range of $w - a_1$ and therefore $w \in \sigma(a_1)$.

Corollary 4.19. *If $(a_1, \dots, a_n)$ is affine then affine combinations of the a_j are closable.*

Proof. To any affine map of $\mathbb{C}^n$ it corresponds a projective transformation of $\mathbb{CP}^n$. Substituting a projective operator, representing $(a_1, \dots, a_n)$, into this map and projecting the result back to $\mathbb{C}^n$ we obtain a closed extension of the affine combination. $\square$

The correspondence between affine tuples and projective tuples is one to one in the following sense.

Theorem 4.20. *Fix $[\alpha], [\beta_1], \dots, [\beta_n] \in \mathbb{CP}^n$ in general position. Then to any affine tuple $(a_1, \dots, a_n)$ it corresponds a unique projective tuple $[b]$ with nonempty $\sigma[b]^*$ and with $[\alpha] \in \sigma[b]^*_{adm}$ such that $a_j = \langle b, \alpha\rangle^{-1}\langle b, \beta_j\rangle$ for $j = 1, \dots, n$.*

Proof. We have already seen that if $[b]$ is a projective tuple with $[\alpha] \in \sigma[b]^*_{adm}$ and $[\alpha], [\beta_1], \dots, [\beta_n]$ in general position then we get a well defined affine tuple

$$(\langle b, \alpha\rangle^{-1}\langle b, \beta_1\rangle, \dots, \langle b, \alpha\rangle^{-1}\langle b, \beta_n\rangle).$$

For the converse we assume $\alpha = [1, 0, \dots, 0]$, $\beta_j = [0, \dots, 0, 1, 0, \dots, 0]$. Theorem 4.17 says that there is a projective tuple $[b]$ with the desired properties. We have to show that if $[\tilde{b}]$ is another projective tuple with these properties then $[b] = [\tilde{b}]$. So we assume that $b_0^{-1}b_j = \tilde{b}_0^{-1}b_j$, $j = 1, \dots, n$. We may assume that b is a representative for $[b]$ such that $e = \langle b, \lambda\rangle$ by Lemma 4.3. From Proposition 4.9 we get

$$\mathscr{D}(b_0^{-1}) = \cap \mathscr{D}(b_0^{-1}b_j) = \cap \mathscr{D}(\tilde{b}_0^{-1}\tilde{b}_j) = \mathscr{D}(\tilde{b}_0^{-1}).$$

Hence $c := \tilde{b}_0^{-1}b_0$ is an invertible bounded operator. Moreover from Lemma 4.2 and the assumption we see

$$\tilde{b}_j c = \tilde{b}_j\tilde{b}_0^{-1}b_0 = \tilde{b}_0^{-1}\tilde{b}_j b_0 = b_0^{-1}b_j b_0 = b_j$$

so $b = \tilde{b}c$. It remains to show that $c \in (\tilde{b})'$. But $e = \langle b, \lambda\rangle = \sum_0^n \lambda_j b_j$ so $c^{-1} = \sum_0^n \lambda_j \tilde{b}_j$ and hence $c \in (\tilde{b})'$. □

Definition 4.21. Let $[\alpha], [\beta_1], \dots, [\beta_n] \in \mathbb{CP}^n$ be fixed in general position. We define $\rho_{\alpha,\beta}$ to be the mapping

$$[z] \mapsto (\langle z, \alpha\rangle^{-1}\langle z, \beta_1\rangle, \dots, \langle z, \alpha\rangle^{-1}\langle z, \beta_n\rangle).$$

The one to one correspondence can now be stated as: The mapping

$$\rho_{\alpha,\beta} : \{[b]; \sigma[b]^* \neq \emptyset,\ [\alpha] \in \sigma[b]^*_{adm}\} \to \{a; a \text{ is affine}\}$$

is one to one and onto.

5. Spectra of affine operators

We define the spectrum of an affine operator a, corresponding to a projective operator $[b]$ via $\rho_{\alpha,\beta}([b]) = a$, and show that $\rho_{\alpha,\beta}(\sigma[b]) = \sigma(a)$. Throughout this section we will assume that $\alpha = [1, 0, \dots, 0]$ and $\beta_j = [0, \dots, 1, \dots, 0]$ in the proofs.

Let $a = (a_1, \dots, a_n)$ be an affine operator. For $z \in \mathbb{C}^n$ we let δ_{z-a} denote interior multiplication with $\sum_1^n (z_j - a_j)\frac{\partial}{\partial z_j}$ and the domain of definition $\mathscr{D}(\delta_{z-a})$ for this operator is all forms with coefficients in $\bigcap_1^n \mathscr{D}(a_j)$.

Definition 5.1. Let $a = (a_1, \dots, a_n)$ be an affine operator. We define $\sigma(a) \subseteq \mathbb{C}^n$ by specifying its complement: $z \notin \sigma(a)$ if and only if for all $f^k \in \mathscr{N}(\delta_{z-a})$ it exists a $k+1$-form f^{k+1} with coefficients in $\bigcap_{j,k=1}^n \mathscr{D}(a_j a_k)$ such that $f^k = \delta_{z-a} f^{k+1}$.

Remark 5.2. There are other definitions of the Taylor spectrum for unbounded operators, see e.g. [6] and [12].

We denote the set of all forms with coefficients in $\bigcap_{j,k=1}^n \mathscr{D}(a_j a_k)$ by $\mathscr{D}^2$.

Lemma 5.3. *Let $[b]$ be a projective operator and assume that $[1, 0, \dots, 0]$ is an admissible hyperplane and that $\sigma[b]^*$ is nonempty. Put $b' = (b_1, \dots, b_n)$ and let $a = (b_0^{-1} b_1, \dots, b_0^{-1} b_n)$. Then $K_\bullet(\delta_{b'}, X)$ is exact if and only if for all $f^k \in \mathscr{N}(\delta_a)$ it exists an f^{k+1} with coefficients in $\mathscr{D}(b_0^{-2}) = \mathscr{R}(b_0^2)$ such that $f^k = \delta_a f^{k+1}$.*

Proof. Note that $\mathscr{D}(b_0^{-1}) = \bigcap_1^n \mathscr{D}(b_0^{-1} b_j)$ by Proposition 4.9. Assume that $K_\bullet(\delta_{b'}, X)$ is exact and let $f^k \in \mathscr{N}(\delta_a)$. Then $\delta_{b'} b_0^{-1} f^k = 0$ and so there is an $\tilde{f}^{k+1}$ such that $b_0^{-1} f^k = \delta_{b'} \tilde{f}^{k+1}$. But then

$$f^k = \delta_{b'} b_0 \tilde{f}^{k+1} = \delta_a b_0^2 \tilde{f}^{k+1}.$$

Thus $f^{k+1} := b_0^2 \tilde{f}^{k+1}$ has coefficients in $\mathscr{D}(b_0^{-2})$ and $f^k = \delta_a f^{k+1}$.

Now assume that if $f^k \in \mathscr{N}(\delta_a)$ it exists an f^{k+1} with coefficients in $\mathscr{D}(b_0^{-2})$ such that $f^k = \delta_a f^{k+1}$. If $\delta_{b'} \tilde{f}^k = 0$ then clearly $b_0 \tilde{f}^k \in \mathscr{N}(\delta_a)$ and so there is an $\tilde{f}^{k+1}$ with coefficients in $\mathscr{D}(b_0^{-2})$ such that

$$b_0 \tilde{f}^k = \delta_a \tilde{f}^{k+1} = \delta_{b'} b_0^{-1} \tilde{f}^{k+1}.$$

Hence $\tilde{f}^k = \delta_{b'} b_0^{-2} \tilde{f}^{k+1}$ and so $K_\bullet(\delta_{b'}, X)$ is exact. □

Theorem 5.4. *Let a be an affine operator and $[b]$ a projective one with nonempty $\sigma[b]^*$ and $[\alpha] \in \sigma[b]^*_{adm}$ such that $a = \rho_{\alpha,\beta}([b])$. Then*

$$\sigma(a) = \rho_{\alpha,\beta}(\sigma[b]).$$

Proof. Under our assumptions on $[\alpha]$ and $[\beta]$ we have that $\rho_{\alpha,\beta}$ is the mapping $[z] \mapsto (z_1/z_0, \dots, z_n/z_0)$. We will show that $[1, 0, \dots, 0] \notin \sigma[b]$ if and only if $0 \notin \sigma(a)$. By the Spectral Mapping Theorem we get that the line through the origin and $(1, 0, \dots, 0)$ in $\mathbb{C}^{n+1}$ does not intersect $\sigma(b)$ if and only if $0 \notin \sigma(b_1, \dots, b_n)$. Thus what we have to show is that $0 \notin \sigma(b_1, \dots, b_n)$ if and only if $0 \notin \sigma(a)$. But this is exactly the statement in Lemma 5.3 and so the only thing left in order to prove the theorem is to check that $\mathscr{D}(b_0^{-2}) = \bigcap_{j,k=1}^n \mathscr{D}(a_j a_k)$. Since $a_j a_k = b_0^{-1} b_j b_0^{-1} b_k \supseteq b_j b_k b_0^{-2}$

the inclusion $\subseteq$ is clear. Conversely assume $x \in \bigcap_{j,k=1}^n \mathscr{D}(a_j a_k)$. Then, at least $x \in \bigcap_j^n \mathscr{D}(a_j) = \mathscr{D}(b_0^{-1})$ by Proposition 4.9. Thus $x = b_0 y$ for some y. The assumption on x now implies that $b_k y = b_0^{-1} b_k x \in \bigcap_j^n \mathscr{D}(a_j) = \mathscr{D}(b_0^{-1})$ for $k = 1, \dots, n$. Since we may assume that $e = \sum_0^n \lambda_k b_k$ we get $y = \sum_0^n \lambda_k b_k y \in \mathscr{D}(b_0^{-1})$. Thus $x = b_0 y \in \mathscr{D}(b_0^{-2})$ and we are done. □

Theorem 4.20 implies that to an affine operator a we have a unique projective operator $[b]$ such that $a = \rho_{\alpha,\beta}([b])$ for some fixed choice of $[\alpha], [\beta_1], \dots, [\beta_n]$ in general position. So applying Theorem 5.4 we see that $\sigma(a)$ has a well defined, invariant and closed extension $\hat{\sigma}(a) \subseteq \mathbb{CP}^n$ defined by

$$\hat{\sigma}(a) = \sigma[b].$$

6. Cayley transforms

We summarise our results to see that the affine tuples are precisely those tuples which are Cayley transforms of bounded ones and that the Spectral Mapping Theorem holds.

Let $a = (a_1, \dots, a_n)$ be affine and let $[\lambda] \in \mathbb{CP}^n$ be such that that condition (i) in Definition 4.12 is fulfilled. Then if $a_0 = \lambda_0 + \sum_1^n \lambda_j a_j$, the projective operator

$$[b] = [a_0^{-1}, a_1 a_0^{-1}, \dots, a_n a_0^{-1}]$$

projects to a and $[\lambda] \notin \sigma[b]^*$ by Theorem 4.17 and its proof. Let $[\beta_1], \dots, [\beta_n]$ be points in $\mathbb{CP}^n$ such that $[\lambda], [\beta_1], \dots, [\beta_n]$ are in general position. Applying the projection $\rho_{\lambda,\beta}$ to $[b]$ we get the bounded commuting tuple

$$\rho_{\lambda,\beta}([b]) = ((\beta_{1,0} + \sum_1^n \beta_{1,j} a_j) a_0^{-1}, \dots, (\beta_{n,0} + \sum_1^n \beta_{n,j} a_j) a_0^{-1})$$

and

$$\sigma(\rho_{\lambda,\beta}([b])) = \rho_{\lambda,\beta}(\sigma[b])$$

by Theorem 3.8. Hence if ϕ is the corresponding rational fractional transformation we see that $\phi(a) = \rho_{\lambda,\beta}([b])$ is a bounded commuting tuple and by Theorem 5.4 we have $\sigma(\phi(a)) = \phi(\hat{\sigma}(a))$ naturally interpreted.

Conversely assume that a tuple of closed operators $a = (a_1, \dots, a_n)$ is the Cayley transform of a bounded commuting tuple $(b_1, \dots, b_n)$, that is

$$a_k = (\lambda_{0,0} + \sum_1^n \lambda_{0,j} b_j)^{-1} (\lambda_{k,0} + \sum_1^n \lambda_{k,j} b_j),$$

where $(\lambda_{j,k})$ is an invertible matrix and $\lambda_{0,0} + \sum_1^n \lambda_{0,j} b_j$ is injective, i.e. the affine hyperplane $\{z \in \mathbb{C}^n ; \langle z, \lambda_0 \rangle = 0\}$ is admissible. Then clearly $[e, b_1, \dots, b_n]$ is a projective operator and $[1, 0, \dots, 0] \in \sigma[e, b_1, \dots, b_n]^*$.

Moreover, the hyperplane $[\lambda_{0,0}, \dots, \lambda_{0,n}]$ has to be admissible and so a is the projection of a projective operator from an admissible hyperplane. Since the spectrum of the projective operator also has a nonempty dual complement it follows from Theorem 4.17 that a is affine.

7. Integral formulas for the analytic functional calculus of projective tuples

We provide integral formulas realising the functional calculus described in Section 3. Analogously to [1] we will construct a $\bar{\partial}$-closed $(n, n-1)$-form, $\omega_b^n x$, with values in $X \otimes L^n$, defined in $U \setminus \sigma[b]$, where L^{-1} is the tautological line bundle and U is $\mathbb{CP}^n$ minus some hyperplane, such that if $f \in \mathscr{O}(\sigma[b])$, then

$$f([b])x = \int_{\partial D} f \frac{\langle b, \lambda \rangle^n}{\langle z, \lambda \rangle^n} \omega_b^n x$$

where $\lambda \in \sigma[b]^*$ and D is a suitable neighbourhood of $\sigma[b]$.

We let δ_z denote contraction with the vector field $\sum_0^n z_j \frac{\partial}{\partial z_j}$. Letting f be a k-homogeneous $(p, 0)$-form in some cone in $\mathbb{C}^{n+1}$ then f is the pullback of an L^k-valued $(p, 0)$-form in the projection of the cone in $\mathbb{CP}^n$ if and only if $\delta_z f = 0$. The statement is local and we may verify it when $z_0 \neq 0$. If f is the pullback of an L^k-valued $(p, 0)$-form then f is k-homogeneous and can be written as:

$$f = \sum_I f_I d(z_{I_1}/z_0) \wedge \cdots \wedge d(z_{I_p}/z_0).$$

Since $\delta_z d(z_i/z_0) = \delta_z(dz_i/z_0 - z_i/z_0^2 dz_0) = z_i/z_0 - z_0 z_i/z_0^2 = 0$ we have $\delta_z f = 0$. Conversely, a straight-forward calculation shows that if $f = \sum f_I dz_I$ is any k-homogeneous $(p, 0)$-form then

$$f = z_0^p \sum_{0 \notin I} f_I d(z_{I_1}/z_0) \wedge \cdots \wedge d(z_{I_p}/z_0) + \frac{(-1)^p}{z_0} (\delta_z f) \wedge dz_0.$$

So if $\delta_z f = 0$ then clearly f is the pullback of a $(p, 0)$-form which has to have values in L^k since f is k-homogeneous. In what follows we will identify the space of $X \otimes L^k$ valued $(p, 0)$-forms on some subset of $\mathbb{CP}^n$ with the space of k-homogeneous X-valued δ_z-closed $(p, 0)$-forms on the cone over this subset in $\mathbb{C}^{n+1}$.

We let δ_b denote interior multiplication with $\sum_0^n b_j \frac{\partial}{\partial z_j}$. This operator commutes with δ_z so it maps δ_z-closed X-valued forms to δ_z-closed X-valued forms. However, δ_b reduces the homogeneity one step and therefore δ_b maps k-homogeneous k-forms to $k-1$-homogeneous $k-1$-forms. Moreover b is commuting so we have $\delta_b \circ \delta_b = 0$, and we get the complex

$$K_\bullet(X \otimes L^\bullet \otimes \Lambda^{\bullet,0} T^* \mathbb{CP}^n_{[z]}, \delta_b). \tag{12}$$

The operator δ_b depends on the choice of representative for $[b]$ but nevertheless we have the following proposition.

Proposition 7.1. *Let $[b]$ be a projective tuple and b any representative. Then $[z] \notin \sigma[b]$ if and only if the complex*

$$K_\bullet(X \otimes L^\bullet \otimes \Lambda^{\bullet,0} T^* \mathbb{CP}^n_{[z]}, \delta_b)$$

is exact.

Proof. We may assume that $[z] = [1, 0, \dots, 0]$. We first claim that $[1, 0, \dots, 0] \notin \sigma[b]$ if and only if $0 \notin \sigma(b_1, \dots, b_n)$. Actually, if $0 \notin \sigma(b_1, \dots, b_n)$, that is $(b_1, \dots, b_n)$ is nonsingular, then $(z_0 - b_0, b_1, \dots, b_n)$ is nonsingular for all $z_0 \in \mathbb{C}$, see [10]. Hence $(z_0, 0, \dots, 0) \notin \sigma(b_0, \dots, b_n)$ for all $z_0 \in \mathbb{C}$, which means that $[1, 0, \dots, 0] \notin \sigma[b]$. On the other hand, if $[1, 0, \dots, 0] \notin \sigma[b]$ then $(z_0, 0, \dots, 0) \notin \sigma(b_0, \dots, b_n)$ for all $z_0 \in \mathbb{C}$. From the projection property for the Taylor spectrum, see [10], we conclude that $0 \notin \sigma(b_1, \dots, b_n)$.

To finish the proof we show that $0 \notin \sigma(b_1, \dots, b_n)$ if and only if the complex (12) is exact for $[z] = [1, 0, \dots, 0]$. Note that for any $f \in X \otimes L^k \otimes \Lambda^{k,0} T^* \mathbb{CP}^n_{[1,0,\dots,0]}$ we have $\delta_{[1,0,\dots,0]} f = z_0 \frac{\partial}{\partial z_0} f = 0$ so f does not contain any dz_0. Hence δ_b acts just as interior multiplication with $\sum_1^n b_j \frac{\partial}{\partial z_j}$, which we denote by $\delta_{b'}$, and we can identify the complex (12) with the complex

$$0 \longleftarrow \Lambda^0 X \xleftarrow{\delta_{b'}} \Lambda^1 X \xleftarrow{\delta_{b'}} \cdots \xleftarrow{\delta_{b'}} \Lambda^n X \longleftarrow 0 \, .$$

However, by definition, this complex is exact precisely when $0 \notin \sigma(b_1, \dots, b_n)$, and we are done. $\square$

Assume $[1, 0, \dots, 0] \in \sigma[b]^*$ and let $(\zeta_1, \dots, \zeta_n) = (z_1/z_0, \dots, z_n/z_0)$ be local coordinates round $[1, 0, \dots, 0]$. In these local coordinates δ_b is interior multiplication with

$$b_0 \sum_1^n (b_0^{-1} b_j - \zeta_j) \frac{\partial}{\partial \zeta_j}$$

and we abbreviate this $b_0 \delta_{b_0^{-1} b - \zeta}$.

Proposition 7.2. *Let $[b]$ be a projective tuple with $\sigma[b]^*$ nonempty and let U be a neighbourhood of $\sigma[b]$ which does not intersect a hyperplane. Then for any q the following complex is exact:*

$$K_\bullet(\mathcal{E}_{\bullet,q}(U \setminus \sigma[b], X \otimes L^\bullet), \delta_b).$$

Proof. We may assume that U does not intersect the hyperplane $[1, 0, \dots, 0]$. We know that pointwise for $[z] \in U \setminus \sigma[b]$ the complex

$$K_\bullet(X \otimes L^\bullet \otimes \Lambda^{\bullet,0} T^* \mathbb{CP}^n_{[z]}, \delta_b)$$

is exact. In the local coordinates $(\zeta_1, \ldots, \zeta_n) = (z_1/z_0, \ldots, z_n/z_0)$ this means that the complex

$$K_\bullet(X \otimes \Lambda^{\bullet,0} T^*\mathbb{C}^n, b_0 \delta_{b_0^{-1}b-\zeta})$$

is exact for $\zeta \in U \setminus \sigma[b]$. From the theory of parametrised complexes it follows that

$$K_\bullet(\mathcal{E}_{\bullet,0}(U \setminus \sigma[b], X), b_0 \delta_{b_0^{-1}b-\zeta})$$

is exact, see e.g. [12]. But this is the statement in the proposition (in local coordinates) for $q = 0$. Taking exterior products with barred differentials does not affect exactness since δ_b commutes with this operation. Hence the statement is true for any q. □

We now construct the integral representation of the functional calculus. Let $f \in \mathscr{O}(U)$ where U is a neighbourhood of $\sigma[b]$ that avoids a hyperplane. Let x be the function which is identically x in $U \setminus \sigma[b]$. From Proposition 7.2 we see that there is a form $\omega_b^1 x \in \mathcal{E}_{1,0}(U \setminus \sigma[b], X \otimes L^1)$ such that $x = \delta_b \omega_b^1 x$. Now δ_b and $\bar\partial$ anti-commute and so $\delta_b \bar\partial \omega_b^1 x = -\bar\partial \delta_b \omega_b^1 x = -\bar\partial x = 0$. Hence by Proposition 7.2 there is a form $\omega_b^2 \in \mathcal{E}_{2,1}(U \setminus \sigma[b], X \otimes L^2)$ such that $\bar\partial \omega_b^2 x = \delta_b \omega_b^1 x$. Continuing in this way and successively solving the equations $\bar\partial \omega_b^j x = \delta_b \omega_b^{j+1} x$ we finally arrive at a form $\omega_b^n x \in \mathcal{E}_{n,n-1}(U \setminus \sigma[b], X \otimes L^n)$. This form is $\bar\partial$-closed because, as above $\delta_b \bar\partial \omega_b^n x = 0$ and since δ_b is injective on this level we must have $\bar\partial \omega_b^n x = 0$. If we start with another solution $x = \delta_b \tilde\omega_b^1 x$ and solve the equations $\bar\partial \tilde\omega_b^j x = \delta_b \tilde\omega_b^{j+1} x$ then $\omega_b^n x$ and $\tilde\omega_b^n x$ define the same $\bar\partial$-cohomology class. In fact, since $\delta_b(\omega_b^2 x - \tilde\omega_b^2 x) = \bar\partial(\omega_b^1 x - \tilde\omega_b^1 x)$ and $\delta_b(\omega_b^1 x - \tilde\omega_b^1 x) = 0$ we get from Proposition 7.2 that $\delta_b(\omega_b^2 x - \tilde\omega_b^2 x) = \bar\partial \delta_b w^1 = -\delta_b \bar\partial\, w^1$, that is $\delta_b(\omega_b^2 x - \tilde\omega_b^2 x + \bar\partial w^1) = 0$, for some w^1. Inductively we obtain $\delta_b(\omega_b^n x - \tilde\omega_b^n x + \bar\partial w^{n-1}) = 0$ and since δ_b is injective on that level we get $\omega_b^n x - \tilde\omega_b^n x + \bar\partial w^{n-1} = 0$. Hence we get a well defined mapping (depending on the representative b)

$$x \mapsto [\omega_b^n x]_{\bar\partial}.$$

From the construction it is clear that this map is linear in x.

Proposition 7.3. *Let b be a projective tuple and assume $[\lambda] \in \sigma[b]^*$. Then*

$$\Big[\frac{\langle b, \lambda\rangle^n}{\langle z, \lambda\rangle^n} \omega_b^n x\Big]_{\bar\partial}$$

does not depend on the representative for $[b]$.

Proof. Clearly $\langle z, \lambda\rangle \langle b, \lambda\rangle^{-1} \delta_b$ does not depend on the representative. Let $\tilde\omega^j$, $j = 1, \ldots, n$ be solutions to the equations $x = \langle z, \lambda\rangle \langle b, \lambda\rangle^{-1} \delta_b\, \tilde\omega^1$, $\bar\partial \tilde\omega^j = \langle z, \lambda\rangle \langle b, \lambda\rangle^{-1} \delta_b\, \tilde\omega^{j+1}$ in $U \setminus \sigma[b]$. Then $\tilde\omega^j$ can not depend on the representative. Moreover $\omega^j := \langle z,\lambda\rangle^j \langle b,\lambda\rangle^{-j}\tilde{\omega}^j$, $j = 1,\dots,n$ must satisfy the equations $x = \delta_b\omega^1$, $\bar\partial\omega^j = \delta_b\omega^{j+1}$ in $U\setminus\sigma[b]$. Hence we have

$$\big[\frac{\langle b,\lambda\rangle^n}{\langle z,\lambda\rangle^n}\omega_b^n x\big]_{\bar\partial} = [\tilde{\omega}^n]_{\bar\partial}. \qquad \square$$

Theorem 7.4. *Let $[b]$ be a projective tuple with $[\lambda]\in\sigma[b]^*$. If $f\in\mathscr{O}(\sigma[b])$ and D is a neighbourhood of $\sigma[b]$ whose closure is contained in an open set, which avoids some hyperplane and in which f is holomorphic. Then*

$$f([b])x = \int_{\partial D} f\frac{\langle b,\lambda\rangle^n}{\langle z,\lambda\rangle^n}\omega_b^n x.$$

Proof. After a projective transformation we can assume that $[\lambda] = [1,0,\dots,0]$ and since the $\bar\partial$-cohomology class of $\langle b,\lambda\rangle^n\langle z,\lambda\rangle^{-n}\omega_b^n x$ does not depend on the representative we may assume that b is the representative such that $e = \langle b,\lambda\rangle = b_0$ given by Proposition 4.3. We recapitulate the definition of $f([b])$. Let $\tilde f$ be the function that makes the following diagram commute:

$$\begin{tikzcd} \mathbb{C}^{n+1} \arrow[d] \arrow[dr, "\tilde f"] & \\ \mathbb{CP}^n \arrow[r, "f"'] & \mathbb{C}. \end{tikzcd}$$

Then $f([b])x = \tilde f(b)x$. Let p denote the mapping $V = \{z\in\mathbb{C}^{n+1}; z_0\neq 0\}\to\mathbb{C}^n$ given by $(z_0,\dots,z_n)\mapsto(z_1/z_0,\dots,z_n/z_0)$ and let ϕ be the local chart $(\zeta_1,\dots,\zeta_n)\mapsto[1,\zeta_1,\dots,\zeta_n]$. Then

$$\begin{tikzcd} V \arrow[d, "p"] \arrow[dr, "\tilde f"] & \\ \mathbb{C}^n \arrow[r, "\phi^* f"'] & \mathbb{C} \end{tikzcd}$$

must commute. From the Composition Rule in Taylors functional calculus we get that

$$\tilde f(b) = \phi^* f(b_1,\dots,b_n).$$

We will show that

$$\int_{\partial D} f\omega_b^n x = \phi^* f(b_1,\dots,b_n)x.$$

In the local chart ϕ, δ_b is the operator $\delta_{b'-\zeta}$ where $b' = (b_1, \ldots, b_n)$ because of our choice of b. So our solutions $\omega_b^j x$ to the δ_b-equations must satisfy

$$\begin{aligned} x &= \delta_{b'-\zeta}\phi^*(\omega_b^1 x) \\ \bar\partial\phi^*(\omega_b^1 x) &= \delta_{b'-\zeta}\phi^*(\omega_b^2 x) \\ &\vdots \\ \bar\partial\phi^*(\omega_b^{n-1} x) &= \delta_{b'-\zeta}\phi^*(\omega_b^n x) \end{aligned}$$

in $\phi^{-1}(U \setminus \sigma[b])$. But from the Spectral Mapping Theorem $\phi^{-1}(U \setminus \sigma[b]) = \phi^{-1}(U) \setminus \sigma(b')$. Hence $[\phi^*(\omega_b^n x)]_{\bar\partial}$ must be the same $\bar\partial$-cohomology class as the resolvent class Andersson defines in [1] corresponding to b'. Moreover it is shown in [1] that integrating against this resolvent realises the functional calculus. Thus we obtain

$$\phi^* f(b_1, \ldots, b_n)x = \int \phi^* f \phi^*(\omega_b^n x) = \int \phi^*(f\omega_b^n x) = \int f\omega_b^n x. \qquad \square$$

We have seen that the resolvent, that is the $\bar\partial$-cohomology class determined by

$$\frac{\langle b, \lambda\rangle^n}{\langle z, \lambda\rangle^n}\omega_b^n x,$$

does not depend on the representative for $[b]$ and that the functional calculus is realised by integrating against it. Actually, the resolvent is even independ on the choice of $[\lambda] \in \sigma[b]^*$ in the following sense.

Theorem 7.5. *Let $[b]$ be a projective tuple and assume that $[\lambda]$, $[\tilde\lambda] \in \sigma[b]^*$. Let U be a pseudo convex neighbourhood of $\sigma[b]$ such that none of the hyperplanes $[\lambda]$ and $[\tilde\lambda]$ intersect U. Then*

$$\frac{\langle b, \lambda\rangle^n}{\langle z, \lambda\rangle^n}\omega_b^n x$$

and

$$\frac{\langle b, \tilde\lambda\rangle^n}{\langle z, \tilde\lambda\rangle^n}\omega_b^n x$$

are $\bar\partial$-cohomologous in $U \setminus \sigma[b]$.

In order to prove Theorem 7.5 we have to look more closely at the relation between the homological construction of the functional calculus and the integral construction. We recapitulate the homological construction. Let $c = (c_1, \ldots, c_n)$ be a commuting tuple of bounded operators on X. We let $\mathcal{E}_{p,q}(U, X)$ denote the set of smooth X-valued (p, q)-forms in $U \subseteq \mathbb{C}^n$ and we put

$$\mathscr{L}^k(U, X) = \bigoplus_{q-p=k} \mathcal{E}_{p,q}(U, X).$$

The operator $\nabla_{z-c} = \delta_{z-c} - \bar\partial$ is an anti-derivative on $\bigoplus_k \mathscr{L}^k(U,X)$ and maps $\mathscr{L}^k(U,X)$ to $\mathscr{L}^{k+1}(U,X)$. Moreover $\nabla_{z-c}\circ\nabla_{z-c} = 0$ and we get the complex $Tot\mathscr{L}(U,X)$:

$$\cdots \xrightarrow{\nabla_{z-c}} \mathscr{L}^{k-1}(U,X) \xrightarrow{\nabla_{z-c}} \mathscr{L}^{k}(U,X) \xrightarrow{\nabla_{z-c}} \mathscr{L}^{k+1}(U,X) \xrightarrow{\nabla_{z-c}} \cdots .$$

This complex is exact if U is disjoint with $\sigma(c)$ since the Koszul-complex is exact outside of $\sigma(c)$. The crucial part of the homological construction of the functional calculus for c is to show that for any neighbourhood U of $\sigma(c)$ we have that X and $H_0(Tot\mathscr{L}(U,X))$ are isomorphic as $\mathscr{O}(\mathbb{C}^n)$-modules. Since $H_0(Tot\mathscr{L}(U,X))$ has a natural $\mathscr{O}(U)$-module structure, which extends the $\mathscr{O}(\mathbb{C}^n)$-module structure, the isomorphism yields a $\mathscr{O}(U)$-module structure on X extending the $\mathscr{O}(\mathbb{C}^n)$-module structure. Furthermore one shows that if $U' \subseteq U$ are neighbourhoods of $\sigma(c)$ then the $\mathscr{O}(U')$-module structure on X extends the $\mathscr{O}(U)$-module structure. Hence we get a $\mathscr{O}(\sigma(c))$-module structure on X and this is our functional calculus. Given a function $f \in \mathscr{O}(U)$ (U a neighbourhood of $\sigma(c)$) the X-valued function $z \mapsto xf(z)$ determines an element in $H_0(Tot\mathscr{L}(U,X))$ and the isomorphism maps this element to $f(c)x$ by definition. This construction is due to Taylor see [10] and [9].

The integral construction of $f(c)x$ is first to solve the equation $\nabla_{z-c}\omega_{z-c}x = x$ in $U\setminus\sigma(c)$, then identifying the component, $\omega^n_{z-c}x$, of $\omega_{z-c}x$ of bidegree $(n,n-1)$, and put

$$f(c)x = \int_{\partial D} f(z)\,\omega^n_{z-c}x.$$

Note that for bidegree reasons, solving $\nabla_{z-c}\omega_{z-c}x = x$ is exactly the same as solving the equations $x = \delta_{z-c}\,\omega^1_{z-c}x$, $\bar\partial\,\omega^k_{z-c}x = \delta_{z-c}\,\omega^{k+1}_{z-c}x$, $k = 1,\ldots,n-1$. In [1] Andersson shows that the two definitions of $f(c)x$ coincide. The crucial step in proving Theorem 7.5 is the following lemma.

Lemma 7.6. *Let $c = (c_1,\ldots,c_n)$ be bounded commuting operators on X and let U be a pseudo convex neighbourhood of $\sigma(c)$. If $f \in \mathscr{O}(U)$ and $f(c) = 0$ then*

$$[f(z)\,\omega^n_{z-c}x]_{\bar\partial} = 0,$$

where $\omega^n_{z-c}x$ is the component of bidegree $(n,n-1)$ of a solution $\omega_{z-c}x$ to $\nabla_{z-c}\,\omega_{z-c}x = x$ in $U\setminus\sigma(c)$.

Proof. Clearly we have $\nabla_{z-c}\,f(z)\omega_{z-c}x = f(z)x$ in $U\setminus\sigma(c)$. From the homological construction we see that $xf(z)$ must be ∇_{z-c}-exact in U since $f(c)x = 0$. Hence $xf(z) = \nabla_{z-c}\,u(z)$ for some $u \in \mathscr{L}^{-1}(U,X)$. Thus $u - f(z)\omega_{z-c}x$ is ∇_{z-c}-closed in $U\setminus\sigma(c)$. Since $Tot\mathscr{L}(U\setminus\sigma(c),X)$ is exact there is a $v \in \mathscr{L}^{-2}(U\setminus\sigma(c),X)$ such that

$$u(z) - f(z)\omega_{z-c}x = \nabla_{z-c}\,v(z)$$

in $U \setminus \sigma(c)$. Identifying terms of bidegree $(n, n-1)$ we see that

$$u_{n,n-1} - f(z)\omega^n_{z-c}x = \bar\partial v_{n,n-2} \tag{13}$$

in $U \setminus \sigma(c)$. Moreover, $\nabla_{z-c}\, u = xf(z)$ so for bidegree reasons $\bar\partial\, u_{n,n-1} = 0$. Since U is pseudo convex $u_{n,n-1}$ is actually $\bar\partial$-exact and letting $u_{n,n-1} = \bar\partial\, \tilde v_{n,n-2}$ we get from (13) that

$$f(z)\omega^n_{z-c}x = \bar\partial(\tilde v_{n,n-2} - v_{n,n-2})$$

in $U \setminus \sigma(c)$ which is what we wanted to show. □

We proceed and prove Theorem 7.5.

Proof of Theorem 7.5. From Theorem 7.4 we know that both the forms $\langle b, \lambda\rangle^n \langle z, \lambda\rangle^{-n}\omega^n_b x$ and $\langle b, \tilde\lambda\rangle^n \langle z, \tilde\lambda\rangle^{-n}\omega^n_b x$ represent the functional calculus. We have to show that they are $\bar\partial$-cohomologous in $U \setminus \sigma[b]$. We let ρ be a projections from $[\lambda]$. From the proof of Theorem 7.4 we see that $\rho_*(\langle b, \lambda\rangle^n \langle z, \lambda\rangle^{-n}\omega^n_b x)$ defines the resolvent class $\omega_{\zeta-\rho([b])}$ corresponding to $\rho([b])$ if we choose $b \in [b]$ such that $\langle b, \lambda\rangle = e$. Hence in the local coordinates $\zeta = \rho([z])$ the difference between the two forms has to be on the form

$$(1 - f(\zeta))\omega_{\zeta-\rho([b])}$$

where f is holomorphic in $\rho(U)$. Now since both of the forms realise the functional calculus we must have $f(\rho([b])) = 0$. Hence from Lemma 7.6 we see that in the local coordinates, the two forms has to be $\bar\partial$-cohomologuos in $\rho(U) \setminus \sigma(\rho([b]))$. □

The function $f(\zeta)$ is the function $\langle b, \tilde\lambda\rangle^n \langle z, \tilde\lambda\rangle^{-n}$ in the local coordinates ζ. Hence we see that making a change of variables by a rational fractional transform of $\mathbb{C}^n$, computing the resolvent in the new coordinates and pulling it back, we get $\langle b, \tilde\lambda\rangle^n \langle z(\zeta), \tilde\lambda\rangle^{-n}$ times the resolvent we get if we compute it directely. Theorem 7.5 implies that the two forms are $\bar\partial$-cohomologuos in suitable domains.

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