Thesis for the Degree of Master of Science in Physics

Meta-Stable Vacua in Supersymmetric QCD

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Abstract

This masters thesis discusses some aspects of four dimensional supersymmetric gauge theories. More specifically, it discusses a method to break supersymmetry spontaneously in certain gauge theories in a way that bypasses some phenomenological problems.

We start by giving a short introduction to supersymmetry and spontaneous supersymmetry breaking. We then discuss a supersymmetric version of QCD and the phenomenon of Seiberg duality. Using these tools we show, following K. Intriligator, N. Seiberg and D. Shih, that supersymmetric QCD has a supersymmetry breaking metastable vacuum for certain number of flavors. We conclude by showing that this metastable vacuum can be made parametrically long lived.

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Introduction

With the Large Hadron Collider going online one of the most fascinating discoveries would be that of a symmetry called supersymmetry. Supersymmetry has many intriguing qualities. Besides being the only reasonable expansion to the Poincaré group, solving the so called hierarchy problem and giving a candidate to the dark matter, its sheer beauty should not be forgotten. One of the properties of supersymmetric theories is that each fermion/boson should have an equally heavy bosonic/fermionic partner. When we look at the real world, we see that this is not the case. If supersymmetry exists at higher energies it must therefore be spontaneously broken.

By looking at the superalgebra it is easy to show that unbroken supersymmetry is equivalent to having vanishing vacuum energy. Most effort to build a theory that breaks supersymmetry spontaneously for low energies has therefore been directed towards building theories without vacua with vanishing vacuum energy. This puts strong constraints on the possible theories. One is related to a chiral global symmetry called R-symmetry. The rule of thumb is that if a theory does not have R-symmetry then it generically has a supersymmetric vacuum. For supersymmetry to be spontaneously broken we therefore want the Lagrangian to have R-symmetry. One inconvenient property of R-symmetry is that the fermionic partner of the vector gauge boson must have charge $+1$. Because a charged fermion can not get a Majorana mass the gaugino must be massless. Since no massless superpartners of the vector bosons corresponding to unbroken gauge symmetries are observed in nature, the R-symmetry seems to be broken in the low energy theory. However, if R-symmetry breaks spontaneously the theory gets a massless goldstone boson, which is not observed. Although this problem might be solvable it illustrates how problematic it can be to require that the theory does not have any supersymmetric vacuum. By permitting the theory to have supersymmetric vacua as long as it also has a parametrically long lived meta stable supersymmetry breaking vacuum, new candidates for high energy theories become available [1].

One of these is SQCD, which is the supersymmetrized version of QCD. If the theory has N_C colors and N_F flavors it is asymptotically free for $N_F < 3N_C$. Although the low energy theory is not weakly coupled, the constraints supersymmetry imposes make it possible to describe the low energy degrees of freedom in a simple manifestly supersymmetric manner. For $N_F > N_C + 1$ it turns out that there exists a duality relation between the high energy theory, often called the electric theory, and a low energy theory with gauge group $SU(N_F - N_C)$, called the magnetic theory. This duality is called Seiberg duality. For $N_F \leq \frac{3}{2}N_C$ the magnetic theory is IR free and Seiberg duality offers a simple way to study the low energy dynamics. Although massive SQCD is known to have N_C supersymmetric vacua, these are not visible classically in the magnetic theory. Taking the perturbative correction to the potential into account one finds a compact manifold of supersymmetry breaking ground states in a region of field space where the coupling constant is small. Also taking nonperturbative effects into account resurrects the supersymmetric vacua of the electric theory, making the previous vacuum only metastable. These true supersymmetric vacua are however far from the false supersymmetry breaking metastable one and one can show that, by choosing the parameters in an appropriate way, the false vacuum can be made parametrically long lived. Although this particular theory clearly is not phenomenologically viable it is also clear that the prejudice that supersymmetry breaking forbids a theory form having any supersymmetric vacua is unnecessarily restrictive.

1.1 Outline

In the second chapter, we give a short introduction to supersymmetry. We start by writing down the superalgebra and look at how it can be represented using superspace. We then write down the invariants that can be used in a Lagrangian, first using chiral superfields and then using vector superfields. From here we go on to write down the Lagrangian of a general supersymmetrized Yang-Mills theory. We end the chapter by deriving the mass matrix and stating some results regarding loop corrections.

In the third chapter, we begin by discussing the correspondence between unbroken supersymmetry and vanishing vacuum energy. After showing why the classical moduli can not be lifted by perturbative corrections we give some examples of how supersymmetry can be broken. We then show why a spontaneously broken global R-symmetry indicates that supersymmetry is broken. After discussing some of the implications of this fact, we finish the chapter by giving some examples.

The fourth chapter begins with some general remarks regarding SQCD. We then give a more detailed description of SQCD theories with an increasing number of flavors, starting with zero. For each number of flavors we introduce a low energy theory describing the low energy degrees of freedom. We test these theories by going to weakly coupled limits, checking decoupling properties, anomaly matching etc.

In chapter five, we begin by studying the classical flat directions of the magnetic dual of massive SQCD with $N_C + 1 < N_C \leq \frac{3}{2} N_C$. We then calculate the masses of all fields as a function of these flat directions. Using these masses we calculate the one loop correction to the potential and conclude that some of these flat directions (the non-compact ones, called pseudo moduli) are lifted and one is left with a compact vacuum manifold spanned by the Goldstone bosons. Taking nonperturbative effects into account we then show how supersymmetry is restored for certain large fields. We conclude the chapter by showing that even though the vacuum found earlier is only metastable, it is parametrically long lived.

2

Supersymmetry

If one wants to extend the Poincaré algebra and also wants the S-matrix and mass spectrum to behave in a reasonable way, it has been shown, see [2], that the only option is a class of algebras called superpoincaré algebras. These algebras are labeled by the number of fermionic generators added (the supercharges). The simplest is called $\mathcal{N} = 1$ and is the one we will focus on. Many extensions of the standard model have been proposed using this symmetry. The simplest one is the Minimal Supersymmetric Standard Model, MSSM. This theory has many intriguing properties. First of all, it explains why the higgs mass is so much smaller than the energy scale of some unifying theory. Secondly, looking at how the coupling constants runs with increasing energy one sees that they coincide at a certain point, perhaps signaling some kind of unification. Lastly, MSSM introduces many new particles, some of which might be candidates to explain dark matter. With LHC going online soon, the possibility to discover supersymmetry is bigger than ever!

2.1 The Superalgebra and the Superspace

The Lie algebra of the $\mathcal{N} = 1$ superpoincaré group consists of the Poincaré algebra and the two new generators Q_α and $\bar{Q}_{\dot{\alpha}}$, where α and $\dot{\alpha}$ are Weyl spinor indices [3, 4]. These new generators obey, using the metric $\eta_{\mu\nu} \sim (-1, 1, 1, 1)$,

$$\left\{Q_\alpha, \bar{Q}_{\dot{\alpha}}\right\} = 2\sigma^\mu_{\alpha\dot{\alpha}} P_\mu \ , \ \left\{Q_\alpha, Q_\beta\right\} = \left\{\bar{Q}_{\dot{\alpha}}, \bar{Q}_{\dot{\beta}}\right\} = 0 \tag{2.1}$$

and transform as spinors under the Poincaré algebra. In particular, Q_α and $\bar{Q}_{\dot{\alpha}}$ commute with the momentum operator P_μ. This means that all states in a supersymmetry representation must have the same mass. Another interesting property of supersymmetric theories is that they contain equally many bosonic and fermionic degrees of freedom. One way to show this is to define an operator $(-1)^{N_F}$ which has eigenvalue $+1$ when acting on a bosonic state and eigenvalue -1 when acting on a fermionic one. Since Q_α and $\bar{Q}_{\dot{\alpha}}$ change a bosonic/fermionic state to a fermionic/bosonic, $(-1)^{N_F}$ must satisfy

$$(-1)^{N_F} Q_\alpha = -Q_\alpha (-1)^{N_F} \,. \tag{2.2}$$

Choosing a particular momentum vector and using the cyclicity of the trace gives

$$\begin{aligned} 2\sigma^\mu_{\alpha\dot{\alpha}} P_\mu \sum \langle\psi| (-1)^{N_F} |\psi\rangle &= \sum \langle\psi| (-1)^{N_F} 2\sigma^\mu_{\alpha\dot{\alpha}} P_\mu |\psi\rangle = \\ &= \sum \langle\psi| (-1)^{N_F} \left(Q_\alpha \bar{Q}_{\dot{\alpha}} + \bar{Q}_{\dot{\alpha}} Q_\alpha\right) |\psi\rangle = \\ &= \sum \langle\psi| - Q_\alpha (-1)^{N_F} \bar{Q}_{\dot{\alpha}} + Q_\alpha (-1)^{N_F} \bar{Q}_{\dot{\alpha}} |\psi\rangle = 0 \,, \end{aligned} \tag{2.3}$$

where the sums run over all states $|\psi\rangle$ in a multiplet. This means that $\sum \langle\psi| (-1)^{N_F} |\psi\rangle = 0$ and hence that the multiplet contains equally many fermionic and bosonic states.

Our goal in this chapter is to find a way to write expressions that are manifestly invariant under supersymmetry. One good way to do this is to use what is called the superspace. This is an extension of the ordinary space and is parameterized by the usual four spatial coordinates x^μ and two new complex Grassmann coordinates θ^α and $\bar{\theta}^{\dot{\alpha}} = (\theta^\alpha)^*$, [5, 6, 7]. θ^α and $\bar{\theta}^{\dot{\alpha}}$ transform as Weyl spinors. A field that lives in superspace is called a superfield. The Poincaré subgroup acts on the spatial coordinates in the usual way. To get the generators of translations in the field space representation we use the fact that

$$\Phi(x + a, \theta^\alpha, \bar{\theta}^{\dot{\alpha}}) = e^{-ia^\mu \mathcal{P}_\mu} \Phi(x, \theta^\alpha, \bar{\theta}^{\dot{\alpha}}) e^{ia^\mu \mathcal{P}_\mu}, \tag{2.4}$$

where $\mathcal{P}$ is the generator of the translation in the coordinate space. By Taylor expanding both sides to first order in a^μ and using that the relation must be valid for any a^μ we get

$$\left[\mathcal{P}, \Phi(x, \theta^\alpha, \bar{\theta}^{\dot{\alpha}})\right] = i\partial_\mu \Phi(x, \theta^\alpha, \bar{\theta}^{\dot{\alpha}}) \,. \tag{2.5}$$

The operator that generates translation in field space can therefore be represented as $P_\mu = i\partial_\mu$. How do we then interpret Q_α and $\bar{Q}_{\dot{\alpha}}$? In a similar way that $-ia^\mu \mathcal{P}_\mu$ generated $x^\mu \to x^\mu + a^\mu$ we let $\xi^\alpha \mathcal{Q}_\alpha$ and $\xi^{\dot{\alpha}} \bar{\mathcal{Q}}_{\dot{\alpha}}$

generate

$$\begin{aligned}\theta^\alpha &\to \theta^\alpha + \xi^\alpha \\ \bar{\theta}^{\dot{\alpha}} &\to \bar{\theta}^{\dot{\alpha}} + \bar{\xi}^{\dot{\alpha}} \\ x^\mu &\to x^\mu + i\theta\sigma^\mu\bar{\xi} - i\xi\sigma^\mu\bar{\theta}\,. \end{aligned} \tag{2.6}$$

In terms of superfields this can be written as

$$\begin{aligned}&\Phi(x^\mu + i\theta\sigma^\mu\bar{\xi} - i\xi\sigma^\mu\bar{\theta}, \theta^\alpha + \xi^\alpha, \bar{\theta}^{\dot{\alpha}} + \bar{\xi}^{\dot{\alpha}}) = \\ &\quad e^{i(\xi^\alpha \mathcal{Q}_\alpha + \xi^{\dot{\alpha}}\bar{\mathcal{Q}}_{\dot{\alpha}})}\Phi(x^\mu, \xi^\alpha, \bar{\xi}^{\dot{\alpha}})e^{-i(\xi^\alpha \mathcal{Q}_\alpha + \xi^{\dot{\alpha}}\bar{\mathcal{Q}}_{\dot{\alpha}})}\,. \end{aligned} \tag{2.7}$$

By Taylor expanding to first order as before we get

$$\begin{aligned}[\mathcal{Q}_\alpha, \Phi] &= (\partial_\alpha - i\sigma^\mu_{\alpha\dot{\alpha}}\bar{\theta}^{\dot{\alpha}}\partial_\mu)\Phi = Q_\alpha\Phi, \\ \left[\bar{\mathcal{Q}}_{\dot{\alpha}}, \Phi\right] &= (\partial_{\dot{\alpha}} - i\theta^\alpha\sigma^\mu_{\alpha\dot{\alpha}}\partial_\mu)\Phi = \bar{Q}_{\dot{\alpha}}\Phi, \end{aligned} \tag{2.8}$$

where Q_α and $\bar{Q}_{\dot{\alpha}}$ are the generating operators in field space and we have assumed that Φ is bosonic. These operators satisfy

$$\left\{Q_\alpha, \bar{Q}_{\dot{\alpha}}\right\} = 2i\sigma^\mu_{\alpha\dot{\alpha}}\partial_\mu, \tag{2.9}$$

which, using $P_\mu = i\partial_\mu$, is identical to (2.1). This means that we indeed have a representation. Because Q_α and $\bar{Q}_{\dot{\alpha}}$ only induce a translation in θ^α and $\bar{\theta}^{\dot{\alpha}}$ and an x independent shift in x^μ, the determinant of the transformation is 1. Thus

$$\int d^4x d^2\theta d^2\bar{\theta}\Phi \tag{2.10}$$

is manifestly invariant under supersymmetry transformations. If we Taylor expand Φ we get

$$\begin{aligned}\Phi(x^\mu, \theta^\alpha, \bar{\theta}^{\dot{\alpha}}) =& \phi(x) + \theta\psi + \bar{\theta}\bar{\chi} + \theta\theta m(x) + \bar{\theta}\bar{\theta}n(x) + \\ & \theta\sigma^\mu\bar{\theta}v_\mu(x) + \theta\theta\bar{\theta}\bar{\lambda}(x) + \bar{\theta}\bar{\theta}\theta\eta(x) + \theta\theta\bar{\theta}\bar{\theta}d(x), \end{aligned} \tag{2.11}$$

where ϕ, m, n, v_μ and d are complex bosons and ψ, χ, λ and η are spinors. Contracted spinor indices are omitted. Because Grassmann numbers anticommute, all higher powers in θ and $\bar{\theta}$ vanish. Under supersymmetry transformations these fields transform among themselves. The representation is however reducible. To get an irreducible representation we have to impose some suitable constraint on the superfield. The two kinds of superfields we will use are called chiral and vector superfields.

2.1.1 Chiral Superfields

The chiral superfields use the fact that the operators

$$D_\alpha = \partial_\alpha + i\sigma^\mu_{\alpha\dot\alpha}\bar\theta^{\dot\alpha}\partial_\mu \ , \ \bar D_{\dot\alpha} = -\partial_{\dot\alpha} - i\theta^\alpha\sigma^\mu_{\alpha\dot\alpha}\partial_\mu \tag{2.12}$$

obey

$$\left\{D_\alpha, \bar D_{\dot\alpha}\right\} = -2i\sigma^\mu_{\alpha\dot\alpha}\partial_\mu \,. \tag{2.13}$$

and anticommute with all other D_α, $\bar D_{\dot\alpha}$, Q_α, $\bar Q_{\dot\alpha}$. If a superfield obeys the constraint $\bar D\Phi = 0$, the fact that $\bar D$ anticommutes with the supersymmetry generators insures that the constraint is still fulfilled after a supersymmetry transformation. It is therefore a suitable constraint to impose in order to reduce the supersymmetry representation. These fields are called chiral superfields and it has been shown that they in fact form an irreducible representation. The constraint is especially simple in the coordinate system $\left\{y^\mu = x^\mu + i\theta\sigma^\mu\bar\theta\,,\, \theta^\alpha\,,\, \bar\theta^{\dot\alpha}\right\}$: $\bar D_{\dot\alpha} = -\partial_{\dot\alpha}$. In this base the chiral superfield must therefore be independent of $\bar\theta^{\dot\alpha}$ and can thus be written as

$$\begin{aligned}\Phi =&\phi(y) + \sqrt{2}\theta\psi(y) + \theta\theta F(y) = \phi(x) + i\theta\sigma^\mu\bar\theta\partial_\mu\phi(x) + \frac{1}{4}\theta\theta\bar\theta\bar\theta\Box\phi(x)\\ &+ \sqrt{2}\theta\psi(x) - \frac{i}{\sqrt{2}}\theta\theta\partial_\mu\psi(x)\sigma^\mu\bar\theta + \theta\theta F(x)\,.\end{aligned} \tag{2.14}$$

Since ψ has mass dimension $3/2$, θ must have mass dimension $-1/2$. The component fields ϕ ,ψ and F transform under (2.6) as

$$\begin{aligned}&\phi \to \phi + \sqrt{2}\xi\psi\\ &\psi \to \psi + i\sqrt{2}\sigma^\mu\bar\xi\partial_\mu\phi + \sqrt{2}\xi F\\ &F \to F + i\sqrt{2}\bar\xi\bar\sigma^\mu\partial_\mu\psi\,.\end{aligned} \tag{2.15}$$

Two important invariants that we will use are built by chiral superfields. The first is, by using integration by part,

$$\int d^4xd^2\theta d^2\bar\theta\Phi^\dagger\Phi = \int d^4xF^*F - \partial_\mu\phi^*\partial^\mu\phi - i\bar\psi\bar\sigma^\mu\partial_\mu\psi\,. \tag{2.16}$$

The ϕ terms in the expression above will later work as kinetic terms for the scalar field and the ψ terms will do the same job for the fermions. These terms can more generally be written as $\int d^4xd^2\theta d^2\bar\theta K(\Phi,\bar\Phi)$ where $K(\Phi,\bar\Phi)$ is called the Kähler potential.

The second invariant uses the fact that any holomorphic function of a chiral superfield $W(\Phi)$ is itself a chiral superfield. Since chiral superfields are independent of $\bar{\theta}^{\dot{\alpha}}$ when written in terms of y^μ, integrals of the type above will vanish. If we instead only integrate over y^μ and θ^α we get a non vanishing integral:

$$\int d^4y d^2\theta W(\Phi(y,\theta)) = \int d^4y \left(\frac{\partial W}{\partial \phi} F + \frac{1}{2}\frac{\partial^2 W}{\partial \phi^2}\psi\psi \right) , \tag{2.17}$$

where the derivatives act on the scalar part of the superpotential. This is invariant because y^μ transforms as $y^\mu \to y^\mu + 2i\theta\sigma^\mu\bar{\xi}$, which is still independent of $\bar{\theta}^{\dot{\alpha}}$. The terms coming from the expression above contain no derivatives and usually act as a kind of potentials in supersymmetric Lagrangians. The function W is called the superpotential. Since θ^α has mass dimension $-1/2$ the integral above is only dimensionless if W has mass dimension 3. If we want the theory to be renormalizable and all coupling constants to have non negative mass dimension, W must be at most a polynomial of degree 3 in Φ.

2.1.2 Vector Superfields

Vector superfields are defined by the constraint $V^\dagger = V$. If expanded in θ and $\bar{\theta}$ the general vectorfield can be written as

$$\begin{aligned} V(x,\theta,\bar{\theta}) =& C(x) + i\theta\chi(x) - i\bar{\theta}\bar{\chi}(x) + \frac{i}{2}\theta\theta\left[M(x) + iN(x)\right] \\ &- \frac{i}{2}\bar{\theta}\bar{\theta}\left[M(x) - iN(x)\right] - \theta\sigma^\mu\bar{\theta}v_\mu(x) \\ &+ i\theta\theta\bar{\theta}\left[\bar{\lambda}(x) + \frac{i}{2}\bar{\sigma}^\mu\partial^\mu\chi(x)\right] - i\bar{\theta}\bar{\theta}\theta\left[\lambda(x) + \frac{i}{2}\sigma^\mu\partial_\mu\bar{\chi}(x)\right] \\ &+ \frac{1}{2}\theta\theta\bar{\theta}\bar{\theta}\left[D(x) + \frac{1}{2}\Box C(x)\right] , \end{aligned} \tag{2.18}$$

where the bosonic functions are real. Vector superfields are often used as gauge fields in supersymmetric theories. The gauge transformation is then parameterized by a chiral field Λ and acts as $V \to V + \Lambda + \Lambda^\dagger + \ldots$. By making an appropriate gauge choice it is always possible to put C, χ, M and N to zero. This is the so called Wess-Zumino(WZ) gauge. There is then only one real gauge parameter left: $v_\mu \to v_\mu - i\partial_\mu(A - A^*)$. In WZ-gauge, the vector superfield can be written as

$$V = -\theta\sigma^\mu\bar{\theta}v_\mu(x) + i\theta\theta\bar{\theta}\bar{\lambda}(x) - i\bar{\theta}\bar{\theta}\theta\lambda(x) + \frac{1}{2}\theta\theta\bar{\theta}\bar{\theta}D(x) . \tag{2.19}$$

Supersymmetry transformations will not keep vector superfields in the WZ-gauge. However, one can always make a new gauge transformation and get back into WZ gauge. Supersymmetry is therefore closed up to a gauge transformation.

2.2 Supersymmetric Gauge Theories

One strength of superspace is that Yang-Mills Lagrangians are generated in a simple and compact way, [8,9]. Assume that we want to build a Yang-Mills theory with a gauge group G having dimension D. As indicated in the previous section, a multiplet of chiral superfields transforms as

$$\Phi \to e^{-i\Lambda^A T_A}\Phi\,, \tag{2.20}$$

where Λ^A are chiral superfields and T_A are the generators of some representation R with dimension d. The superpotential is constructed in the same way as for any other symmetry. The condition on the superpotential to be gauge invariant can be written as

$$\frac{\partial W}{\partial \phi^a}(T_A\phi)^a = 0\,, \tag{2.21}$$

where the a is summed over. Since chiral fields are not real there is no easy way to make a gauge invariant kinetic term:

$$\Phi^\dagger\Phi \to \Phi^\dagger e^{i(\Lambda^{A\dagger}-\Lambda^A)T_A}\Phi\,. \tag{2.22}$$

However, if we introduce a vectorfield V_A that is defined to transforms as

$$e^{V^A T_A} \to e^{-i\Lambda^{A\dagger}T_A}e^{V^A T_A}e^{\Lambda^A T_A}\,, \tag{2.23}$$

the combination $\Phi^\dagger e^{V^A T_A}\Phi$ is invariant under gauge transformations. V^A here has the same function as the connection in ordinary Yang-Mills theories. The first terms in the expansion of (2.23) are

$$V^A \to V^A + i(\Lambda - \Lambda^\dagger) + ... \tag{2.24}$$

which is the same as the transformation discussed in the previous section. Doing the θ^α and $\bar\theta^{\dot\alpha}$ integrals explicitly gives

$$\int d^4x d^2\theta d^2\bar\theta \Phi^\dagger e^{V^A T_A}\Phi = \int d^4x \Big(-(D_\mu\phi)^\dagger D^\mu\phi - i\bar\psi\bar\sigma^\mu D_\mu\psi \\ +F^*F + \sqrt{2}i(\bar\phi T_A\psi\lambda^A - \psi T_A\phi\chi\bar\lambda^A) + \bar\phi T_A\phi D^A\Big)\,, \tag{2.25}$$

where D is the usual covariant derivative. Again, a more general kinetic term can be constructed by replacing $\Phi^\dagger e^{V^A T_A}\Phi$ with some gauge invariant Kähler function $K(\Phi^\dagger, e^{V^A T_A}\Phi)$ in the integral above.

What about kinetic terms for the gauge fields? A chiral field can be constructed from the gauge field:

$$\mathcal{W}_\alpha = -\frac{1}{4}\bar D\bar D e^{-V^A T_A}D_\alpha e^{V^A T_A}\,. \tag{2.26}$$

The field above transforms as $\mathcal{W}_\alpha \to e^{-\Lambda^A T_A}\mathcal{W}_\alpha e^{\Lambda^A T_A}$. If the gauge group representation obeys $\mathrm{Tr}T_A T_B = \frac{1}{2}\delta_{AB}$, a gauge invariant supersymmetric kinetic term for the vector gauge field can be written as

$$\frac{-i\tau}{16\pi}\int d^4x d^2\theta \mathrm{Tr}\mathcal{W}^\alpha\mathcal{W}_\alpha + c.c. =$$
$$\int d^4x \left[\frac{1}{g^2}\left(-\frac{1}{4}F_A^{\mu\nu}F_{\mu\nu}^A - i\bar{\lambda}^A\bar{\sigma}^\mu D_\mu\lambda^A + \frac{1}{2}D_A^2\right) + \frac{\theta}{32\pi^2}\tilde{F}_A^{\mu\nu}F_{\mu\nu}^A\right] \quad (2.27)$$

where

$$\tau = \left(\frac{\theta}{2\pi} + i\frac{4\pi}{g^2}\right) \quad (2.28)$$

and $F_A^{\mu\nu}F_{\mu\nu}^A$ is the usual kinetic term for the gauge field in Yang-Mills theories. The full Yang-Mills action is then the sum of (2.17), (2.25) and (2.27):

$$\mathcal{L} = \frac{-i\tau}{16\pi}\int d^4x d^2\theta \mathrm{Tr}\mathcal{W}^\alpha\mathcal{W}_\alpha + c.c. + \int d^4x d^2\theta d^2\bar{\theta}\Phi^\dagger e^{V^A T_A}\Phi$$
$$+ \int d^4y d^2\theta W(\Phi(y,\theta)) + c.c. \quad (2.29)$$

The equations of motion can be solved for the auxiliary fields:

$$\bar{F}_a = -\frac{\partial W}{\partial\phi^a} = -W_a \text{ and } D_A = -g^2\phi^\dagger T_A\phi\,. \quad (2.30)$$

These fields can then be put back into (2.29) and one gets the Lagrangian

$$\begin{aligned}\mathcal{L} =& \frac{1}{g^2}\left[-\frac{1}{4}F_A^{\mu\nu}F_{\mu\nu}^A - i\bar{\lambda}^A\bar{\sigma}^\mu D_\mu\lambda^A\right] + \frac{\theta}{32\pi^2}\tilde{F}_A^{\mu\nu}F_{\mu\nu}^A - (D_\mu\phi)^\dagger D^\mu\phi \\ & - i\bar{\psi}\bar{\sigma}^\mu \not{D}_\mu\psi + \sqrt{2}i(\bar{\phi}T_A\psi\lambda^A - \phi T_A\bar{\chi}\bar{\lambda}^A) + \frac{1}{2}W_{ab}\psi_a\psi_b + \frac{1}{2}\bar{W}^{ab}\bar{\psi}_a\bar{\psi}_b \\ & - V_F - V_D\,,\end{aligned} \quad (2.31)$$

where $V_F = W_a\bar{W}^a$, $V_D = \frac{g^2}{2}(\phi^\dagger T_A\phi)^2$, $W_{a\ldots b} = \frac{\partial^n W}{\partial\phi_a\ldots\partial\phi_b}$ and $\bar{W}^{a\ldots b} = \frac{\partial^n\bar{W}}{\partial\bar{\phi}_a\ldots\partial\bar{\phi}_b}$

2.2.1 The Mass Matrix

Since the theory must have the same vacuum expectation value in any inertial system, all non scalar fields must have vanishing vacuum expectation value. The vacuum expectation value of ϕ is determined by minimizing the potentials above. Assume the minima is at ϕ_0. If we expand the scalar field as $\phi = \phi_0 + \delta\phi$, what is then the mass matrix? The kinetic term of the scalar field decomposes as

$$\begin{aligned}-(D_\mu\phi)^\dagger(D^\mu\phi) = -(D_\mu\delta\phi)^\dagger(D^\mu\delta\phi) &+ 2\mathrm{Im}\left[\partial^\mu\delta\phi^T\,(T_A\phi_0)^*\right] v^A_\mu \\ &- (\phi_0^\dagger T_A T_B\phi_0) v^A_\mu v^{B\mu}\,. \end{aligned} \tag{2.32}$$

The second term is a potential problem. However, as we wrote above, we still have one free gauge parameter, ImA, after choosing WZ-gauge. By using this freedom we can go to the so called unitary gauge. In this gauge the fields satisfy

$$\mathrm{Im}\left[\phi^\dagger(T_A\phi_0)\right] = 0, \tag{2.33}$$

and the second term in (2.32) vanishes. This is the usual Higgs mechanism where a scalar field is 'eaten' by a vector field making it massive. The third term gives the vector bosons, after symmetrization, the squared mass matrix

$$M_1^2 = g^2\left(\phi_0^\dagger\left\{T_A, T_B\right\}\phi_0\right)\,. \tag{2.34}$$

To get the mass matrix for the scalar fields it is useful to expand the potentials as

$$\begin{aligned} V_F &= \frac{1}{2}W_{abc}\bar{W}^c\delta\phi^a\delta\phi^b + \frac{1}{2}W_c\bar{W}^{abc}\delta\bar{\phi}_a\delta\bar{\phi}_b + W_{ac}\bar{W}^{bc}\delta\phi^a\delta\bar{\phi}_b \\ V_D &= \sum_A\left((\phi_0+\delta\phi)^\dagger T_A(\phi_0+\delta\phi)\right)\left((\phi_0+\delta\phi)^\dagger T_A(\phi_0+\delta\phi)\right)\,. \end{aligned} \tag{2.35}$$

The linear terms disappear since we are at a minimum. The quadratic terms can be written in a matrix form as

$$M_0^2 = \begin{pmatrix} \bar{W}^{ac}W_{cb} + \mathcal{A} + gD_{A0}{T_A}^a{}_b & \bar{W}^{abc}W_c + \mathcal{B} \\ W_{abc}\bar{W}^c + \mathcal{B}^* & W_{ac}\bar{W}^{cb} + \mathcal{A}^* + gD_{A0}{T_A}^b{}_a \end{pmatrix}, \tag{2.36}$$

where $\mathcal{A} = \sum_A g^2(T_A\phi_0)(T_A\phi_0)^\dagger$, $\mathcal{B} = \sum_A g^2(T_A\phi_0)(T_A\phi_0)^T$ and D_{A0} is the D-term for the vacuum. The terms giving a mass to the fermions are $\sqrt{2}i\bar{\phi}_0 T_A\psi\lambda^A + \frac{1}{2}W_{ab}\psi_a\psi_b + c.c.$, which on matrix form is

$$M_{1/2} = \begin{pmatrix} W_{ab} & \sqrt{2}i(T_B\phi_0)^*_a \\ \sqrt{2}i(T_A\phi_0)^*_b & 0 \end{pmatrix}\,. \tag{2.37}$$

When we square this matrix we get terms of the type $-\sqrt{2}iW_{ab}(T_B\phi_o)^b$. By using the derivative of (2.21) we can write these terms as $\sqrt{2}iW_c{T_A}^c{}_b$ and the squared fermion mass matrix is

$$M_{1/2}^2 = \begin{pmatrix} W^{\dagger ac}W_{cb} + 2g^2\sum_A\left(T^A\phi_0\right)^a\left(T^A\phi_0\right)^*_b & ig\sqrt{2}W^{\dagger c}{T_A}^a{}_c \\ -ig\sqrt{2}W_c{T_B}^c{}_b & 2\phi_0^\dagger\left\{T_A, T_B\right\}\phi_0 \end{pmatrix}\,. \tag{2.38}$$

2.3 Loop Corrections

When studying the quantum effects of the theory it is important to know how the coupling constant runs. From [10] we know that for an arbitrary (non supersymmetric) gauge theory the one loop coupling constant in terms of the dimensional parameter Λ is

$$\alpha_S = \frac{g^2}{4\pi} = \frac{2\pi}{b_0 \log\frac{\sqrt{s}}{\Lambda}} \tag{2.39}$$

and that

$$b_0 = \frac{11}{3}C_2(\text{Adj}) - \sum_{reps.}\left(\frac{2}{3}C(\psi_i) + \frac{1}{6}C(\phi_i)\right), \tag{2.40}$$

where $C_2(\text{Adj})$ is the Casimir contribution coming from the gauge vector field, $C(\psi_i)$ is the index of the irreducible representation that the i'th Weyl spinor transforms in and $C(\phi)$ refers to the index of the i'th real field. In a supersymmetric gauge theory the gauge vector superfield is composed of one vector, which is the same as the vector gauge field in regular Yang-Mills theories, and one Weyl spinor, both transforming in the adjoint representation. Each chiral fields contains one Weyl spinor and one complex (two real) field. Because $SU(N)$ has

$$C(\text{adj}) = C_2(\text{adj}) = N, \tag{2.41}$$

one gets

$$b_0 = 3N - \sum_{reps.} C(\phi_i). \tag{2.42}$$

for $SU(N)$.

The formula for the one loop contribution to the effective potential is

$$V_{\text{eff}}^{(1)} = \frac{1}{64\pi^2}\text{STr}M^4\log\frac{M^2}{\Lambda^2}, \tag{2.43}$$

where the supertrace, STr, is the same as ordinary trace but for a minus sign for fermionic degrees of freedom. This formula is called the Coleman-Weinberg potential [11]. Since a multiplet contains equally many bosonic and fermionic degrees of freedom we have $\text{Str}\mathbf{1} = 0$. Because supersymmetry might be spontaneously broken there is nothing that guarantees that the low energy fermionic and bosonic degrees of freedom are equally heavy. However, taking the supertrace of the mass matrices of the previous section (keeping in mind that the fermions have two degrees of freedom and that the vector particles have three) shows that even if there is no low energy supersymmetry we still have $\text{STr}\mathcal{M}^2 = 0$. Although

Equation (2.43) might require some work to derive, given that we must use STr instead of Tr, it is not surprising that it is not more divergent. This is because both possible terms, $\Lambda^4\mathrm{STr}\mathbf{1}$ and $\Lambda^2\mathrm{STr}\mathcal{M}^2$, vanish by the properties discussed above.

3

Spontaneous Supersymmetry Breaking

As we saw in the previous chapter, in theories with unbroken supersymmetry each fermionic/bosonic particle must have an equally heavy bosonic/fermionic partner. Looking at the nature we see that this is not the case. This means that if nature is supersymmetric, supersymmetry must be spontaneously broken.

In this chapter we start by discussing and giving examples of the connection between supersymmetry breaking and vanishing vacuum energy. We then discuss the relation between supersymmetry breaking and R-symmetry. We also give some examples to illustrate different types of supersymmetry breaking.

3.1 F- and D-flatness

When studying the low energy properties of a theory, it is useful to split the superfields into a vacuum part and a perturbative part: $\Phi = \Phi_{\text{vacuum}} + \delta\Phi$. For the low energy theory (which is written in terms of $\delta\Phi$) to have an unbroken supersymmetry, Φ and $\delta\Phi$ must have the same transformation properties. For this to be the case the vacuum part of the fields must be invariant under supersymmetry. It is therefore crucial to know how the vacuum transforms under supersymmetry transformations. A key observation is that by contracting the non trivial anticommutation relation for the supersymmetry generators with $\bar{\sigma}^0$ one gets

$$-\{Q_1, \bar{Q}_{\dot{1}}\} - \{Q_2, Q_{\dot{2}}\} = \{Q_\alpha, Q_{\dot{\beta}}\}\bar{\sigma}^{0\dot{\beta}\alpha} = 2\sigma^\mu_{\alpha\dot{\beta}} P_\mu \bar{\sigma}^{0\dot{\beta}\alpha} = -4P_0\,. \quad (3.1)$$

Since P_0 is the energy, we can write the Hamiltonian as

$$H = \frac{1}{4}\left(Q_1\bar{Q}_{\dot{1}} + \bar{Q}_{\dot{1}}Q_1 + Q_2\bar{Q}_{\dot{2}} + \bar{Q}_{\dot{2}}Q_2\right) . \tag{3.2}$$

The expectation value of the vacuum energy can thus be written as

$$\langle\Omega|H|\Omega\rangle = \frac{1}{4}\left[\left|Q_1\left|\Omega\right\rangle\right|^2 + \left|\bar{Q}_{\dot{1}}\left|\Omega\right\rangle\right|^2 + \left|Q_2\left|\Omega\right\rangle\right|^2 + \left|\bar{Q}_{\dot{2}}\left|\Omega\right\rangle\right|^2\right] \geq 0\,, \tag{3.3}$$

where $|\Omega\rangle$ is the vacuum state. This is greater or equal to zero, equality holding when $Q_1|\Omega\rangle = \bar{Q}_{\dot{1}}|\Omega\rangle = Q_2|\Omega\rangle = \bar{Q}_{\dot{2}}|\Omega\rangle = 0$. But this is also the condition for the vacuum state to be invariant under supersymmetry transformations. This means that a necessary and sufficient condition for the vacuum to be supersymmetric is for it to have vanishing energy. In terms of the Lagrangian (2.31), the vanishing of the vacuum energy can be written as $V_F + V_D = 0$ which implies

$$\begin{aligned} F_a =& -\frac{\partial\bar{W}}{\partial\bar{\phi}^a} = 0 \\ D_A =& -g^2\phi^\dagger T_A\phi = 0. \end{aligned} \tag{3.4}$$

These conditions are called F-flatness and D-flatness conditions respectively. Note that the first condition is complex whereas the second is real. The fact that F must be zero could also have been seen from (2.15) - in order for a chiral field to be invariant, ψ and F must be zero and ϕ must be constant. If one works out the transformation properties of the vector superfield one sees that the same is true for D. By looking at (2.36) and (2.38) one sees that the F-flatness condition is necessary to put the mass matrices on block diagonal form. If one also puts D to zero the mass matrix becomes supersymmetric.

Since (3.4) imposes $2d + D$ real constraints, where d is the dimension of the representation and D is the dimension of the group, on $2d$ real scalar fields, one might be led to think that most theories do not have a supersymmetric vacuum and that spontaneous supersymmetry breaking is elementary. However, since the superpotential must be invariant under the gauge transformations, Equation (2.21) has to be obeyed. Assume for simplicity that a generic point in field space completely breaks the gauge symmetry. Equation (2.21) then makes $2D$ of the real F-flatness conditions redundant. We therefore have $2d - D$ independent real constraints on $2d$ real scalar fields. Such equation systems generically have D dimensional solutions. However, because the gauge symmetry was completely broken, the D dimensions corresponds to gauge degrees of freedom. The degeneracy can thus be removed by making an appropriate choice of

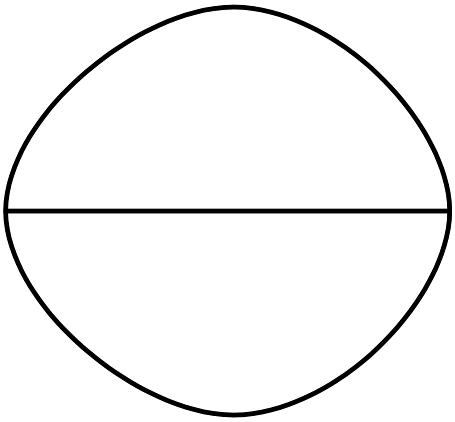

Figure 3.1: A typical two loop diagram. All vacuum diagrams with more than one loop must have one or more interactions.

gauge. To sum up, a generic superpotential has a supersymmetric vacuum.

One very interesting property of supersymmetric theories is that if the vacuum is supersymmetric at tree level, no higher loop corrections can break supersymmetry. A simple proof of this property is given in [12]. As we showed in the previous chapter, an unbroken supersymmetry guarantees that each bosonic degree of freedom has a corresponding equally heavy fermionic partner. The supertrace in the one loop effective potential (2.43) therefore vanishes. Higher loop corrections to the vacuum energy have one or more vertices and will be of the type in Figure 3.1, where the lines are some kind of propagators of the superfields. Even though we do not specify the exact form of the propagators, because of supersymmetry we know that the final expression will be some contraction of them and an integral $d^4xd^2\theta d^2\bar{\theta}$ over each vertex. If we integrate out all but one vertex we will have an expression on the form

$$\int d^4xd^2\theta d^2\bar{\theta} f(x,\theta,\bar{\theta}), \tag{3.5}$$

where $f(x,\theta,\bar{\theta})$ is some unknown function. Since superspace is homogenous and all points are equivalent, $f(x,\theta,\bar{\theta})$ can only be a constant. However, integrals of constant functions over Grassmann numbers vanish and the diagrams give no contribution to the vacuum energy.

If a vacuum has a non vanishing vacuum energy the argument above does not apply. If we have a classical manifold of vacua, some directions might correspond to Goldstone bosons. These are protected from perturbative corrections. The directions that are not protected by global symmetries are called pseudo-moduli. In general, peturbative corrections make these directions massive or tachyonic.

Supersymmetry breaking introduces a massless particle in a way analogous to the ordinary Goldstone bosons. Goldstone bosons are particles corresponding to the shift $\delta\phi$ introduced by a global symmetry transformation; $\phi \to \phi + \delta\phi$. Since the two states are equivalent and have the same energy the shift relating them must be massless. In a similar way, if we have a vacuum with $F \neq 0$ the supersymmetry transformation (2.15) will shift the fermion part of the chiral field by $\sqrt{2}\xi F$. Because the new state has the same energy as the original ones, the shift must correspond to a massless fermion. Since this massless fermion originates from a spontaneously broken global symmetry in a similar way as Goldstone bosons, it is called a Goldstino. It is possible to show that nonperturbative corrections can not give the Goldstino a mass. This might seem problematic since no such particle has been observed. However, it turns out that when supergravity is taken into account the Goldstino is eaten by the gravitino in a similar way as scalar fields are eaten by the vector bosons in the higgs mechanism.

3.1.1 Example: $W = c\Phi$

One of the simplest ways to break supersymmetry is to have an ungauged theory with the superpotential $W = c\Phi$. The equation of motion for the auxiliary field gives $F = -c^*$ which obviously can not be put to zero. The potential, $V_F = |c|^2$, is constant and the field is massless. $\langle X \rangle$ is therefore a pseudo-moduli.

3.1.2 Example: The Basic O'Raifeartaigh Model

O'Raifeartaigh models are a group of ungauged theories in which the F-flatness conditions are over-constrained and hence have no solution [13]. The basic O'Raifeartaigh model has the three superfields X, Φ_1 and Φ_2 and the superpotential

$$W = \frac{1}{2}hX\Phi_1^2 + \mu\Phi_1\Phi_2 + fX, \tag{3.6}$$

where h, μ and f can be chosen to be real numbers. The F-terms are then

$$-\bar{F}_X = \frac{1}{2}h\phi_1^2 + f\ ,\ -\bar{F}_{\Phi_1} = hX\phi_1 + \mu\phi_2 \text{ and } -\bar{F}_{\Phi_2} = \mu\phi_1\,. \tag{3.7}$$

Here F_X and F_{Φ_2} can not simultaneously be put to zero and supersymmetry is thus broken. F_{Φ_1} can however be put to zero for every ϕ_1. The solution is $\langle \phi_2 \rangle = -\frac{h}{\mu} \langle X\phi_1 \rangle$ with X arbitrary. To find the pseudo moduli we have to minimize

$$|F_X|^2 + |F_{\Phi_2}|^2 = f^2 + \frac{h^2}{4}|\phi_1|^4 + |\phi_1|^2 + \frac{hf}{2}(\phi_1^2 + \phi_1^{*2}) \, . \tag{3.8}$$

The only terms which can be negative are the last two. They are the most negative, compared to their absolute value, if ϕ_1 is purely imaginary. We therefore put $\phi_1 = i\phi'$, where ϕ' is real, and the potential becomes $f^2 + \frac{h^2}{4}\phi'^4 + (\mu^2 - hf)\phi'^2$. If $y \equiv \frac{hf}{\mu^2} < 1$ the potential has a minimum at $\phi_1 = 0$, which also puts ϕ_2 to zero. X is still arbitrary and spans the pseudo-moduli space. It is therefore massless at the classical level. As y grows and becomes larger than one, the potential is bent down on both sides of the real axis and two new vacua appears at $\phi_1 = \pm i\sqrt{\frac{2f}{h}(1 - 1/y)}$. We will concentrate on the case $y < 1$.

Since X is a pseudo moduli, quantum corrections will induce a mass for it. To calculate such a mass we need to know how the masses of the components in Φ_1 and Φ_2 depend on the vacuum expectation value $\langle X \rangle$. The mass matrix for the scalar bosons is

$$M_0^2 = \begin{pmatrix} \mu^2 + |hX|^2 & \mu hX & fh & 0 \\ \mu hX^* & \mu^2 & 0 & 0 \\ fh & 0 & \mu^2 + |hX|^2 & \mu hX^* \\ 0 & 0 & \mu hX & \mu^2 \end{pmatrix} \begin{matrix} \phi_1 \\ \phi_2 \\ \phi_1^* \\ \phi_2^* \end{matrix} \tag{3.9}$$

which has the eigenvalues

$$m_0^2 = \frac{1}{2}\left(\nu fh + h^2|X|^2 + 2\mu^2 \pm h\sqrt{f^2 + 2\nu fh|X|^2 + h^2|X|^2 + 4|X|^4\mu^2}\right), \tag{3.10}$$

where $\nu = \pm 1$. Similarly, the mass matrix for the fermions is

$$M_{1/2}^2 = \begin{pmatrix} \mu^2 + |hX|^2 & \mu hX \\ \mu hX & \mu^2 \end{pmatrix} \begin{matrix} \psi_{\Phi_1} \\ \psi_{\Phi_2} \end{matrix} \tag{3.11}$$

which has the eigenvalues

$$m_{1/2}^2 = \frac{1}{2}\left(h^2|X|^2 + 2\mu^2 \pm \sqrt{h^2|X|^2 + 4\mu^2}\right) \, . \tag{3.12}$$

Plugging the masses into the expression for the one loop effective potential (2.43) gives

$$V_{\text{eff}}^{(1)} = V_0 + M_X^2 X^2 \tag{3.13}$$

where

$$V_0 = \frac{\mu^2}{64\pi^2}\left(-2\log\frac{\mu^2}{\Lambda} + (1-y)^2\log\frac{1-y}{\Lambda} + (1+y)^2\log\frac{1+y}{\Lambda}\right) \tag{3.14}$$

and

$$M_X^2 = \frac{h^2\mu^2}{32\pi^2 y}\left(-2y - (1-y)^2\log(1-y) + (1+y)^2\log(1+y)\right). \quad (3.15)$$

The terms containing $\log\Lambda$ canceled in M_X^2. In V_0, the $\log\Lambda$ terms can be absorbed into a running coupling constant but we are not interested in the exact result. The interesting result is that M_X^2 is positive. For small y we have $M_X^2 \simeq \frac{h^2\mu^2}{48\pi^2}y^2$.

3.2 R Symmetry

When analyzing supersymmetry breaking it is important to consider a possible global $U(1)_R$ symmetry called 'R symmetry'. The formulation of manifestly supersymmetric Lagrangians using superspace suggests that the action could be invariant under rotations of the Grassmann coordinates in the complex plane: $\theta \to e^{-i\alpha}\theta$, $\bar\theta \to e^{i\alpha}\bar\theta$. For super fields, R symmetry transformations are made up by the transformations above combined with a chiral rotation:

$$\Phi(x^\mu, \theta^\alpha, \bar\theta^{\dot\alpha}) \to e^{in\alpha}\Phi(x^\mu, e^{-i\alpha}\theta^\alpha, e^{i\alpha}\bar\theta^{\dot\alpha}). \quad (3.16)$$

The kinetic terms for the chiral fields (2.25) are invariant under these transformations for any n if the vector gauge fields have $n = 0$. In terms of the components this means that

$$\begin{array}{c} \phi \to e^{in\alpha}\phi \\ \psi \to e^{i(n-1)\alpha}\psi \\ F \to e^{i(n-2)\alpha}F \end{array} \quad \text{and} \quad \begin{array}{c} D^A \to D^A \\ \lambda^A \to e^{i\alpha}\lambda^A \\ v^A \to v^A \end{array}. \quad (3.17)$$

The potential term for the chiral fields comes from

$$\mathcal{L}_{\text{potential}} = \int d^4x d^2\theta W + c.c. = \left(\text{coefficient of } \theta\theta \text{ in } \int d^4x W\right). \quad (3.18)$$

For this to be invariant, W must transform as $W \to e^{2i\alpha}W$, i.e. have $n = 2$. The kinetic term of the vector gauge fields is also a integral of chiral fields, this time $\mathcal{W}_\alpha$. A short calculation shows that $\mathcal{W}_\alpha$ has $n = 1$. This means that the integrand in (2.27) has $n = 2$ and therefore is invariant. The Lagrangian thus has R-symmetry if and only if one can assign R charges to each irreducible representation such that the superpotential has charge 2 and the symmetry is anomaly free.

As discussed in the beginning of this chapter, after choice of gauge, the F- and D- flatness conditions put $2d - D$ constraints on the $2d - D$

real scalar fields in the theory. Let us for simplicity consider an ungauged theory. For a generic superpotential W this equation system should have a solution and therefore a supersymmetric vacuum. To break supersymmetry some degrees of freedom must be over-constrained. Is there an easy way to get an indication of whether this is the case? A regular global $U(1)$ symmetry makes it possible to write the superpotential in terms of invariants:

$$W = f(\Phi_1^{n_2}/\Phi_2^{n_1}, \dots, \Phi_1^{n_d}/\Phi_d^{n_1}). \tag{3.19}$$

Here the number of complex unknowns is reduced to $d-1$. However, since the number of independent complex equations also is reduced to $d-1$ we still generically do not have any degree of freedom over constrained. This global U(1) does not do the job.

If we have a spontaneously broken R-symmetry the theory can be formulated with a field Φ_1 with $n_1 \neq 0$ and $\langle \phi_1 \rangle \neq 0$ and superpotential

$$W = \Phi_1^{2/n_1} f\left(\frac{\Phi_2}{\Phi_1^{n_2/n_1}}, \dots, \frac{\Phi_d}{\Phi_1^{n_d/n_1}}\right), \tag{3.20}$$

see [14]. Solving the F-flatness conditions puts d conditions on the $d-1$ unknowns in f and we have in general an over constrained system. This suggests that if there is a spontaneously broken R-symmetry the supersymmetry is also spontaneously broken. This is not a universal rule but more of a rule of thumb.

One example of when the rule might not work is when W contains a factor Φ_i^p, $p > 1$. In this case just having $\Phi_i = 0$ puts all F terms to zero without constraining the other fields. If some charged unconstrained field then gets a vacuum expectation value the R-symmetry is spontaneously broken but supersymmetry is not. This is the case for the superpotential $W = \Phi_1\Phi_2^2 + \Phi_2^3$. Here the F-terms put ϕ_2 to zero. Since ϕ_1 can take any value and since it has R charge $+2/3$, R-symmetry is generically broken.

As we have seen above, if a theory has a R-symmetry the gauginos have charge $+1$. The only way a gaugino can get a mass without breaking gauge invariance is trough a Majorana mass. Such a term has R-charge 2 and is thus forbidden by R-symmetry. If the gauge symmetry is broken, the gaugino mixes with a quark and gets a Dirac mass. However, since we have low energy unbroken gauge symmetries all gauginos can not get a mass in this way and the situation looks problematic. One solution is of course that the R-symmetry is spontaneously broken. This is however also problematic since we then get a massless Golstone boson, referred to as a R-axion, from the symmetry breaking. Although all this might be solvable, it indicates how problematic it is to find a phenomenologically acceptable model of supersymmetry breaking.

3.2.1 Example: $W = c\Phi$ continued

As we saw in the first example in this chapter, in a theory with the superpotential $W = c\Phi$ the supersymmetry is spontaneously broken. By giving Φ the charge 2 we see that this theory indeed has a R symmetry that is spontaneously broken at $\langle\phi\rangle \neq 0$ and behaves as expected. One way to break the R symmetry explicitly is to add a term $\frac{1}{2}\epsilon\Phi^2$ to the superpotential. The F flatness condition is then $c + \epsilon\phi = 0$, which has a zero at $\phi = -c/\epsilon$. This means that as ϵ increases and R symmetry is more and more broken, a supersymmetric vacuum approaches from infinity.

3.2.2 Example: The Basic O'Raifeartaigh Model cont.

The superpotential in (3.6) respects R symmetry if $n_X = 2$, $n_{\Phi_1} = 0$ and $n_{\Phi_2} = 2$. Since the model spontaneously breaks supersymmetry this is just what we expect. The R symmetry is explicitly broken if we add a term proportional to Φ_2^2 to the superpotential;

$$W = \frac{1}{2}hX\Phi_1^2 + \mu\Phi_1\Phi_2 + fX + \frac{1}{2}\epsilon\mu\Phi_2^2 \,. \tag{3.21}$$

The F terms

$$-\bar{F}_X = \frac{1}{2}h\phi_1^2 + f \ , \ -\bar{F}_{\phi_1} = hX\phi_1 + \mu\phi_2 \text{ and } -\bar{F}_{\phi_2} = \mu\phi_1 + \epsilon\mu\phi_2 \tag{3.22}$$

can now all simultaneously be put to zero and we have a supersymmetric vacuum at

$$\phi_1 = \pm\sqrt{-\frac{2f}{h}} \ , \ \phi_2 = \mp\frac{1}{\epsilon}\sqrt{-\frac{2f}{h}} \text{ and } X = \frac{\mu}{h\epsilon}. \tag{3.23}$$

In the same way as in the previous example, as ϵ grows, a vacuum approaches from infinity. Once again, the connection between supersymmetry breaking and R symmetry holds. If $y < 1$, $\epsilon \ll 1$ and we study the potential close to the origin (in particular, we take ϕ_2 to be small) we do not expect the new term to give a big contribution to the physics of the old pseudo moduli. Calculating the masses as a function of the vacuum expectation value of X gives the boson masses

$$m_0^2 = \frac{1}{2}\Big(\nu fh + h^2|X|^2 + (2+\epsilon^2)\mu^2 \\ \pm\sqrt{h^2(h|X|^2+\nu f)^2 + 4|h\mu X + \epsilon\mu^2|^2 + \epsilon^2\mu^2(\epsilon^2\mu^2 - 2h^2|X|^2 - 2\nu hf)}\Big) \,, \tag{3.24}$$

where $\nu = \pm 1$, and fermion masses

$$m_{1/2}^2 = \frac{1}{2}\left(h^2|X|^2 + (2+\epsilon^2)\mu^2 \right. \\ \left. \pm\sqrt{h^4|X|^4 + 4\left|h\mu X + \epsilon\mu^2\right|^2 + \epsilon^2\mu^2\left(\mu^2\epsilon^2 - 2h^2|X|^2\right)}\right) . \tag{3.25}$$

Putting these masses into (2.43) and Taylor expanding gives

$$V_{\text{eff}}^{(1)} = V_0(\epsilon) + V_1\epsilon \text{Re}X + M_X^2|X|^2 + O(X^{4-n}\epsilon^n) = V_0'(\epsilon) \\ + M_X^2\left|X + \frac{V_1}{2M_X^2}\epsilon\right|^2 + O(X^{4-n}\epsilon^n)\,, \tag{3.26}$$

where M_X^2 is the same as for the supersymmetric case,

$$V_1 = \frac{\mu^3 h}{16\pi^2 y}\left(-2y + (1-y)\log(1-y) + (1+y)\log(1+y)\right), \tag{3.27}$$

and $V_0(\epsilon)$ and $V_0'(\epsilon)$ are some non zero second order polynomials that we won't specify. Although the position of the minima is shifted as ϵ grows, the mass, and hence the stability, of the X field remains unchanged to lowest orders.

This is very interesting. It shows that although the supersymmetric vacua always are the stable points with the lowest energy it is sometimes possible to construct a theory that also has a metastable supersymmetry breaking vacuum. In order for this to be an acceptable way of breaking supersymmetry the lifetime of the metastable vacua has to be parametrically long. This way of breaking supersymmetry sidesteps the limitations R-symmetry and Witten index impose on model building.

4

Supersymmetric QCD and the Seiberg Duality

Supersymmetric QCD is a very interesting group of theories. It is a supersymmetric generalization of usual QCD and is, for sufficiently small numbers of flavors, asymptotically free. One fascinating property is that the strongly coupled low energy theory can be described in a simple and explicitly supersymmetric fashion.

We begin the chapter with some general remarks and then systematically work our way trough increasing number of flavors, starting with zero. For each numbers of flavors we introduce a low energy theory which we test using different methods, such as decoupling, weakly coupling limits and anomaly matching.

4.1 Supersymmetric QCD

SQCD has, not surprisingly, $SU(N_C)$ as its gauge group. The vector gauge fields of ordinary QCD have in SQCD been promoted to vector superfields, still transforming in the adjoint representation of the gauge group [15]. In a similar way, the N_F left and right handed quarks of ordinary QCD are promoted to chiral superfields, Q and $\tilde{Q}$, transforming in the fundamental and anti fundamental representations. In terms of superfields, the lagrangian can be written as

$$\mathcal{L} = \frac{-i}{16\pi}\int d^2\theta\tau\mathcal{W}^{\alpha a}\mathcal{W}^a_\alpha + c.c. + \int d^2\theta d^2\bar{\theta}\mathrm{Tr}\left[Q^\dagger e^{iT_aV^a}Q\right] + \int d^2\theta d^2\bar{\theta}\mathrm{Tr}\left[\tilde{Q}^\dagger e^{i\tilde{T}_aV^a}\tilde{Q}\right] + \int d^2\theta W(Q,\tilde{Q}) + c.c. \quad (4.1)$$

where $\tilde{T} = -T^*$. If $W = 0$ this theory has the symmetries

	$SU(N_C)$	$SU(N_F)_L$	$SU(N_F)_R$	$U(1)_B$	$U(1)_R$	$U(1)_A$
Q	$\square$	$\square$	1	1	$1-\frac{N_C}{N_F}$	1
$\tilde{Q}$	$\bar{\square}$	1	$\bar{\square}$	-1	$1-\frac{N_C}{N_F}$	1

, (4.2)

where the R-charges have been chosen such that $U(1)_R$ is anomaly free. The R-charges above give ψ_Q and $\psi_{\tilde{Q}}$ R-charge $-N_C/N_F$ and since $C(\square) = C(\bar{\square}) = \frac{1}{2}$ the anomaly coefficient coming from the quarks is

$$(2N_F)(2C(\square))\left(-\frac{N_C}{N_F}\right) = -2N_C. \tag{4.3}$$

This exactly cancels the $2C(Adj) = 2N_C$ coming from the gauginos. In the table above, $U(1)_A$ is the only anomalous symmetry.

Using (2.42) and the values of the Casimir operators above gives $b_0 = 3N_C - N_F$. The theory is thus asymptotically free for $N_F < 3N_C$. If N_F is between $\frac{3}{2}N_C$ and $3N_C$, it turns out that, although the theory is weakly coupled at high energies, the theory does not get strongly coupled at low but instead approaches a fixed point [16, 17]. If N_F is less than or equal to $\frac{3}{2}N_C$ the theory gets strongly coupled at low energies. We might thus expect the theory to behave in a similar confining way as QCD. This will turn out to be true in some cases. It has also been shown that massive SQCD has Witten index N_C [18]. This means that the low energy theory has supersymmetric vacua and it will therefore be possible to describe it in a supersymmetric way. In particular, since the effective superpotential has to be holomorphic, it will be practically available to computations.

4.2 $N_F = 0$

If the theory does not have any matter fields ($N_F = 0$) it is called pure super Yang-Mills(SYM). The Lagrangian is then especially simple:

$$\mathcal{L} = -\frac{1}{4g^2}\,(F^a_{\mu\nu})^2 - \frac{1}{g^2}\,\bar{\lambda}^a i\not{D}\lambda^a + \frac{\theta}{32\pi^2}\,F^a_{\mu\nu}\tilde{F}^{a\mu\nu}. \tag{4.4}$$

This is very similar to ordinary QCD. The only difference is that the fermions transform in the adjoint representation instead of the fundamental and anti fundamental. For this theory, $b_0 = 3N_C$ and the coupling constant runs as

$$e^{2\pi i\tau_{\text{eff}}} = e^{-8\pi^2/g^2(E)+i\theta} = \left(\frac{\Lambda}{E}\right)^{3N_C}. \tag{4.5}$$

What is then the effective superpotential when the gauge fields and gauginos have been integrated out? The only thing the low energy superpotential can depend on is the effective coupling constant τ_{eff} at some energy M (or equivalently Λ). In many applications this coupling constant will depend on the vacuum expectation value of some chiral superfield Φ. It might thus be appropriate to also look at τ_{eff} as a chiral superfield. This is the context in which we later will use these calculations. Since the low energy potential depends on a chiral superfield and since we are interested in the physics around a supersymmetric vacuum, the effective potential can be written (off shell) as

$$\Gamma = \int d^2\theta W_{eff}(\tau_{eff}) + c.c. \,, \tag{4.6}$$

where W_{eff} is a holomorphic function of τ_{eff}. Our goal is thus to find W_{eff}.

Because the theory have supersymmetric vacua we do not expect the theory to have R-symmetry. However, we will be able to use a modification of the R-symmetry to find the W_{eff}. Since there are no quarks in the theory we can not use their charges to make the R-symmetry anomaly free. Because λ^a has charge 1 the anomal R-current is $\partial_\mu J^\mu = \frac{C(Adj)}{16\pi^2} F^a_{\mu\nu} F^{a\mu\nu} = \frac{N_C}{16\pi^2} F^a_{\mu\nu} F^{a\mu\nu}$. This means that if $\lambda \to \lambda e^{i\alpha}$ we have

$$\mathcal{L} \to \mathcal{L} - \alpha\partial_\mu J^\mu \to \mathcal{L} - 2N_C\alpha \frac{1}{32\pi^2} F^a_{\mu\nu}\tilde{F}^{a\mu\nu}. \tag{4.7}$$

When we calculate

$$e^{i\Gamma(\tau)} = \int \mathcal{D}v^a \mathcal{D}\lambda^a e^{i\int d^4x \mathcal{L}(\tau)}, \tag{4.8}$$

the instanton sector with $\int d^4x F^a_{\mu\nu}\tilde{F}^{\mu\nu}_a \neq 0$ will therefore not be invariant under R transformations. However, if we combine the anomalous R-symmetry transformations with the shift $\theta \to \theta + 2N_C\alpha$, or equivalently $\tau \to \tau + \frac{2N_C}{2\pi}\alpha$, we find a symmetry of the effective theory. By using (4.6) one sees that W_{eff} must have charge 2 with respect to this modified R-symmetry.

Equation (4.5) makes it possible to write the modified R-symmetry as $\Lambda(\tau_{eff}, M)^{3N_C} \to \Lambda(\tau_{eff}, M)^{3N_C} e^{i2N_C\alpha}$. Since the effective potential can only depend on τ_{eff} and M, or equivalently Λ, the only possible superpotential with R-charge 2 and mass dimension 3 is

$$W_{eff} = aM^3 e^{2\pi i\tau/N_C} = a\Lambda(\tau_{\text{eff}}, M)^3, \tag{4.9}$$

where a is some number. Using instanton calculations it has been shown that a is N_C [19].

Does SQCD have a gaugino condensate in a similar ways as the quarks condense in QCD? This can be answered by calculating $\langle\lambda^{\alpha a}\lambda^a_\alpha\rangle$. A key observation is that the lower component of $\mathcal{W}^{\alpha a}$ in the WZ gauge is $-i\lambda^{\alpha a}$. This means that the scalar part of $\langle -\mathcal{W}^{\alpha a}\mathcal{W}_{\alpha a}\rangle$ is identical to the vacuum expectation value of the fermion bilinear we are looking for. Treating the coupling constant of the microscopic theory, τ, as a chiral superfield makes it possible to get the vacuum expectation value of the gaugino bilinear by a simple derivation:

$$\langle\lambda^{\alpha a}\lambda^a_\alpha\rangle = -16\pi\frac{\partial}{\partial F_\tau}\log Z \text{ were } Z = \int \mathcal{D}v^a\mathcal{D}\lambda^a e^{i\int L}. \tag{4.10}$$

When we calculated the effective action we obtained it as a function of τ_{eff} and not of τ. However, using Equation (4.5) it is easy to show that the two coupling constants are related to each other by an additive constant only depending on energy difference. Now using

$$\log Z = i\Gamma = i\int d^2\theta W_{eff}(\tau_{eff}) + c.c. \tag{4.11}$$

the gaugino condensate can be calculated in terms of the effective superpotential:

$$\begin{aligned}\langle\lambda^{\alpha a}\lambda^a_\alpha\rangle &= -16\pi i\,\frac{\partial F_{\tau_{\text{eff}}}}{\partial F_\tau}\frac{\partial}{\partial F_{\tau_{\text{eff}}}}\int d^2\theta W_{\text{eff}}(\tau_{\text{eff}})\\ &= -16\pi i\,\frac{\partial}{\partial F_{\tau_{\text{eff}}}}\left[F_{\tau_{\text{eff}}}\frac{\partial W_{\text{eff}}}{\partial\tau_{\text{eff}}} + \psi^2_{\tau_{\text{eff}}}\frac{\partial^2 W_{\text{eff}}}{\partial\tau^2_{\text{eff}}}\right]\\ -&= 16\pi i\,\frac{\partial W_{\text{eff}}}{\partial\tau_{\text{eff}}} = 32\pi^2\Lambda^3\,.\end{aligned} \tag{4.12}$$

The gauginos therefore condensate in similar fashion as the quarks in QCD.

If α is equal to a multiple of $2\pi/2N_C$ the anomalous R-symmetry induces a 2π shift in θ. This has no effect on a Yang-Mills theory and hence, for these α, the R-symmetry is unbroken. This means that a Z_{2N_C} subgroup of $U(1)_R$ must survive in the effective superpotential. By looking at (4.9) we see that this is the case. Is this symmetry spontaneously broken when the gauginos get a vacuum expectation value? Under the Z_{2N_C} subgroup of the anomalous R-symmetry, the gaugino bilinear transforms as $\lambda^{a\alpha}\lambda^a_\alpha \to \lambda^{a\alpha}\lambda^a_\alpha e^{i2\pi m/N_C}$, where m is an integer. Z_{2N_C} is thus spontaneously broken down to Z_2 and we have N_C different vacua, just as the Witten index indicated.

One application of the effective superpotential above is in the study of pure SYM as a low energy limit of massive SQCD with $N_F \neq 0$ and energy scale Λ. This can be done by adding a mass term to the chiral fields:

$$W_{\text{tree}} = \text{Tr}m^i_j M^j_i = \text{Tr}m^i_j Q^c_i \tilde{Q}^j_c, \tag{4.13}$$

where we have defined M^i_j to be the gauge invariant one gets when the gauge indices of Q and $\tilde{Q}$ are contracted. For generic m this gives a mass to all fields in the chiral multiplets. If the mass scale is much larger than Λ, the massive quarks decouple when the theory is still weakly coupled. If we integrate them out we are thus left with the pure SYM theory we studied above. We call the energy scale of this theory Λ_L. Because the coupling constants of the high and low energy theory must be identical at decoupling, Λ and Λ_L can be related. If all masses are approximately the same, decoupling happens at energy $(\text{det}m)^{1/N_F}$. Since the high energy theory has $b_0 = 3N_C - N_F$ and the low energy theory has $b_0 = 3N_C$ the matching condition is

$$\left(\frac{\Lambda_L}{(\text{det}m)^{1/N_F}}\right)^{3N_C} = \left(\frac{\Lambda}{(\text{det}m)^{1/N_F}}\right)^{3N_C - N_F}, \tag{4.14}$$

which can be simplified to $\Lambda_L^{3N_C} = \Lambda^{3N_C - N_F}\text{det}m$. Expressing the effective superpotential of the low energy theory in terms of Λ and m gives

$$W_{eff} = N_C \left(\Lambda^{3N_C - N_F}\text{det}m\right)^{1/N_C}. \tag{4.15}$$

In a similar way as derivation of W_{eff} with respect to τ_{eff} gave $\langle -\lambda^{\alpha a}\lambda^a_\alpha\rangle$, derivation with respect to m^i_j gives $\left\langle M^j_i \right\rangle = \left\langle Q_i \cdot \tilde{Q}^j \right\rangle$. Working this out, see [20], gives

$$\langle M \rangle = \left(\Lambda^{3N_C - N_F}\text{det}m\right)^{1/N_C} m^{-1}. \tag{4.16}$$

4.3 $0 < N_F < N_C$

One useful property of SYM is that the solution to the D-flatness conditions can be parameterized by the gauge invariant polynomials of the theory [21]. Hence, at the classical level, the moduli space of the theory can be parameterized by M_{ij} which contains N_F^2 complex degrees of freedom. Classically we therefore have N_F^2 massless bosonic fields.

Another way to see this is to note that by using gauge and flavor transformations it is always possible to put Q into the form

$$\langle Q_{ik} \rangle = \begin{pmatrix} a_1 & & & & 0 & \dots \\ & a_2 & & & 0 & \\ & & \dots & & & \\ & & & a_{N_F-1} & 0 & \\ & & & & a_{N_F} \; 0 & \dots \end{pmatrix}, \tag{4.17}$$

where k denotes gauge index. Inserting this into the D-flatness condition

$$D^a = \text{Tr}\left(QT^aQ^\dagger - \tilde{Q}^*T^a\tilde{Q}^T\right) = 0 \tag{4.18}$$

gives $\langle Q\rangle = \pm\left\langle \tilde{Q}^*\right\rangle$. The general vacuum (up to gauge transformations) can then be constructed by choosing N_F complex scalars a_i and acting on the vacuum with two flavor $SU(N_F)$ transformations. However, the flavor rotations corresponding to diagonal generators T_A generate transformations that are equivalent to choosing new a_i and therefore do not introduce any new degrees of freedom. This leaves us with a subgroup generated by $2N_F(N_F-1)$ generators. For generic diagonal elements, gauge transformations ca not cancel the off diagonal flavor transformations. The vacua therefore have $N_F + \frac{1}{2}(2N_F(N_F-1)) = N_F^2$ complex dimensions.

A third way to reach this result is to note that if the D^a are zero, the only effect V_D has on the mass matrix is the $\mathcal{A}$ and $\mathcal{B}$ terms in (2.36). These terms only affect bosons corresponding to broken gauge symmetries. As we will see later, this part of the mass matrix splits the affected complex scalar bosons into one massless part and one massive. The massless part is eaten in the higgs mechanism. Each broken gauge symmetry therefore reduces the number of complex massless bosons by one. By looking at (4.17) one sees that a generic vacuum breaks $SU(N_C) \to SU(N_C - N_F)$. Since the number of broken symmetries is just the difference between the two groups, $(N_C^2-1)-((N_C-N_F)^2-1) = 2N_FN_C - N_F^2$, and since we originally had $2N_CN_F$ massless complex fields, we end up with our good old N_F^2.

What do quantum effects do to the classical moduli? For $N_F < N_C$ it turns out that the low energy theory can be described in terms of the gauge invariant M. This means that when the theory gets strongly coupled, the two quarks combine into a gauge invariant meson in a similar way as for QCD. If we are in a supersymmetric vacuum, the theory will be supersymmetric and we can describe the mesons in terms of chiral superfields. Although the strongly coupled behavior is extremely hard to calculate explicitly there is a number of ways to check if the picture is consistent. However, the first thing we have to do is to get an expression for the effective potential.

In the last section we assigned a charge to Λ to compensate for the anomaly introduced by the R-symmetry. In this case $U(1)_R$ is non anomalous. However, the $U(1)_A$ symmetry is anomalous. Using the usual technique to 'handle' it using Λ gives

	$SU(N_C)$	$SU(N_F)_L$	$SU(N_F)_R$	$U(1)_B$	$U(1)_R$	$U(1)_A$
$\Lambda^{3N_C-N_F}$	**1**	**1**	**1**	0	0	$2N_F$
M	**1**	$\square$	$\bar{\square}$	0	$2-\frac{2N_C}{N_F}$	2
m	**1**	$\bar{\square}$	$\square$	0	$\frac{2N_C}{N_F}$	-2

(4.19)

We also listed the transformation properties of M and the transformation properties a matrix m must have in order to make the tree level superpotential $W_{tree} = mM$, which we will use later, invariant. Since Λ and M are the only quantities that can appear in the effective superpotential, all we have to do is to find out what combinations of these that have charge 2 with respect to the modified $U(1)_R$ and are invariant with respect to the other symmetries. Because only M is R charged and because only the determinant is invariant under $SU(N_F)_L \times SU(N_F)_R$ we must have a factor $(\det M)^{1/(N_F-N_C)}$ in W_{eff}. After taking $U(1)_A$ into account the only possible effective superpotential is

$$W_{eff} = C_{N_F,N_C}\left(\frac{\Lambda^{(3N_C-N_F)}}{\det M}\right)^{1/(N_C-N_F)}. \tag{4.20}$$

This is the so called Affleck-Dine-Seiberg superpotential, see [19]. Because M has mass dimension 2, W_{eff} has mass dimension 3, exactly as it should.

The F-flatness conditions of this superpotential have no solutions. However, the potential energy goes to zero as $\langle M\rangle$ goes to infinity. The interpretation is that the theory does not have a stable ground state.

Is this superpotential sensible? One check is to give one of the flavors a big mass and integrate it out. In the original high energy theory it was clear that this would give us an SQCD theory with $N_F - 1$ flavors and an energy scale related to the original one in the usual way. In order for the low energy theory to be consistent it must behave in the same way. The N_F'th flavor becomes massive if we use the tree level superpotential

$$W_{\text{tree}} = mQ^c_{N_F}\tilde{Q}^{N_F}_c = mM^{N_F}_{N_F}. \tag{4.21}$$

Note that this mass term breaks the global symmetry down to

$$SU(N_F-1)_L \times SU(N_F-1)_R \times U(1)_B \times U(1)_R \times U(1)_A. \tag{4.22}$$

It is possible to show that the only effective low energy superpotential that is consistent when letting Λ and m go to zero in different ways is

$$W_{eff} = C_{N_C,N_F}\left(\frac{\Lambda^{(3N_C-N_F)}}{\det M}\right)^{1/(N_C-N_F)} + mM_{N_FN_F}. \tag{4.23}$$

As $Q^c_{N_F}$ and $\tilde{Q}^{N_F}_c$ become very heavy and we integrate them out, we have to solve the corresponding F-flatness conditions. Since $M^i_{N_F}$ and $M^{N_F}_i$ contain massive quarks, these composite fields become massive and can also be integrated out. Although the reason is not as obvious as in the high energy case, we still have to solve the F-flatness conditions corresponding to the degrees of freedom we integrate out. The F-terms for the meson fields are

$$-F^\dagger_{M^i_j} = -\frac{C_{N_C,N_F}}{N_C - N_F}\left(\frac{\Lambda^{(3N_C-N_F)}}{\det M}\right)^{1/(N_C-N_F)} M^{-1}{}^i_j + m\delta^i_{N_F}\delta^{N_F}_j \,. \quad (4.24)$$

Putting $F_{M^i_{N_F}}$ and $F_{M^{N_F}_i}$, $i < N_F$, to zero makes M^{-1}, and hence M, block diagonal. In particular this means that $M^{-1}{}^{N_F}_{N_F} = 1/M^{N_F}_{N_F}$ and that $\det M = M^{N_F}_{N_F}\det M'$, where M' is the block corresponding to the $N_F - 1$ first flavors. Solving $F_{M^{N_F}_{N_F}} = 0$ gives

$$M^{N_F}_{N_F} = \left(\frac{C_{N_C,N_F}}{m(N_C - N_F)}\left(\frac{\Lambda^{3N_C-N_F}}{\det M'}\right)^{\frac{1}{N_F-N_C}}\right)^{\frac{N_C-N_F}{N_C-N_F+1}} \quad (4.25)$$

which, when put back into (4.23), yields

$$W = C'\left(\frac{\Lambda'^{3N_C-N_F+1}}{\det M'}\right)^{\frac{1}{N_C-N_F+1}} \quad \text{where } \Lambda'^{3N_C-N_F+1} = m\Lambda^{3N_C-N_F}$$
$$\text{and } C' = \left(C^{N_C-N_F}_{N_C,N_F}(N_F - N_C)\right)^{\frac{1}{N_C-N_F+1}} + \left(\frac{C_{N_C,N_F}}{N_C - N_F}\right)^{\frac{N_C-N_F}{N_C-N_F+1}} \,. \quad (4.26)$$

This is the effective potential for $N_F - 1$ flavor. If $C_{N_C,N_F} = N_C - N_F$ we get $C' = N_C - N_F + 1$. Note that the new energy scale is consistent with coupling constant matching at energy m:

$$\left(\frac{\Lambda_{N_F}}{m}\right)^{3N_C-N_F} = \left(\frac{\Lambda_{N_F-1}}{m}\right)^{3N_C-N_F+1} \,. \quad (4.27)$$

This shows that the effective potential is consistent, at least when it comes to reducing the number of flavors.

For $N_F = N_C - 1$, giving all a_i in (4.17) a big vacuum expectation value breaks the $SU(N_C)$ gauge symmetry completely and the coupling constant does not become large for small energies. This makes it possible to calculate the effective superpotential exactly using instantons. Affleck, Dine and Seiberg did this and showed that $C_{N_C,N_F} = N_C - N_F$.

A last illustrative check is to make all quarks massive by using the tree level superpotential $W = m^i_j M^j_i$, where m is a non singular matrix. This term breaks the R-symmetry explicitly and we therefore expect to find a supersymmetric vacuum. The effective superpotential of this theory is $W_{eff} = (N_C - N_F)\left(\frac{\Lambda^{(3N_C-N_F)}}{\det M}\right)^{1/(N_C-N_F)} + m^j_i M^i_j$, which has the F-flatness conditions

$$-F^\dagger_{M^i_j} = -\left(\frac{\Lambda^{3N_C-N_F}}{\det M}\right)^{1/(N_C-N_F)} \left(M^{-1}\right)^j_i + m^j_i = 0\,. \tag{4.28}$$

These equations have the solution $M^i_j = \left(\det m \Lambda^{3N_C-N_F}\right)^{1/N_C} \left(m^{-1}\right)^i_j$, which exactly coincides with (4.16). This vacuum expectation value behaves in a similar way those in the last two examples of Chapter 3: as the R-symmetry breaking term m grows, a supersymmetric vacuum approaches from infinity.

4.4 $N_F = N_C$

To solve the D-flatness condition for $N_F \geq N_C$ it is convenient to use gauge and flavor symmetries to put Q on diagonal form and $\tilde{Q}$ on upper triangular form. The D-flatness conditions corresponding to off diagonal generators put the off diagonal part of $\tilde{Q}^T\tilde{Q}^*$ to zero. Since $\tilde{Q}$ is upper triangular this implies that $\tilde{Q}$ is diagonal. Solving the D-flatness conditions corresponding to diagonal generators gives us $Q^\dagger Q - \tilde{Q}^T\tilde{Q}^* = c\mathbf{1}$, for some constant c. The vacua can therefore be written as

$$Q = \begin{pmatrix} a_1 & & & \\ & a_2 & & \\ & & ... & \\ & & & a_{N_C} \\ & & & \end{pmatrix} \text{ and } \tilde{Q} = \begin{pmatrix} b_1 & & & \\ & b_2 & & \\ & & ... & \\ & & & b_{N_C} \\ & & & \end{pmatrix} \tag{4.29}$$

with $|a_i|^2 - |b_i|^2 = c$ for some constant c which is independent of i. Since the vacuum completely breaks the $SU(N_C)$ gauge symmetry, $N_C^2 - 1$ massless complex fields are eaten. This leaves us with $2N_FN_C - (N_C^2 - 1)$ complex dimensions. For $N_F = N_C$ the moduli space thus has $N_F^2 + 1$ complex dimensions. M is therefore not enough to parameterize the moduli and the theory must have some new gauge invariant. As it turns out, the theory does not have one new invariant but two:

$$\begin{aligned} B &= \frac{1}{n!}\epsilon_{i_1...i_{N_C}}\epsilon^{n_1...n_{N_C}} Q^{i_1}_{n_{N_C}} ... Q^{i_{N_C}}_{n_{N_C}} = \det Q \\ \tilde{B} &= \frac{1}{n!}\epsilon^{i_1...i_{N_C}}\epsilon_{n_1...n_{N_C}} \tilde{Q}^{n_{N_C}}_{i_1} ... \bar{Q}^{n_{N_C}}_{i_{N_C}} = \det\tilde{Q}\,, \end{aligned} \tag{4.30}$$

where the i's are gauge indices and the n's are flavor indices. These quantities are, however, not independent of M_n^m. By using the determinant product rule one gets

$$\det M = \det Q\tilde{Q} = \det Q \det\tilde{Q} = B\tilde{B}\,. \tag{4.31}$$

and the number of independent invariants matches the complex dimension of the moduli space. Can the low energy strongly coupled theory be described in terms of these gauge invariants in a similar way as in the previous case? What form can the effective superpotential have? The $U(1)_A$ symmetry is anomalous. This can be 'handled' in the usual way by giving Λ^{2N_C} charge $2N_C$. By looking at Equation (4.2) one sees that Q and $\tilde{Q}$ have charge zero under the anomaly free R symmetry. Since the symmetry is anomaly free, Λ is also uncharged. This means that it is impossible to construct a superpotential with charge 2 from the available invariants.

One way to check whether the quantum moduli space is identical to the classical one is to use the tree level superpotential $W_{tree} = mM$ and study the vacuum expectation values as m goes to zero. Using Equation (4.16) one sees that $\langle \det M \rangle = \Lambda^{2N_C}$ for $N_F = N_C$. By looking at Equation (4.19) we deuce that it's impossible to construct a non zero expression with B-charge N_C using Λ and m and we therefore have $\langle B \rangle = \left\langle \tilde{B} \right\rangle = 0$. This point in field space is not a part of the classical moduli space. There is however no symmetry preventing us from modifying Equation (4.31) to

$$\det M - B\tilde{B} = \Lambda^{2N_C}, \tag{4.32}$$

see [22].

If we give Q and $\tilde{Q}$ the vacuum expectation value $a_i = b_i = v$, the entire gauge symmetry is broken. The D-term potential $V_D = \sum (D^A)^2$ will then give a mass of order v^2 to the higgs bosons, their fermionic superpartners and the vector bosons. At energies below v we therefore have a theory consisting of the massless ungauged fields living on the moduli while we at energies above v have the original theory. Since the theory is asymptotically free, a big vacuum expectation value of Q and $\tilde{Q}$ forces the gauge part to decouple at high energy where it is weakly coupled. We should therefore not expect big nonperturbative quantum modifications to the moduli in this region. For theories with $N_F \geq N_C$ this gives us one new important consistency check: does the moduli space of the low energy effective theory reduce to the classical moduli space when the quarks are given big vacuum expectation values? For this to be the case, the classical algebraic relations between the gauge invariants must be imposed in this limit. Looking at (4.32) we see that when M,

B and $\tilde{B}$ are very large (compared to Λ) we effectively have the classical relation (4.31).

One way to impose the constraint (4.32) is to introduce a Lagrange multiplier in the superpotential

$$W = X\left(\det M - B\tilde{B} - \Lambda^{2N_C}\right). \tag{4.33}$$

This superpotential is not the real Wilsonian superpotential but only a way to enforce the constraint. Adding a mass term for Q^{N_F} and $\tilde{Q}_{N_F}$ yields

$$W = X\left(\det M - B\tilde{B} - \Lambda^{2N_C}\right) + mM^{N_F}_{N_F}. \tag{4.34}$$

Solving the F-flatness conditions for $M^i_{N_F}$ makes M block diagonal in the same way as for $N_F < N_C$ (we call the blocks M' and $t = M^{N_F}_{N_F}$). Solving the other F-term conditions gives

$$-F^\dagger_B = 0 \Rightarrow X\tilde{B} = 0 \qquad -F^\dagger_{M^{N_F}_{N_F}} = 0 \Rightarrow X = -\frac{m}{\det M'} \tag{4.35}$$

$$-F^\dagger_{\tilde{B}} = 0 \Rightarrow XB = 0 \quad -F^\dagger_X = 0 \Rightarrow \det M' t - B\tilde{B} - \Lambda^{2N_C} = 0. \tag{4.36}$$

Inserting these conditions into the effective superpotential gives

$$W = \frac{m\Lambda^{2N_C}}{\det M'}, \tag{4.37}$$

which is the Affleck-Dine-Seiberg superpotential for $N_F = N_C - 1$ if we use the usual relation to relate the energy scales.

The calculations above suggest that the low energy degrees of freedoms are the mesons, composite particles consisting of combination of a quark and a antiquark, and the baryons, composite particles consisting of an antisymmetric combination of quarks. Since the low energy theory is supersymmetric, each massless boson must have a massless fermion partner. One criteria that such fermionic low energy degrees of freedom must fulfill is the 't Hooft anomaly matching condition [23].

The condition can be stated in the following way. Assume that the low energy theory has the anomaly free global symmetry group G and that these symmetries have the associated currents J^a_μ, J^b_μ and J^c_μ. These currents are then contracted into a triangle diagram. The matching condition states that one must get the same anomaly coefficients when calculating these triangle diagrams in terms of the low energy composite fermions and when calculating them in terms of the high energy quarks. At most points on the moduli space the global symmetry group is broken down to $U(1)_R$ and the constraint is weaker. However, at certain points a bigger symmetry group survives and here the 't Hooft anomaly matching condition is stronger. One such point fulfilling the quantum moduli constraint is $M^i_j = \Lambda^2\delta^i_j$, $B = 0$, $\tilde{B} = 0$. The meson field breaks the flavor symmetry down to the diagonal subgroup:

$$\begin{aligned} &SU(N_F)_L \times SU(N_F)_R \times U(1)_R \times U(1)_B \\ &\qquad\qquad \to SU(N_F)_{diag} \times U(1)_R \times U(1)_B\,. \end{aligned} \tag{4.38}$$

One important result from representation theory is that $\square \otimes \bar{\square} = \mathrm{Adj} \oplus \mathbf{1}$ for $SU(N)$. Around this point, the meson field can be expanded as $M = \Lambda^2\mathbf{1} + \delta M = \Lambda^2\mathbf{1} + \delta M' + \frac{1}{N_F}\mathrm{Tr}\delta M\mathbf{1}$, where $\delta M'$ is the traceless part. $\mathrm{Tr}\delta M$ is the invariant in the decomposition above. Expanding $\det M$ to first order gives $\det M = \Lambda^{2N_F} + \Lambda^{2(N_F-1)}\mathrm{Tr}\delta M$. After inserting this into the quantum moduli constraint and solving for $\mathrm{Tr}\delta M$ we are left with the independent fields $\delta M'$, B and $\tilde{B}$. The relevant fermions for the anomaly matching thus transform as

	$SU(N_F)$	$U(1)_B$	$U(1)_R$
ψ_B	$\mathbf{1}$	N_F	-1
$\psi_{\tilde{B}}$	$\mathbf{1}$	$-N_F$	-1
$\psi_{\delta M'}$	Adj	0	-1
ψ_Q	$\square$	1	-1
$\psi_{\tilde{Q}}$	$\bar{\square}$	-1	-1
λ^A	$\mathbf{1}$	0	1

. (4.39)

The anomaly coefficients of the possible triangle diagrams are

	composite	elementary
$SU(N_F)^3$	$A(\mathrm{Adj}) = 0$	$A(\square) + A(\bar{\square}) = 0$
$SU(N_F)^2U(1)_R$	0	0
$SU(N_F)^2U(1)_B$	$-C(\mathrm{Adj}) = -N_F$	$N_C\left(C(\square) + C(\bar{\square})\right)(-1) = -N_F$
$SU(N_F)^1...$	0	0
$U(1)_B^2U(1)_R$	$-2N_F^2$	$-2N_F^2$
$U(1)_BU(1)_R^2$	0	0
$U(1)_B^3$	0	0
$U(1)_R^3$	$-(N_F^2+1)$	$-(N_F^2+1)$
$\mathrm{Tr}U(1)_B$	0	0
$\mathrm{Tr}U(1)_R$	$-(N_F^2+1)$	$-(N_F^2+1)$

, (4.40)

where $SU(N_F)^1...$ summarize all diagram containing only one $SU(N_F)$-current and where we in the two last rows have calculated the coefficients for the gravitational anomaly. As one can see, all coefficients match.

Another point with a large unbroken global symmetry is $M = 0$, $B = -\tilde{B} = \Lambda^{N_C}$. This vacuum expectation value induces the symmetry breaking

$$SU(N_F)_L \times SU(N_F)_R \times U(1)_R \times U(1)_B \\ \rightarrow SU(N_F)_L \times SU(N_F)_R \times U(1)_R. \quad (4.41)$$

By expanding B and $\tilde{B}$ and solving the moduli condition one can eliminate $\delta\tilde{B}$. The relevant fermions then transform as

	$SU(N_F)_L$	$SU(N_F)_R$	$U(1)_R$
$\psi_{\delta B}$	$\mathbf{1}$	$\mathbf{1}$	-1
ψ_M	$\square$	$\bar{\square}$	-1
ψ_Q	$\square$	$\mathbf{1}$	-1
$\psi_{\tilde{Q}}$	$\mathbf{1}$	$\bar{\square}$	-1
λ^A	$\mathbf{1}$	$\mathbf{1}$	1

(4.42)

The anomaly coefficients of the triangle diagrams are, up to $SU(N_F)_L \leftrightarrow SU(N_F)_R$,

	composite	elementary
$SU(N_F)^3$	$N_F A(\square)$	$N_F A(\square)$
$SU(N_F)^2 U(1)_R$	$-N_F/2$	$-N_F/2$
$SU(N_F)^1 ...$	0	0
$U(1)_R^3$	$-(N_F^2+1)$	$-(N_F^2+1)$
$\mathrm{Tr}U(1)_R$	$-(N_F^2+1)$	$-(N_F^2+1)$

(4.43)

Once again the anomalies match. All this strengthen our earlier assertion that the degrees of freedom of the low energy theory are M, B and $\tilde{B}$ subjected to $\det M - B\tilde{B} = \Lambda^{2N_C}$.

4.5 $N_F = N_C + 1$

For $N_F = N_C + 1$, the gauge invariants are

$$M^i_j = Q^c_j \tilde{Q}^i_c \;,\; B^i = \frac{1}{N_C!}\epsilon^{ij_1 \ldots j_{N_C}} \epsilon_{c_1 \ldots c_{N_C}} Q^{c_1}_{j_1} \ldots Q^{c_{N_C}}_{j_{N_C}} \\ \text{and } \tilde{B}_i = \frac{1}{N_C!}\epsilon_{ij_1 \ldots j_{N_C}} \epsilon^{c_1 \ldots c_{N_C}} \tilde{Q}^{j_1}_{c_1} \ldots \tilde{Q}^{j_{N_C}}_{c_{N_C}}, \quad (4.44)$$

where B^i transforms in $(\bar{\square}, \mathbf{1})$ and $\tilde{B}_j$ transforms in $(\mathbf{1}, \square)$ with respect to the flavor symmetries $SU(N_F)_L \times SU(N_F)_R$. Since $N_F > N_C$, we have that $B^i Q^c_i = B_j \tilde{Q}^j_c = 0$ by antisymmetry. This implies that the classical moduli must fulfill $M^i_j B^j = M^i_j \tilde{B}_i = 0$. One way to look at B^i and $\tilde{B}_i$ is as (up to a sign) the determinant of the square matrices one gets when eliminating the i'th flavor from Q and $\tilde{Q}$ respectively. By using the determinant product rule one sees that $B^i\tilde{B}_j$ is the determinant of $Q_k^c\tilde{Q}_c^l = (-1)^{l+k}M_k^l$ with the i'th row and the j'th column eliminated. This is also known as the minor of M. Using that the inverse of a matrix can be written as $(M^{-1})_j^i = \frac{(\text{minor}M)_j^i}{\det M}$, the classical moduli space conditions can be summed up as

$$B^i\tilde{B}_j = \det M(M^{-1})_j^i \text{ and } M_j^iB^j = M_j^i\tilde{B}_i = 0. \tag{4.45}$$

Since the second constraint says that M has a null-vector, the determinant of M is zero. The mathematically well defined way to look at $\det M(M^{-1})_j^i$ is thus as the minor of M.

Using the result regarding the complex dimension of the moduli space from the previous section, we know that there must be N_F^2 independent complex gauge invariants. To fulfill the second constraint the determinant of M must be zero. This means that, only taking the second constraint into consideration, M has $N_F^2 - 1$ complex degrees of freedom. A generic matrix M with zero determinant only has one null vector (the same is true for M^T). Hence B and $\tilde{B}$ are determined by the second and third constraint up to one complex normalization constant each. If we only had the second and third condition we would therefore have $N_F^2 + 1$ independent complex invariants.

The first constraint is more tricky to impose. A generic matrix can be expanded in terms of its eigenvectors: $M = \sum_i \lambda_i v_i^T \hat{v}_i$, where λ_i are the eigenvalues, v_i are the eigenvectors and $\hat{v}_i$ are the covectors to the eigenvectors. In the same language, the inverse can be written as $M^{-1} = \sum_i \frac{1}{\lambda_i} v_i^T \hat{v}_i$. If λ_0 goes to zero, the corresponding term in this sum goes to infinity. This can be fixed by multiplying with the determinant. All terms corresponding to non zero eigenvalues then vanish and we have $\det M(M^{-1}) = \left(\prod_{i\neq 0}\lambda_i\right) v_0\hat{v}_0$. Since B and $\tilde{B}$ are the eigenvector and the coeigenvector for the zero eigenvalue, the only thing (4.45) does is to normalize $B\tilde{B}$. Hence, one complex degree of freedom is removed and there are N_F^2 independent gauge invariants. This exactly matches the complex dimension of the moduli space.

What about quantum modifications to the theory? By once again using (4.16) and $N_F = N_C + 1$ we see that $\langle \det M\, M^{-1}\rangle = \Lambda^{2N_C-1}m$, which vanishes as m goes to zero. Using the transformation properties in (4.19) gives $\langle B\rangle = \left\langle \tilde{B}\right\rangle = 0$. This point in field space belongs to the classical moduli space and we therefore do not expect any quantum modifications to the moduli space. It turns out, see [22], that the effective superpotential is

$$W_{eff} = \frac{1}{\Lambda^{2N_C-1}} \left(M^i_j B^j \tilde{B}_i - \mathrm{det} M \right). \tag{4.46}$$

This superpotential has R charge 2 and correct mass dimension. Note that the quotient between the two terms is totaly invariant and has mass dimension zero. This means that symmetry arguments alone are not sufficient to say that this is the only form the superpotential can have. Also note that F-flatness conditions are

$$-F^\dagger_{B^j} = \frac{1}{\Lambda^{2N_C-1}} M^i_j \tilde{B}_i = 0 \; - F^\dagger_{\tilde{B}_i} = \frac{1}{\Lambda^{2N_C-1}} M^i_j B^j = 0 \tag{4.47}$$

$$-F^\dagger_{M^i_j} = \frac{1}{\Lambda^{2N_C-1}} \left(B^j \tilde{B}_i - \mathrm{det} M \left(M^{-1}\right)^j_i \right) = 0, \tag{4.48}$$

which indeed give the expected moduli space. In the limit where M, B and $\tilde{B}$ are big, the quantum description must reduce to the weakly coupled theory with the classical algebraic relations between the fields. This turns out to be the case, since, for big vacuum expectation values (or equivalently small Λ) fields corresponding to non pseudo moduli directions becomes heavy and can be integrated out.

Another check of the quantum picture is to add a mass to the N_F'th flavor in the usual way. Combining the two F-flatness conditions

$$- F^\dagger_{\tilde{B}_j} = \frac{1}{\Lambda^{2N_C-1}} M^j_i B^i = 0, \; j < N_C \text{ and} \tag{4.49}$$

$$- F^\dagger_{M^{N_F}_i} = \frac{1}{\Lambda^{2N_C-1}} \left(B^i \tilde{B}_{N_F} - \mathrm{det} M \left(M^{-1}\right)^i_{N_F} \right) - m\delta^i_{N_F} = 0, \; i \leq N_F \tag{4.50}$$

gives $M^j_{N_F} = 0$ for $j < N_F$. By using $-F^\dagger_{B^i} = 0$ and $-F^\dagger_{M^i_{N_F}} = 0$ instead we get $M^{N_F}_i = 0$ for $i < N_F$. Once again the meson matrix is reduced to block diagonal form. In a similar way, a short calculation shows that $B^i = \tilde{B}_i = 0$ for $i < N_F$. The condition $-F^\dagger_{M^{N_F}_{N_F}} = 0$ is then

$$\frac{1}{\Lambda^{2N_C-1}} \left(B^{N_F} \tilde{B}_{N_F} - \mathrm{det} M' \right) - m = 0, \tag{4.51}$$

which, if the scale Λ decouples in the usual way, exactly is the quantum moduli constraint for $N_F = N_C$ (4.32). We see that none of the F-flatness conditions constrain $M^{N_F}_{N_F}$ and that it therefore in some sense still is free. Putting the results back into the effective action yields

$$W = \frac{M^{N_F}_{N_F}}{\Lambda^{2N_C-1}} \left(B^{N_F} \tilde{B}_{N_F} - \mathrm{det} M' + m\Lambda^{2N_C-1} \right), \tag{4.52}$$

which, if we treat $M^{N_F}_{N_F}$ as a Lagrange multiplier, is equivalent to (4.33).

In a similar way as for $N_F < N_C$, we can also check the effective theory by adding a mass to all quarks: $W_{tree} = m^i_j M^j_i$, where m^i_j is a non singular matrix. This gives the effective superpotential

$$W_{eff} = \frac{1}{\Lambda^{2N_C-1}} \left(M^i_j B^j \tilde{B}_i - \det M \right) + m^i_j M^j_i \,. \tag{4.53}$$

The F-flatness conditions for this superpotential are (4.47) and

$$- F^\dagger_{M^i_j} = \frac{1}{\Lambda^{2N_C-1}} \left(B^j \tilde{B}_i - \det M \left(M^{-1}\right)^j_i \right) + m^j_i = 0 \,. \tag{4.54}$$

Using that m is non singular one finds the solution $B^i = \tilde{B}_i = 0$ and $M^i_j = \left(\Lambda^{(2N_C-1)} \det m\right)^{1/(N_F-1)} \left(m^{-1}\right)^i_j$, which agrees with (4.16).

A last way to check if the low energy degrees of freedom and the effective potential make sense is trough 't Hooft anomaly matching. This time, the point $M = B = \tilde{B} = 0$ belongs to the quantum moduli space. The global symmetry is unbroken at this point. Note that this is an example of a theory displaying confinement without chiral symmetry breaking. Since the remaining global symmetry group is big, the matching condition is extra restrictive. At this point in field space, the first order expansion of (4.45) is zero and can not be used to eliminate any field. The relevant fermions thus transform as

	$SU(N_F)_L$	$SU(N_F)_R$	$U(1)_R$	$U(1)_B$
ψ_M	$\square$	$\bar{\square}$	$-1+2/N_F$	0
ψ_B	$\bar{\square}$	$\mathbf{1}$	$-1/N_F$	N_F-1
$\psi_{\tilde{B}}$	$\mathbf{1}$	$\square$	$-1/N_F$	$-N_F+1$
ψ_Q	$\square$	$\mathbf{1}$	$-1+1/N_F$	1
$\psi_{\tilde{Q}}$	$\mathbf{1}$	$\bar{\square}$	$-1+1/N_F$	-1
λ^A	$\mathbf{1}$	$\mathbf{1}$	1	0

. (4.55)

Up to the interchange $SU(N_F)_L \leftrightarrow SU(N_F)_R$, the anomaly coefficients of the triangle diagrams are

	composite	elementary
$(SU(N_F)_L)^3$	$N_F A(\square) + A(\bar{\square}) = N_C A(\square)$	$N_C A(\square)$
$(SU(N_F)_L)^2 U(1)_B$	$N_C C(\square)$	$N_C C(\square)$
$(SU(N_F)_L)^2 U(1)_R$	$(-1 + 1/N_F) C(\square)$	$(-1 + 1/N_F) C(\square)$
$SU(N_C)_L ...$	0	0
$(U(1)_R)^3$	$-2/N_F^2 - \frac{(N_F-2)^3}{N_F}$	$-2/N_F^2 - \frac{(N_F-2)^3}{N_F}$,
$(U(1)_R)^2 U(1)_B$	0	0
$U(1)_R (U(1)_B)^2$	$-2(N_F-1)^2$	$-2(N_F-1)^2$
$(U(1)_B)^3$	0	0
$\mathrm{tr} U(1)_R$	$-N_F^2 + 2N_F - 2$	$-N_F^2 + 2N_F - 2$
$\mathrm{tr} U(1)_B$	0	0

(4.56)

which match perfectly. If one is not convinced by the matching at this point, one can look at $B = \tilde{B} = 0$ and $M = \mathrm{diag}(0, ..., 0, 1, ..., 1)\Lambda^2$, where there are n non zero elements, $N_F - n > 2$. This breaks the symmetry to

$$\begin{aligned} &SU(N_F)_L \times SU(N_F)_R \times U(1)_R \times U(1)_B \\ &\to SU(N_F - n)_L \times SU(N_F - n)_R \times SU(n)_{\mathrm{diag}} \times U(1)_B, \end{aligned} \quad (4.57)$$

where $SU(N_F - n)$ affects the $N_F - n$ first elements in the flavor multiplet and $SU(n)_{\mathrm{diag}}$ is the diagonal subgroup affecting the last n. Expanding the (4.45) to first order yields $B^i = \tilde{B}_i = 0$ for $i \geq N_F - n$. Using the decompositions

$$\begin{aligned} M = \begin{pmatrix} M_1 & M_2 \\ M_3 & M_4 \end{pmatrix}, B = \begin{pmatrix} B_1 \\ 0 \end{pmatrix}, \tilde{B} = \begin{pmatrix} \tilde{B}_1 \\ 0 \end{pmatrix}, \\ Q = \begin{pmatrix} Q_1 \\ Q_2 \end{pmatrix} \text{ and } \tilde{Q} = \begin{pmatrix} \tilde{Q}_1 \\ \tilde{Q}_2 \end{pmatrix}, \end{aligned} \quad (4.58)$$

the sub-blocks transform as

	$SU(N_F-n)_L$	$SU(N_F-n)_R$	$SU(n)_{\text{diag}}$	$U(1)_B$
ψ_{M_1}	$\square$	$\bar{\square}$	$\mathbf{1}$	0
ψ_{M_2}	$\square$	$\mathbf{1}$	$\bar{\square}$	0
ψ_{M_3}	$\mathbf{1}$	$\bar{\square}$	$\square$	0
ψ_{M_4}	$\mathbf{1}$	$\mathbf{1}$	$\square\otimes\bar{\square}=\text{Adj}\oplus\mathbf{1}$	0
ψ_{B_1}	$\bar{\square}$	$\mathbf{1}$	$\mathbf{1}$	1
$\psi_{\tilde{B}_1}$	$\mathbf{1}$	$\square$	$\mathbf{1}$	-1
ψ_{Q_1}	$\square$	$\mathbf{1}$	$\mathbf{1}$	1
ψ_{Q_2}	$\mathbf{1}$	$\mathbf{1}$	$\square$	1
$\psi_{\tilde{Q}_1}$	$\mathbf{1}$	$\bar{\square}$	$\mathbf{1}$	-1
$\psi_{\tilde{Q}_2}$	$\mathbf{1}$	$\mathbf{1}$	$\bar{\square}$	-1

(4.59)

and the anomaly coefficients are

	composite	elementary
$(SU(N_F-n)_L)^3$	$(N_F-1)A(\square)$	$(N_F-1)A(\square)$
$(SU(N_F-n)_L)^2\,U(1)_B$	$(N_F-1)C(\bar{\square})$	$(N_F-1)C(\square)$
$(SU(n)_{\text{diag}})^3$	0	0
$(SU(n)_{\text{diag}})^2\,U(1)_B$	0	0
$(U(1)_B)^3$	0	0
$\text{Tr}U(1)_B$	0	0

. (4.60)

All this together strongly suggests that the low energy theory is described by the baryons B^i and B_j and the mesons M^i_j.

4.6 $N_F > N_C + 1$ and the Seiberg Duality

For $N_F > N_C + 1$ the picture is not, at least initially, as pretty as in the previous cases. The invariants are

$$M_i^j = Q_i^c\tilde{Q}_c^j\,,\, B^{j_1..j_n} = \frac{1}{N_C!}\epsilon^{j_1..j_n i_1..i_{N_C}}\epsilon_{c_1..c_{N_C}}Q_{i_1}^{c_1}..Q_{i_{N_C}}^{c_{N_C}} \text{ and}$$
$$\tilde{B}_{j_1..j_n} = \frac{1}{N_C!}\epsilon_{j_1..j_n i_1..i_{N_C}}\epsilon^{c_1..c_{N_C}}\tilde{Q}^{i_1}_{c_1}..\tilde{Q}^{i_{N_C}}_{c_{N_C}}\,, \qquad (4.61)$$

where $n = N_F - N_C$. B transforms in $\overline{\begin{array}{|c|}\hline \\ \hline \vdots \\ \hline \\ \hline\end{array}}$ of $SU(N_F)_L$ and $\tilde{B}$ transforms in $\begin{array}{|c|}\hline \\ \hline \vdots \\ \hline \\ \hline\end{array}$, of $SU(N_F)_R$. The classical moduli conditions are similar to the ones of the previous section:

$$\begin{aligned} &M^j_{i_1} B^{i_1 \dots i_n} = 0 \ , \ M^{i_1}_j \tilde{B}_{i_1 \dots i_n} = 0 \text{ and} \\ &\quad B^{i_1 \dots i_n} \tilde{B}_{j_1 \dots j_n} = \text{minor} M^{i_1 \dots i_n}_{j_1 \dots j_n} = \frac{\partial^n \det M}{\partial M^{i_1}_{j_1} \dots \partial M^{i_n}_{j_n}}, \end{aligned} \tag{4.62}$$

where the "minor" is (up to a sign) the determinant of M with the rows $i_1, ..., i_n$ and columns $j_1, ..., j_n$ removed. Two flavor invariants are $\frac{1}{\Lambda^{b_0}} \det M$ and $\frac{1}{\Lambda^{b_0}} B^{i_1 \dots i_n} M^{i_1}_{j_1} \dots M^{i_n}_{j_n} \tilde{B}_{j_1 \dots j_n}$. Both have R charge $2(N_F - N_C)$. Since the quotient between these is invariant under the entire symmetry group, symmetry arguments are not sufficient to determine the effective superpotential. In the spirit of the previous section one might guess that the effective superpotential is something similar to

$$W \sim \frac{1}{\Lambda^{b_0}} \left(\det M - \frac{1}{n} B^{i_1 \dots i_n} M^{i_1}_{j_1} \dots M^{i_n}_{j_n} \tilde{B}_{j_1 \dots j_n} \right). \tag{4.63}$$

However, one immediately realizes that this does not work. The above superpotential has the F-flatness conditions

$$\begin{aligned} -F^\dagger_{M^i_j} &= \frac{1}{\Lambda^{b_0}} \left(\frac{\partial \det M}{\partial M^i_j} - B^{j i_2 \dots i_n} M^{j_2}_{i_2} \dots M^{j_n}_{i_n} \tilde{B}_{i j_2 \dots j_n} \right) = 0 \\ -F^\dagger_{B^{i_1 \dots i_n}} &= \frac{1}{n \Lambda^{b_0}} M^{j_1}_{i_1} \dots M^{j_n}_{i_n} \tilde{B}_{j_1 \dots j_n} = 0 \\ -F^\dagger_{\tilde{B}_{j_1 \dots j_n}} &= \frac{1}{n \Lambda^{b_0}} B^{i_1 \dots i_n} M^{j_1}_{i_1} \dots M^{j_n}_{i_n} = 0. \end{aligned} \tag{4.64}$$

These conditions are not the same as the classical. This is not surprising since the last of the classical moduli conditions transforms in $\begin{array}{|c|}\hline \\ \hline \vdots \\ \hline \\ \hline \end{array} \times \begin{array}{|c|}\hline \\ \hline \vdots \\ \hline \\ \hline \end{array}$ and no F-terms, regardless of superpotential, transform in that representation. One might argue that a more cleaver choice of superpotential might give F-flatness conditions that are mathematically equivalent to (4.62), although not in the same representation.

By once again using (4.16) and (4.19) we see that $\langle M \rangle \sim \Lambda^{3 - \frac{N_F}{N_C}} m^{\frac{N_F}{N_C} - 1}$, which vanishes as m goes to zero, and $\langle B \rangle = \left\langle \tilde{B} \right\rangle = 0$. It is therefore reasonable to check if the anomalies match at this point in field space.

The relevant fermions transform as

	$SU(N_F)_L$	$SU(N_F)_R$	$U(1)_R$	$U(1)_B$
$\psi_{M^i_j}$	$\square$	$\bar{\square}$	$1-2\frac{N_C}{N_F}$	0
$\psi_{B^{i_1\ldots i_n}}$	$\overline{\begin{smallmatrix}\square\\ \vdots\\ \square\end{smallmatrix}}$	$\mathbf{1}$	$N_C-1-\frac{N_C^2}{N_F}$	N_C
$\psi_{\tilde{B}_{i_1\ldots i_n}}$	$\mathbf{1}$	$\begin{smallmatrix}\square\\ \vdots\\ \square\end{smallmatrix}$	$N_C-1-\frac{N_C^2}{N_F}$	$-N_C$
ψ_Q	$\square$	$\mathbf{1}$	$-\frac{N_C}{N_F}$	1
$\psi_{\tilde{Q}}$	$\mathbf{1}$	$\bar{\square}$	$-\frac{N_C}{N_F}$	-1
λ	$\mathbf{1}$	$\mathbf{1}$	1	0

. (4.65)

Using this we calculate the anomaly coefficients to be

	composite	elementary
$(SU(N_F)_L)^3$	$N_C a$	$N_C A(\square)$
$(SU(N_F)_L)^2\,U(1)_R$	$cR_{\psi_B}-N_F C(\square)R_{\psi_M}$	$N_C C(\square)R_{\psi_Q}$
$(SU(N_F)_L)^2\,U(1)_B$	cN_C	$N_C C(\square)$
$SU(N_F)_L\ldots$	0	0
$(U(1)_R)^3$	$2d_1 R^3_{B_\psi}+N_F^2 R^3_{\psi_M}$	$d_2-2N_F N_C R^3_{\psi_Q}$
$(U(1)_R)^2\,U(1)_B$	0	0
$U(1)_R\,(U(1)_B)^2$	$2d_1 R_{B_\psi}N_C^2$	$2N_F N_C R_{\psi_Q}$
$(U(1)_B)^3$	0	0
$\mathrm{Tr}U(1)_R$	$2d_1 R_{B_\psi}+N_F^2 R_{\psi_M}$	$d_2-2N_F N_C R_{\psi_Q}$
$\mathrm{Tr}U(1)_B$	0	0

. (4.66)

where

$$
\begin{aligned}
c &= C\left(\begin{smallmatrix}\square\\ \vdots\\ \square\end{smallmatrix}\right), & d_1 &= \dim\left(\begin{smallmatrix}\square\\ \vdots\\ \square\end{smallmatrix}\right) = \tfrac{N_F!}{n!(N_F-n)!},\\
d_2 &= \dim Adj = (N_C^2-1), & a &= A\left(\begin{smallmatrix}\square\\ \vdots\\ \square\end{smallmatrix}\right) = \tfrac{(N_F-3)!(N_F-2n)}{(N_F-n-1)!(n-1)},
\end{aligned}
\tag{4.67}
$$

$R_{B_\psi} = N_C - 1 - \frac{N_C^2}{N_F}$, $R_{\psi_Q} = \left(-\frac{N_C}{N_F}\right)$ and $R_{\psi_M} = \left(1-2\frac{N_F}{N_F}\right)$. The anomalies clearly does not match. Looking at the whole picture, the situation for $N_F > N_C+1$ is clearly not as simple as in previous sections.

In fact, the solution to the problem is something even more interesting - the so called Seiberg duality [16]. The idea is that the baryons $B^{i_1\ldots i_{N_F-N_C}}$ and $\tilde{B}_{i_1\ldots i_{N_F-N_C}}$ and the mesons M^i_j are the gauge invariants of quarks in a dual theory with gauge group $SU(N)$, $N = N_F - N_C$. The dual theory has N_F flavors of the quarks φ_c^i and $\tilde{\varphi}_i^c$, $N_F \times N_F$ uncharged fields ${M_m}_j^i$, the superpotential

$$W = \varphi_c^f {M_m}_f^h \tilde{\varphi}_h^c \tag{4.68}$$

and the characteristic scale $\tilde{\Lambda}$. The original theory is usually called the electric theory whereas the dual is called the magnetic theory. The gauge invariants of the electric and magnetic theory are related by

$$\epsilon_{c_1 \ldots c_{N_C}} \epsilon^{j_1 \ldots j_N i_1 \ldots i_{N_C}} Q_{i_1}^{c_1} \ldots Q_{i_{N_C}}^{c_{N_C}} = C \epsilon^{c_1 \ldots c_N} \varphi_{c_1}^{j_1} \ldots \varphi_{c_N}^{j_N}$$
$$\text{where } C = \sqrt{-(-\hat{\Lambda})^{N_C - N_F} \Lambda^{3N_C - N_F}}, \tag{4.69}$$

and similar for $\tilde{Q}$ and $\tilde{\varphi}$. We also have the relation ${M_m}_j^i = \frac{1}{\hat{\Lambda}} Q_j^c \tilde{Q}_c^i$ where $\hat{\Lambda}$ is some scale that fixes the dimension. The scales Λ, $\tilde{\Lambda}$ and $\hat{\Lambda}$ are related trough

$$\Lambda^{3N_C - N_F} \tilde{\Lambda}^{3(N_F - N_C) - N_F} = (-1)^{N_F - N_C} \hat{\Lambda}^{N_F}. \tag{4.70}$$

Let us consider the coupling constant of the electric theory. As we mentioned in the beginning of the chapter, SQCD is IR free for $N_F \geq 3N_C$. This means that as long as we are below the Landau pole at Λ the theory behaves nicely and can be described by perturbation theory. For $N_F < 3N_C$ the theory is asymptotically free. One might expect that the theory gets strongly coupled at energies close to Λ. This turns out to be true for $N_F \leq \frac{3}{2} N_C$. However, for $\frac{3}{2} N_C < N_F < 3N_C$ the two loop contribution to the running of the coupling constant is big enough to stop the running and stabilize the low energy coupling constant at a fixed point [16].

The magnetic theory behaves in an analog way but reversed: for $N_F \leq \frac{3}{2} N_C$ (or equivalently $N_F \geq 3N$) the theory is IR free, for $\frac{3}{2} N_C < N_F < 3N_C$ (or equivalently $\frac{3}{2} N < N_C < 3N$) the theory is asymptotically free with an IR fixed point and for $N_F \geq 3N_C$ (or equivalently $N_F \leq \frac{3}{2} N$) the theory is asymptotically free and strongly coupled for low energies. The fixed points for the electric and magnetic theory have been calculated and for the superpotential above they match, indicating that the magnetic theory indeed is a dual way of describing the same physics.

If the magnetic theory is IR-free, the Kähler potential is

$$K = \frac{1}{\beta} \left(\varphi^\dagger \varphi + \tilde{\varphi}^\dagger \tilde{\varphi} \right) + \frac{1}{\alpha |\Lambda|^2} M^\dagger M = \frac{1}{\beta} \left(\varphi^\dagger \varphi + \tilde{\varphi}^\dagger \tilde{\varphi} \right) + \frac{|\hat{\Lambda}|^2}{\alpha |\Lambda|^2} {M_m}^\dagger M_m, \tag{4.71}$$

where the Λ is needed for dimensional reasons and α and β are unknown numbers of order one. One property of the duality is that one is allowed to rescale φ and $\tilde{\varphi}$ if one changes $\tilde{\Lambda}$ and $\hat{\Lambda}$ in an appropriate way. This rescaling can be used to put $\beta = 1$. Alternatively it could have been used to put $C = 1$, but this is not the normalization we will use here. The α can not be put to one in this way. We can, however, write our theory in terms of the field $\Phi^i_j = \frac{|\hat{\Lambda}|}{\sqrt{\alpha}|\Lambda|} M_m$. Using this notation, the Kähler potential and superpotential are

$$K = \mathrm{Tr}\left(\varphi^\dagger \varphi + \tilde{\varphi}^\dagger \tilde{\varphi}\right) + \mathrm{Tr}\Phi^\dagger \Phi, \; W = h \varphi^i_c \Phi^j_i \tilde{\varphi}^c_i, \; h = \frac{\sqrt{\alpha}|\Lambda|}{|\hat{\Lambda}|}. \tag{4.72}$$

If Seiberg duality is to be an actual duality, performing it twice should give the original theory. We start with the usual theory consisting of Q, $\tilde{Q}$, $W = 0$, energy scale Λ and gauge group $SU(N_C)$. Seiberg duality then gives a theory consisting of the quarks φ and $\tilde{\varphi}$ transforming in the gauge group $SU(N_F - N_C)$, the neutral field $M_m = \frac{Q\tilde{Q}}{\hat{\Lambda}}$, the superpotential $W = \varphi M_m \tilde{\varphi}$ and the energy scale $\tilde{\Lambda}$. $\tilde{\Lambda}$ is related to Λ and $\hat{\Lambda}$ by (4.70) and we have put $\beta = 1$. Performing the duality transformation again gives a theory with the original gauge group $SU(N_C)$, the quarks d and $\tilde{d}$, uncharged fields M_m and $N = \frac{\varphi\tilde{\varphi}}{\hat{\Lambda}'}$, energy scale $\bar{\Lambda}$ and the superpotential

$$W = \mathrm{Tr}\varphi M_m \tilde{\varphi} + \mathrm{Tr} d N \tilde{d} = \mathrm{Tr} N(\hat{\Lambda} M_m + \tilde{d} d). \tag{4.73}$$

The new energy scale $\bar{\Lambda}$ can be put equal to Λ if we choose $\hat{\Lambda}' = -\hat{\Lambda}$. Choosing the energy scales in this way, it is not obvious that β' of the new theory is equal to one. The first term in the superpotential above gives a mass to the uncharged chiral fields N and M_m. Integrating them out and solving the F-flatness conditions gives

$$-F^\dagger_N = \alpha'(-\hat{\Lambda}' M_m + d\tilde{d}) = 0 \text{ and } -F^\dagger_{M_m} = -\alpha \hat{\Lambda}' N = 0, \tag{4.74}$$

where α and α' are the coefficients of the uncharged fields in the Kähler potential. The first condition tells us that the quarks d and $\tilde{d}$ are identical to Q and $\tilde{Q}$. By imposing the conditions (4.74), the superpotential is put to zero and we are left with the same theory we started with. Note that the minus sign in the energy scale relation (4.70) turned out to be necessary for the duality to work.

The magnetic theory has the global symmetries

	$SU(N_F)_L$	$SU(N_F)_R$	$U(1)_B$	$U(1)_R$	$U(1)_A$
φ	$\bar{\Box}$	$\mathbf{1}$	$\frac{N_C}{N}$	$1 - \frac{N}{N_F}$	1
$\tilde{\varphi}$	$\mathbf{1}$	$\Box$	$-\frac{N_C}{N}$	$1 - \frac{N}{N_F}$	1
M_m	$\Box$	$\bar{\Box}$	0	$2\frac{N}{N_F}$	-2
$\tilde{\Lambda}^{3N-N_F}$	$\mathbf{1}$	$\mathbf{1}$	0	0	$2N_F$

(4.75)

which is the same global symmetry group as in the electric theory. Here $U(1)_A$ is anomalous and has been 'fixed' in the usual way. Using the relations between the electric and magnetic fields we see that the symmetry transformations above are the ones corresponding to the global symmetry transformations in the electric theory. For the $U(1)_A$ in the two theories to correspond to each other, we must give $\hat{\Lambda}$ charge 4.

As usual, we check the theory by adding a mass to the N_F'th flavor and look at how it decouples. In the magnetic theory, this gives the effective superpotential

$$W = h\varphi_c^f \Phi_f^h \tilde{\varphi}_h^c + hm\hat{\Lambda}\Phi_{N_F}^{N_F} = h\varphi_c^f \Phi_f^h \tilde{\varphi}_h^c - h\mu^2 \Phi_{N_F}^{N_F},\ \mu^2 \equiv -m\hat{\Lambda}. \quad (4.76)$$

Since $\Phi_i^{N_F}$ and $\Phi_{N_F}^i$ contain one massive quark each, they become massive. Solving their F-flatness conditions yields

$$\varphi_c^{N_F}\tilde{\varphi}_{N_F}^c - \mu^2 = 0,\ \varphi_c^i \tilde{\varphi}_{N_F}^c = 0,\ i < N_F. \quad (4.77)$$

Here $\varphi_c^{N_F}$ receives a vacuum expectation value and breaks the gauge symmetry down to $SU(N-1)$. The higgs mechanism then gives a mass to the N_F'th quark flavor and they can be integrated out. Solving their F-flatness condition, $\varphi_c^i \Phi_i^{N_F} = \Phi_{N_F}^i \varphi_i^c = 0$ for $i \leq N_F$, leads to $\Phi_{N_F}^i = \Phi_i^{N_F} = 0$. Putting all this back into the superpotential gives the dual theory for $N_F - 1$ flavors, which is just what we needed. For $N_F = N_C + 2$ Equation (4.77) breaks the gauge symmetry completely and the instanton contribution can be calculated. This has been done and the result is the effective potential for $N_F = N_C + 1$ flavors.

What about 't Hooft anomaly matching? Clearly $\varphi = \tilde{\varphi} = \Phi = 0$ is a part of the moduli space. The relevant fermions then transform as

	$SU(N_F)_L$	$SU(N_F)_R$	$U(1)_B$	$U(1)_R$
ψ_φ	$\bar{\square}$	$\mathbf{1}$	$\frac{N_C}{N_F - N_C}$	$-1 + \frac{N_C}{N_F}$
$\psi_{\tilde{\varphi}}$	$\mathbf{1}$	$\square$	$-\frac{N_C}{N_F - N_C}$	$-1 + \frac{N_C}{N_F}$
ψ_Φ	$\square$	$\bar{\square}$	0	$1 - 2\frac{N_C}{N_F}$
$\tilde{\lambda}$	$\mathbf{1}$	$\mathbf{1}$	0	1
ψ_Q	$\square$	$\mathbf{1}$	1	$-\frac{N_C}{N_F}$
$\psi_{\tilde{Q}}$	$\mathbf{1}$	$\bar{\square}$	-1	$-\frac{N_C}{N_F}$
λ	$\mathbf{1}$	$\mathbf{1}$	0	1

(4.78)

Using these representations one gets the anomaly coefficients

$$\begin{array}{c|cc}
 & \text{electric} & \text{magnetic} \\
\hline
\left(SU(N_F)_L\right)^3 & N_C A(\Box) & N_C A(\Box) \\
\left(SU(N_F)_L\right)^2 U(1)_B & C(\Box) N_C & C(\Box) N_C \\
\left(SU(N_F)_L\right)^2 U(1)_R & -\frac{N_C^2}{N_F} C(\Box) & -\frac{N_C^2}{N_F} C(\Box) \\
SU(N_F)_L ... & 0 & 0 \\
\left(U(1)_R\right)^3 & -1 + N_C^2 - \frac{2N_C^4}{N_F^2} & -1 + N_C^2 - \frac{2N_C^4}{N_F^2} \\
\left(U(1)_R\right)^2 U(1)_B & 0 & 0 \\
U(1)_R \left(U(1)_B\right)^2 & -2N_C^2 & -2N_C^2 \\
\left(U(1)_B\right)^3 & 0 & 0 \\
\mathrm{Tr} U(1)_R & -1 - N_C^2 & -1 - N_C^2 \\
\mathrm{Tr} U(1)_B & 0 & 0
\end{array} , \tag{4.79}$$

which match perfectly, offering a far from trivial test of Seiberg duality.

5

The Metastable Vacuum of the Magnetic Theory

As we saw in the previous section, for $N_C + 1 < N_F \leq \frac{3}{2}N_C$ (or equivalently $3 < 3N \leq N_F$) the magnetic theory is IR free. This means that we have a way to handle the strongly coupled low energy dynamics of the electric theory. We start this chapter by studying the pseudo moduli space of the magnetic theory with an added mass term. It turns out that the magnetic theory not only has Goldstone bosons, but also pseudo flat directions. We then calculate the masses of all fields as a function of the pseudo flat directions. Using these masses, we calculate the one loop effective potential and show that the one loop masses of the pseudo moduli are positive [1]. We then study how supersymmetry is restored in some parts of the field space, thus making the supersymmetry breaking vacuum only meta stable.

5.1 The Massive Magnetic Dual

If we add an equal mass m to all quarks in the electric theory, the superpotential becomes

$$W(\varphi, \tilde{\varphi}, \Phi) = h\mathrm{Tr}\varphi^T \Phi \tilde{\varphi} - h\mu^2 \mathrm{Tr}\Phi \tag{5.1}$$

in the magnetic theory, where $\mu^2 = -m\hat{\Lambda}$. For sake of simplicity, we will take μ to be real. Since the only energy scale in the superpotential is μ, the tree level masses must be of the same order of magnitude. h (together with the gauge coupling constant g) is an overall dimensionless coupling constant. The tree level masses will contain one power of one of these coupling constants whereas the one loop masses will contain two. A last important dimensionless constant is $\epsilon = \frac{\mu}{\tilde{\Lambda}}$, where $\tilde{\Lambda}$ is the energy scale of the magnetic theory. This quantity will parameterize how strong the perturpative dynamics is compared to the non perturbative.

The mass term explicitly breaks the $U(1)_A$ symmetry and we are left with the global symmetry group

$$SU(N_F) \times U(1)_B \times U(1)_R \,, \tag{5.2}$$

where the R-symmetry is anomalous, and the gauge group $SU(N)$. The fields transform as

	SU(N)	SU(N_F)	U(1)$_B$	U(1)$_R$
Φ	1	$\square \times \bar{\square}$	0	2
φ	$\square$	$\bar{\square}$	1	0
$\tilde{\varphi}$	$\bar{\square}$	$\square$	-1	0

. (5.3)

Since the R-symmetry is classically unbroken we do not expect to find a tree level supersymmetric vacuum. This might seem strange since the full electric theory has Witten index N_C. However, in the end of the chapter we will see that non-peturbative effects restore supersymmetry at certain points in field space.

The F-terms are

$$-F_\Phi^\dagger = h(\varphi\tilde{\varphi}^T - \mu^2 \mathbf{1}_{N_F})\,, -F_{\tilde{\varphi}}^\dagger = h\Phi^T\varphi \text{ and } -F_\varphi^\dagger = h\Phi\tilde{\varphi}\,. \tag{5.4}$$

Since $\varphi\tilde{\varphi}^T$ at most can have rank N and $\mathbf{1}_{N_F}$ has rank $N_F > N$, all F-terms can not simultaneously be put to zero and supersymmetry is indeed spontaneously broken at tree level. To find the minima of V_F, it is convenient to note that, by using the flavor and gauge symmetries, an arbitrary field configuration can always be transformed into

$$\phi_0 = \left\{ \varphi_0 = \begin{pmatrix} \varphi_1 \\ 0 \end{pmatrix}, \tilde{\varphi}_0 = \begin{pmatrix} \tilde{\varphi}_1 \\ 0 \end{pmatrix}, \Phi_0 = \begin{pmatrix} \Phi_{11} & \Phi_{12} \\ \Phi_{21} & \Phi_{22} \end{pmatrix} \right\}, \tag{5.5}$$

where $\tilde{\varphi}_1$ is an $N \times N$ diagonal matrix, φ_1 is an $N \times N$ upper triangular matrix, Φ_{11} is an $N \times N$ matrix, Φ_{21} and Φ_{12}^T are $(N_F - N) \times N$ matrices and Φ_{22} is an $(N_F - N) \times (N_F - N)$ matrix. Using this, the F-terms can be rewritten as

$$-F_\Phi^\dagger = h \begin{pmatrix} \varphi_1\tilde{\varphi}_1^T - \mu^2\mathbf{1}_N & 0 \\ 0 & -\mu^2\mathbf{1}_{N_F-N} \end{pmatrix}, -F_{\tilde{\varphi}}^\dagger = h \begin{pmatrix} \Phi_{11}^T\varphi_1 \\ \Phi_{12}^T\varphi_1 \end{pmatrix},$$
$$-F_\varphi^\dagger = h \begin{pmatrix} \Phi_{11}\tilde{\varphi}_1 \\ \Phi_{21}\tilde{\varphi}_1 \end{pmatrix}. \tag{5.6}$$

First of all, it is possible to put all F-terms, except those in $-F^\dagger_{\Phi_{22}}$, to zero. Doing so corresponds to finding the vacuum. Putting $-F^\dagger_{\Phi_{11}}$ to zero implies $\varphi_1\tilde{\varphi}_1^T = \mu^2 \mathbf{1}_N$, which, since $\tilde{\varphi}_1$ is diagonal, implies that φ_1 also is diagonal. φ_1 and $\tilde{\varphi}_1$ therefore have maximum rank. Because of this, the conditions $\Phi_{11}\tilde{\varphi}_1 = 0$, $\Phi_{21}\tilde{\varphi}_1 = 0$ and $\Phi_{12}^T\varphi_1 = 0$ imply that $\Phi_{11} = \Phi_{21} = \Phi_{12} = 0$. A generic vacuum state can therefore be reached by choosing a matrix Φ_{22} and two matrices φ and $\tilde{\varphi}$ fulfilling the constraint above and performing the appropriate flavor and gauge transformations. In the minima, the potential energy (coming from the F-terms) is

$$V_{\min} = h^2\mu^4(N_F - N)\,. \tag{5.7}$$

We will focus on the maximally symmetric vacuum

$$\phi_0 = \left\{ \varphi_0 = \begin{pmatrix} \mu\mathbf{1}_N \\ 0 \end{pmatrix}, \tilde{\varphi}_0 = \begin{pmatrix} \mu\mathbf{1}_N \\ 0 \end{pmatrix}, \Phi_0 = \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix} \right\}. \tag{5.8}$$

This vacuum has some interesting properties. First of all, it is invariant under the global symmetry $SU(N)_{diag} \times SU(N_F - N) \times U(1)_{B'} \times U(1)_R$ as well as under interchange of φ and $\tilde{\varphi}$. Again, the $U(1)_R$ is anomalous. Here the $SU(N_F)$ flavor symmetry has first been broken down to one $SU(N)$ acting on the N first components in the flavor multiplets and one $SU(N_F - N)$ acting on the last $N_F - N$ ones. The flavor $SU(N)$ and its gauge counterpart have then been broken down to their diagonal subgroup, denoted by $SU(N)_{diag}$ above. The $U(1)_B$ has been modified so it only acts on the $N_F - N$ last components in the flavor multiplet, denoted above by $U(1)_{B'}$. Note that this is a combination of $U(1)_B$ and some flavor transformation. When studying the physics around this vacuum we will parameterize the perturbations with

$$\delta\phi = \left\{ \delta\varphi = \begin{pmatrix} \chi \\ \rho \end{pmatrix}, \delta\tilde{\varphi} = \begin{pmatrix} \tilde{\chi} \\ \tilde{\rho} \end{pmatrix}, \delta\Phi = \begin{pmatrix} Y & Z^T \\ \tilde{Z} & \hat{\Phi} \end{pmatrix} \right\}. \tag{5.9}$$

The transformation properties of these fields under the unbroken global symmetry group of the vacuum are

	$SU(N)_{diag}$	$SU(N_F - N)$	$U(1)_{B'}$
Y	$\square \times \bar{\square}$	1	0
Z	$\square$	$\bar{\square}$	-1
$\tilde{Z}$	$\bar{\square}$	$\square$	1
$\hat{\Phi}$	1	$\square \times \bar{\square}$	0
χ	$\bar{\square} \times \square$	1	0
$\tilde{\chi}$	$\square \times \bar{\square}$	1	0
ρ	$\square$	$\bar{\square}$	1
$\tilde{\rho}$	$\bar{\square}$	$\square$	-1

. (5.10)

What about the D-terms? The gauge generators act on ϕ_0 as

$$T_A\phi_0 = \left\{\varphi_0 t_A^T, \tilde{\varphi}_0(-t_A), \Phi\right\} = \left\{\left(\begin{array}{c} \mu t_A^T \\ 0 \end{array}\right), \left(\begin{array}{c} -\mu t_A \\ 0 \end{array}\right), 0\right\}, \tag{5.11}$$

where ϕ_0 again denotes the entire matter multiplet (c.f.r. (5.5) and (5.8)) and t_A are the generators of the fundamental representation of $SU(N)$. This means that the D-terms, $D_{A0} = -g^2\phi_0^\dagger T_A \phi_0$, are traces of the generators. Because the generators are traceless, the D-terms vanish and the vacuum studied is indeed (to tree level) the point in field space with lowest energy.

What goldstone bosons are coming from the symmetry breaking in this particular vacuum? One expression for the goldstone bosons is $(T_a^{SU(N_F)}\phi_0)^\dagger\delta\phi - (T_a^{SU(N_F)*}\phi_0^*)^\dagger\delta\phi^*$, where $T_a^{SU(N_F)}$ are the generators for the flavor symmetry transformations of ϕ. $T_a^{SU(N_F)}$ acts on ϕ_0 as

$$T_a^{SU(N_F)}\phi_0 = \left\{-t_a^{SU(N_F)T}\varphi_0\,, t_a^{SU(N_F)}\tilde{\varphi}_0\,, t_a^{SU(N_F)}\Phi_0 - \Phi_0 t_a^{SU(N_F)T}\right\}. \tag{5.12}$$

It is now convenient to split the generators of the fundamental representation into the blocks

$$t_a^{SU(N_F)} = \left(\begin{array}{cc} t_a^{11} & t_a^{21\dagger} \\ t_a^{21} & t_a^{22} \end{array}\right). \tag{5.13}$$

Since $\Phi_0 = 0$, the Goldstone bosons can be written as

$$\begin{aligned} &\mathrm{Tr}\left[\left(-t_a^{SU(N_F)T}\varphi_0\right)^\dagger\delta\varphi + \left(t_a^{SU(N_F)}\tilde{\varphi}_0\right)^\dagger\delta\tilde{\varphi}\right] \\ &\quad - \mathrm{Tr}\left[\left((-t_a^{SU(N_F)^T})^*\varphi_0^*\right)^\dagger\delta\varphi^* + \left(t_a^{SU(N_F)*}\tilde{\varphi}_0^*\right)^\dagger\delta\tilde{\varphi}^*\right] \\ &= \mu\mathrm{Tr}\left[-t_a^{11^\dagger}\left(\chi^T - \tilde{\chi}\right) + t_a^{11}\left(\chi^* - \tilde{\chi}^\dagger\right)\right. \\ &\qquad\qquad \left. + t_a^{21}\left(\rho^* + \tilde{\rho}\right) - t_a^{21*}\left(\rho + \tilde{\rho}^*\right)\right]. \end{aligned} \tag{5.14}$$

Let us first turn to t_a^{11}. This generator can either be an $SU(N)$ generator or proportional to the identity matrix (coming from $U(1)_B$). Using that any matrix can be written as $\sum c^a t_a + c^d\mathbf{1}$, for some complex constants c, one sees that the Goldstone bosons coming from t_a^{11} are the complex fields

$$(\chi - \tilde{\chi}^T) - h.c. \tag{5.15}$$

The generators t_a^{21} can be chosen to have at most one non zero element. This element can either be 1 or i and can be anywhere in the matrix by choosing an appropriate a. The real Goldstone bosons corresponding to the real and the imaginary generators can thus be combined to

$$\rho + \tilde{\rho}^* \in \mathbb{C}\,. \tag{5.16}$$

Because the vacuum is an invariant under $SU(N_F)_{diag}$, the traceless Goldstone bosons derived above will be eaten in the Higgs mechanism.

Does the theory have pseudo moduli? The fact that we stay in the vacuum for arbitrary Φ_{22} makes $\hat{\Phi}$ massless. Since $\hat{\Phi}$ is not the Goldstone boson of any broken symmetry it spans a pseudo moduli space. The constraint $\varphi_1 \tilde{\varphi}_1^T = \mu^2 \mathbf{1}_N$ is not only unaffected under $SU(N)$ transformations (in which the generators are made up by $N \times N$ hermitian traceless matrices) but also under transformations generated antihermitian matrices. These generators give rise to the massless fields

$$\hat{\chi} = (\chi - \tilde{\chi}^T) + h.c.\,. \tag{5.17}$$

The super Higgs mechanism will give the off diagonal part of $\hat{\chi}$ a mass but $\mathrm{Tr}\hat{\chi}$ will be massless at tree level.

Because neither $\hat{\Phi}$ nor $\mathrm{Tr}\hat{\chi}$ are protected by Goldstone's theorem and because the vacuum is not supersymmetric, both fields are expected to acquire a mass to one loop. To calculate this mass we have to parameterize the pseudo moduli space around the maximally symmetric vacuum. Since $\mathrm{Tr}\hat{\chi} = 2\mathrm{TrRe}\,[\chi - \tilde{\chi}]$, a good parametrization of this massless direction is $\varphi_1 = \mu e^{\theta} = \mathbf{1}_N$ and $\tilde{\varphi}_1 = \mu e^{-\theta}\mathbf{1}_N$, where θ is a real number. The $\hat{\Phi}$ direction is parameterized by $\Phi_{22} = X_0 \mathbf{1}_{N_F - N}$, where also X_0 is chosen to be a real number for simplicity. One then gets the ϕ expansion

$$\begin{aligned} \phi = \phi_0 + \delta\phi = \Bigg\{ \Phi = \begin{pmatrix} Y & Z^T \\ \tilde{Z} & X_0 \mathbf{1}_{N_F-N} + \hat{\Phi} \end{pmatrix}, \\ \varphi = \begin{pmatrix} \mu e^{\theta}\mathbf{1}_N + \chi \\ \rho \end{pmatrix}, \tilde{\varphi} = \begin{pmatrix} \mu e^{-\theta}\mathbf{1}_N + \tilde{\chi} \\ \tilde{\rho} \end{pmatrix} \Bigg\}\,. \end{aligned} \tag{5.18}$$

When these fields are put into the superpotential (5.1) one gets

$$\begin{aligned} W =& h\mathrm{Tr}\left[\mu e^{\theta} Y \tilde{\chi} + \mu e^{\theta} Z^T \tilde{\rho} + \mu e^{-\theta} \chi^T Y + \chi^T Y \tilde{\chi} + \chi^T Z^T \tilde{\rho} \right. \\ & \left. + \mu e^{-\theta} \rho^T \tilde{Z} + \chi^T Z^T \tilde{\rho} + \rho^T (X_0 \mathbf{1} + \hat{\Phi}) \tilde{\rho} \right] - h\mu^2 \mathrm{Tr}\hat{\Phi}\,. \end{aligned} \tag{5.19}$$

The only way in which superpotential terms involving three fields could occur in the mass matrices is, if W_c is non zero, in $W^{\dagger abc} W_c$. But, because the only linear term in W is $h\mu^2 \mathrm{Tr}\hat{\Phi}$, the terms $\chi^T Z^T \tilde{\rho}$, $\chi^T Z^T \tilde{\rho}$, $\chi^T Y \tilde{\chi}$ and $\rho^T (X_0 \mathbf{1} + \hat{\Phi})_{\text{Off diag}} \tilde{\rho}$ give no contribution to the mass matrices and can be discarded from the superpotential without affecting the masses.

One easily sees that the remaining superpotential can be split in two parts: one part containing the fields $\hat{\Phi}$, ρ, $\tilde{\rho}$, Z and $\tilde{Z}$,

$$W_{\hat{\Phi}\rho\tilde{\rho}Z\tilde{Z}} = h\mathrm{Tr}\left[\mu e^{\theta}\tilde{\rho}Z^T + \mu e^{-\theta}\tilde{Z}\rho^T + (X_0\mathbf{1} + \hat{\Phi})_{\mathrm{diag}}\tilde{\rho}\rho^T\right] - h\mu^2\mathrm{Tr}\hat{\Phi}_{\mathrm{diag}}, \tag{5.20}$$

and one part containing χ, $\tilde{\chi}$, Y and $\tilde{Y}$,

$$W_{\chi\tilde{\chi}Y} = h\mathrm{Tr}\left[\mu e^{\theta}Y\tilde{\chi} + \mu e^{-\theta}\chi^T Y\right] . \tag{5.21}$$

What about the contributions to the mass matrices due to the gauge couplings? First of all, because D_{A0} is zero, the term $\sum_A D_{A0}T_A$ in (2.36) gives no contribution to the boson mass matrix. Secondly, W^c is only non zero for $\hat{\Phi}$. Because $\hat{\Phi}$ is uncharged under the gauge group, this means that the term $W_c {T_A}^c{}_b$ in (2.38) is zero, implying that the gauginos do not mix with the matter fermions. Finally, $T_A\phi_0$ only has components corresponding to χ and $\tilde{\chi}$. In particular, the fields $\hat{\Phi}$, ρ, $\tilde{\rho}$, Z and $\tilde{Z}$ are not affected by the gauge contributions of the form $\sum (T_A\phi_0)(T_A\phi_0)^{\dagger}$. This, together with the fact that they do not mix with the other fields in the superpotential, makes it possible to put the mass matrix in a block diagonal form with the mentioned field in one block and χ, $\tilde{\chi}$ and Y in the other.

5.2 Masses of $\hat{\Phi}$, ρ,$\tilde{\rho}$, Z and $\tilde{Z}$

A key observation is that the superpotential can be rewritten as

$$W_{\hat{\Phi}\rho\tilde{\rho}Z\tilde{Z}} = h\sum_{f=1}^{N_F-N}\Bigg(\left(X_0\mathbf{1} + \hat{\Phi}\right)_{ff}\left(\tilde{\rho}\rho^T\right)_{ff} + \mu e^{\theta}\left(\tilde{\rho}Z^T\right)_{ff} + \mu e^{-\theta}\left(\tilde{Z}\rho^T\right) - \mu^2\left(X_0 + \hat{\Phi}_{ff}\right)\Bigg) . \tag{5.22}$$

Because the terms with different f do not interact, the superpotential above can be seen as $N_F - N$ models of the form

$$W'_f = h\left(X\phi_1\cdot\phi_2 + \mu e^{-\theta}\phi_1\cdot\phi_3 + \mu e^{\theta}\phi_2\cdot\phi_4\right) + \mu^2 X\,, \tag{5.23}$$

where $\phi_{1i} = \rho_{fi}$, $\phi_{2i} = \tilde{\rho}_{fi}$, $\phi_{3i} = \tilde{Z}_{fi}$, $\phi_{4i} = Z_{fi}$, $X = X_0 + \hat{\Phi}_{ff}$, $\langle X\rangle = X_0$ and $i = 1 \ldots N$.

5.2.1 Scalar Bosons

The boson mass matrix for the fields studied in this section is

$$M_0^2 = \begin{pmatrix} W^{\dagger ac}W_{cb} & W^{\dagger abc}W_c \\ W_{abc}W^{\dagger c} & W_{ac}W^{\dagger cb}\end{pmatrix} = \begin{pmatrix} M_{11} & M_{12} \\ M_{21} & M_{22}\end{pmatrix}, \tag{5.24}$$

where $M_{11} = M_{11}^\dagger = M_{22}^*$ and $M_{21} = M_{21}^T = M_{12}^*$. Looking at W' one sees that fields with different i's do not couple to each other. This decoupling, combined with the one in previous section, means that the mass matrix has been block diagonalized all the way down to the individual components in fields ρ, $\tilde{\rho}$ Z and $\tilde{Z}$. This is not surprising since these fields have the unbroken flavor symmetry $SU(N_F - N) \times U(1)_{B'} = U(N_F - N)$. Using this gives the mass matrices

$$M_{11} = h^2\mu^2 \begin{pmatrix} (e^{-2\theta} + \frac{X_0}{\mu^2}^2) & 0 & 0 & \frac{X_0}{\mu}e^{\theta} & 0 \\ 0 & (e^{2\theta} + \frac{X_0}{\mu^2}^2) & \frac{X_0}{\mu}e^{-\theta} & 0 & 0 \\ 0 & \frac{X_0}{\mu}e^{-\theta} & e^{-2\theta} & 0 & 0 \\ \frac{X_0}{\mu}e^{\theta} & 0 & 0 & e^{2\theta} & 0 \\ 0 & 0 & 0 & 0 & 0 \end{pmatrix} \quad (5.25)$$

and

$$M_{21} = h^2 \begin{pmatrix} 0 & -\mu^2 & 0 & 0 & 0 \\ -\mu^2 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{pmatrix}, \quad (5.26)$$

where the rows correspond to ρ_{fi}, $\tilde{\rho}_{fi}$ $\tilde{Z}_{fi}$, Z_{fi} and X_{ff} respectively. First of all, one sees that X_{ff} is massless. Because $\hat{\Phi}_{\text{off diag}}$ does not appear in $W_{\hat{\Phi}\rho\tilde{\rho}Z\tilde{Z}}$, the off diagonal part is also massless. Hence, the entire field $\hat{\Phi}$ is massless. This is exactly what we expect since arbitrary $\hat{\Phi}$ belongs to the pseudo moduli space.

Secondly, because the fields making up an eigenvector must transform in the same representation, M_0^2 must become block diagonal if we permute the rows and columns into groups of fields transforming in the same representation. By looking at table (5.10) one sees that the only such combinations are $(\rho, \tilde{\rho}^*, Z, \tilde{Z}^*)$ and $(\rho^*, \tilde{\rho}, Z^*, \tilde{Z})$. The block of the first combination is

$$h^2 \begin{pmatrix} \mu^2 e^{-2\theta} + X_0^2 & X_0\mu e^{\theta} & -\mu^2 & 0 \\ X_0\mu e^{\theta} & \mu^2 e^{2\theta} & 0 & 0 \\ -\mu^2 & 0 & \mu^2 e^{2\theta} + X_0^2 & X_0\mu e^{-\theta} \\ 0 & 0 & X_0\mu e^{-\theta} & \mu^2 e^{-2\theta} \end{pmatrix} \begin{matrix} \rho \\ Z \\ \tilde{\rho}^* \\ \tilde{Z}^* \end{matrix} . \quad (5.27)$$

The eigenvalues and eigenvectors of this matrix can be calculated exactly. A factor $\sqrt{X_0^4 + 4\theta^2\mu^4}$ in the eigenvalues makes Taylor expansions around $X_0 = \theta = 0$ ill defined. Taking this into account and expanding to second order gives the masses

$$m_1^2 = 0\,,\; m_2^2 = 2h^2\left(X_0^2 + \mu^2(1+2\theta^2)\right)\,,$$
$$m_{3,4}^2 = h^2\left((1+2\theta^2)\mu^2 \pm \sqrt{X_0^4 + 4\theta^2\mu^4}\right) \tag{5.28}$$

and eigenvectors (unnormalized for simplicity)

$$v_1 = \begin{pmatrix} (1+\theta)\mu \\ -X_0 \\ (1-\theta)\mu \\ -X_0 \end{pmatrix},\; v_2 = \begin{pmatrix} -(1+\theta)\mu \\ -X_0(1+4\theta) \\ (1+3\theta)\mu \\ X_0 \end{pmatrix},$$
$$v_{3,4} = \begin{pmatrix} 0 \\ 2\theta(X_0^2+\mu^2) \pm \sqrt{X_0^4 + 4\theta\mu^4} \\ X_0\left(2\theta\mu + \frac{\sqrt{X_0^4+4\theta^2\mu^4}}{\mu}\right) \\ X_0^2 \end{pmatrix}. \tag{5.29}$$

By calculating the block in the mass matrix corresponding to ρ^*, $\tilde{\rho}$, Z^* and $\tilde{Z}$, one sees that it is almost identical to the first one (up to an exchange $\theta \to -\theta$). This does not affect the masses since θ appears quadratically and means that, for $\theta = 0$, the eigenvectors can be combined into complex fields. In particular, for $\theta = X_0 = 0$, the first eigenvector can be written as $\rho + \tilde{\rho}^*$, which exactly is the Goldstone boson in equation (5.16).

5.2.2 Fermions

As we mentioned earlier, the matter fermions do not couple to the gauginos, implying that the fermion mass matrix is equal to M_{11} in Equation (5.25). The fermions $\psi_{\hat{\Phi}}$ are massless for the same reason as for the bosons $\hat{\Phi}$. Because the only non zero F-terms are in the diagonal part of $F_{\hat{\Phi}}$, $\mathrm{Tr}\psi_{\hat{\Phi}}$ corresponds to the Goldstino. By once again looking at Table (5.10) we see that by permuting the fields into the pairs (ψ_ρ, ψ_Z) and $(\psi_{\tilde{\rho}}, \psi_{\tilde{Z}})$ we get the remaining matrix into a block diagonal form:

$$h^2 \begin{pmatrix} \mu^2 e^{-2\theta} + X_0^2 & X_0\mu e^{\theta} & 0 & 0 \\ X_0\mu e^{\theta} & \mu^2 e^{2\theta} & 0 & 0 \\ 0 & 0 & \mu^2 e^{2\theta} + X_0^2 & X_0\mu e^{-\theta} \\ 0 & 0 & X_0\mu e^{-\theta} & \mu^2 e^{-2\theta} \end{pmatrix} \begin{matrix} \psi_\rho \\ \psi_Z \\ \psi_{\tilde{\rho}} \\ \psi_{\tilde{Z}} \end{matrix}\,. \tag{5.30}$$

This matrix has the eigenvalues

$$m_{1,2}^2 = h^2\left(\mu^2 + \frac{X_0^2}{2} + 2\theta^2\mu^2 - \sqrt{X_0^2 + 4\theta^2\mu^2}\right)$$
$$m_{3,4}^2 = h^2\left(\mu^2 + \frac{X_0^2}{2} + 2\theta^2\mu^2 + \sqrt{X_0^2 + 4\theta^2\mu^2}\right) \quad (5.31)$$

and eigenvectors

$$v_{1,3} = \begin{pmatrix} X_0^2 - 4\theta\mu^2 \mp 2\mu\sqrt{X_0^2 + 4\theta^2\mu^2} \\ 2X_0\mu(1+\theta) \\ 0 \\ 0 \end{pmatrix},$$
$$v_{2,4} = \begin{pmatrix} 0 \\ 0 \\ X_0^2 + 4\theta\mu^2 \mp 2\mu\sqrt{X_0^2 + 4\theta^2\mu^2} \\ 2X_0\mu(1-\theta) \end{pmatrix}. \quad (5.32)$$

5.3 The Masses of χ, $\tilde{\chi}$ and Y

5.3.1 Scalar Bosons

By looking at Equation (5.21) one sees that $W^{\dagger abc}W_c$ is zero and that

$$W^{\dagger ac}W_{cb} = h^2 \begin{pmatrix} \mu^2 e^{-2\theta}\mathbf{1} & \mu^2\mathbf{1} & 0 \\ \mu^2\mathbf{1} & \mu^2 e^{2\theta}\mathbf{1} & 0 \\ 0 & 0 & \mu^2\left(e^{2\theta} + e^{-2\theta}\right)\mathbf{1} \end{pmatrix} \begin{matrix} \chi \\ \tilde{\chi}^T \\ Y \end{matrix}. \quad (5.33)$$

Since Y is not charged with respect to the gauge symmetry, it will not get a contribution from the gauge part of the mass matrix. This means that

$$Y \in \mathbb{C} \text{ with } m^2 = 2h^2\mu^2\cosh[2\theta]. \quad (5.34)$$

The situation for χ and $\tilde{\chi}$ is slightly more complicated. This is because M_{11} and M_{21} receive D-term contributions such as $g^2\sum_A(T_A\phi_0) \times (T_A\phi_0)^\dagger$. By looking at (5.10) we see that χ, $\chi^\dagger$, $\tilde{\chi}^*$ and $\tilde{\chi}^T$ transform under the $\bar{\Box} \times \Box$-representation of SU(N) flavor symmetry. From representation theory we know that this representation can be split into the adjoint and the trivial representation, where the trivial representation corresponds to χ and $\tilde{\chi}$ being proportional to $\mathbf{1}$. When we calculated the D-term for ϕ_0, we saw that the contribution to V_D from this kind of fields vanishes. It thus makes sense to choose a basis where the two representations are seen explicitly. One convenient decomposition is

$$\chi^T = \chi' + \frac{1}{\sqrt{N}}\frac{\mathrm{Tr}\chi}{\sqrt{N}}\mathbf{1} = \sum_A \sqrt{2}t_A\chi_A + \frac{1}{\sqrt{N}}\chi_D\mathbf{1} \quad (5.35)$$

and

$$\tilde{\chi} = \sum_A \sqrt{2} t_A \tilde{\chi}_A + \frac{1}{\sqrt{N}} \tilde{\chi}_D \mathbf{1} \,, \tag{5.36}$$

where $\chi_D = \mathrm{Tr}\chi/\sqrt{N}$ and similar for $\tilde{\chi}_D$. This change of coordinates is unitary since $\mathrm{Tr} t_A t_B = \frac{1}{2}\delta_{AB}$, $\mathrm{Tr} t_A = 0$ and $\mathrm{Tr}\mathbf{1} = N$.

One way to see what the gauge contribution to M_{11} looks like in the new basis is to write out the mass term

$$\begin{aligned} g^2 \sum_A \delta\phi^\dagger (T_A \phi_0)(T_A\phi_0)^\dagger \delta\phi &= \\ = g^2 \sum_A \Big(\mu e^\theta \mathrm{Tr}\left[\chi^\dagger t_A^T\right] &+ \mu e^{-\theta} \mathrm{Tr}\left[\tilde{\chi}^\dagger (-t_A^\dagger)\right]\Big) \\ &\times \left(\mu e^\theta \mathrm{Tr}\left[t_A^* \chi\right] + \mu e^{-\theta}\mathrm{Tr}\left[(-t_A)\tilde{\chi}\right]\right) \\ = g^2 \sum_A \frac{\mu^2}{2} \left(e^\theta \chi_A^* - e^{-\theta}\tilde{\chi}_A^*\right) &\left(e^\theta \chi_A - e^{-\theta}\tilde{\chi}_A\right), \end{aligned} \tag{5.37}$$

where we used a generalization of (5.11) for the pseudo moduli. Because the coordinate transformation is unitary and because $W^{\dagger ac}W_{cb}$ is diagonal, the superpotential contribution to M_{11} is not affected by the change of basis. M_{11} can therefore be written as

$$M_{11} = \mu^2 h^2 \begin{pmatrix} \left(e^{-2\theta} + \frac{1}{2}\frac{g^2}{h^2} e^{2\theta}\right)\mathbf{1} & \left(1 - \frac{1}{2}\frac{g^2}{h^2}\right)\mathbf{1} & 0 & 0 \\ \left(1 - \frac{1}{2}\frac{g^2}{h^2}\right)\mathbf{1} & \left(e^{2\theta} + \frac{1}{2}\frac{g^2}{h^2} e^{-2\theta}\right)\mathbf{1} & 0 & 0 \\ 0 & 0 & e^{-2\theta} & 1 \\ 0 & 0 & 1 & e^{2\theta} \end{pmatrix} \begin{matrix} \chi_A \\ \tilde{\chi}_A \\ \chi_D \\ \tilde{\chi}_D \end{matrix} \tag{5.38}$$

in the new basis.

Since χ_D and $\tilde{\chi}_D$ are not affected by the D-term potential, their mass matrix will be block diagonal. This means that the fields will be complex and that the only thing we have to do is to find the eigenvalues and eigenvectors of the trace block in M_{11}. This gives

$$\begin{aligned} &\frac{e^\theta \chi_D - e^{-\theta}\tilde{\chi}_D}{\sqrt{e^{2\theta} + e^{-2\theta}}} \in \mathbb{C} \text{ with } m^2 = 0 \\ &\frac{e^{-\theta} \chi_D + e^{\theta}\tilde{\chi}_D}{\sqrt{e^{2\theta} + e^{-2\theta}}} \in \mathbb{C} \text{ with } m^2 = 2h^2\mu^2 \cos[2\theta] \,. \end{aligned} \tag{5.39}$$

The imaginary part of the massless field is a Goldstone boson corresponding to the trace of (5.15). The real part corresponds to the tree level massless field $\mathrm{Tr}\hat{\chi}$ of (5.17). This is one of the pseudo moduli fields who's one loop mass we are calculating.

Let us now turn to $M_{21} = g^2 \sum_A (T_A\phi_0)^*(T_A\phi_0)^\dagger$. In the new basis, this matrix will be the same as the V_D-contribution to M_{11}. The complete mass matrix for the fields transforming in the adjoint representation is thus

$$M_A^2 = \mu^2 h^2 \begin{pmatrix} e^{-2\theta} + \frac{g^2 e^{2\theta}}{2h^2} & 1 - \frac{1}{2}\frac{g^2}{h^2} & \frac{1}{2}\frac{g^2}{h^2}e^{2\theta} & -\frac{1}{2}\frac{g^2}{h^2} \\ 1 - \frac{1}{2}\frac{g^2}{h^2} & e^{2\theta} + \frac{g^2 e^{-2\theta}}{2h^2} & -\frac{1}{2}\frac{g^2}{h^2} & \frac{1}{2}\frac{g^2}{h^2}e^{2\theta} \\ \frac{1}{2}\frac{g^2}{h^2}e^{2\theta} & -\frac{1}{2}\frac{g^2}{h^2} & e^{-2\theta} + \frac{g^2 e^{2\theta}}{2h^2} & 1 - \frac{1}{2}\frac{g^2}{h^2} \\ -\frac{1}{2}\frac{g^2}{h^2} & \frac{1}{2}\frac{g^2}{h^2}e^{-2\theta} & 1 - \frac{1}{2}\frac{g^2}{h^2} & e^{2\theta} + \frac{g^2 e^{-2\theta}}{2h^2} \end{pmatrix}, \tag{5.40}$$

where the rows correspond to χ_A, $\tilde{\chi}_A$, χ_A^* and $\tilde{\chi}_A^*$ respectively. This matrix has the eigenvectors and eigenvalues

$$\begin{aligned} \frac{\sqrt{2}\mathrm{Im}\left[e^\theta \chi_A - e^{-\theta}\tilde{\chi}_A\right]}{\sqrt{e^{2\theta} + e^{-2\theta}}} &\in \mathbb{R} \text{ with } m_1^2 = 0 \\ \frac{\sqrt{2}\mathrm{Re}\left[e^\theta \chi_A - e^{-\theta}\tilde{\chi}_A\right]}{\sqrt{e^{2\theta} + e^{-2\theta}}} &\in \mathbb{R} \text{ with } m_2^2 = 2g^2\mu^2 \cosh[2\theta] \\ \frac{e^{-\theta}\chi_A + e^{\theta}\tilde{\chi}_A}{\sqrt{e^{2\theta} + e^{-2\theta}}} &\in \mathbb{C} \text{ with } m_3^2 = 2h^2\mu^2 \cosh[2\theta]\,. \end{aligned} \tag{5.41}$$

A couple of points are to be made here. First of all, the traceless part of the Goldstone boson in Equation (5.15) can, when $\mu \to \mu e^\theta$, be written as

$$e^\theta \chi' - e^{-\theta}\tilde{\chi}' - h.c. = \sum_A \sqrt{2} t_A^T \mathrm{Im}\left[e^\theta \chi_A - e^{-\theta}\tilde{\chi}_A\right], \tag{5.42}$$

where χ' and $\tilde{\chi}'$ are the traceless parts of corresponding fields. This coincides with the massless field above.

Secondly, from Equation (2.33) we see that the unitary gauge condition can be written as:

$$\begin{aligned} 0 =& \sum_n \mathrm{Im}\left[(\phi_0 + \delta\phi)^\dagger (T_A\phi_0)\right] = \\ =& \sqrt{2}\mu \sum_B \mathrm{Im}\left[e^\theta \chi_B^* \mathrm{Tr}\left[t_B^T t_A^T\right] + e^{-\theta}\tilde{\chi}_B^* \mathrm{Tr}\left[-t_A t_B\right]\right] = \\ =& \frac{\mu}{\sqrt{2}} \mathrm{Im}\left[e^\theta \chi_A - e^{-\theta}\tilde{\chi}_A\right], \end{aligned} \tag{5.43}$$

which corresponds to setting the Goldstone boson discussed above to zero. This is nothing other than the usual Higgs mechanism. Thirdly, the second field in Equation (5.41) corresponds to the traceless part of (5.17).

Lastly, because the last field in (5.39) and the last field in (5.41) have the same mass and the same dependence on χ and $\tilde{\chi}$, they can be combined back into the original matrix field:

$$\frac{e^{-\theta}\chi + e^{\theta}\tilde{\chi}^T}{\sqrt{e^{2\theta}+e^{-2\theta}}} \in \mathbb{C} \text{ with } m_3^2 = 2h^2\mu^2\cosh[2\theta]\,. \tag{5.44}$$

This might seem strange since there is no unbroken global $U(N)$ symmetry that forces fields in the two irreducible representations to have the same mass. However, the superpotential (5.21) has a 'fake' global $U(N)\times U(N)$ symmetry. This is not a real global symmetry since it only appeared after we discarded the higher order interaction terms that did not affect the mass matrix. Note that this symmetry does not guarantee the gauge masses to behave nicely, something $\hat{\chi}$ bitterly experienced.

5.3.2 Fermions

The square mass matrix for the fermions is (5.38) with $g^2/2 \to g^2$, see (2.38). This matrix has eigenvectors and masses

$$\begin{aligned}
\frac{e^{\theta}\psi_{\chi_D} - e^{-\theta}\psi_{\tilde{\chi}_D}}{\sqrt{e^{2\theta}+e^{-2\theta}}} &\text{ with } m^2 = 0\\
\frac{e^{\theta}\psi_{\chi_A} - e^{-\theta}\psi_{\tilde{\chi}_A}}{\sqrt{e^{2\theta}+e^{-2\theta}}} &\text{ with } m^2 = 2g^2\mu^2\cosh[2\theta]\\
\frac{e^{-\theta}\psi_{\chi} + e^{\theta}\psi_{\tilde{\chi}}}{\sqrt{e^{2\theta}+e^{-2\theta}}} &\text{ with } m^2 = 2h^2\mu^2\cosh[2\theta],
\end{aligned} \tag{5.45}$$

where the last field has been recombined for the same reason as in the previous section. We also have

$$\psi_Y \text{ with } m^2 = 2h^2\mu^2\cosh[2\theta] \tag{5.46}$$

5.4 Gauge Fields

As one can see in (2.38), the gauge fermions (gauginos) have the diagonal mass matrix

$$\begin{aligned}
M^2_{1/2(\text{gauge})} = 2g^2\left(T_A\phi_0\right)^{\dagger}\left(T_B\phi_0\right) &= g^2\mu^2\left(e^{2\theta}+e^{-2\theta}\right)\delta_{AB}\\
&= 2g^2\mu^2\cosh[2\theta]\delta_{AB}.
\end{aligned} \tag{5.47}$$

The gauge vector bosons have the mass matrix

$$M^2_{1(\text{gauge})} = g^2\phi_0^{\dagger}\left\{T_A, T_B\right\}\phi_0 = 2g^2\mu^2\cosh[2\theta]\delta_{AB}\,. \tag{5.48}$$

All fields that acquire a mass trough gauge interactions are equally heavy and their contribution to the one loop potential will vanish.

5.5 One Loop Effective Potential

Now that we have all the masses, calculating the one loop effective potential is a simple thing. A summary of the masses and the number of scalar, spinor and vector degrees of freedom can be found in Table 5.5. One might argue that assuming μ and X_0 to be real is not very natural. Redoing the calculations for complex variables is not much harder but it will give expressions for the eigenvectors that are more complicated. However, it is not hard to convince oneself that all characteristic polynomials will only contain $|\mu|^2$ and $|X_0|^2$. Hence, the more general masses are obtained by doing the substitution $\mu^2 \to |\mu|^2$ and $X_0^2 \to |X_0|^2$. When these masses are put into (2.43) and the potential is taylor expand to second order in X_0 and θ one gets the mass term

$$V_{\text{eff}}^{(1)} = \frac{h^4 |\mu|^2 N(N_F - N)(\log 4 - 1)}{8\pi^2} \left(|X_0|^2 + 2\theta^2 |\mu|^2\right) . \qquad (5.49)$$

Note the important point that $V^(1)_{eff} > 0$, indicating the absence of a tachyonic instability. Up until now we have calculated the potential for the background $\varphi_1 = \mu e^{\theta}\mathbf{1}$, $\tilde{\varphi}_1 = \mu e^{-\theta}\mathbf{1}$ and $\Phi_{22} = X_0\mathbf{1}$. Since the kinetic terms for the pseudo moduli fields are written in terms of $\hat{\chi}_D$ and $\hat{\Phi}$, we have to translate the effective potential above into these variables.

To translate θ we must split the background into one vacuum part, $\varphi_1 = \mu\mathbf{1}$ and $\tilde{\varphi}_1 = \mu\mathbf{1}$, and one perturbation part, $\chi_0 = \mu(e^{\theta} - 1)\mathbf{1}$ and $\tilde{\chi}_0 = \mu(e^{-\theta} - 1)\mathbf{1}$. Doing the usual unitary change of basis gives

$$\hat{\chi}_D = \frac{\text{Tr}\left[\chi_0 - \tilde{\chi}_0 + \chi_0^* - \tilde{\chi}_0^*\right]}{2\sqrt{N}} = 2\mu\sqrt{N}\theta + O(\theta^2) \qquad (5.50)$$

and also $O(\theta^2)$ contributions to some of the other orthogonal fields. This means that we can the let $\theta^2 \to \hat{\chi}_D^2/4\,|\mu|^2\,N$ in (5.49).

Since the superpotential $W_{\hat{\Phi}\rho\tilde{\rho}Z\tilde{Z}}$ decouple to $N_F - N$ independent terms (for which the background can be chosen independently) and since we have an unbroken $SU(N_F - N)$ global symmetry, the only possible mass for $\hat{\Phi}$ term is of the type $\text{Tr}\hat{\Phi}^\dagger\hat{\Phi}$. Using

$$\text{Tr}\hat{\Phi}^\dagger\hat{\Phi} = (N_F - N)X_0^2 \qquad (5.51)$$

we can write the final one loop potential as

$$V_{\text{eff}}^{(1)} = \frac{h^4 |\mu|^2 (\log 4 - 1)}{8\pi^2} \left[N\text{Tr}\hat{\Phi}^\dagger\hat{\Phi} + \frac{(N_F - N)\hat{\chi}_D^2}{2}\right] . \qquad (5.52)$$

This means that the pseudo moduli space is lifted to one loop. Since the only massless fields are the Goldstone bosons, the vacuum space is compact.

Mass	Bosonic DOF	Fermionic DOF	Vector DOF
0	$2N_F(N_F - N) + 2$	$2 + 2(N_F - N)^2$	
$2h^2\left(X_0^2 + \mu(1+2\theta^2)\right)$	$2N(N_F - N)$		
$h^2\left(\mu^2(1+2\theta^2) + \sqrt{X_0^4 + 4\mu^4\theta^2}\right)$	$2N(N_F - N)$		
$h^2\left(\mu^2(1+2\theta^2) - \sqrt{X_0^4 + 4\mu^4\theta^2}\right)$	$2N(N_F - N)$		
$h^2\left(\mu^2(1+2\theta^2) + \frac{X_0^2}{2} + \sqrt{\mu^2 X_0^2 + 4\mu^4\theta^2}\right)$		$4N(N_F - N)$	
$h^2\left(\mu^2(1+2\theta^2) + \frac{X_0^2}{2} - \sqrt{\mu^2 X_0^2 + 4\mu^4\theta^2}\right)$		$4N(N_F - N)$	
$2h^2\mu^2\cosh[2\theta]$	$4N^2$	$4N^2$	
$2g^2\mu^2\cosh[2\theta]$	$(N^2 - 1)$	$4(N^2 - 1)$	$3(N^2 - 1)$

Table 5.1: Table summarizing the masses and number of bosonic, fermion and vector degrees of freedom. Notice that the number of bosonic and fermionic degrees of freedom match.

5.6 Dynamical Supersymmetry Restoration

In the magnetic theory we have N_F quarks in the $\Box$-representation of the gauge group and N_F quarks in the $\bar{\Box}$-representation. For this matter content we have $b_0 = 3N - N_F$ and the coupling constant

$$e^{-8\pi^2/g^2(E)+i\theta} = \left(\frac{E}{\tilde{\Lambda}}\right)^{N_F-3N} . \tag{5.53}$$

As we have discussed earlier, the magnetic theory is IR-free. Let us study this theory further by giving the meson matrix a big vacuum expectation value with eigenvalues of the same order of magnitude. The term $h\mathrm{Tr}\varphi^T\Phi\tilde{\varphi}$ in the superpotential will then give φ and $\tilde{\varphi}$ a mass of order $h\Phi$. If we integrate out these fields and the energy is below $h\Phi$, all propagators are of order $1/(h\Phi)^2$ and their contribution to the effective theory is negligible. After the quarks have been integrated out we have a theory consisting of a pure super Yang-Mills theory and an uncharged meson field. The coupling constant of this low energy theory runs as

$$e^{-8\pi^2/g^2(E)+i\theta} = \left(\frac{E}{\Lambda_L}\right)^{-3N} , \tag{5.54}$$

where Λ_L is the characteristic scale for the low energy theory. Matching the two expressions for the coupling constants at energy $h\Phi$, which can be written in a $SU(N_F)$ invariant way as $E_{\text{match}}^{N_F} = h^{N_F}\det\Phi$, gives

$$\left(\frac{h\,(\det\Phi)^{1/N_F}}{\Lambda_L}\right)^{-3N} = \left(\frac{h\,(\det\Phi)^{1/N_F}}{\tilde{\Lambda}}\right)^{N_F-3N}$$
$$\Rightarrow \Lambda_L^{3N} = h^{N_F}\,(\det\Phi)\,\tilde{\Lambda}^{-(N_F-3N)}. \tag{5.55}$$

What is then the effective potential? As we saw in the beginning of Chapter 4, the gauginos condense in the strongly coupled low energy region. Using the effective potential for the pure Yang-Mills theory (4.9) together with the remaining term in the superpotential gives

$$W_{eff} = N\Lambda_L^3 - h\mu^2\mathrm{Tr}\Phi = N\left(h^{N_F}(\det\Phi)\tilde{\Lambda}^{-(N_F-3N)}\right)^{1/N} - h\mu^2\mathrm{Tr}\Phi\,. \tag{5.56}$$

Since the first term in the effective superpotential has R-charge $2N_F/N$, R-symmetry is explicitly broken. We therefore expect to find a solution to the F-flatness conditions. Where is this vacuum? Since Φ transforms under $\Box\times\bar{\Box}$, it is possible to transform Φ into an upper triangular matrix. This is because in this representation a flavor transformation corresponds to a change of basis and by using the Gram-Schmidt method it is possible (if the matrix can be decomposed into a sum of eigenvectors) to choose a base in which Φ is upper triangular. The superpotential is then simplified to

$$W_{eff} = N\left(h^{N_F}(\Phi_{11}\Phi_{22}...\Phi_{N_FN_F})\tilde{\Lambda}^{-(N_F-3N)}\right)^{1/N} - h\mu^2\left(\Phi_{11}+...+\Phi_{N_FN_F}\right). \quad (5.57)$$

Calculating the F terms and putting all of them to zero gives

$$\Phi_{11} = ... = \Phi_{N_FN_F} = \frac{1}{h}\mu\frac{1}{\left(\frac{\mu}{\tilde{\Lambda}}\right)^{\frac{N_F-3N}{N_F-N}}}. \quad (5.58)$$

Since we were able to put all F terms to zero and since we have no D terms to worry about, this is a supersymmetric vacuum. Because this new vacuum has vanishing vacuum energy, the supersymmetry breaking 'vacuum' found earlier is only metastable. This is very interesting. As we saw in the previous chapter, massive SQCD has no R-symmetry and has supersymmetric vacua. Naively, SQCD is not a good model to break supersymmetry spontaneously. However, this metastable vacuum totally sidesteps the limitations R-symmetry and Witten index impose and give us a new way to break supersymmetry.

The supersymmetric vacuum can also be written as

$$\langle h\Phi\rangle = \mu\frac{1}{\epsilon^{\frac{N_F-3N}{N_F-N}}} = \tilde{\Lambda}\epsilon^{\frac{2N}{N_F-N}}, \quad (5.59)$$

showing that the supersymmetric vacuum is far below the Landau pole and far from the scale of the metastable supersymmetry breaking vacuum. Equation (5.59) is exactly the same as one gets using Equation (4.16) with $\mu^2 = -m\hat{\Lambda}$, $h\Phi = M/\hat{\Lambda}$ and the relation between the different scales in Equation (4.70). The duality therefore survives another non trivial test. From the electric theory we knew that the theory had N_C supersymmetric vacua. These are not visible in the magnetic theory using only perturbation theory but appear only when taking non-perturbative effects into account. As pointed out earlier, the $U(1)_R$ symmetry is anomalous. It should therefore not be surprising that supersymmetry is restored by non-perturbative effects.

At this point it is important to check whether the physics at scales above $\tilde{\Lambda}$ (where the magnetic theory becomes strongly coupled) can alter our conclusions. Typically, the biggest effects will be in the Kähler potential and have the form (omitting the traces for ease of notation)

$$K = \varphi^\dagger\varphi + \tilde{\varphi}^\dagger\tilde{\varphi} + \Phi^\dagger\Phi + \frac{c}{|\tilde{\Lambda}|^2}\left(\Phi^\dagger\Phi\right)^2 + ..., \quad (5.60)$$

where c is some unknown constant. After integrating the Kähler potential over the Grassmann coordinates, the Lagrangian gets a factor of the form $\left(1+c\frac{\Phi^\dagger\Phi}{|\tilde\Lambda|^2}\right)$ in front of the ordinary kinetic terms (omitting higher order terms in $\Phi^\dagger\Phi/|\tilde\Lambda|^2$). This factor then, trough the F-terms, changes the effective potential. Omitting higher powers, the change can schematically be written as

$$V \rightsquigarrow V\left(1-c\frac{\Phi^\dagger\Phi}{|\tilde\Lambda|^2}\right) \sim V_0+V^{(1)}_{eff}-\frac{cV_0}{|\tilde\Lambda|^2}\Phi^\dagger\Phi \sim V_0+V^{(1)}_{eff}-c|\mu^2\epsilon^2|\Phi^\dagger\Phi, \tag{5.61}$$

where we have used that V_0 is of order $|\mu|^4$. As one can see, the change in masses due to high energy effects is of order $|\mu^2\epsilon^2|$, which, by choosing a sufficiently small ϵ, can be made much smaller than the one loop mass derived above. We therefore do not have to worry about high energy corrections making the masses tachyonic.

What about the nonperturbative effective superpotential derived above? Does not the $\tilde\Lambda$ in (5.57) signal high energy physics at work? The answer is no. The $\tilde\Lambda$ entered as a way to describe the running of the coupling constant at energies of order $\langle h\Phi\rangle$. Since this is much smaller than $\tilde\Lambda$, the appearance of $\tilde\Lambda$ does not have anything to do with high energy dynamics. Can the contribution due to the Kähler potential invalidate the conclusions regarding dynamical supersymmetry restoration? Calculating the potential from (5.56) gives

$$V \sim |\mu^2|^2+|\mu|^2\frac{\Phi^{\frac{N_F-N}{N}}}{\tilde\Lambda^{\frac{N_F-3N}{N}}}. \tag{5.62}$$

We see that the contributions are equally big (or tiny, $\delta V\sim\mu^4\epsilon^{\frac{4N}{N_F-3N}}$) when $\Phi\sim\tilde\Lambda\epsilon^{\frac{2N}{N_F-3N}}\ll\langle\Phi\rangle_{SUSY}$. This means that in the region of dynamical supersymmetry restoration, the contribution coming from high energy effects is negligible and our conclusions stand unaltered.

5.7 Lifetime of the Metastable Vacuum

In order for the metastable vacuum found earlier in this chapter to be phenomenologically acceptable it is important that its lifetime can be made arbitrary large by choosing ϵ sufficiently small. It is therefore important to calculate how the lifetime depends on ϵ.

From ordinary quantum mechanics we know that a particle moving in a one dimensional potential $V(x)$ can tunnel trough a classically inaccessible region with a probability proportional to e^{-B}, where

$B = 2\int dx\sqrt{2V(x)}$ and where the integration runs over the classically inaccessible region. A similar expression is used for a multidimensional potential. The subtlety is that the integral now runs along the specific path minimizing B. The path corresponding to the minimal B must fulfill the Euler-Lagrange equations derived from the expression $\delta B = 0$. By massaging these expressions it has been shown [24] that one can write B as

$$B = \int_{-\infty}^{\infty} d\tau \left(\frac{1}{2}\frac{d\vec{q}}{d\tau} \cdot \frac{d\vec{q}}{d\tau} + V(\vec{q}) \right) , \tag{5.63}$$

which is equal to the action of a classical particle rolling in the 'up side down' potential from the entering point to the exit point and back (τ here has the function of time). B is therefore called the bounce action. The equation of motion coming from B is related to their classical counterparts trough $\tau = it$.

Let us now consider a scalar quantum field with the Lagrangian $\mathcal{L} = \frac{1}{2}\partial_\mu\phi\partial^\mu\phi - V(\phi)$. This can be seen as an infinite dimensional generalization of the previous case with the potential $V(\phi) + (\vec{\nabla}\phi)^2$ and where the scalar product has been replaced with a integration over space. The scalar field generalization of the bounce action is therefore

$$B = \int d^3x d\tau \left(\frac{1}{2}\left(\frac{d\phi}{d\tau}\right)^2 + \frac{1}{2}\left(\vec{\nabla}\phi\right)^2 + V(\phi) \right) . \tag{5.64}$$

For a generic potential $V(\phi)$, it turns out that finding the minimal B is a hard task. However, we are not interested in the exact value of B but only care about how it depends on ϵ. A qualitative version of the effective potential is therefore sufficient for our purposes.

As discussed earlier, the supersymmetry breaking metastable vacuum is at $\Phi = \varphi_2 = \tilde{\varphi}_2 = 0$ and $\varphi_1 = \tilde{\varphi}_1 = \mu\mathbf{1}_N$ and has the vacuum energy $V_{meta} = h^2(N_F - N)\mu^4$. The supersymmetric vacuum is at $\varphi = \tilde{\varphi} = 0$ and $\Phi = \mu/h\epsilon^{\frac{N_F-3N}{N_F-N}}$. To find the path with minimal bounce action between these points we have to take a look at the classical potential (we omit the one loop correction calculated earlier since it is small and only applies around the metastable vacuum):

$$V_{cl} = \left|h\left(\varphi\tilde{\varphi} - \mu^2\mathbf{1}_{N_F}\right)\right|^2 + \left|h\Phi^T\varphi\right| + \left|h\Phi\tilde{\varphi}\right|^2 \tag{5.65}$$

We see that the last two terms become large for large Φ if we do not have $\varphi = \tilde{\varphi} = 0$. We should therefore decrease φ and $\tilde{\varphi}$ before Φ becomes too large. However, we are not allowed to decrease φ and $\tilde{\varphi}$ too much because of the first term. The path of minimal potential turns out to be approximately $\varphi = \tilde{\varphi} = \sqrt{\mu^2 - 2\Phi^2}$ for $\Phi^2 < \mu^2/2$. This path has its

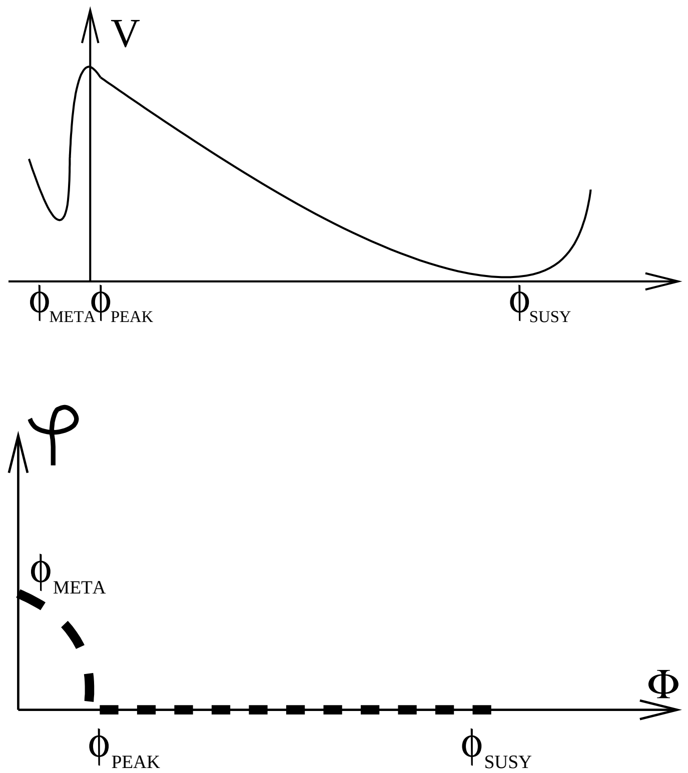

Figure 5.1: Top: A qualitative figure showing how the potential varies during the path with minimal potential. Bottom: A qualitative figure of the approximative path with minimal potential in the φ-Φ-plane.

peak value at $\varphi = \tilde{\varphi} = 0$ and $\Phi = \mu^2/2$ where $V_{peak} = h^2 N_F \mu^4$. From this point the potential slowly decreases because of the non-perturbative effects until it finally reaches the supersymmetric vacuum. A schematic picture of the potential and the path can be seen in Figure 5.1.

It is convenient to approximate the barrier with a triangle since the analytic expression for the bouncing action is known for this case, see [25]. The triangular barrier is characterized by the initial energy (V_+) and field (ϕ_+), the final energy (V_-) and field (ϕ_-) and the peak energy (V_T) and field (ϕ_T). If

$$\sqrt{\frac{\Delta V_-}{\Delta V_+}} \geq \frac{2\Delta\phi_-}{\Delta\phi_- - \Delta\phi_+}\,, \tag{5.66}$$

where $\Delta\phi_\pm = \pm(\phi_T - \phi_\pm)$ and $\Delta V_\pm = V_T - V_\pm$, the bounce action is

$$B^{(1)} = \frac{32\pi^2}{3} \frac{1+c}{\left(\sqrt{1+c}-1\right)^4} \frac{\Delta\phi_+^4}{\Delta V_+}\,, \tag{5.67}$$

where $c = \frac{\Delta V_- \Delta\phi_+}{\Delta V_+ \Delta\phi_-}$. If the inequality is not satisfied, the bounce action is

$$B^{(2)} = \frac{1}{96}\pi^2 \lambda_+^2 R_T^3 \left(-\beta_+^3 + 3c\beta_+^2\beta_- + 3c\beta_+\beta_-^2 - c^2\beta_-^3\right) , \tag{5.68}$$

where $\lambda_\pm = \Delta V_\pm/\Delta\phi_\pm$, $\beta_\pm = \sqrt{8\Delta\phi_\pm/\lambda_\pm}$ and $R_T = \frac{1}{2}\left(\frac{\beta_+^2 + c\beta_-^2}{c\beta_- - \beta_+}\right)$. In our case we have $\Delta V_+ = h^2\mu^4 N$, $\Delta V_- = h^2\mu^4 N_F$, $\Delta\phi_+ = \mu$ and $\Delta\phi_- = \mu/h\epsilon^{\frac{N_F-3N}{N_F-N}}$. The condition (5.66) can, for small ϵ, be simplified to $\sqrt{\frac{N_F}{N}} \geq 2$. Since we only know that $N_F > 3N_C$ we have to compute both bounce actions. These turn out to be

$$B^{(1)} = \frac{512\pi^2 N^3}{3N_F^4 h^6 \epsilon^{4\frac{N_F-3N}{N_F-N}}} \text{ and } B^{(2)} = \frac{2\pi^2}{3h^6\epsilon^{4\frac{N_F-3N}{N_F-N}}} \frac{3\sqrt{N} - \sqrt{N_F}}{\left(\sqrt{N_F} - \sqrt{N}\right)^3}. \tag{5.69}$$

Although these quantities differ with some numerical factor, they are both proportional to $1/\epsilon^{4\frac{N_F-3N}{N_F-N}}$. This means that the tunneling probability goes as $\text{Exp}\left(-A \middle/ \epsilon^{4\frac{N_F-3N}{N_F-N}}\right)$ for some constant A. The metastable vacuum can hence be made arbitrarily long lived by choosing a sufficiently small ϵ.

6

Conclusions

In Chapter 3 we discussed spontaneous supersymmetry breaking and some of the phenomenological difficulties tied to it. In particular, we discussed the way the absence of a spontaneously broken R-symmetry generically signals the existence of a supersymmetric vacuum. Given that massive SQCD does not have an anomaly free R-symmetry (not to mention the fact that it has a non zero Witten index), it is not, at least at first glance, an attractive candidate for supersymmetry breaking. However, using the dual theory we showed that, in addition to the supersymmetric vacua, the theory has a parametrically long-lived metastable supersymmetry breaking vacuum.

This clearly opens the door to a new family of supersymmetric gauge theories. In fact, in [26] Intriligator et. al. argue that metastable supersymmetry breaking is inevitable. The argument can be summarized by the following. If gravitational effects are not sufficient to give the R-axion a sufficiently big mass, a small explicit R-symmetry breaking term of order ϵ has to be added to the superpotential. Since such terms tend to restore supersymmetry for field strengthes proportional to ϵ^{-1}, the supersymmetry breaking state is now only metastable at best. This would indeed be an intriguing development.

Does this way of breaking supersymmetry solve the phenomenological problems associated with the R-symmetry? Because no R-symmetry is broken, there are no R-axions. Are then some gauginos forced to be massless? Since the gauge degrees of freedom are integrated out, there are no gauginos for which the mass problem can occur. The situation is therefore rather similar to the basic O'Raifeartaigh model, which to one loop did not have neither R-axions nor gauginos. Nevertheless, the metastable supersymmetry breaking theory has the advantage that the

R-symmetry is anomalous. If further gauge symmetries are introduced, we might therefore be in a better situation with regard to the gaugino masses.

This method is used in certain types of supersymmetry breaking. In supersymmetry breaking by direct mediation the theory has one supersymmetry breaking sector and one sector corresponding to the minimal supersymmetric standard model. A subgroup of the global symmetry group of the supersymmetry breaking sector is then gauged and identified with (a subgroup of) the standard model gauge group. It would very interesting to study this further. Another interesting topic to study is whether the metastable vacuum is favored in the early rapidly expanding universe [27, 28, 29].

It should be stressed that the model studied in this thesis is just a toy model and that a great deal of work is needed in order to (possibly) construct a phenomenologically acceptable model. Nevertheless, metastable supersymmetry breaking offers a new much needed method of spontaneous supersymmetry breaking.

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