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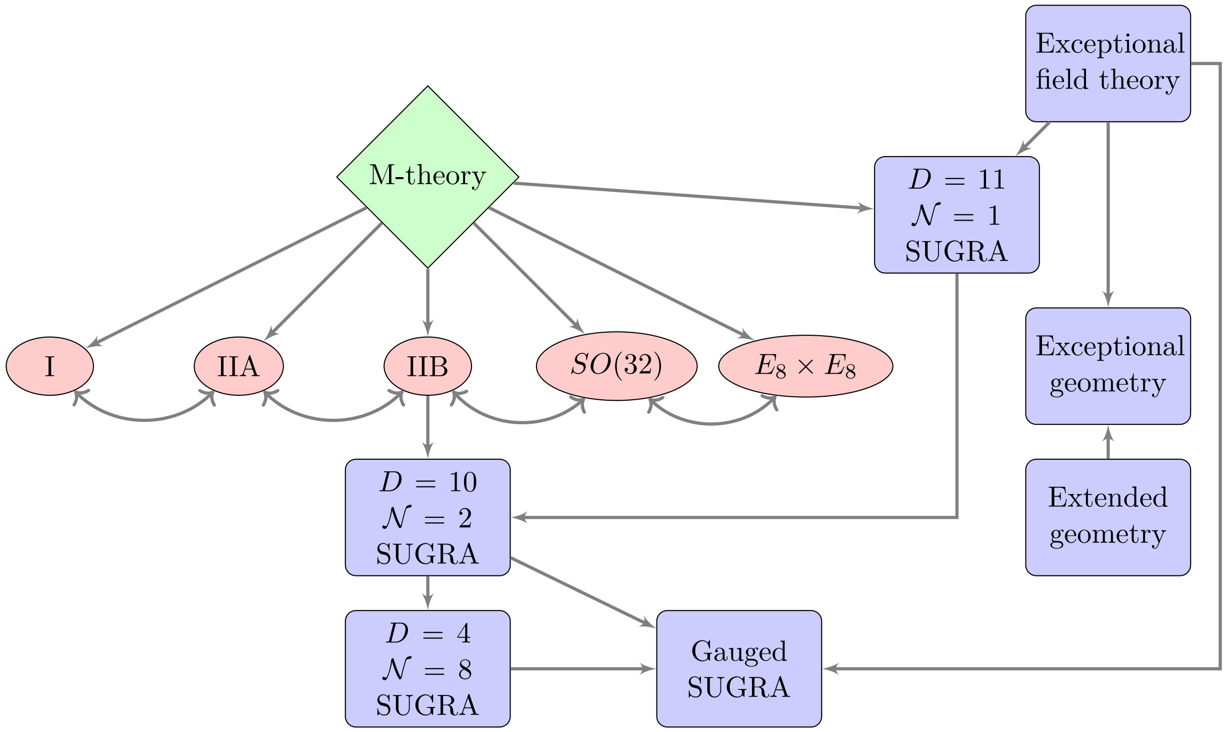

Extended geometries

Master's thesis in Physics and Astronomy

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CHALMERS UNIVERSITY OF TECHNOLOGY
Gothenburg, Sweden 2018

Master's thesis 2018

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Extensions of geometry motivated by the symmetries of string theory/M-theory

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Cover: Schematic diagram of *some* of the relations between M-theory, string theories (ellipses), supergravity theories and extended geometries.

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Abstract

The low-energy effective field theories of string theory/M-theory compactified on torii possess “hidden” symmetries given by the exceptional Lie groups. Moreover a discrete version of these, U-duality, appears to be unbroken in the full theory. Generically such transformations mix gravitational and non-gravitational degrees of freedom and a covariant formalism calls for a merger of the underlying fields. The goal of extended geometries is therefore to generalise ordinary geometry in order to include non-gravitational degrees of freedom as a part of an extended geometry. This is done by introducing space-time coordinates in a module of an arbitrary structure group and by the introduction of generalised diffeomorphisms. The physically most interesting cases, being double geometry and exceptional geometry based on even-dimensional orthogonal groups and exceptional groups respectively, are reviewed before the general construction of extended geometries is introduced. Especially we focus on closure of the algebra of generalised diffeomorphisms which for consistency requires a local embedding of ordinary geometry specified by the so-called section constraint. Moreover, an invariant action for the generalised metric is derived as well as geometrical objects such as torsion and a generalised Ricci-tensor. Lastly exceptional geometry based on the rank seven exceptional group with a global solution to the section constraint is used to study properties of flux compactifications from eleven-dimensional supergravity to four dimensions.

Keywords: Extended geometry, U-duality, M-theory, Symmetry.

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Robin Karlsson, Gothenburg, June 2018

Contents

1

Introduction

During the 20th century fundamental physics was split into two overarching categories: quantum physics governing the subatomic world through the strong, weak and electromagnetic forces and the theory of general relativity describing gravity. The latter tells the story of gravity warping space-time such that an object in it follows the shortest path in a curved space-time. On the other hand we have the other three forces. These are described by quantum field theories in which particles arise as excitations of quantized fields. This framework works extraordinarily well to high precision tested by particle accelerators around the world, e.g. at the Large Hadron Collider at CERN Geneva. Quantum field theory is conventionally placed in a flat space-time, in agreement with Einsteins ingenious equivalence principle, stating roughly that in a sufficiently small region space-time looks flat, and the fact that the force of gravity is many orders of magnitudes weaker than the other forces. However, a black hole presents a region of space-time where gravity is very strong and one should start to worry about quantum effects and gravity being of similar strength. Stephen Hawking thought about this and reached the conclusion that quantum effects would lead to evaporation of the black hole, in stark contradiction to the classical result. Except for gravity the development has been to take a theory, say electromagnetism coupled to matter, and quantise the fields of the theory. One would therefore try to do the same with the gravitational field. However, this turns out to be incredibly hard and pathological; quantising gravity turns out to give a non-renormalisable theory unable to predict physical observables in contrast to the renormalisable theories of the other forces.

A common tool for fundamental physics during the last century has been the use of symmetries. Notably the standard model of particle physics relies on the gauge group $SU(3) \times SU(2) \times U(1)$ and without this symmetry the theory would lose its predictive power. In fact, even the particle spectrum is in some sense determined, or at least restrained, by ensuring that these symmetries hold non-perturbatively through so-called anomaly cancellations. General relativity on the other hand is governed by the symmetry of diffeomorphism invariance, which roughly speaking implies that the laws of physics look the same in an arbitrary coordinate system.

Finding a consistent quantisation of gravity, quantum gravity, has been an ongoing field of research for many years, the most promising theory being string theory. String theory replaces the notion of the most fundamental objects as pointlike to be one-dimensional extended objects, strings. Oscillations of such strings are interpreted as different particles of which it turns out that the graviton, the particle mediating gravity, is present in any consistent theory of quantised strings. It turns out that extended objects perceive the physical world quite differently than particles which seems to make the theory more well behaved than their pointlike counterparts. However, one striking implication of string theory is that it can be consistently formulated only in ten space-time dimensions, which seems to be in contradiction with our everyday experience of only four dimensions. One way of dealing with this is the idea of compactification where certain spacelike directions are "small" compared to the others, such that the effective dimension perceived is lower. Moreover, five consistent theories of strings have been developed: Type I, Type IIA/B and two heterotic string theories. These were thought to be five inequivalent theories and the hope was that one of them should describe our physical world. However, it was argued by Witten [1] that they all were in fact related to each other through dualities. Even more so, their description at low energies can be derived from a theory in eleven dimensions dubbed M-theory. This theory in one dimension higher is known at low energies while the full theory still remains unknown.

1.1 Dualities and extended geometries

A key feature of string theory dualities is that gravitational and non-gravitational degrees of freedom are mixed. However, gravity describes the geometry of space-time which is conceptually different from other fields living in this space-time. In order to make the dualities manifest symmetries and unify these degrees of freedom we should either generalise the notion of geometry and space-time or in some way change our perspective of gravity. In this thesis we follow the former and try to extend the ordinary notion of geometry to also include non-gravitational degrees of freedom as part of the geometry and space-time.

Compactification of eleven-dimensional supergravity, the low energy effective field theory of M-theory, on a torus T^d has been known for a long time to possess a rigid global hidden symmetry group[1] $E_{d(d)}$[2, 3, 4], which turns out to be the continuous version of the discrete duality groups. The goal of exceptional geometry is to make the action of the duality group manifest in the formulation of the theory prior to any compactification. This is done by replacing the structure group $GL(d)$ by $E_{d(d)} \times \mathbb{R}$ as well as to consider an enlarged space-time with coordinates transforming in a module of the structure group. To relate these exceptional geometries to a theory based on ordinary geometry a choice of a "physical subspace" of space-time is chosen by the so-called section constraint. In other words, by defining a local embedding $GL(d) \hookrightarrow E_{d(d)} \times \mathbb{R}^+$ the ordinary notion of geometry is obtained. Solving the section constraint is crucial for the consistency of the theory and importantly this is done in a covariant manner.

Extended geometry [5] is a general formalism for describing a geometry with a structure algebra that is a split real form of a Kac-Moody algebra $\mathfrak{g} \times \mathbb{R}$, which in turn exponentiates to a structure group $G \times \mathbb{R}^+$. One then introduce generalised diffeomorphisms, in analogy to ordinary diffeomorphisms, and demand that the theory is invariant under such transformations. Especially, in order to ensure covariance the algebra of generalised diffeomorphisms needs to close which as in the exceptional case demands a section constraint. The dynamics of a generalised metric can then be formulated in terms of an invariant action and geometrical objects such as connection, torsion and a generalised Ricci tensor can be constructed. Note that as special cases this formalism includes both double geometry and exceptional geometry which are the two physically most interesting cases.

[1] This group will be called the duality group and it should be clear from context whether this concerns the continuous or the discrete version.

Moreover, the construction of extended geometries hints about a deep connection between geometry on one side and certain algebraic structures on the other. Typically such algebraic structures arise from extensions of the structure algebra; important examples are given by adding an even or odd simple root to the Dynkin diagram of $\mathfrak{g}$ such that one obtains a Kac-Moody algebra or a Borcherds superalgebra respectively [6, 7]. Such extensions are often infinite-dimensional and as such encode a rich algebraic structure of the theory. Even larger structures which seems to have even deeper connection to extended geometries are the tensor hierarchy algebra [8, 9] and L_∞ algebras [10, 11].

1.2 Flux compactifications and generalised geometry

String theory and M-theory are well defined in ten and eleven dimensions respectively and at face value seems far-fetched from our observed reality. However, assuming that space-time is divided into an external and an internal space-time offers a solution to this problem. Since the extra dimensions of the compact internal space-time is not observed at low energies the characteristic length scale is supposedly small compared to the length scale with which we currently can probe nature.

From an external point of view when compactifying a theory the purely internal degrees of freedom are described by scalar fields. These scalar fields do e.g. determine the geometry of the internal space as well as any internal flux of gauge fields. Typically the simplest compactifications, e.g. on torii, produces an excess number of massless such scalar fields, or moduli, that leads to many different problems, one of which is that they would give rise to a long-ranged fifth force that is not observed experimentally. To obtain a phenomenologically more interesting theory the internal geometry either needs to be more complicated or one could allow for gauge fields with fluxes along the internal directions. Flux compactifications [12] have received much interest in the last decade leading to the notion of a large string theory landscape. However, flux compactifications are rather difficult to deal with while the purely geometrical backgrounds can be dealt with more easily using geometrical tools.

Flux compactifications it turns out can be studied effectively using (exceptional) generalised geometry. These are geometries obtained by solving the section constraint globally, i.e. choosing a physical subspace of coordinates globally, and which by construction has a manifest action of the exceptional groups. Especially, this formalism naturally includes the presence of non-trivial gauge fields as part of the geometry. For this reason there exists appropriate generalisations of the geometrical tools used in the flux-less case. These allow us to study compactifications with fluxes using again a (generalised) geometrical language [13, 14, 15, 16].

1.3 Outline

The second chapter provides a short introduction to Lie algebras that will be crucial for the remaining part of the thesis. Moreover, Kac-Moody algebras as well as Borcherds superalgebras are introduced by two examples that are of much importance for the construction of extended geometries. Chapter 3 introduces the most basic notions of string theory with the goal of motivating the appearance of T-dualities of the bosonic string.

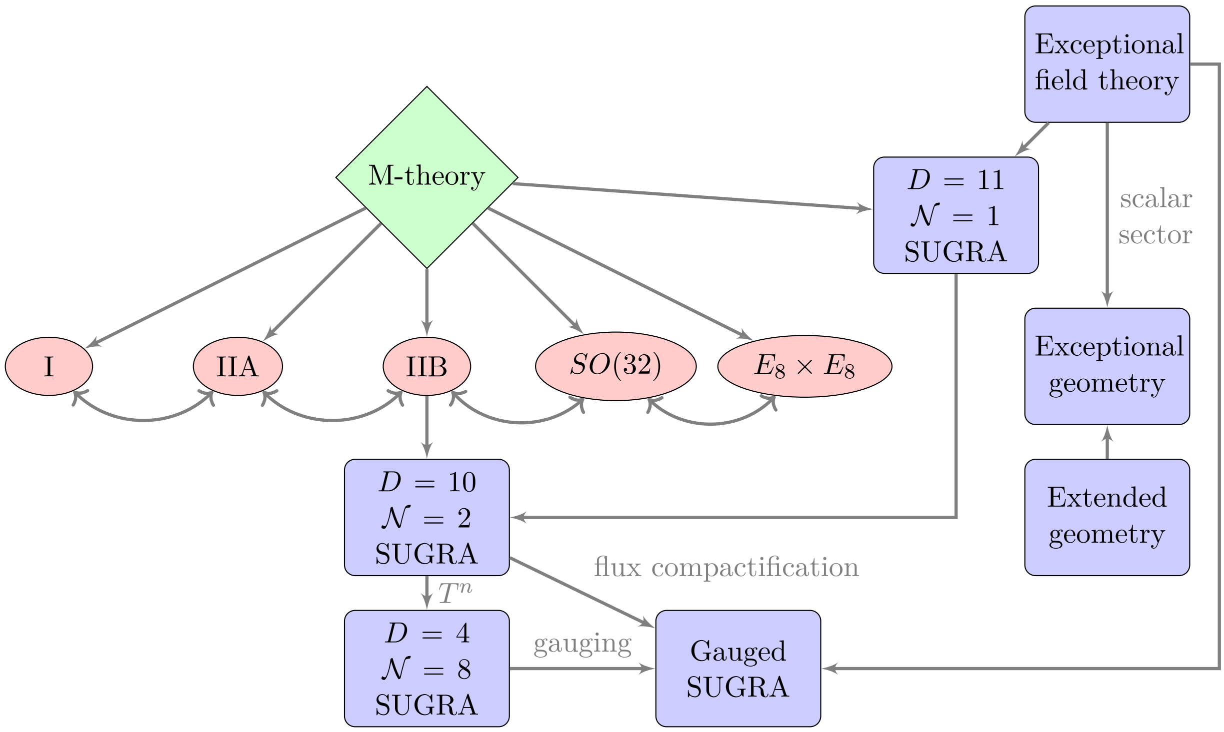

Figure 1.1: Schematic diagram of *some* of the relations between M-theory, string theories (ellipses), supergravity theories and extended geometries.

Furthermore the field content of supergravity in ten and eleven dimensions is introduced as well as some instructive examples of compactifications. The goal of this is to put into context the somewhat abstract ideas of extended geometries that this thesis introduce. Lastly in this chapter the notion of "dualities" in general is discussed in more detail. In Chapters 4 and 5 double field theory and exceptional field theory respectively are introduced. These theories are formulated with a manifest action of the duality groups and one retrieves ordinary supergravity theories as certain solutions of the section constraint. Especially they provide the two physically most interesting examples of extended geometries and, moreover, also show how to include the external space-time which is not included in the pure formulation of extended geometries. The main part of this thesis concerns the formulation of extended geometries introduced in Chapter 6. By analysing the consistency of generalised diffeomorphisms a complete list of extended geometries without so-called ancillary transformations are found. Furthermore, a pseudo-action invariant under generalised diffeomorphisms encoding the dynamics of the generalised metric is formulated. We then continue by introducing geometrical objects such as a connection, torsion and generalised Ricci tensor characterising an extended geometry. The goal of the last chapter is to study flux compactifications with a (exceptional) generalised geometrical formulation. To do this some ordinary geometrical tools used for fluxless compactifications are introduced as well as the basic concepts of generalised geometry. A schematic diagram of the relations between some of the different theories discussed in this thesis is presented in Figure 1.1.

2

Symmetry algebras

A cornerstone of extended geometries and any fundamental theory of physics is symmetry. Hence, this chapter aims to introduce the corresponding mathematics describing these symmetries, especially Lie algebras and their representation theory. In order to construct extended geometries we need, in particular, to understand Dynkin diagrams, weights and representation theory. Moreover, we extend the finite-dimensional Lie algebras to certain infinite-dimensional algebras, Kac-Moody algebras as well as Borcherds superalgebras, which encode important information about the theory.

Finite-dimensional simple Lie algebras have been completely classified and come in four different (infinite) families

$$\mathfrak{a}_{n-1} \cong \mathfrak{sl}_n, \quad \mathfrak{b}_n \cong \mathfrak{so}_{2n+1}, \quad \mathfrak{c}_n \cong \mathfrak{sp}_{2n}, \quad \mathfrak{d}_n \cong \mathfrak{so}_{2n},$$

together with five exceptional algebras

$$\mathfrak{g}_2, \quad \mathfrak{f}_4, \quad \mathfrak{e}_6, \quad \mathfrak{e}_7, \quad \mathfrak{e}_8.$$

The two main examples that are of interest in this thesis are the $\mathfrak{so}_{2n}$ family and the so-called $\mathfrak{e}_n$-series. The general construction of extended geometries is, however, based on an arbitrary Kac-Moody algebra $\mathfrak{g}$.

A complete introduction to the theory of Lie algebras is beyond the scope of this thesis. Instead the goal is to introduce main concepts that are of interest later on. For a thorough introduction to finite-dimensional Lie algebras we refer to [17, 18, 19] and for infinite-dimensional algebras [20]. In Section 2.1 Lie groups, representations and how to retrieve the corresponding Lie algebra is reviewed. The theory of Lie algebras is then introduced in Section 2.2. Extensions of finite-dimensional Lie algebras to Kac-Moody algebras and to Borcherds superalgebras are then described in Section 2.3.1 and 2.3.2 respectively. Two specific examples of extended algebras are introduced which is used later on for the construction of extended geometries. The appearance of these kinds of extended symmetry algebras in string theory and supergravity is further presented in [6, 7, 21, 22].

2.1 Lie groups and their connection to Lie algebras

A group is simply a set G together with an associative multiplication rule '$\cdot$' such that the following properties hold[1] (for any $g, h, k \in G$)

$$
\begin{aligned}
&& g \cdot h &\in G, && \text{closure}\\
&& g \cdot (h \cdot k) &= (g \cdot h) \cdot k, && \text{(associativity)}\\
\exists\, e \in G \quad & \text{s.t.} & e \cdot g &= g = g \cdot e, && \text{(unit element)}\\
\exists\, g^{-1} \in G \quad & \text{s.t.} & g^{-1} \cdot g &= e = g \cdot g^{-1}. && \text{(inverse element)}
\end{aligned}
$$

These axioms are natural properties of symmetry transformations which motivates the use of groups to describe symmetries. The main interest in this thesis will be in continuous infinite-dimensional groups called *Lie groups*.

A Lie group G is a group which is also a differentiable manifold together with a group operation which is smooth. A typical example of such a group is $SO(3)$ describing rotations in three dimensions. Elements in the group acts as symmetry operations on physical objects such as fields and how they act are described by *representations* $(\rho, \mathbb{V})$[2], where $\mathbb{V}$ is a module (vector space) on which the group acts according to the linear map ρ. The map ρ is further a homomorphism such that for $g_1, g_2 \in G$

$$\rho : G \to \mathrm{Aut}(\mathbb{V}) \qquad \rho(g_1 \cdot g_2) = \rho(g_1) \circ \rho(g_2), \tag{2.1}$$

where '$\cdot$' denotes group multiplication and '$\circ$' composition of linear maps (matrix multiplication for finite-dimensional $\mathbb{V}$). A subrepresentation $\mathbb{W} \subset \mathbb{V}$ is a subspace which is preserved under the action of G, i.e.

$$\rho(g)w \in \mathbb{W} \qquad \text{for any} \quad g \in G, \quad w \in \mathbb{W}. \tag{2.2}$$

Importantly, a representation is called irreducible if it does not contain any subrepresentation except for the trivial representation $\{0\}$ and $\mathbb{V}$ itself. Given two representations $(\rho_1, \mathbb{V}_1)$ and $(\rho_2, \mathbb{V}_2)$ it is possible to build two new representations, the direct sum representation and the tensor product representation

$$(\mathbb{V}_1 \oplus \mathbb{V}_2, \rho_\oplus), \qquad \rho_\oplus : g \mapsto \rho_1(g) \oplus \rho_2(g), \tag{2.3}$$
$$(\mathbb{V}_1 \otimes \mathbb{V}_2, \rho_\otimes), \qquad \rho_\otimes : g \mapsto \rho_1(g) \otimes \rho_2(g). \tag{2.4}$$

Moreover, a representation is called indecomposable if it can not be described as a direct sum of representations. Irreducible and indecomposable representations are thus "building blocks" for a general representation. Note that a decomposable representation is reducible while a reducible representation need not be decomposable. However, in the cases we are looking at reducible representations will be decomposable. A given tensor product representation of two irreducible representations R_1 and R_2 can then be decomposed as a direct sum of irreducible representations R_i as

$$R_1 \otimes R_2 = \bigoplus_i R_i. \tag{2.5}$$

[1]We denote groups with capital letters G and algebras with the $\mathfrak{mathfrak}$ font, e.g. $\mathfrak{g}$.

[2]Quite often a "representation" is interchangeably used for ρ and $\mathbb{V}$, we adopt to this as it is usually clear from context what is meant.

In general a Lie group is a curved manifold which makes it somewhat difficult to work with. Hence, it is often beneficial to linearise and look at infinitesimal transformations at the identity element. Consider the case when $G \in \mathrm{Aut}(\mathbb{V})$, i.e. G is a matrix group, and take $t \in [-a, a]$ with $a \in \mathbb{R}_+$ and the set of curves $\gamma(t) : [-a, a] \to G$ such that $\gamma(0) = e \in G$. The tangent space at the identity $T_\mathrm{e}G$ then consist of elements $\tilde{g}$ in the corresponding *Lie algebra*

$$\tilde{g} = \frac{\mathrm{d}}{\mathrm{d}t}\gamma(t)|_{t=0}. \tag{2.6}$$

What is gained is that the tangent space $T_\mathrm{e}G$ is a vector space which in most situations are easier to deal with. Moreover, the dimension of the Lie algebra is determined by the dimensions of the manifold G. On the other hand one also introduce the exponential map $\exp : \mathfrak{g} \to G$ in order to retrieve the group. For matrix groups this map is simply the exponential of matrices in the usual sense, hence the name. Note that the topology of the group becomes important when going from the algebra to the group. For a compact connected group G the exponential map is onto such that any element in G can be written using elements of the algebra. On the other hand, for a compact group that is not connected only the connected part can be reached.

The group G carries a representation on the Lie algebra called the adjoint representation $\mathrm{Ad} : G \to \mathrm{Aut}(\mathfrak{g})$ by $g \in G$ and $\tilde{g} \in \mathfrak{g}$ acting as

$$\mathrm{Ad}_g(\tilde{g}) = g\tilde{g}g^{-1}. \tag{2.7}$$

This is well-defined since the elements of the group acts naturally on the curve γ used to define $\tilde{g}$. In the same spirit we can look at the action of a curve passing through the identity element acting on $\mathfrak{g}$ with $h, \tilde{g} \in \mathfrak{g}$ which defines the adjoint action of the algebra

$$\mathrm{ad}(h) : \mathfrak{g} \to \mathrm{End}(\mathfrak{g}), \qquad h \mapsto \mathrm{ad}(h)(\tilde{g}), \tag{2.8}$$

according to

$$\mathrm{ad}(h)(g) = \frac{\mathrm{d}}{\mathrm{d}t}\gamma_h(t)\tilde{g}\gamma_h^{-1}(t)|_{t=0}. \tag{2.9}$$

It is easily seen that on matrix algebras the adjoint action is given by the commutator

$$\mathrm{ad}(h)(g) = [h, g], \tag{2.10}$$

which equips the tangent space with an anti-symmetric bilinear product, and hence defining an algebra.

2.2 Lie algebras

A Lie algebra $\mathfrak{g}$ is a vector space with an anti-symmetric bilinear product called the Lie bracket $[\![\cdot,\cdot]\!] : \mathfrak{g} \times \mathfrak{g} \to \mathfrak{g}$ that satisfies the Jacobi identity $a, b, c \in \mathfrak{g}$

$$[\![a, [\![b, c]\!]]\!] + [\![b, [\![c, a]\!]]\!] + [\![c, [\![a, b]\!]]\!] = 0, \tag{2.11}$$

which can easily be seen to be true for matrix commutators. Note that the Lie bracket is non-associative, in fact the Jacobi identity can be seen as a condition to ensure associativity of the Lie group. A representation of the algebra is given by a module $\mathbb{V}$ and a linear map $\rho : \mathfrak{g} \to \mathrm{End}(\mathbb{V})$ which satisfies

$$\rho([\![a, b]\!]) = [\rho(a), \rho(b)], \tag{2.12}$$

where $[\cdot,\cdot]$ denotes the matrix commutator. This differs from the property of a representation for the group, but follows from consistency with eq. (2.7). As for groups there is a tensor product representation for Lie algebras. Given two representations the tensor product representation is defined by the module $\mathbb{V}_1 \otimes \mathbb{V}_2$, with an action given by

$$\begin{aligned} \rho_\otimes : \quad & \mathfrak{g} \to \mathrm{End}\left(\mathbb{V}_1 \otimes \mathbb{V}_2\right), \\ & g \mapsto \rho_1(g) \otimes \mathbf{1} + \mathbf{1} \otimes \rho_2(g), \end{aligned} \tag{2.13}$$

where $\mathbf{1}$ is the identity matrix.

Since $\mathfrak{g}$ is a vector space it is possible to introduce a basis T^a, where $a = 1, 2, \ldots, \dim \mathfrak{g}$, and write the Lie bracket as

$$[\![T^a, T^b]\!] = f^{ab}{}_c T^c, \tag{2.14}$$

where repeated indices are summed over. The $f^{ab}{}_c$ are called the structure constants and specify the algebra. If the structure constants vanish the algebra is called abelian.

A linear subspace $\mathfrak{h} \subset \mathfrak{g}$ is a subalgebra of $\mathfrak{g}$ if

$$[\![\mathfrak{h}, \mathfrak{h}]\!] \subseteq \mathfrak{h}, \tag{2.15}$$

and it is called an invariant subalgebra, or ideal, if also

$$[\![\mathfrak{h}, \mathfrak{g}]\!] \subseteq \mathfrak{h}. \tag{2.16}$$

This give the important notion of simple and semi-simple algebras: a semi-simple algebra is an algebra with no abelian ideals and a simple algebra, on the other hand, is an algebra containing no proper ideals (i.e. no ideals other than the $\{0\}$ and $\mathfrak{g}$ itself). Simple algebras can then be used as building blocks for more general algebras, for example the Lie algebra related to the standard model of particle physics with gauge group $SU(3) \times SU(2) \times U(1)$ is given by $\mathfrak{g} = \mathfrak{su}(3) \oplus \mathfrak{su}(2) \oplus \mathfrak{u}(1)$, with each component being a simple algebra.

So far the algebras have been implicitly assumed real, i.e. real linear combinations of elements in the vector space. Note, however, that the matrix algebras such as $\mathfrak{su}(2)$ still contains complex entries. It is often convenient to extend the field ($\mathbb{R} \to \mathbb{C}$) and allow for complex linear combinations instead, which we will do from now on unless specified otherwise. One consequence of this is that algebras that are inequivalent over $\mathbb{R}$ could be equivalent upon complexification, e.g. $\mathfrak{su}(2)_\mathbb{C} \cong \mathfrak{sl}(2,\mathbb{C})$ while $\mathfrak{su}(2)_\mathbb{R} \ncong \mathfrak{sl}(2,\mathbb{R})$.

An important object is the symmetric invariant bilinear form of a simple Lie algebra called the Killing form. Given a representation ρ the Killing form is given by

$$\kappa(T^a, T^b) := \kappa^{ab} = \frac{1}{I_\rho} \mathrm{Tr}\left(\rho(T^a)\rho(T^b)\right), \tag{2.17}$$

where I_ρ is a normalisation constant that depends on the representation. Invariance of the Killing form means that for any $g, h, k \in \mathfrak{g}$ it satisfies

$$\kappa([\![g, h]\!], k) = \kappa(h, [\![g, k]\!]). \tag{2.18}$$

Interesting properties can be read off from the Killing form, e.g. a semi-simple algebra implies that the Killing form is non-degenerate. Moreover, if κ is negative definite, then the algebra is called compact meaning in turn that the corresponding exponentiated Lie group is a compact manifold. Importantly, one can also use the Killing form and its inverse to raise and lower adjoint indices.

2.2.1 Example – Chevalley set

As an example we will study $\mathfrak{a}_1 \cong \mathfrak{sl}(2,\mathbb{C})$ which is an important algebra because, as we will see later, more complicated algebras can be constructed as a collection of "interacting" $\mathfrak{a}_1$ subalgebras. This algebra consists of 2×2 traceless matrices over $\mathbb{C}$ and a possible choice of basis is

$$h = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}, \qquad e = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}, \qquad f = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}. \tag{2.19}$$

The structure constants in this basis are easily calculated to

$$[h,e] = 2e, \qquad [h,f] = -2e, \qquad [e,f] = h. \tag{2.20}$$

Observe that the elements e, f can thus be seen as step operators increasing(decreasing) the eigenvalue of h by ± 2. From a physicists point of view this is reminiscent of the $\{J_3, J_+, J_-\}$ basis of angular momentum in quantum mechanics and the action of the algebra is displayed in Figure 2.1. The Killing form in the fundamental representation, this is the representation with $\rho(g) = g$ for matrix algebras, is easily calculated and one finds

$$\kappa = \begin{pmatrix} 2 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix}, \tag{2.21}$$

with an ordering $\{h, e, f\}$. Indeed we find that $\det(\kappa) \neq 0$ as claimed for a simple Lie algebra.

Suppose that we have a finite-dimensional irreducible representation $\mathbb{V}$. Then due to the commutation relations one finds that h acts diagonally on vectors $v \in \mathbb{V}$ and can therefore be decomposed into eigenspaces of h according to

$$\mathbb{V} = \bigoplus_{\alpha} \mathbb{V}_\alpha, \tag{2.22}$$

where any $v \in \mathbb{V}_\alpha$ is an eigenvector of h such that[3] $hv_\alpha = \alpha v$. As for a spin representation in quantum mechanics one deduce that possible eigenvalues are integers symmetrically distributed around the origin. A representation for $\mathfrak{a}_1$ can thus be specified by a vector v_λ, where λ is the largest eigenvalue of h on $\mathbb{V}$. The representation is then spanned by states obtained by acting with the lowering operator f on v_λ

$$\mathbb{V} = \{v_\lambda, fv_\lambda, f^2 v_\lambda, \ldots, f^n v_\lambda\}. \tag{2.23}$$

Note that $ev_\lambda = 0$ as well as $f(f)^n v_\lambda = 0$, the vectors v_λ and $(f)^n v_n$ are therefore called highest and lowest weight vectors respectively. The eigenvalues of h on $\mathbb{V}$ are called weights, and especially, λ is called the highest weight.

2.2.2 Roots, weights and representations

Based on the $\mathfrak{a}_1$ example we will now build up the representation theory for an arbitrary simple Lie algebra $\mathfrak{g}$. The starting point is to find a maximal commuting subalgebra $\mathfrak{h}$

[3] Here we dropped the explicit use of $\rho(h)v := hv$, and we will continue to do so when convenient.

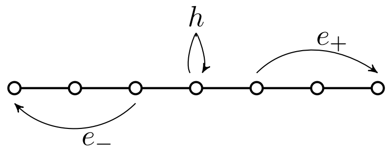

Figure 2.1: Illustration of the "weight" lattice for $\mathfrak{a}_1$. Action of the algebra is indicated.

called the Cartan subalgebra, which in the example above was just given by $\{h\}$. The Cartan subalgebra then act diagonally on any irreducible finite-dimensional representation $\mathbb{V}$. This property is analogous to the fact that commuting operators in quantum mechanics can be simultaneously diagonalisable. Hence, the representation can be decomposed as

$$\mathbb{V} = \bigoplus_{\alpha} \mathbb{V}_\alpha. \tag{2.24}$$

Compared to the case where $\mathfrak{h}$ were one-dimensional, the eigenvalue α in this case is rather a vector containing the eigenvalues for each element in $\mathfrak{h}$. Another way to put it is that $\alpha \in \mathfrak{h}^*$, i.e. α is a linear functional $\alpha : \mathfrak{h} \to \mathbb{C}$. The dimension r of the Cartan subalgebra is called the rank and is often denoted as a subscript on the algebra, e.g. $\mathfrak{g}_r$, $\mathfrak{a}_r$, $\mathfrak{e}_8$ and so on. Especially, decomposing the adjoint representation we find

$$\mathfrak{g} = \mathfrak{h} \oplus \bigoplus_{\alpha \in \Delta} \mathfrak{g}_\alpha, \tag{2.25}$$

where, by definition, for any $h \in \mathfrak{h}$ and $v_\alpha \in \mathfrak{g}_\alpha$

$$\mathrm{ad}_h(v_\alpha) = \alpha(h) v_\alpha. \tag{2.26}$$

This choice of basis is called the Cartan-Weyl basis. The sum in (2.25) is over a finite set $\Delta \subset \mathfrak{h}^*$ called the root system and the elements $\alpha \in \Delta$ are called roots. Moreover, it follows from the Jacobi identity that

$$\mathrm{ad}_{\mathfrak{g}_\alpha}(\mathfrak{g}_\beta) = [\![\mathfrak{g}_\alpha, \mathfrak{g}_\beta]\!] \subset \mathfrak{g}_{\alpha+\beta}. \tag{2.27}$$

We thus see that by acting with an element in $\mathfrak{g}_\alpha$ on some other element in $\mathfrak{g}_\beta$, one can "move" around in the algebra, or equivalently in the adjoint representation. Acting with the Cartan subalgebra on the other hand preserves the subspace $\mathfrak{g}_\beta$. One then defines the root lattice Λ_R, which simply is an r-dimensional lattice spanned by integral linear combinations of the roots $\alpha \in \Delta$. Furthermore, we split the root system Δ into positive and negative roots as $\Delta = \Delta^+ \cup \Delta^-$. This can be seen as a choice of a hyperplane in the root lattice such that it contains no roots, the set of roots thus gets divided in two disjoint unions. Intuitively the positive roots corresponds to raising operators while the negative roots acts as lowering operators. There are some arbitrariness in this choice, however, the particular choice is without significance as long as one sticks to it. There is a canonical choice of basis for the root lattice called simple roots; these are r positive roots such that any positive root is obtained by non-negative linear combinations of such roots.

The discussion above concerned the adjoint representation, let us continue to some arbitrary finite-dimensional irreducible representation $\mathbb{V}$. As above this can be decomposed into eigenspaces of $\mathfrak{h}$

$$\mathbb{V} = \bigoplus_{\beta} \mathbb{V}_\beta, \tag{2.28}$$

where β, called weights, lie in some finite subset of $\mathfrak{h}^*$. Using the defining property of Lie algebra representations one finds that by applying an element $e_\alpha \in \mathfrak{g}$ on $v_\alpha \in \mathbb{V}_\beta$ that

$$\rho(e_\alpha)v_\beta \in \mathbb{V}_{\alpha+\beta}. \tag{2.29}$$

Hence, root vectors (a root vector e_α is a vector in the vector space with the root α as eigenvalue) can again be used to "jump between" vectors in $\mathbb{V}$ with different eigenvalues/weights as in the $\mathfrak{a}_1$ case. Moreover, using non-degeneracy of the Killing form, one can show that if α is a root, then also $-\alpha$ is a root. This is important since one can then construct a collection of $\mathfrak{a}_1$ subalgebras of $\mathfrak{g}$ as

$$\mathfrak{a}_1^\alpha = e_\alpha \oplus e_{-\alpha} \oplus [\![e_\alpha, e_{-\alpha}]\!]. \tag{2.30}$$

Any representation $\mathbb{V}$ of $\mathfrak{g}$ must also be a representation of the subalgebras $\mathfrak{a}_1^\alpha$, and since the weights of $\mathfrak{a}_1$ are integer-valued it follow that also the weights of any representation $\mathbb{V}$ of $\mathfrak{g}$ are integer valued, i.e. a weight $\beta \in \mathfrak{h}^*$ satisfies $\beta(h) \in \mathbb{Z}$ for any $h \in \mathfrak{h}$. With this one defines the weight lattice Λ_W, which is a rank r lattice containing all weights β such that $\beta(\mathfrak{h}) \in \mathbb{Z}$. Any representation is therefore equivalent to a set of weights in Λ_W.

Importantly, any irreducible finite-dimensional representation can be characterised by a highest weight λ, as such we call the representation a highest weight representation. A highest weight λ and the corresponding highest weight vector is defined such that acting with any root vector corresponding to a positive root $\alpha \in \Delta^+$ annihilates the vector

$$e_\alpha v_\lambda = 0. \tag{2.31}$$

In the $\mathfrak{a}_1$ analogy with quantum mechanics this simply corresponds to the largest J_3 eigenvalue with the generalisation that λ is now a "vector of quantum numbers". Given any highest weight vector v_λ the corresponding highest weight representation $\mathbb{V}$ consist of elements obtained by acting with root vectors corresponding to negative roots $\alpha \in \Delta^-$ on v_λ. Since by assumption the representation is finite-dimensional and irreducible this construction eventually terminates.

The Killing form for semi-simple Lie algebras was introduced above as a non-degenerate symmetric bilinear form on the algebra. Being non-degenerate implies that it defines an inner product on $\mathfrak{g}$, and hence also an isomorphism between algebra $\mathfrak{g}$ and its dual space $\mathfrak{g}^*$. Therefore, since $\alpha \in \mathfrak{h}^*$, any root α is associated with an element $h_\alpha := \alpha^\vee$ called a coroot in the Cartan subalgebra such that

$$\beta(\alpha^\vee) = \frac{2(\alpha, \beta)}{(\alpha, \alpha)}, \tag{2.32}$$

for any $\beta \in \mathfrak{h}^*$. Moreover, this in turn can be used to define a non-degenerate inner product $(\cdot, \cdot)$ on $\mathfrak{h}^*$ according to

$$(\alpha, \beta) := c_\alpha c_\beta \kappa(h_\alpha, h_\beta), \tag{2.33}$$

for some constants $c_{\alpha,\beta}$. The coroots of simple roots α_i are called simple coroots $\alpha_i^\vee$. Using simple coroots a canonical basis on the weight space is given by the so called fundamental weights Λ_i satisfying

$$\Lambda_i(\alpha_j^\vee) = \delta_{ij}. \tag{2.34}$$

Any weight vector can thus be expanded in fundamental weights

$$\lambda = \sum_i \lambda_i \Lambda_j, \tag{2.35}$$

where the coefficients λ_i are called Dynkin labels. An irreducible finite-dimensional module of a semi-simple Lie algebra is uniquely defined by an integral dominant highest weight λ of multiplicity one, here integral dominant means that $\lambda_i \in \mathbb{Z}_{\geq 0}$.

A partial ordering is introduced between weights as follows: given two weights λ and μ, then $\lambda > \mu$ if $\lambda - \mu$ is expressible as a non-negative linear combination of positive roots. Moreover, the height h of a root β is defined as

$$\beta = \sum_{i=1}^{r} a_i \alpha_i \implies h = \sum_{i=1}^{r} a_i. \tag{2.36}$$

For a simple Lie algebra there exists a unique highest root, typically denoted θ, such that $h_\theta > h_\alpha$, for any other root α. Moreover, one defines the so-called Coxeter labels a_i and dual Coxeter labels $a_i^\vee$ as the coefficients (up to scaling) of the highest root in simple roots and coroots respectively according to

$$\theta = \sum_{i=1}^{r} a_i \alpha_i, \qquad \frac{2}{(\theta, \theta)} \theta = \sum_{i=1}^{r} a_i^\vee \alpha_i^\vee. \tag{2.37}$$

Yet another important object that is of interest is the so-called Weyl vector ρ. The Weyl vector is defined as

$$\rho := \frac{1}{2} \sum_{\alpha \in \Delta^+} \alpha, \tag{2.38}$$

and one can show that it can also be rewritten as

$$\rho = \sum_{i=1}^{r} \Lambda_i \tag{2.39}$$

2.2.3 Cartan matrix and Dynkin diagram

We have seen that a Lie algebra is specified by its root system. Another way to characterise a Lie algebra which is convenient for further extensions later on, is through its Cartan matrix. The $r \times r$-dimensional Cartan matrix can be obtained from the root system as

$$a_{ij} := \alpha_j(\alpha_i^\vee) = \frac{2}{(\alpha_i, \alpha_i)} (\alpha_i, \alpha_j), \tag{2.40}$$

where α_i and $\alpha_j^\vee$ are simple roots and simple coroots respectively. In fact, a finite-dimensional simple Lie algebra is completely characterised by a Cartan matrix (up to permutations) with the following properties:

$$\begin{aligned} a_{ii} &= 2, \\ a_{ij} = 0 &\Longleftrightarrow a_{ji} = 0, \\ a_{ij} \in \mathbb{Z}_{\leq 0} \quad \text{for} &\quad i \neq j, \\ \det a &> 0, \end{aligned} \tag{2.41}$$

and the restriction that it can not be written as the direct sum of such matrices. Classification of finite-dimensional simple Lie algebras thus boils down to specifying matrices with these properties.

Starting with a Cartan matrix the corresponding algebra is then built up by $3r$ generators $\{h_i, e_i, f_i \mid i = 1, 2, \ldots, r\}$ subject to the constraints

$$\begin{aligned}
[\![h_i, h_j]\!] &= 0, \\
[\![h_i, e_j]\!] &= a_{ij} e_j, \\
[\![h_i, f_j]\!] &= -a_{ij} f_j, \\
[\![e_i, f_j]\!] &= \delta_{ij} h_i, \\
(\mathrm{ad}_{e_i})^{1-a_{ij}} e_j = (\mathrm{ad}_{f_i})^{1-a_{ij}} f_j &= 0,
\end{aligned} \tag{2.42}$$

with $i \neq j$ in the last relation. This is the so-called Chevalley-Serre basis which is a special example of a Cartan-Weyl basis. Moreover, the last line in (2.42) contains the Serre relations and the rest being the Chevalley relations. A Lie algebra built from a Cartan matrix a will typically be denoted $\mathfrak{g}(a)$. To give an example, $\mathfrak{a}_2 \cong \mathfrak{sl}(3)$ has the Cartan matrix

$$a_{\mathfrak{a}_2} = \begin{bmatrix} 2 & -1 \\ -1 & 2 \end{bmatrix}, \tag{2.43}$$

which is easily seen to fulfill the properties listed above.

The Cartan matrix for a finite-dimensional simple Lie algebra is symmetrisable, i.e. there exists a diagonal matrix d such that

$$S = da, \tag{2.44}$$

with S a symmetric matrix. The matrix S defines an inner product $(\cdot, \cdot)$ on $\mathfrak{h}^*$ and since $a_{ii} = 2$ we find

$$d_i = \frac{(\alpha_i, \alpha_i)}{2}, \tag{2.45}$$

such that

$$S_{ij} = \frac{a_{ij}(\alpha_i, \alpha_i)}{2}. \tag{2.46}$$

This is defined up to an overall scale which we fix by setting $(\alpha_i, \alpha_i) = 2$ for the longest simple root α_i[4]. The length of the other simple roots are then determined by

$$(\alpha_j, \alpha_j) = \frac{2a_{ij}}{a_{ji}} < 2, \tag{2.47}$$

if i denotes the longest simple root. If all simple roots have the same length the Cartan matrix is already symmetric and the algebra is called simply-laced. Since S is non-degenerate this defines an isomorphism between $\mathfrak{h}$ and its dual space according to (2.32) and an inner product on $\mathfrak{h}^*$

$$(\alpha_i^\vee, \alpha_j^\vee) := \frac{2\alpha_j(\alpha_i^\vee)}{(\alpha_j, \alpha_j)}. \tag{2.48}$$

Since the pairing $\alpha_j(\alpha_i^\vee) = a_{ij}$ we find

$$(\alpha_i^\vee, \alpha_j^\vee) = (ad^{-1})_{ij} =: \hat{a}_{ij}, \tag{2.49}$$

[4] This is a common convention.

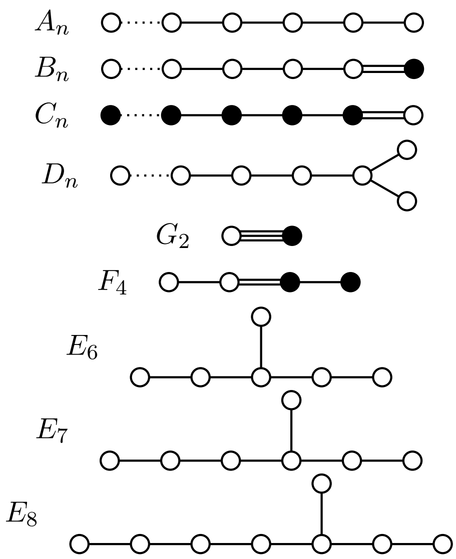

Figure 2.2: Dynkin diagrams for finite-dimensional simple Lie algebras.

the inner product on the Cartan subalgebra is thus given by the Cartan matrix symmetrised with d^{-1} "from the right". Invariance of the inner product $(\cdot,\cdot)$ on $\mathfrak{h}$ can then be used to extend the inner product to the whole algebra $\mathfrak{g}(A)$. This essentially completes the reconstruction of a finite-dimensional simple Lie algebra from its Cartan matrix. However, it is also convenient to define an inner product on the weight space. The fundamental weights were defined as the dual basis to $\alpha_i^\vee$ in (2.34) and the corresponding inner product on the weight space is given by the inverse matrix $\hat{a}^{-1}$

$$(\Lambda_i, \Lambda_j) = (\hat{a}^{-1})_{ij}. \tag{2.50}$$

The inner product of two weights λ and μ is therefore given by

$$(\lambda, \mu) = \sum_{ij} (\hat{a}^{-1})_{ij} \lambda_i \mu_j, \tag{2.51}$$

where λ_i and μ_j are Dynkin labels. The matrix $\hat{a}^{-1}$ is often called the quadratic form matrix and can be found for finite simple Lie algebras in e.g. [17].

A description of a Lie algebra in terms of its Cartan matrix also enables specifying an algebra through its so-called Dynkin diagram. A Dynkin diagram is a diagram with r number of nodes connected by $\max\{|a_{ij}|, |a_{ji}|\}$ lines. For non simply-laced algebras an arrow is denoted from i to j if $|a_{ij}| > |a_{ji}|$, i.e. towards the short root; another common convention is an open dot at long roots and a filled dot at short roots. Any simple finite-dimensional Lie algebra is thus described by its Dynkin diagram in Figure 2.2 and Chevalley-Serre relations (2.42).

2.2.4 Real forms

The analysis above used $\mathbb{C}$ as the underlying field. However, it is often of interest to look only at real linear combinations of elements in the algebra. Choosing a basis T^a, $a = 1, 2, \ldots, \dim \mathfrak{g}$, it is clear that if the structure constants $f^{ab}{}_c$ are real, a restriction to real linear combinations is consistent. More generally, a real form $\hat{\mathfrak{g}}$ of an algebra $\mathfrak{g}$ satisfies

$$\mathfrak{g} \cong \hat{\mathfrak{g}} \oplus i\hat{\mathfrak{g}}, \tag{2.52}$$

i.e. the complexification of $\hat{\mathfrak{g}}$ is isomorphic to $\mathfrak{g}$. A generic Lie algebra $\mathfrak{g}$ typically has several inequivalent real forms. Two real forms that can be constructed for any Lie algebra are the split real form and the compact real form.

The structure constants in the Chevalley-Serre basis are real, hence, by restricting to real linear combinations we get a real form called the *split real form*. A second real form for any Lie algebra can be found by noting that on $\mathfrak{g}$ the Killing form is non-degenerate and one can introduce a basis such that

$$\kappa = \begin{pmatrix} \mathbf{1}_p & 0 \\ 0 & -\mathbf{1}_{d-p} \end{pmatrix}, \tag{2.53}$$

where d is the dimension of the algebra. It is always possible to choose $p = 0$ such that $\kappa^{ab} = -\delta^{ab}$. Upon restriction to real linear combinations this defines the compact real form of $\mathfrak{g}$. Furthermore, given a real form $\hat{\mathfrak{g}}$ the $(d-p)$-dimensional subspace on which κ is negative definite is actually a subalgebra of $\hat{\mathfrak{g}}$. This is called the *maximal compact subalgebra*, typically denoted $\mathfrak{k}$, of $\hat{\mathfrak{g}}$ and it is by construction a real form. Note that in this case upon restriction to a real form we can no longer make "Wick-rotations" to change the signature of κ.

Later on we will deal with so called non-linear sigma models where physical fields take values in the coset group

$$G/H, \tag{2.54}$$

where G is the group obtained by exponentiating a split real form $\mathfrak{g}$ and H is likewise the group corresponding to the maximal compact subalgebra $\mathfrak{k}$ of $\mathfrak{g}$.

Another way to define the maximal compact subalgebra is by the so-called Chevalley involution[5] ω. The Chevalley involution is defined through its action on Chevalley generators

$$\omega(h^i) = -h^i, \qquad \omega(e_i) = -e_i, \qquad \omega(f_i) = -f_-. \tag{2.55}$$

The maximal compact subalgebra $\mathfrak{k}$ of the real form $\mathfrak{g}$ can then be defined as the subset that is pointwise fixed under ω

$$\mathfrak{k} = \{g \in \mathfrak{g} \,|\, \omega(g) = g\}. \tag{2.56}$$

With the definition of the Chevalley involution one sees that the maximal compact subalgebra is therefore spanned by $e_i - f_i$, $i = 1, \ldots, r$. The algebra $\mathfrak{g}$ can thus be decomposed as

$$\mathfrak{g} = \mathfrak{p} \oplus \mathfrak{k}, \tag{2.57}$$

where ω acts as ∓ 1 on $\mathfrak{p}$ and $\mathfrak{k}$ respectively. Note that $\mathfrak{p}$ is not a subalgebra, instead it transform in a representation of $\mathfrak{k}$

$$[\![\mathfrak{p}, \mathfrak{p}]\!] \subset \mathfrak{k}, \qquad [\![\mathfrak{k}, \mathfrak{p}]\!] \subset \mathfrak{p}, \qquad [\![\mathfrak{k}, \mathfrak{k}]\!] \subset \mathfrak{k}. \tag{2.58}$$

Moreover, one could also decompose the algebra according to the Iwasawa decomposition

$$\mathfrak{g} = \mathfrak{k} \oplus \mathfrak{h} \oplus \mathfrak{n}_+, \tag{2.59}$$

where $\mathfrak{n}_+$ is the subspace of positive roots. The subspace $\mathfrak{h} \oplus \mathfrak{n}_+$ is called a (positive) Borel subalgebra.

[5] An involution is a Lie algebra automorphism with eigenvalue ± 1.

2.2.5 Invariant tensors

Given a gauge symmetry neither the lagrangian nor any physical observable should depend on the gauge. As such one has to construct objects that transforms in the trivial representation (singlet) of the gauge group, or equivalently the corresponding algebra. Hence, in order to have a non-trivial theory one needs to extract the singlet representation from tensor product representations. This is precisely done by invariant tensors, or intertwiners. Two important examples of such invariant tensors are the Killing form and the structure constants.

Suppose we have a field ϕ^N, $N = 1, 2, \ldots, \dim R_1$, in a representation R_1 and a field ψ^α, $\alpha = 1, 2, \ldots, \dim R_2$, in another representation R_2. By definition these transform as

$$\phi^M \mapsto R_1(T^a)^M{}_N \phi^N \qquad \psi^\alpha \mapsto R_2(T^a)^\alpha{}_\beta \psi^\beta. \tag{2.60}$$

Then the singlet contribution from the tensor product $R_1 \otimes R_2$ is given by

$$t_{N\alpha}\phi^N\psi^\alpha \mapsto R_1(T^a)^M{}_N t_{M\alpha}\phi^N\psi^\alpha + R_2(T^a)^\alpha{}_\beta t_{N\alpha}\phi^N\psi^\beta, \tag{2.61}$$

if the tensor t satisfies

$$R_1(T^a)^M{}_N t_{M\alpha} + R_2(T^a)^\alpha{}_\beta t_{N\alpha} = 0, \tag{2.62}$$

and t is called an invariant tensor. The condition (2.62) is easily extended to an arbitrary tensor product representation. Note, however, that finding such an invariant tensor is not always possible since not all tensor product representations include a singlet representation. In this way an underlying symmetry reduce the number of possible terms in a lagrangian.

Given a representation R and its conjugate dual $\overline{R}$ with index ^{m} and ${}_m$ respectively, it is possible to create new invariant tensors from given invariant tensors by summing, taking products and contracting indices

$$\begin{aligned} t^{n_1\ldots}{}_{m_1\ldots} + \tilde{t}^{n_1\ldots}{}_{m_1\ldots}, \\ t^{n_1\ldots}{}_{m_1\ldots}\tilde{t}^{k_1\ldots}{}_{j_1\ldots}, \\ t^{n_1\ldots j\ldots}{}_{m_1\ldots j\ldots}. \end{aligned} \tag{2.63}$$

The minimal set of invariant tensors needed for constructing any invariant tensors are called primitive invariants. For the fundamental representation and its dual one finds that δ^i_j, $\epsilon^{i_1\ldots i_n}$ and $\epsilon_{i_1\ldots i_n}$ are primitive invariants for any algebra. For the $\mathfrak{a}_n$ these are actually the only primitive invariants, for the other finite simple Lie algebras the remaining ones are listed in Table 2.1.

Using invariant tensors one can construct projection operators $\mathbb{P}$, which projects a tensor product representation on to one of its irreducible subrepresentations. A projection operator $\mathbb{P}$ is defined such that

$$\mathbb{P}\mathbb{P} = \mathbb{P} \quad \text{and} \quad \mathbb{P}\mathbb{Q} = 0, \tag{2.64}$$

if $\mathbb{Q}$ is a projection operator onto some other subrepresentation. Projection operators are therefore useful to project tensor product representations onto some subspace corresponding to one (or more) of its irreducible subrepresentations.

Table 2.1: Primitive invariants for finite simple Lie algebras are listed. We use $d^{\cdots}$ and $f^{\cdots}$ for completely symmetric and anti-symmetric tensors respectively. For $\mathfrak{e}_8$ there exists at least one primitive tensor $t^{ijkl\cdots}$, but its definite form is unknown.

$\mathfrak{g}$	t
$\mathfrak{b}_r$,$\mathfrak{d}_r$	δ^{ij}
$\mathfrak{c}_r$	f^{ij}
$\mathfrak{e}_6$	d^{ijk}
$\mathfrak{e}_7$	f^{ij}, d^{ijkl}
$\mathfrak{e}_8$	$\delta^{ij}, f^{ijk}, t^{ijkl\ldots}$
$\mathfrak{f}_4$	δ^{ij}, d^{ijk}
$\mathfrak{g}_2$	δ^{ij}, f^{ijk}

Another important example of invariant tensors are Casimir operators. These are elements in the center of the universal enveloping algebra $U(\mathfrak{g})$ of $\mathfrak{g}$, i.e. they commute with any element in $\mathfrak{g}$. As such Casimir operators takes the same value for any element in an irreducible representation. For example, the quadratic Casimir C_2 is an element in $\mathfrak{g} \otimes \mathfrak{g}$ such that

$$(\mathrm{ad}(x) \otimes \mathbf{1} + \mathbf{1} \otimes \mathrm{ad}(x))\, C_2 = 0, \tag{2.65}$$

with an analogous invariance condition for the n-th Casimir operator $C_n \in \mathfrak{g}^{\otimes n}$. By a version of Schur's lemma for Lie algebras it then follows that an irreducible representation of a Casimir operator is proportional to the identity. A well-known example of a quadratic Casimir is the J^2 spin operator in quantum mechanics with an eigenvalue $j(j+1)$ on a spin-j representation. Generally the quadratic Casimir is given by (up to scaling)

$$C_2 = \frac{1}{2}\kappa_{ab} T^a T^b, \tag{2.66}$$

where κ_{ab} is the inverse Killing form. To evaluate this we choose a Cartan-Weyl basis

$$\mathfrak{g} = \mathfrak{h} \oplus \bigoplus_{\alpha \in \Delta} e_\alpha, \tag{2.67}$$

and due to invariance and non-degenaracy of the Killing form one can show that

$$\kappa(e_\alpha, e_\beta) = c_\alpha \delta_{\alpha,-\beta}, \tag{2.68}$$

with a constant c_α. The Casimir operator can then be rewritten using the commutation relations and the definition of the Weyl vector as

$$\frac{1}{2}\kappa_{ab} T^a T^b = \frac{1}{2}(h,h) + (h,\rho) + \sum_{\alpha \in \Delta^+} c_\alpha e_\alpha e_{-\alpha}, \tag{2.69}$$

where (h,h) denotes the part restricted to the Cartan subalgebra. Without loss of generality this can be evaluated on the highest weight state of an irreducible representation $R(\lambda)$, which is particularly easy in the rewritten form above, and we find

$$C_2(R(\lambda)) = \frac{1}{2}(\lambda, \lambda + 2\rho). \tag{2.70}$$

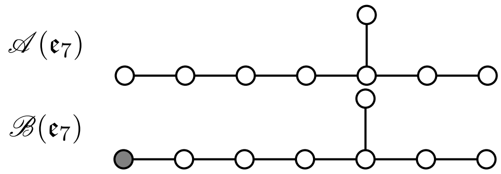

Figure 2.3: Two extensions of the $\mathfrak{e}_7$ structure algebra. The upper diagram is equivalent to $\mathfrak{e}_8$ while the lower denotes a super-extension to a Borcherds superalgebra.

2.3 Extended algebras

Below extensions of finite simple Lie algebras are introduced. In this thesis they mainly provide another way of deriving the invariant Y-tensor and provide a rather simple way to solve the so-called section constraint introduced in Chapter 6. The Y-tensor together with the section constraint plays a crucial rôle in extended geometries. However, there are further signs that these extended algebras, and actually even larger algebras derived from these, such as the tensor hierarchy algebra [8] and L_∞ algebras [10], are needed for a complete formulation of extended geometries. As we will see below these extended algebras can be visualised in terms of Dynkin diagrams and interesting such extensions are given in Figure 2.3.

2.3.1 Kac-Moody algebras

In Section 2.2 we saw that a finite-dimensional simple Lie algebra is characterised by an indecomposable Cartan matrix with the properties

$$\begin{aligned} a_{ii} &= 2, \\ a_{ij} = 0 &\Longleftrightarrow a_{ji} = 0, \\ a_{ij} \in \mathbb{Z}_{\leq 0} \quad \text{for} &\quad i \neq j, \\ \det a &> 0, \end{aligned} \tag{2.71}$$

and the Chevalley-Serre generators satisfying

$$\begin{aligned} [\![h_i, h_j]\!] &= 0, \\ [\![h_i, e_j]\!] &= a_{ij} e_j, \\ [\![h_i, f_j]\!] &= -a_{ij} f_j, \\ [\![e_i, f_j]\!] &= \delta_{ij} h_i, \\ (\mathrm{ad}_{e_i})^{1-a_{ij}} e_j = (\mathrm{ad}_{f_i})^{1-a_{ij}} f_j &= 0, \end{aligned} \tag{2.72}$$

for $i \neq j$ in the Serre relations. This serves as a convenient starting point for extensions to a more general class of symmetry algebras since we will relax the condition $\det a > 0$. We will moreover assume that the generalised Cartan matrix A is symmetrisable which implies the existence of a symmetric invariant bilinear form. Relaxing the requirement of a positive-definite Cartan matrix three classes of Kac-Moody algebras are obtained,

- if $\det A > 0$, then $\mathfrak{g}(A)$ is a simple finite-dimensional Lie algebra,
- if $\det A = 0$ with one negative eigenvalue, then $\mathfrak{g}(A)$ is an *affine* algebra,

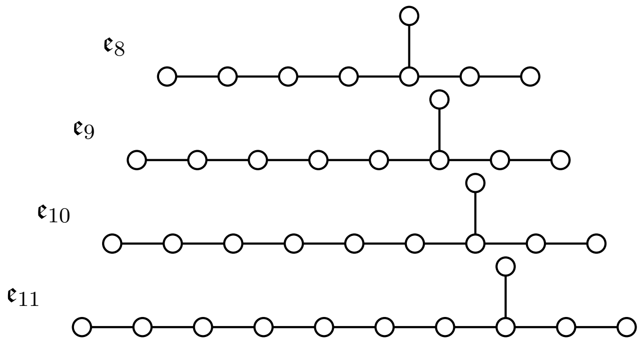

Figure 2.4: Dynkin diagram of $\mathfrak{e}_8$ and its extensions $\mathfrak{e}_9$, $\mathfrak{e}_{10}$ and $\mathfrak{e}_{11}$.

- if A does not satisfy either of the above constraints then it is an indefinite Kac-Moody algebra.

The affine algebras are infinite-dimensional but a complete classification is still possible. Indefinite Kac-Moody algebras are also infinite-dimensional but have not been completely classified. A subclass of indefinite KM algebras are Lorentzian algebras, these are characterised by having a non-degenerate symmetric bilinear form with precisely one negative eigenvalue.

The extension of a simple finite Lie algebra can be achieved by adding a set of Chevalley generators $\{h_0, e_0, f_0\}$ to the existing $3r$ Chevalley generators $\{h_i, e_i, f_i\}_{(i=1,2,\ldots,r)}$. This amounts to the addition of one node to the Dynkin diagram according to the generalised Cartan matrix A_{IJ}, where $I, J = 0, 1, 2, \ldots, r$. For the affine algebras A has one zero eigenvalue and r strictly positive, this is equivalent, to upon removing any one node of the extended Dynkin diagram, one should obtain a Dynkin diagram representing a direct sum of finite-dimensional Lie algebras. One also defines hyperbolic Kac-Moody algebras as Lorentzian KM algebras of indefinite type that upon removing any node in the Dynkin diagram, a direct sum of finite-dimensional and at most one affine Kac-Moody algebra are obtained. Kac-Moody algebras are then constructed in the same way as finite-dimensional Lie algebras through the Chevalley-Serre relations in (2.72) by replacing $i, j \to I, J = 0, 1, 2, \ldots, r$ and $a_{ij} \to A_{IJ}$. However, in the affine case the Cartan matrix is not invertible and one also needs to add a central element and a derivation, we refer to the literature for details. Moreover, much of the representation theory of simple finite Lie algebras carry over to the infinite-dimensional algebras but there are important exceptions.

Interesting examples of extended algebras of physical interest are obtained by extending the exceptional Lie algebra $\mathfrak{e}_8$. First we can extend $\mathfrak{e}_8$ to an affine algebra $\mathfrak{e}_9$ as shown in Figure 2.4. One can then go even further and successively add two more nodes to obtain a hyperbolic algebra $\mathfrak{e}_{10}$ and a Lorentzian, but not hyperbolic, algebra $\mathfrak{e}_{11}$ also displayed in Figure 2.4. For further discussion about these algebras see [23, 24]. This so-called $\mathfrak{e}_n$-series appears as "hidden" symmetries of eleven-dimensional supergravity compactified on a torus T^n for $n \leq 9$, which for $n = 9$ are infinite-dimensional. It is further believed that a discrete version of $\mathfrak{e}_n$ survives quantisation. Furthermore, it is conjectured [25] that the discrete version of $\mathfrak{e}_{11}$ is a symmetry for the full M-theory. The main motivation behind extended geometries presented in this thesis is to make these symmetries inherent in the theory prior to any compactification.

We now construct a specific example of Kac-Moody algebras based on some other Kac-

Moody algebra $\mathfrak{g} = \mathfrak{g}(a)$ and a module $R(\lambda)$ following that of [5]. Assume that a_{ij}, $i, j = 1, 2, \ldots, r$ is an invertible symmetrisable generalised Cartan matrix. We then extend this further to a generalised Cartan matrix A_{IJ}, $I, J = 0, 1, \ldots, r$, by adding one row and one column as

$$A_{00} = 2, \qquad A_{i0} = -\lambda_i, \qquad A_{ij} = a_{ij}, \tag{2.73}$$

such that the matrix A is symmetrisable using a diagonal matrix D_I, with $D_0 = 1/k$ and $D_i = \frac{(\alpha_i, \alpha_i)}{2}$. It then follows that $A_{0i} = kD_iA_{i0}$. The algebra $\mathscr{A}(\mathfrak{g})$ is then constructed from A with the Chevalley relations

$$[\![h_I, e_J]\!] = A_{IJ}e_J, \qquad [\![h_I, f_J]\!] = -A_{IJ}f_J, \qquad [\![e_I, f_J]\!] = \delta_{IJ}h_J, \tag{2.74}$$

and the Serre relations

$$(\mathrm{ad}_{e_i})^{1-A_{iJ}}e_J = (\mathrm{ad}_{f_i})^{1-A_{iJ}}f_J = 0 \quad \text{for} \quad i \neq J. \tag{2.75}$$

Moreover, since A is symmetrisable we can construct a symmetric invariant bilinear form on the root space as $(\alpha_I, \alpha_J) := (DA)_{IJ}$, with α_I simple roots.

For example, if the underlying algebra $\mathfrak{g}(a)$ is chosen to be $\mathfrak{e}_r$ together with the module corresponding to the highest weight $\lambda = \Lambda_1$, then this construction gives $\mathscr{A}(\mathfrak{e}_t) = \mathfrak{e}_{r+1}$. For $10 > r \geq 8$ the corresponding Dynkin diagrams are given in Figure 2.4.

2.3.2 Borcherds superalgebras

We will here give a brief set of definitions from [26] for a yet larger class of algebras, super Borcherds algebras, also called generalised Kac-Moody algebras or Borcherds-Kac-Moody algebras (BKM). A particular example of a BKM algebra which is of relevance in the construction of extended geometries is then introduced. The extension is again done by relaxing one of the conditions of the Cartan matrix, or since in this case we extend Kac-Moody algebras, the generalised Cartan matrix. In words one allow for the possibility of simple imaginary roots such that the diagonal in the generalised Cartan matrix is of indefinite sign.

Before moving on to the construction of Borcherds superalgebras the concept of a superalgebra needs to be introduced. A superalgebra is a $\mathbb{Z}_2$-graded algebra with a bilinear product that respects this grading. In other words, a superalgebra $\mathfrak{g}$ can be decomposed into "even" and "odd" subspaces $\mathfrak{g}_0$ and $\mathfrak{g}_1$ according to

$$\mathfrak{g} = \mathfrak{g}_0 \oplus \mathfrak{g}_1, \tag{2.76}$$

such that the bilinear product $[\![\cdot, \cdot]\!]$ respects this in the sense

$$[\![\mathfrak{g}_p, \mathfrak{g}_q]\!] \subseteq \mathfrak{g}_{p+q}, \tag{2.77}$$

where $p + q = p + q \,(\mathrm{mod}\, 2)$. Typically "even" and "odd" are used interchangeably with "bosonic" and "fermionic" respectively. Generally a gradation of an algebra by the discrete group $\mathbb{Z}$ is defined as

$$\mathfrak{g} = \bigoplus_{p \in \mathbb{Z}} \mathfrak{g}_p, \tag{2.78}$$

and it is *consistent* if

$$[\![\mathfrak{g}_p, \mathfrak{g}_q]\!] \subseteq \mathfrak{g}_{p+q}. \tag{2.79}$$

In a supergraded algebra the Lie bracket becomes a supercommutator such that for $x, y \in \mathfrak{g}$

$$[\![x, y]\!] = (-1)^{|x||y|} [\![y, x]\!], \tag{2.80}$$

where $|\cdot| = 0, 1$ if the element is even or odd respectively. The Jacobi identity then becomes the super-Jacobi identity

$$(-1)^{|x||z|} [\![x, [\![y, z]\!]]\!] + (-1)^{|y||x|} [\![y, [\![z, x]\!]]\!] + (-1)^{|z||y|} [\![z, [\![x, y]\!]]\!] = 0. \tag{2.81}$$

Now to the construction of Borcherds superalgebras. Let I be an index set $I = 1, 2, \ldots, N$, where $S \subset I$ denotes indices corresponding to fermionic indices. A Borcherds superalgebra is then defined by a non-degenerate symmetric generalised Cartan matrix B_{ij} that satisfis the following relations ($i, j \in I$):

$$\begin{aligned} &B_{ij} \leq 0 \quad \text{for} \quad i \neq j, \\ \text{if} \quad &B_{ij} > 0 \implies \frac{2B_{ij}}{B_{ii}} \in \mathbb{Z}, \\ \text{if} \quad &B_{ii} > 0 \quad \text{and} \quad i \in S \implies \frac{B_{ij}}{B_{ii}} \in \mathbb{Z} \quad \text{for all} \quad j \in I. \end{aligned} \tag{2.82}$$

The corresponding Borcherds superalgebra $\mathscr{B} = \mathfrak{g}(B)$ is then generated by $3N$ Chevalley-generators $\{h_i, e_i, f_i\}_{(i \in I)}$ with the following Chevalley-Serre relations

$$[\![h_i, h_J]\!] = 0, \tag{2.83a}$$

$$[\![h_i, e_j]\!] = B_{ij} e_j, \qquad [\![h_i, f_j]\!] = -B_{ij} f_j, \qquad [\![e_i, f_j]\!] = \delta_{ij} h_j, \tag{2.83b}$$

$$\deg e_i = 0 = \deg f_i \quad \text{if} \quad \notin S, \qquad \deg e_i = 1 = \deg f_i \quad \text{if} \quad i \in S, \tag{2.83c}$$

$$(\mathrm{ad}_{e_i})^{1 - \frac{2B_{ij}}{B_{ii}}} e_j = (\mathrm{ad}_{f_i})^{1 - \frac{2B_{ij}}{B_{ii}}} f_j = 0 \quad \text{if} \quad B_{ii} > 0 \quad \text{and} \quad i \neq j. \tag{2.83d}$$

The BKM algebras constitute a large class of algebras and the general definition above was mainly given for the interested reader. Instead we consider the construction of a specific type of BKM superalgebras relevant for extended geometries, which again follows that of [5]. As for the Kac-Moody example above start with an invertible and symmetrisable generalised Cartan matrix a_{ij} with $i, j = 1, 2, \ldots, r$. As above, add one row and one column and form a symmetrisable matrix B_{IJ} with $I, J = 0, 1, \ldots r$, according to

$$B_{00} = 0, \qquad B_{i0} = -\lambda_i, \qquad B_{ij} = a_{ij}. \tag{2.84}$$

We assume, as in the KM-case, that B is symmetrised with the diagonal matrix $D_0 = 1/k$ and $D_i = \frac{(\alpha_i, \alpha_i)}{2}$ from which it follows that $B_{0i} = k D_i B_{i0}$. As we have seen adding a row and column to a_{ij} is equivalent to adding a set of Chevalley generators $\{h_0, e_0, f_0\}$, however, the elements e_0 and f_0 are now odd generators. The construction goes through in much the same way using the Chevalley-Serre relations with the ordinary Lie bracket replaced with the supercommutator

$$[\![h_I, e_J]\!] = B_{IJ}, \qquad [\![h_I, f_J]\!] = -B_{IJ} f_J, \qquad [\![e_I, f_J]\!] = \delta_{IJ} h_J, \tag{2.85}$$

and

$$[\![e_0, e_0]\!] = [\![f_0, f_0]\!] = 0, \qquad (\mathrm{ad}_{e_i})^{1 - B_{iJ}} e_J = (\mathrm{ad}_{f_i})^{1 - B_{iJ}} f_J = 0 \quad \text{for} \quad i \neq J. \tag{2.86}$$

The algebra $\mathscr{B}$ has a consistent $\mathbb{Z}$-grading and can thus be decomposed as

$$\mathscr{B} = \bigoplus_{p\in\mathbb{Z}} \mathscr{B}_p, \tag{2.87}$$

where $e_0 \in \mathscr{B}_1$, $f_0 \in \mathscr{B}_{-1}$ and the remaining $3r+1$ Chevalley generators in $\mathscr{B}_0$.

The subspace at $p = 0$ is even and given by

$$\mathscr{B}_0 = \mathfrak{g} \oplus c, \tag{2.88}$$

where c is a central element containing h_0. The normalisation of c is chosen according to

$$c = \sum_{I=0}^{r} (B^{-1})_{0I} h_I \implies [\![c, e_0]\!] = e_0. \tag{2.89}$$

The grading thus corresponds to the eigenvalue of c, it is easily shown using the Jacobi-identity that f_0 has eigenvalue -1 which it must since $[\![e_0, f_0]\!] \in \mathscr{B}_0$. Moreover, $[\![\mathscr{B}_r, \mathscr{B}_p]\!] \subseteq \mathscr{B}_{p+r}$ from which it follows especially that

$$[\![\mathscr{B}_0, \mathscr{B}_p]\!] \subseteq \mathscr{B}_p. \tag{2.90}$$

The subspaces $\mathscr{B}_p$ are thus irreducible modules of $\mathfrak{g}$ under the adjoint action. Since the c by definition commutes with the adjoint action we indeed see that the grading is consistent. In terms of the weight lattice Λ_W the element c measures the projection of a weight onto a line orthogonal to the $\Lambda_{W_\mathfrak{g}} \subset \Lambda_W$ sub-lattice. Furthermore, from the Serre relations it follows that $f_0 \in \mathscr{B}_{-1}$ is a highest weight vector of $\mathfrak{g}$

$$[\![e_i, f_0]\!] = 0, \tag{2.91}$$

and, moreover, the Dynkin labels are given by

$$[\![h_i, f_0]\!] = -B_{i0} f_0 = \lambda_i f_0. \tag{2.92}$$

Likewise e_0 is a lowest weight vector for $\mathscr{B}_1$ with Dynkin labels

$$[\![h_i, e_0]\!] = -\lambda_i f_0. \tag{2.93}$$

We thus see that the subspaces $\mathscr{B}_{-1}$ and $\mathscr{B}_1$ transforms in $R_1 = R(\lambda)$ and $R_{-1} = \overline{R(\lambda)}$ of $\mathfrak{g}$ respectively. Furthermore, we let R_p denote the representation of $\mathfrak{g}$ corresponding to $\mathscr{B}_{\mp p}$. In order to determine R_2 note that this is an even subspace and as such $\mathscr{B}_{-2}$ transforms in the symmetric product $\vee^2 R(\lambda)$ under $\mathfrak{g}$. However, the highest weight vector of $\vee^2 R(\lambda)$ is $[\![f_0, f_0]\!]$ which vanish due to the Serre relations and therefore

$$R_2 = \vee^2 R(\lambda) \ominus R(2\lambda). \tag{2.94}$$

Given the odd simple root β_0 and the even simple roots $\beta_i = \alpha_i$ of $\mathfrak{g}$ it is possible to construct a metric on the root space as usual

$$(\beta_I, \beta_J) = (DB)_{IJ}. \tag{2.95}$$

As for simple Lie algebras the inner product on the root space induce an inner product on the Cartan subalgebra which can be further extended to the whole algebra $\mathscr{B}$ using invariance property. Note, however, that while this inner product is symmetric on (e_i, f_J) it is anti-symmetric on $(e_0, f_0) = -(f_0, e_0) = k$ due to the fact that e_0 and f_0 are odd roots.

2.3.3 The invariant Y-tensor

Below we will use the extended algebras $\mathscr{A}$ and $\mathscr{B}$ to derive the Y-tensor that plays a crucial rôle in extended geometries. Moreover, we will assume that the respective generalised Cartan matrices A and B are invertible and symmetrisable. This construction was given in [5] where the assumption of invertibility was relaxed.

Starting with the purely bosonic algebra $\mathscr{A}$ we note that just as in the case for the algebra $\mathscr{B}$ there is a consistent $\mathbb{Z}$-grading; hence the algebra can be decomposed as

$$\mathscr{A} = \bigoplus_{p\in\mathbb{Z}} \mathscr{A}_p. \tag{2.96}$$

By the definition of a consistent grading the subspaces $\mathscr{A}_p$ are again modules under the $\mathscr{A}_0 = \mathfrak{g} \oplus \tilde{c}$ subalgebra. Especially, at level ± 1 these modules are isomorphic to $R(\lambda)$ and $\overline{R(\lambda)}$ of $\mathfrak{g}$ respectively which follows from the Serre relations in (2.75). Introducing a basis $\tilde{E}^M$ for $R(\lambda)$ and $\tilde{F}_M$ for $\overline{R(\lambda)}$ the algebra $\mathscr{B}_0$ acts as

$$\begin{cases} [\![T^\alpha, \tilde{E}^M]\!] = -T^{\alpha M}{}_N \tilde{E}^M, & [\![\tilde{c}, \tilde{E}^M]\!] = \tilde{E}^M \\ [\![T^\alpha, \tilde{F}_M]\!] = T^{\alpha N}{}_M \tilde{F}_N, & [\![\tilde{c}, \tilde{F}_N]\!] = -\tilde{F}_N. \end{cases} \tag{2.97}$$

Note that there exists an isomorphism between $\mathscr{A}_{\pm1}$ and $\mathscr{B}_{\pm1}$. This is however not true at higher levels.

We then look at the commutator

$$[\![\tilde{E}^M, \tilde{F}_N]\!] = \eta_{\alpha\beta} T^{\alpha M}{}_N X^\beta + b\tilde{c}\delta^M_N, \tag{2.98}$$

where b is a constant and we used the grading of $\mathscr{A}$. Choosing a normalisation of the basis such that $(\tilde{E}^M, \tilde{F}_N) = \delta^M_N$ and the invariance of $(\cdot,\cdot)$ we find

$$([\![\tilde{E}^M, \tilde{F}_N]\!], T^\alpha) = (\tilde{E}^N, [\![\tilde{F}_N, T^\alpha]\!]) = -T^{\alpha M}{}_N, \tag{2.99}$$

and by inserting the ansatz (2.98)

$$\eta_{\gamma\beta} T^{\gamma M}{}_N (X^\beta, T^\alpha) + b(c, T^\alpha) = -T^{\alpha M}{}_N. \tag{2.100}$$

Using $(\tilde{c}, T^\alpha) = 0$ and $(T^\alpha, T^\beta) = \eta^{\alpha\beta}$ we find $X^\beta = -T^\beta$. Similarly we have

$$([\![\tilde{E}^M, \tilde{F}_N]\!], c) = b\delta^M_N (c, c) = -(\tilde{E}^M, [\![\tilde{c}, \tilde{F}_N]\!]) = \delta^M_N. \tag{2.101}$$

By inserting the explicit normalisation for $\tilde{c}$ we find $(\tilde{c}, \tilde{c}) = \sum_{IJ} (A^{-1})_{0I} (A^{-1})_{0J} (h_I, h_J) = k(A^{-1})_{00}$ which gives

$$[\![\tilde{E}^M, \tilde{F}_N]\!] = -\eta_{\alpha\beta} T^{\alpha M}{}_N T^\beta + \frac{1}{k(A^{-1})_{00}} \delta^M_N c. \tag{2.102}$$

Consider now the invariant tensor

$$\tilde{f}^{MN}{}_{PQ} = ([\![[\![\tilde{E}^M, \tilde{F}_P]\!], \tilde{E}^N]\!], \tilde{F}_Q). \tag{2.103}$$

Inserting the explicit expression for $[\![\tilde{E}^M, \tilde{F}_P]\!]$ found above we get

$$\begin{aligned} \tilde{f}^{MN}{}_{PQ} &= -\eta_{\alpha\beta} T^{\alpha M}{}_P ([\![T^\beta, \tilde{E}^N]\!], \tilde{F}_Q) + \frac{1}{k(A^{-1})_{00}} \delta^M_P ([\![c, \tilde{E}^N]\!], \tilde{F}_Q) \\ &= \eta_{\alpha\beta} T^{\alpha M}{}_P T^{\beta N}{}_Q + \frac{1}{k(A^{-1})_{00}} \delta^M_P \delta^N_Q. \end{aligned} \tag{2.104}$$

Using the Jacobi identity and invariance of the inner product it is straightforward to derive from $\tilde{f}^{MN}{}_{PQ}$ that $\tilde{g}^{MN}{}_{PQ} := 2\tilde{f}^{[MN]}{}_{PQ}$ is also given by

$$\tilde{g}^{MN}{}_{PQ} := ([\![\tilde{E}^M, \tilde{E}^N]\!], [\![\tilde{F}_P, \tilde{F}_Q]\!]). \tag{2.105}$$

Analogous expressions are easily derived from the BKM algebra $\mathscr{B}$. We have already noted that $\mathscr{B}$ is consistently graded and that the subspaces at level ± 1 are isomorphic to $R(\lambda)$ and $\overline{R(\lambda)}$ modules of the $\mathfrak{g} \subset \mathscr{B}_0$ subalgebra. As in the purely bosonic case we introduce a basis E^M and F_M at level ± 1 respectively. Note, however, that elements at level ± 1 are odd. Especially, this implies that $(E^M, F_N) = -(F_N, E^M) = -1$, that $[\![E^M, F_N]\!]$ is symmetric and we have to use the super-Jacobi identity.

The $\mathscr{B}_0$ subalgebra act on E^M and F_M as in (2.97) with the replacements $\tilde{E} \mapsto E$ and $\tilde{F} \mapsto F$. Taking the extra sign factors into account it is straightforward following the steps above to find

$$\begin{aligned} f^{MN}{}_{PQ} &:= ([\![[\![E^M, F_P]\!], E^N]\!], F_Q) \\ &= \eta_{\alpha\beta} T^{\alpha M}{}_P T^{\beta N}{}_Q + \frac{1}{k(B^{-1})^{00}} \delta^M_P \delta^N_Q. \end{aligned} \tag{2.106}$$

Furthermore, it is easily seen that $g^{MN}{}_{PQ} := -2f^{(MN)}{}_{PQ}$ is given by

$$g^{MN}{}_{PQ} = ([\![E^M, E^N]\!], [\![F_P, F_Q]\!]). \tag{2.107}$$

Before constructing the Y-tensor we rewrite $f, \tilde{f}$ slightly by expanding the weight λ in simple roots α_i as

$$\lambda = c_j \alpha_j \qquad \text{s.t.} \qquad \lambda_i = \lambda(\alpha_i^\vee). \tag{2.108}$$

Noting that $B_{ij}(B^{-1})_{j0} = \delta_{i0} - B_{i0} = -B_{i0}$ it is found that

$$\lambda = \frac{(B^{-1})^{i0}}{(B^{-1})^{00}} \alpha_i \implies \lambda(\alpha_i^\vee) = B_{ij} \frac{(B^{-1})^{j0}}{(B^{-1})^{00}} = -B_{i0}, \tag{2.109}$$

which by construction fulfills $-B_{i0} = \lambda_i$. We can now express the length of the weight λ as

$$(\lambda, \lambda) = \sum_{i,j=1}^{r} (\alpha_i, \alpha_j) \frac{(B^{-1})_{i0}}{(B^{-1})_{00}} \frac{(B^{-1})_{j0}}{(B^{-1})_{00}}, \tag{2.110}$$

which by definition of the inner product on the root space $(\alpha_i, \alpha_j) = D_i B_{ij}$, and using $D_i B_{i0} = B_{0i}/k$, gives

$$(\lambda, \lambda) = -\frac{1}{k(B^{-1})_{00}}. \tag{2.111}$$

Likewise the factor of $(A^{-1})_{00}$ in $f^{MN}{}_{PQ}$ can be rewritten using $\det A = 2 \det a + \det B$ and $(B^{-1})_{00} = \det a / \det B$ which follows from the general formula for the inverse $B^{-1} = C^T / \det B$, where C^T is the transposed comatrix, and we thus have

$$(\lambda, \lambda) = \frac{2}{k} - \frac{1}{k(A^{-1})_{00}}. \tag{2.112}$$

We are now ready to construct the Y-tensor as

$$\begin{aligned} Y^{MN}{}_{PQ} &= \frac{k}{2} \left(g^{MN}{}_{QP} - \tilde{g}^{MN}{}_{QP} \right) \\ &= -k\eta_{\alpha\beta} T^{\alpha N}{}_P T^{\beta M}{}_Q + [k(\lambda, \lambda) - 1] \, \delta^N_P \delta^M_Q + \delta^N_Q \delta^M_P. \end{aligned} \tag{2.113}$$

The main reason for introducing the extended algebras $\mathscr{A}$ and $\mathscr{B}$ is the construction of the Y-tensor in (2.113) as this will in some sense provide the deviation of extended geometries from ordinary geometry that is based on $\mathfrak{g} = \mathfrak{gl}(n)$. Moreover, this formalism will be useful when determining if so-called ancillary transformations are present or not. However, the full rôle of these extended algebras is still being explored [8, 10, 27].

3

String theory, supergravity & dualities

This chapter aims to put into context the extended geometries introduced in this thesis. The first section introduces some concepts of string theory, with the main point to show the presence of dualities, especially T-duality for the bosonic string. Moreover, the low-energy effective theory, supergravity, related to string theory will be introduced in Section 3.2. This is done in order to motivate the appearance of the continuous version of the duality group in supergravity compactified on torii. In section 3.3 the notion of dualities is introduced more thoroughly. To the interested reader we recommend [28, 29] for an introduction to string theory.

3.1 String theory

In string theory the fundamental constituents are strings, 1-dimensional extended objects, rather than point-like particles. Of course since our usual picture of point-like particles and quantum field theories works extraordinarily well, such a theory of extended objects should have a low-energy description in terms of quantum fields. Put differently, the typical string length should be small compared to distances we could measure at say the LHC so that a string looks effectively pointlike.

Bosonic string theory can be described by the Brink-Howe-Di Vecchia-Polyakov action

$$S = -\frac{T}{2}\int \mathrm{d}^2\sigma \sqrt{\gamma}\gamma_{\alpha\beta}\partial_\alpha X^\mu \partial_\beta X^\nu G_{\mu\nu}(X), \tag{3.1}$$

where T is the string tension, Σ the string worldsheet parameterised by $\sigma^\alpha \in \Sigma$ with $\alpha = 0, 1$ and $X^\mu : \sigma \to \mathcal{M}$ an embedding of the world-sheet into a manifold $\mathcal{M}$ with metric G. The string tension is related to the so-called Regge slope with dimensions of an area $\alpha' = 1/(2\pi T)$. For what follows we consider strings in flat space $G_{\mu\nu} = \eta_{\mu\nu}$.

The string action in flat space has several important symmetries:

$$\text{Global Poincaré:} \qquad \delta X^\mu = \Lambda^\mu_\nu X^\nu + \epsilon^\mu, \tag{3.2}$$

$$\delta\gamma_{\alpha\beta} = 0, \tag{3.3}$$

World-sheet diffeomorphisms:
$$\delta X^\mu = -\xi^\alpha \partial_\alpha X^\mu, \tag{3.4}$$
$$\delta\gamma_{\alpha\beta} = -\xi^\rho \partial_\rho \gamma_{\alpha\beta} - \partial_\alpha \xi^\rho \gamma_{\rho\beta} - \partial_\beta \xi^\rho \gamma_{\alpha\rho}, \tag{3.5}$$
$$\delta\sqrt{-\gamma} = -\partial_\rho(\sqrt{-\gamma}\xi^\rho), \tag{3.6}$$

Weyl rescaling:
$$\delta X^\mu = 0, \tag{3.7}$$
$$\delta\gamma_{\alpha\beta} = 2\Lambda\gamma_{\alpha\beta}. \tag{3.8}$$

In the following, consider a string in a flat Minkowski space with $G_{\mu\nu} = \eta_{\mu\nu}$. Varying the Polyakov action w.r.t. to the world-sheet metric we find

$$\delta S = -\frac{T}{2}\int_\Sigma \mathrm{d}^2\sigma \delta\gamma^{\alpha\beta}\sqrt{-\gamma}\left(\partial_\alpha X^\mu \partial_\beta X_\mu - \frac{1}{2}\gamma_{\alpha\beta}\gamma^{\rho\lambda}\partial_\rho X^\mu \partial_\lambda X_\mu\right). \tag{3.9}$$

The world-sheet stress-energy tensor and the corresponding equations of motion are thus given by (in a suitable normalisation)

$$T_{\alpha\beta} = \frac{4\pi}{\sqrt{-\gamma}}\frac{\delta S}{\delta\gamma^{\alpha\beta}} = -\frac{1}{\alpha'}\left(\partial_\alpha X^\mu \partial_\beta X_\mu - \frac{1}{2}\gamma_{\alpha\beta}\gamma^{\rho\lambda}\partial_\rho X^\mu \partial_\lambda X_\mu\right) = 0. \tag{3.10}$$

It follows that inserting a Weyl-transformation we find $T^\alpha_\alpha = 0$. Moreover, since a general $2d$-metric has 3 d.o.f. Weyl invariance together with world-sheet diffeomorphisms are sufficient to set $\gamma_{\alpha\beta} = \eta_{\alpha\beta}$ locally. Assuming a closed string we can throw away boundary terms to find the equations of motion for the string coordinates X^μ

$$\partial_\alpha\left(\sqrt{-\gamma}\gamma^{\alpha\beta}\partial_\beta X^\mu\right) = 0, \tag{3.11}$$

which in conformal gauge ($\gamma_{\alpha\beta} = \eta_{\alpha\beta}$) simply is a wave equation

$$\partial^\alpha \partial_\alpha X^\mu = 0. \tag{3.12}$$

This is easily solved by $X^\mu(\sigma,\tau) = X^\mu_R(\tau-\sigma) + X^\mu_L(\tau+\sigma)$ ($\tau = \sigma^0$ and $\sigma = \sigma^1$) and in light-cone coordinates $\sigma^\pm = \tau \pm \sigma$

$$\begin{aligned} X^\mu_R(\sigma^-) &= \frac{1}{2}x^\mu + \frac{\alpha'}{2}p^\mu\sigma^- + \mathrm{i}\sqrt{\frac{\alpha'}{2}}\sum_{n\neq 0}\frac{1}{n}\alpha^\mu_n \mathrm{e}^{-\mathrm{i}n\sigma^-} \\ X^\mu_L(\sigma^+) &= \frac{1}{2}x^\mu + \frac{\alpha'}{2}p^\mu\sigma^+ + \mathrm{i}\sqrt{\frac{\alpha'}{2}}\sum_{n\neq 0}\frac{1}{n}\tilde\alpha^\mu_n \mathrm{e}^{-\mathrm{i}n\sigma^+}. \end{aligned} \tag{3.13}$$

The first two terms describe constant motion of the center of mass of the string while the sum denotes oscillations along the string. In order for X^μ to be real we also have $\alpha^\mu_n = (\alpha^\mu_{-n})^*$ and likewise for the left-moving oscillators. Inserting the oscillator expansion into the constraints $T_{\alpha\beta} = 0$ which in light-cone coordinates are given by

$$\partial_+ X^\mu \partial_+ X_\mu = \partial_- X^\mu \partial_- X_\mu = 0, \tag{3.14}$$

we find ($p^\mu_0 := \sqrt{\alpha'/2}p^\mu$)

$$\begin{aligned} \partial_- X_L &= \frac{\alpha'}{2}\sum_{n\in\mathbb{Z}}\alpha^\mu_n \mathrm{e}^{-\mathrm{i}n\sigma^-} \implies \\ \partial_- X^\mu \partial_- X_\mu &= \alpha' \sum_n L_n \mathrm{e}^{-\mathrm{i}n\sigma^-} = 0, \end{aligned} \tag{3.15}$$

with $L_n := \frac{1}{2}\sum_n \alpha^\mu_{n-m}\alpha_{m\mu}$ and similar for the left-movers. Especially interesting is the constraint $L_0 = 0$ since it includes the spacetime momenta $p^\mu p_\mu = -m^2$ from which the mass-formula is derived to be

$$m^2 = \frac{4}{\alpha'}\sum_{m>0}\alpha_{-m}\cdot\alpha_m. \tag{3.16}$$

Imposing canonical quantization conditions on X^μ and the conjugate momentum $\Pi_\mu = \frac{1}{2\pi\alpha'}\frac{\delta\mathcal{L}}{\delta X^\mu}$ as $[X^\mu(\sigma,\tau),\Pi_\nu(\sigma',\tau)] = \mathrm{i}\delta^\mu_\nu\delta(\sigma-\sigma')$ and in terms of the expansion we find

$$[x^\mu, p_\nu] = \mathrm{i}\delta^\mu_\nu \qquad \text{and} \qquad [\alpha^\mu_{m,L,R},\alpha^\nu_{n,L,R}] = m\delta_{m,-n}\eta^{\mu\nu}. \tag{3.17}$$

Since L_n now become operators there are possible order ambigiuities when quantizing and a standard result is that the bosonic string has to live in $D = 26$ dimensions and the mass operator is shifted to

$$m^2 = \frac{4}{\alpha'}\left(\sum_{m>0}\alpha_{-m}\cdot\alpha_m - 1\right). \tag{3.18}$$

From the commutation relations $\alpha_{-m}\alpha_m$ is a number operator up to a factor of m and one usuallly defines the total number operator as the sum $N = \sum_{m>0}\alpha_{-m}\cdot\alpha_m$. Since the exact same result can be found using left-moving oscillators we get the level-matching condition $N - \tilde{N} = 0$. Also we see that the mass of the ground state with no oscillators is negative since $m^2|0\rangle = -|0\rangle$ which indicates the presence of the tachyon and the need for a supersymmetric theory of strings.

Consider the closed bosonic string compactified on $X^{25} \sim X^{25} + 2\pi R$, i.e. on S^1 with radius R along its 25th dimension. It follows immediately that the momenta along this direction becomes quantized as $p^{25} = n/R$ for $n \in \mathbb{Z}$. Going once around the string do not need to come back to its starting point, instead it can be wound w number of times around the circle as $X^{25}(\sigma + 2\pi,\tau) = X^{25}(\sigma,\tau) + 2\pi mR \sim X^{25}(\sigma,\tau)$. The expansion of X^μ_{25} is then given given by

$$\begin{aligned} X^\mu_{25,R} &= \frac{1}{2}x^\mu + \frac{\alpha'}{2}\left(\frac{n}{R} + \frac{wR}{\alpha'}\right)\sigma^+ + \text{osc.}, \\ X^\mu_{25,L} &= \frac{1}{2}x^\mu + \frac{\alpha'}{2}\left(\frac{n}{R} - \frac{wR}{\alpha'}\right)\sigma^- + \text{osc.}. \end{aligned} \tag{3.19}$$

The mass formula for the string states is straightforward to derive and one finds

$$\begin{aligned} m^2 &= \left(\frac{n}{R} - \frac{wR}{\alpha'}\right)^2 + \frac{4}{\alpha'}(N-1) \\ &= \left(\frac{n}{R} + \frac{wR}{\alpha'}\right)^2 + \frac{4}{\alpha'}\left(\tilde{N}-1\right). \end{aligned} \tag{3.20}$$

First of all we see that the level matching condition is altered to $N - \tilde{N} = mn$ and more interestingly the mass formula is given by

$$m^2 = \frac{n^2}{r^2} + \frac{m^2R^2}{\alpha'^2} + \frac{2}{\alpha'}(N + \tilde{N} - 2). \tag{3.21}$$

The first term is simple to understand as the mass contribution from the momenta on S^1 and remembering that the string tension was given by $T = 1/(2\pi\alpha')$, the contribution from the second term is simply $(2\pi RmT)^2$, where m is the number of times the string winds around S^1. Most importantly we find that the mass is invariant if we make the transformation $R \leftrightarrow \alpha'/R$ and $m \leftrightarrow n$, the string thus has the same spectra of states if we compactify on a circle with radius R, or a circle with radius α'/R. This is a purely stringy phenomena and this is the so-called T-duality of string theory. Generalising to compactifications on a torus T^d one finds a larger set of dualities given by the discrete group $O(d, d, \mathbb{Z})$. In the corresponding supergravity theory compactified on a torus the the corresponding continuous version $O(d, d, \mathbb{R})$ will be a symmetry. It is precisely the appearance of the continuous version of this group that we want to make manifest in double field theory introduced in chapter 4.

3.2 Supergravity and Kaluza-Klein compactification

Supergravity describes the low-energy effective theory of string theory in 10 dimensions and M-theory in 11 dimensions. In fact, eleven dimensions are the largest number of dimensions in which supergravity is consistent assuming one graviton and no field with spin higher than 2. The $\mathcal{N} = 1$ $D = 11$ supergravity theory describing the low-energy physics of M-theory contains the following field content

$$\begin{cases} g_{MN} & 44, \\ A_{MNP} & 84, \\ \psi_{M\alpha} & 128, \end{cases} \tag{3.22}$$

i.e. a graviton, a three-form field and a Majorana gravitino. Moreover, we have denoted the on-shell degrees of freedom. The action is given by

$$S_{11} = \frac{1}{2\kappa_{11}^2} \int \mathrm{d}^{11}\mathrm{x}\sqrt{-g} \left(R - \frac{1}{2} F \wedge *F \right) - \frac{1}{6} \int A \wedge F \wedge F + S_F, \tag{3.23}$$

where $F = dA$ is the field strength, S_F the fermionic part of the action and κ_{11}^2 is Newton's gravitational constant in 11 dimensions. Note that the three-form potential A couples naturally to an object with a three-dimensional worldvolume Σ, i.e. a 2-brane, as

$$Q_3 \propto \int_\Sigma A. \tag{3.24}$$

In $D = 10$ there are two $\mathcal{N} = 2$ supergravity theories corresponding to the low-energy effective field theory of type IIA/B string theory respectively. The field content for type IIA is given by

$$IIA : \begin{cases} g_{\mu\nu} & 35 & \text{graviton (NS–NS)} \\ B_{\mu\nu} & 28 & \text{two-form field (NS–NS)} \\ \phi & 1 & \text{dilaton (NS–NS)} \\ C_1 & 8 & \text{one-form field (R–R)} \\ C_3 & 56 & \text{three-form field (R–R)} \\ \psi^{\pm}_{\mu\alpha} & 112 & \text{Majorana-Weyl gravitinos} \\ \lambda^{\pm}_{\alpha} & 16 & \text{Majorana-Weyl dilatinos,} \end{cases}$$

and for type IIB

$$IIB:\begin{cases} g_{\mu\nu} & 35 \quad \text{graviton (NS–NS)} \\ B_{\mu\nu} & 28 \quad \text{two-form field (NS–NS)} \\ \phi & 1 \quad \text{dilaton (NS–NS)} \\ C_0 & 1 \quad \text{axion (R–R)} \\ C_2 & 28 \quad \text{two-form field (R–R)} \\ C_4 & 35 \quad \text{four-form field (R–R)} \\ \psi^{1,2}_{\mu\alpha} & 112 \quad \text{Majorana-Weyl gravitinos} \\ \lambda^{1,2}_{\alpha} & 16 \quad \text{Majorana-Weyl dilatinos.} \end{cases}$$

The NS-NS sector is the same for type IIA/B with an action

$$S=\frac{1}{2\kappa_{10}^2}\int \mathrm{d}^D x\sqrt{-g}\mathrm{e}^{-2\phi}\left(R+4(\partial\phi)^2-\frac{1}{2}H\wedge\star H\right),$$

where $H_3=\mathrm{d}B$ is the field strength, $\star$ the hodge dual and κ_{10}^2 the gravitational constant in 10 dimensions. For further details on supergravity and supersymmetry see [30].

In order to reduce these theories to lower dimensions we will look at Kaluza-Klein compactifications. The idea is to assume a separation between external and internal directions such that the underlying spacetime is given by $\mathcal{M}=\mathcal{M}_{D-n}\times\mathcal{M}_n$, where $\mathcal{M}_{D-n}$ is the external spacetime and $\mathcal{M}_n$ the internal space which typically is assumed to be “small”. Note also that below we will mainly be concerned with the bosonic sector of the theory. The discussion below follows that of [31].

We introduce this by the easiest example of a massless scalar field ϕ compactified on a circle $\mathcal{M}=\mathcal{M}_{D-1}\times S^1$. Due to the periodicity of S^1 we can expand ϕ in a fourier expansion and the equation of motion is given by

$$\left(\partial^2_{D-1}-\frac{n^2}{(2\pi R)^2}\right)\phi=0, \tag{3.25}$$

where $n\in\mathbb{Z}$ is the mode-number of ϕ expanded on a circle with radius R. We thus see that the compactification gives rise to an infinite tower of scalar fields with mass $|n|/(2\pi R)$. In the limit of small R all these fields decouple except for the massless field with $m=0$ and there is no longer any dependence on the internal directions. This a *consistent truncation* in the sense that a solution in the lower-dimensional theory will also be a solution in the full theory. However, consistency for compactifications on more general internal manifolds $\mathcal{M}_n$ is usually quite hard to prove. In fact, one success of the extended theories, double field theory and exceptional field theory, is that they allow a natural uplift for certain lower-dimensional theories that can not be well described by supergravity in $D=10$ and $D=11$.

In order to motivate the appearance of “hidden” symmetry groups that later on will be extended to manifest symmetries in extended geometries we consider pure gravity in D dimensions compactified on a circle $\mathcal{M}_D=\mathcal{M}_{D-1}\times S^1$. Split the coordinates $x^M\to(x^\mu,y)$ and assume independence of the internal coordinate. The metric decompose into a lower-dimensional metric $g_{\mu\nu}$, a vector field A_μ and a scalar ϕ and can be parameterised as

$$g_{MN}=\left(\begin{array}{c|c}\mathrm{e}^{2\alpha\phi}g_{\mu\nu}+\mathrm{e}^{2\beta\phi}A_\mu A_\nu & \mathrm{e}^{2\beta\phi}A_\mu \\ \hline \mathrm{e}^{2\beta\phi}A_\nu & \mathrm{e}^{2\beta\phi}\end{array}\right), \tag{3.26}$$

where $\beta = -(D-2)\alpha$ and α constant. Pure gravity in D dimensions is invariant under diffeomorphisms $\xi^M = \xi^M(x^\mu, y)$ in D dimensions under which the metric transforms as

$$\delta_\xi g_{MN} = \mathcal{L}_\xi g_{MN} = \xi^P \partial_P g_{mn} + \partial_M \xi^P g_{PN} + \partial_N \xi^P g_{MP}, \tag{3.27}$$

where $\mathcal{L}$ is the Lie derivative. In order to preserve the parameterisation (3.26) one finds that diffeomorphisms in D dimensions reduce to transformations of the form

$$\xi^M = \left(\hat{\xi}^\mu(x^\nu), cy + \lambda(x^\nu)\right), \tag{3.28}$$

with some constant c. If $c = 0$ one finds by explicit calculation that the lower-dimensional fields transform as

$$\delta_\xi g_{\mu\nu} = \mathcal{L}_{\hat{\xi}^\rho} g_{\mu\nu}, \qquad \delta_\xi A_\mu = \mathcal{L}_{\hat{\xi}^\rho} A_\mu + \partial_\mu \lambda(x^\nu), \qquad \delta_\xi \phi = \mathcal{L}_{\hat{\xi}^\rho} \phi. \tag{3.29}$$

We thus see that the $(D-1)$-dimensional theory transforms as one would expect under $(D-1)$-dimensional diffeomorphisms and a local $U(1)$ gauge transformation of the vector field. However, if $c \neq 0$ one has to consider that the D dimensional theory also possess a scaling symmetry $\delta g_{MN} = 2a g_{MN}$ at the level of equations of motion which combine with the cy term to the following transformation (discussed further in [31])

$$\beta \delta\phi = a + c, \qquad \delta A_\mu = -cA_\mu \qquad g_{\mu\nu} = 2a g_{\mu\nu} - 2\alpha g_{\mu\nu} \Delta\phi. \tag{3.30}$$

By choosing $\alpha = -\frac{c}{D-1}$ the lower-dimensional metric $\delta g_{\mu\nu}$ becomes invariant under the remaining global scaling symmetry. This remaining scaling symmetry is a common feature of supergravity theories typically referred to as a trombone symmetry.

Compactification on a torus $T^n = S^1 \times S^1 \times \ldots \times S^1$ continues in much the same way by recursively compactifying on circles. We omit the details but note that similarly to the case above one finds that the diffeomorphisms preserving the reduction ansatz, $x^M \to (x^\mu, y^i)$, where $i = 1, 2, \ldots n$, are given by

$$\xi^M = \left(\hat{\xi}^\mu(x^\nu), M^i{}_j y^j + \lambda^i(x^\nu)\right), \tag{3.31}$$

where $M \in SL(n, \mathbb{R}) \times \mathbb{R}$ a constant matrix and λ^i are n $U(1)$ gauge parameters. Again there is a scaling symmetry of the uncompactified theory which can be combined with a $\mathbb{R}$ factor in M. As such we find that compactified gravity on a torus T^n has a global $SL(n) \times \mathbb{R}$ symmetry. However, coupling matter to gravity may enhance this symmetry to even larger groups, especially we want to include form fields which have gauge symmetries parameterised by $(p-1)$-form fields. In order to keep the compactification ansatz that no fields depend on the internal coordinates one finds that scalar parts of a gauge potential $A_{j_1 j_2 \ldots j_p}$ transforms as

$$\begin{aligned} \delta A_{j_1 j_2 \ldots j_p} &= \partial_{[j_1} \lambda_{j_2 \ldots j_p]} \\ &= c_{j_1 j_2 \ldots j_p}, \end{aligned} \tag{3.32}$$

with $c_{j_1 j_2 \ldots j_p}$ being a constant anti-symmetric tensor. These corresponds to constant shifts of the purely internal components of the gauge potential A and, moreover, they mutally commute such that they are elements in R^n, where n is the number of independent components of $c_{i_1 \ldots}$. However, they do not necessarily commute with the global $GL(n, \mathbb{R})$ symmetry and typically there is thus a global symmetry group given by

$$GL(N, \mathbb{R}) \ltimes \mathbb{R}^n. \tag{3.33}$$

Table 3.1: Maximal number of scalars of compactified $D = 11$ supergravity on T^n. Here G denotes the group related to the split real form and K the corresponding maximal compact subgroup. The trombone symmetry is excluded.

n	#Scalars	G	K
3	7	$SL(3) \times SL(2)$	$SO(3) \times SO(2)$
4	14	$SL(5)$	$SO(5)$
5	25	$SO(5,5)$	$SO(5) \times SO(5)$
6	42	$E_{6(6)}$	$USp(8)$
7	70	$E_{7(7)}$	$SU(8)$
8	128	$E_{8(8)}$	$SO(16)$

However, a p-form potential is dual to a $(d-p-2)$-form potential, where $d = D - n$ is the dimension of the external spacetime. By dualising to obtain as many scalar fields as possible the global symmetry group enhances even further when this is possible [4]. In conclusion, the global hidden symmetry groups are remnants from the internal diffeomorphisms and internal form field gauge symmetry that preserves the compactification ansatz.

Furthermore, it turns out that these symmetries can be studied by looking at the scalar sector only, the parts containing "Kaluza-Klein" vectors will, non-trivially, possess the same symmetry as the scalar sector. Note that scalars coming from the internal components of the metric characterise the internal geometry, e.g. compactification on S^1 gives rise to a scalar field ϕ and its vacuum expectation value (classical solution) determines the radius of the circle.

Consider now the bosonic sector of $D = 11$ supergravity with the metric g_{MN} and the three-form A_{MNP}. Split spacetime as $\mathcal{M}_{11} = \mathcal{M}_{11-n} \times \mathcal{M}_n$ with a corresponding split for the coordinates $x^M \to (x^\mu, y^i)$. The fields decompose as

$$\begin{aligned} g_{MN} &\to \begin{cases} g_{\mu\nu}, \\ g_{\mu i}, \\ g_{ij}, \end{cases} \\ A_{MNP} &\to \begin{cases} A_{\mu\nu\rho}, \\ A_{\mu\nu i}, \\ A_{\mu ij}, \\ A_{ijk}. \end{cases} \end{aligned} \tag{3.34}$$

As above, it is sufficient to consider the scalar sector of the theory. The number of scalars originating from the metric are $n(n+1)/2$ and the number of scalars from the 3-form potential are $\binom{n}{3}$. We will also make the choice to dualise as much as possible in order to obtain a theory with as many scalar fields as possible. The number of scalars present when compactifying on T^n and the "hidden" symmetry groups G are listed in Table 3.1.

3.2.1 Non-linear sigma model

We have argued above that the compactification of $D = 11$ supergravity on T^n give rise to a hidden global symmetry G. In order to write down a lagrangian for the scalar sector that is manifestly invariant under this symmetry we will consider a non-linear sigma model. The idea is that the scalar fields can be considered as "coordinates" on a coset space G/K, where K is the maximal compact subgroup of G. It is then easily checked that the dimension of the coset $\dim(G/K) = \dim(G) - \dim(K)$, using *Table* 3.1, equals the number of scalars.

To parameterise this Lagrangian it is convenient to look at the algebra $\mathfrak{g}$ and its maximal compact subalgebra $\mathfrak{k}$ that upon exponentiated give the related groups G and H respectively. As we saw in Section 2.2.4 the algebra $\mathfrak{g}$ can be decomposed as

$$\mathfrak{g} = \mathfrak{k} \oplus \mathfrak{h} \oplus \mathfrak{n}_+, \tag{3.35}$$

where $\mathfrak{h}$ and $\mathfrak{n}^+$ denote the Cartan subalgebra and the positive roots respectively. This can be extended to a decomposition at group level obtained by exponentiating

$$G = K \times H \times N_+. \tag{3.36}$$

An element $\mathcal{V}$ in G/K describing the scalar fields $\{\phi^a, \chi^i\}$ can thus be parameterised by

$$\mathcal{V} = \mathrm{e}^{\phi^a(x)h_a}\mathrm{e}^{\chi^i(x)e_i}, \tag{3.37}$$

where h_i are elements in the Cartan subalgebra, e_i positive root vectors in $\mathfrak{g}$ and we sum over repeated indices. The element $\mathcal{V}$ transforms as

$$\mathcal{V}(x) \to k(x)\mathcal{V}(x)g \sim \mathcal{V}g, \tag{3.38}$$

where $k(x) \in K$ and $g \in G$. Note that the element $k(x)$ represents a local transformation, this is analogous to the case of ordinary gravity where the metric is invariant under local Lorentz transformations. However, the transformation by g potentially destroys the parameterisation in (3.37). This can be remedied by a compensating $k(x)$ transformation which depends on both g and $\mathcal{V}$, hence such a transformation is non-linear.

By taking the differential of $\mathcal{V}$ and multiplying with its inverse we find an element in the algebra (a 1-form in the external space)

$$\mathcal{V}^{-1}\mathrm{d}\mathcal{V} = \mathcal{Q} + \mathcal{P}, \tag{3.39}$$

where $\mathcal{Q} \in \mathfrak{k}$ and $\mathcal{P} = \mathfrak{g} \ominus \mathfrak{k}$. Since the elements in $\mathfrak{k}$ and $\mathfrak{p}$ have eigenvalues 1 and -1 under the Chevalley involution one obtain the different terms as

$$\mathcal{Q} = \frac{1}{2}\left(\mathcal{V}^{-1}\mathrm{d}\mathcal{V} + \omega\left(\mathcal{V}^{-1}\mathrm{d}\mathcal{V}\right)\right), \qquad \mathcal{P} = \frac{1}{2}\left(\mathcal{V}^{-1}\mathrm{d}\mathcal{V} - \omega\left(\mathcal{V}^{-1}\mathrm{d}\mathcal{V}\right)\right). \tag{3.40}$$

In order to write down an object invariant under global G and local K in terms of the physical component $\mathcal{P}$ we need an invariant tensor for the tensor product of two adjoint representations. The Killing form is such an invariant tensor and a manifestly invariant term can thus be written as

$$\mathcal{L} \sim \kappa(\mathcal{P}_\mu, \mathcal{P}_\nu)g^{\mu\nu}\sqrt{-g}, \tag{3.41}$$

where $g^{\mu\nu}$ is the inverse metric of the external space. We have thus found a lagrangian for the scalar sector of the theory that is manifestly invariant under G/K.

3.3 Dualities

Duality is a term that is used in many different ways in physics. What we will be interested in are dualities between two theories $\mathscr{T}$ and $\tilde{\mathscr{T}}$ such that we can find a map to relate them

$$\mathscr{T} \leftrightarrow \tilde{\mathscr{T}}. \tag{3.42}$$

A typical example of such a duality is the duality of electromagnetism in vacuum with a mapping

$$\left(\vec{E}, \vec{B}\right) \rightarrow \left(\vec{B}, -\vec{E}\right), \tag{3.43}$$

under which Maxwell's equations are invariant.

Usually we are interested in interacting field theories where we have to employ perturbation theory in some "small" coupling constant g, such that observables are given as a perturbation series

$$R(g) = \sum_{n=0}^{\infty} a_n g^n. \tag{3.44}$$

These series typically do not converge but in many cases they are believed to be asymptotic series. Non-perturbative effects have a characteristic scaling $\sim \mathrm{e}^{-1/g^n}$ which is not captured by the expansion (3.44), especially instanton effects in ordinary field theories typically scale with $n = 2$. In interacting theories we thus get two kinds of duality transformations: weak-weak duality and weak-strong duality.

- Weak-weak duality: $(\mathscr{T}, g) \leftrightarrow (\mathscr{T}', g')$ with $g, g' \ll 1$.
- Weak-strong duality: $(\mathscr{T}, g) \leftrightarrow (\mathscr{T}'', g'')$ with $g \ll 1$ and $g'' \gg 1$.

Weak-strong dualities are of special interest since it allow us to probe the non-perturbative regime of a theory by using perturbative methods in the weakly coupled dual theory. Since little is usually known about the non-perturbative regime of the strongly coupled theory such a duality is however hard to prove. However, there are quantities that are protected against perturbative corrections and extrapolation to the non-perturbative regime is possible. These play an important rôle in confirming these weak-strong dualities. Perhaps the most prominent weak-strong duality is the AdS/CFT correspondence, first conjectured by Maldacena in [32]. This is a rather special duality in the sense that it relates a string theory (which includes gravity) in a given dimension to a conformal field theory in one spacetime dimension lower, without gravity. Another important weak-strong duality that is more relevant for this thesis is the Montonen-Olive duality. This is an $SL(2, \mathbb{Z})$ duality of $\mathcal{N} = 4$ SYM that typically exchange solitonic objects with fundamental excitations and moreover map the coupling constant to its inverse.

Motivation for the presence of a weak-weak T-duality symmetry when compactifying type II string theories on a torus T^d was introduced above, with the the T-duality group $O(d, d, \mathbb{Z})$. Moreover, type IIB has a weak-strong so-called S-duality symmetry with transformations given by $SL(2, \mathbb{Z})$. It is conjectured that these combine into a larger set of dualities called U-duality [33], and contains $SL(2, \mathbb{Z})$ and $O(d, d, \mathbb{Z})$ as subgroups. For the type II theories the U-duality groups upon compactification on a torus T^n are given in Table 3.2 which are just the discrete versions of the continuous groups encountered in supergravity, however, the agreement are between type IIB on T^d and eleven-dimensional supergravity on T^{d+1}, since in this case the external spacetime has the same dimension.

Table 3.2: U-duality group for type II string theories on T^n.

n	U-duality group
1	$SL(2,\mathbb{Z}) \times \mathbb{Z}_2$
2	$SL(2,\mathbb{Z}) \times SL(3,\mathbb{Z})$
3	$SL(5,\mathbb{Z})$
4	$SO(5,5,\mathbb{Z})$
5	$E_{6(6)}(\mathbb{Z})$
6	$E_{7(7)}(\mathbb{Z})$
7	$E_{8(8)}(\mathbb{Z})$
8	$E_{9(9)}(\mathbb{Z})$

4

Double field theory

The low-energy effective theory for the massless NS-NS fields in string theory contains gravity g_{ij}, a 2-form field B_{ij} and a scalar field (dilaton) ϕ. Together with diffeomorphism invariance the 2-form field also comes with an abelian gauge symmetry and the theory is given by the action

$$S = \frac{1}{2\kappa_{10}^2} \int \mathrm{d}^D x \sqrt{-g} \mathrm{e}^{-2\phi} \left(R + 4(\partial\phi)^2 - \frac{1}{12} H^2 \right), \tag{4.1}$$

where $H = \mathrm{d}B$ is the 3-form field strength tensor of B. In its present form there is no manifest T-duality symmetry.

T-duality of closed strings compactified on a torus T^d exchanges momentum on the torus with winding modes by an $O(d, d, \mathbb{Z})$ transformation. Moreover, upon compactifying supergravity in $D = 10$ on a torus T^d one finds that the continuous version of the same group, i.e. $O(d, d, \mathbb{R})$, is a symmetry of the theory and the goal of double field theory is to make this manifest. In order to do this one extends space-time to include coordinates $\tilde{x}_i$ dual to the winding modes w^i and arrange fields in representations of $O(d, d)$. Doing this diffeomorphisms and the 2-form gauge symmetries merge to a single symmetry transformation called generalised diffeomorphisms. At the same time this unifies the metric and the gauge field into a generalised metric. For more about DFT see [27, 34, 35, 36].

4.1 Generalised diffeomorphisms and the section constraint

Ordinary diffeomorphisms ξ^i are encoded in the Lie derivative acting on the tensor fields as

$$L_\xi g_{ij} = \xi^k \partial_k g_{ij} + 2\partial_{(i}\xi^k g_{j)k} \qquad L_\xi B_{ij} = \xi^k \partial_k B_{ij} + 2\partial_{[i}\xi^k B_{|k|j]} \qquad L_\xi \phi = \xi^i \partial_i \phi. \tag{4.2}$$

This can be viewed as a transport term coming from the Taylor expansion of the argument as well as the adjoint action of an element in $GL(d)$ acting according to the index structure. The generalisation of the Lie derivative plays an import rôle in the extended geometries. Gauge transformations of the form-field on the other hand are parameterised by a scalar function λ as

$$\delta_\lambda B_{ij} = 2\partial_{[i}\lambda_{j]}, \tag{4.3}$$

with the metric and dilaton field inert under this transformation. Note that the gauge transformation comes with a reducibility stemming from the fact that $\lambda_i = \partial_i \chi$, where

χ is a scalar function, is a trivial transformation. Reducibility is again a property that becomes important in the extended cases.

The first step to construct a doubled theory is to extend space-time from d to $2d$ dimensions. In the case of type II supergravity this could be thought of as compactifying all $d = 10$ coordinates and double them, however, the general construction works with arbitrary d without any compact directions. Hence we extend the coordinates of space-time to belong to the $2d$ vector representation of $O(d,d)$ according to

$$x^i \to X^M = (\tilde{x}_i, x^j), \tag{4.4}$$

where $\tilde{x}_i$ denotes the added coordinates and X^M the doubled "space-time" coordinate vector with $M = 1, 2, \ldots, 2d$. The metric and the two-form combine to a generalised metric $\mathcal{H}_{MN}$ given by

$$\mathcal{H}_{MN} = \begin{pmatrix} g^{ij} & -g^{ik}B_{kj} \\ B_{ik}g^{kj} & g_{ij} - B_{ik}g^{kl}B_{lj} \end{pmatrix}. \tag{4.5}$$

Before moving on to the generalised diffeomorphisms we note that the group $O(d,d)$ also comes with the invariant metric η_{MN} such that for an element $h \in O(d,d)$

$$h_M{}^P \eta_{PL} h_N{}^L = \eta_{MN} = \begin{pmatrix} 0 & \mathbb{1} \\ \mathbb{1} & 0 \end{pmatrix}. \tag{4.6}$$

The invariant metric η_{MN} and its inverse η^{MN} can be used to raise and lower $M = 1, 2, \ldots, 2d$ indices.

The generalised Lie derivative on a vector V^M is given by

$$\delta_\xi V^M = \mathscr{L}_\xi V^M = \xi^P \partial_P V^M + (\partial^M \xi_P - \partial_P \xi^M) V^P, \tag{4.7}$$

where we implicitly used the invariant metric η to raise and lower indices. In order to connect to the usual Lie derivative as well as being convenient for other extended geometries this can be rewritten as

$$\mathscr{L}_\xi V^M = L_\xi V^M + Y^{MN}{}_{PQ}\, \partial_N \xi^P V^Q, \tag{4.8}$$

where $L_\xi V^M = \xi^P \partial_P V^M - \partial_P \xi^M V^P$ denotes the ordinary Lie derivative and $Y^{MN}{}_{PQ} = \eta^{MN}\eta_{PQ}$ is an invariant tensor of $O(d,d)$.

In order for the symmetry group to close, two consecutive transformations ξ_1, ξ_2 needs to yield a new transformation with some parameter $\xi(\xi_1, \xi_2)$. To this end we adopt the notation that $\Delta = \delta_\xi - \mathscr{L}_\xi$ denotes the departure of an object transforming as a tensor, i.e. $\delta_\xi X = \mathscr{L}_\xi X$ defines a tensor, and look at

$$-\Delta_{\xi_1}(\mathscr{L}_{\xi_2} X^M) = \left([\mathscr{L}_{\xi_1}, \mathscr{L}_{\xi_2}] - \mathscr{L}_{\xi(\xi_1,\xi_2)} \right) X^M \stackrel{!}{=} 0. \tag{4.9}$$

By doing this explicitly one finds that generalised diffeomorphisms closes if

$$\eta^{MN} \partial_M \otimes \partial_N = 0, \tag{4.10}$$

where $\partial_M \otimes \partial_N$ indicates that each derivative act on an arbitrary field. The combined transformation $\xi(\xi_1, \xi_2)^M$ is given by the C-bracket $[\xi_1, \xi_2]_C^M = 1/2(\mathcal{L}_{\xi_1}\xi_2 - \mathcal{L}_{\xi_2}\xi_1)$

$$\xi(\xi_1, \xi_2)^M = [\xi_1, \xi_2]_C^M = \xi_1^P \partial_P \xi_2^M - \frac{1}{2}\xi_{1P}\partial^M \xi_2^P - (1 \leftrightarrow 2). \tag{4.11}$$

The constraint $\eta^{MN}\partial_M \otimes \partial_N = 0$ is called the (strong) section constraint and a generalised version of this is characteristic for extended geometries. By solving the section constraint one restricts space-time to a d-dimensional subspace of the extended space-time, e.g. one solution is given by $\tilde{\partial}^i = 0$ which implies that no fields can depend on the extended coordinates. In fact, starting with this solution one can get all other possible solutions by applying an $O(d,d)$ transformation.

One can now check that if we solve the section constraint with $\tilde{\partial}^i = 0$ and set $\xi^M = (\lambda_i, \xi^i)$ that the generalised diffeomorphism reduces to the ordinary diffeomorphism and gauge transformation

$$\mathscr{L}_\xi g_{ij} = L_\xi g_{ij} \qquad \mathscr{L}_\xi B_{ij} = L_\xi B_{ij} + 2\partial_{[i}\lambda_{j]}, \qquad \mathcal{L}_\xi \phi = L_\xi \phi \tag{4.12}$$

This shows that the generalised diffeomorphisms merge ordinary diffeomorphisms with 2-form gauge transformations.

4.2 DFT action

In order to write down a two-derivative action for DFT we need to take a derivative of the generalised metric $\partial_L \mathcal{H}^{MN}$ and combine such terms to ensure invariance under generalised diffeomorphisms. However, due to the bare derivatives the invariance will not be manifest. Moreover, the combination $\mathrm{e}^{-2\phi}\sqrt{-g}$ is an $O(D,D)$ scalar density with weight one and can be combined as e^{-2d}. Given this the action for DFT reads

$$\begin{aligned} S_{\mathrm{DFT}} = \int \mathrm{d}^D x \mathrm{d}^D \tilde{x} \mathrm{e}^{-2d} \Big(&\frac{1}{8}\mathcal{H}^{MN}\partial_M \mathcal{H}^{KL}\partial_N \mathcal{H}_{KL} - \frac{1}{2}\mathcal{H}^{MN}\partial_M \mathcal{H}^{KL}\partial_L \mathcal{H}_{KN} \\ &-2\partial_M d \partial_N \mathcal{H}^{MN} + 4\mathcal{H}^{MN}\partial_M d \partial_N d \Big) . \end{aligned} \tag{4.13}$$

Since the indices are contracted properly we only need to check the non-tensorial part, i.e. terms with derivatives, to ensure invariance under generalised diffeomorphisms. To this end we examine

$$\Delta_\xi \partial_K \mathcal{H}^{MN} = \partial_K \left(\delta_\xi \mathcal{H}^{MN}\right) - \mathscr{L}_\xi \left(\partial_K \mathcal{H}^{MN}\right) \tag{4.14}$$

which is given by

$$\Delta_\xi \partial_K \mathcal{H}^{MN} = 2\left(\partial_K \partial^{(M}\xi_P - \partial_K \partial_P \xi^{(M}\right)\mathcal{H}^{N)P}, \tag{4.15}$$

up to a term that vanish by the section constraint. Likewise one finds

$$\Delta_\xi \partial_K \mathcal{H}_{MN} = 2\left(\partial_K \partial_{(M}\xi^P - \partial_K \partial^P \xi_{(M}\right)\mathcal{H}_{N)P}. \tag{4.16}$$

For the first term in (4.13) one finds that

$$\Delta_\xi \left(\frac{1}{8}\mathcal{H}^{MN}\partial_M \mathcal{H}^{KL}\partial_N \mathcal{H}_{KL}\right) = \mathcal{H}^{MN}\partial_M \mathcal{H}^{KL}\partial_N \partial_K \xi^P \mathcal{H}_{LP}, \tag{4.17}$$

where we used $\mathcal{H}^{PL}\partial_N \mathcal{H}_{LK} = -\partial_N \mathcal{H}^{PL}\mathcal{H}_{LK}$. A similar calculation for the second term in the action gives

$$\Delta_\xi \left(-\frac{1}{2}\mathcal{H}^{MN}\partial_M \mathcal{H}^{KL}\partial_L \mathcal{H}_{KN}\right) = -\partial_K \partial_L \xi^M \partial_M \mathcal{H}^{KL} - \partial_K \mathcal{H}^{MN}\partial_M \partial_P \xi^L \mathcal{H}^{KP}\mathcal{H}_{NL}, \tag{4.18}$$

where we see that the second term in this expression cancel the variation in (4.17). For the last two terms we also need

$$\Delta_\xi \partial_M d = -\frac{1}{2}\partial_M \partial_P \xi^P. \tag{4.19}$$

Using this we get for the third term in (4.13)

$$\Delta_\xi(-2\partial_M d \partial_N \mathcal{H}^{MN}) = \partial_M \partial_P \xi^P \partial_N \mathcal{H}^{MN} + 2\partial_M \partial_N \partial_P \xi^M \mathcal{H}^{NP} + 2\partial_M d \partial_N \partial_P \xi^N \mathcal{H}^{NP} \tag{4.20}$$

and for the last term

$$\Delta_\xi(4\mathcal{H}^{MN}\partial_M d \partial_N d) = -4\partial_N d \partial_M \partial_P \xi^P \mathcal{H}^{MN}. \tag{4.21}$$

Integrating by parts (note the dilaton prefactor in the action) one finds that the total variation of S_{DFT} indeed vanish and the action is thus invariant under generalised diffeomorphisms.

With the supergravity solution to the section constraint, $\tilde{\partial}^i = 0$, it is interesting to see that the DFT action reduce to that of supergravity (as it should)

$$S_{\mathrm{DFT},\tilde{\partial}=0} = \int \mathrm{d}^D x \mathrm{e}^{-2\phi}\sqrt{-g}\left(R + 4(\partial\phi)^2 - \frac{1}{12}H^2\right). \tag{4.22}$$

Moreover, T-duality on a circle typically exchanges the metric with its inverse and exchanges $\partial \leftrightarrow \tilde{\partial}$. This invariance can be seen nicely by looking at the action when the dilaton and the 2-form are turned off

$$S_{\mathrm{DFT},d=b=0} = \int \mathrm{d}^D x \mathrm{d}^D \tilde{x}\left(R(g,\partial) + R(g^{-1},\tilde{\partial})\right), \tag{4.23}$$

which clearly shows that the action is invariant under T-duality in this case. When the form field and the dilaton are turned on the action is of course still invariant under T-duality, however, there will be a non-trivial mixing between the metric and the 2-form according to Buscher's rules [28] and a shift in the dilaton field. Mixing between the fields is expected according to the generalised diffeomorphisms introduced above.

5

Exceptional field theory

Double field theory aimed to make the continuous T-duality group act naturally in supergravity prior to compactification. The goal of exceptional field theory on the other hand does the same for the U-duality groups by replacing $O(d,d)$ by $E_{d(d)}$. As we have argued these duality groups $E_{d(d)}$ appear upon compactification of supergravity on a d-dimensional torus and the corresponding discrete versions are the dualities of string theory/M-theory. One inherent difference compared to DFT is that the duality groups are rather different between dimensions making the general analysis more complicated. Moreover, for $d \geq 9$ the duality groups become infinite-dimensional. This can in some sense be understood by noting that in $d \leq 2$ external dimensions scalars become dual to scalars.

We will introduce exceptional field theory by the specific example of $E_{6(6)}$ exceptional field theory following [37]. For a similar construction for $E_{7(7)}$ and $E_{8(8)}$, see [38, 39].

There are two main points that we want to convey in this chapter besides providing another concrete example of an extended geometry. First we want to show some details on how to include external dimensions since this is the main part which differs from double field theory and extended geometries. In the former case all coordinates were doubled while in extended geometries we simply drop the external space and consider only the purely internal sector. Secondly we want to motivate the appearance of a hierarchy of form fields related to the tensor hierarchy of gauged supergravities. This hierarchy seems to be important to understand a complete formulation of extended geometries. Moreover, we will demonstrate three different solutions to the section constraint, the first solution makes the global $E_{d(d)}$ symmetry of supergravity compactified on a torus T^d manifest while the other two clarify the embedding of M-theory and type IIB supergravity respectively in the constructed exceptional field theory.

5.1 E_6 – Exceptional field theory

The bosonic sector of eleven-dimensional supergravity consists of the metric g_{AB} and a three-form A_{ABC}. Compactifying this theory on a T^6 torus we get $3 \times 7 = 21$ scalars from the metric and naively $\binom{6}{3} = 20$ scalars from the three-form. However, it is also possible to dualise a three-form in 5 dimensions to a scalar since $5 - 3 - 2 = 0$ and in total there are thus 42 scalars. This equals the number of generators in the coset $E_{6(6)}/USp(8)$, with $\mathfrak{usp}(8)$ being the maximal compact subalgebra of $\mathfrak{e}_{6(6)}$, in agreement with the discussion in Section 3.2.

What we want to do is to expand spacetime similar to the DFT case by splitting the coordinates $x^A \to (x^\mu, y^i)$, with $\mu = 1, 2, \dots 5$ and $i = 1, 2, \dots 6$, and then extend the internal coordinates to transform in **27** of $E_{6(6)}$ as $y^i \to z^M$, with $M = 1, 2, \dots 27$. The extended spacetime is then $5 + 27$ dimensional and the 41 scalars organise into a coset element $\mathcal{M}_{MN}(x, z) \in E_{6(6)}/USp(8)$.

Just as in DFT we introduce generalised diffeomorphisms that encode the $E_{6(6)}$ transformations that we are "geometrising" by its action on a vector $V^M \in \mathbf{27}$ with weight w as

$$\delta_\xi V^M = \mathscr{L}_\xi V^M = \xi^P \partial_P V^M + P^{MN}{}_{PQ}\, \partial_N \xi^P W^Q + w \partial_P \xi^P V^M, \tag{5.1}$$

with the $E_{6(6)}$ invariant tensor

$$\begin{aligned} P^{MN}{}_{PQ} &= -6 \eta_{\alpha\beta} T^{\alpha M}{}_Q T^{\beta N}{}_P \\ &= \frac{1}{3} \delta^M_Q \delta^N_P - \delta^M_P \delta^N_Q + 10 d_{PQS} d^{MNS}, \end{aligned} \tag{5.2}$$

and d_{MNS}, d^{MNS} totally symmetric invariant tensors of $E_{6(6)}$ which we normalise as $d_{MNS} d^{PNS} = \delta^P_M$. Imposing Leibniz property and that the element $(V^N W_N)$, where W_N is a covector with weight w', transforms as a scalar density of weight $w + w'$ the action on W_N is easily found. This, moreover, generalise straightforward to a tensor with any number of indices. Importantly we define the section constraint for $E_{6(6)}$ exceptional field theory as

$$d^{MNS} \partial_N \otimes \partial_S = 0, \tag{5.3}$$

which as in the DFT case is crucial for a consistent theory. Especially, the section constraint is needed to show that the algebra of generalised diffeomorphisms close according to [37] (these results are also derived in the general case in Chapter 6)

$$[\mathscr{L}_{\xi_1}, \mathscr{L}_{\xi_2}] = \mathscr{L}_{\xi_{1,2}}, \tag{5.4}$$

where the transformation parameter $\xi_{1,2} = [\xi_1, \xi_2]_{\mathrm{E}}$ is given by the E-bracket

$$\begin{aligned} [\xi_1, \xi_2]^M_{\mathrm{E}} &= \frac{1}{2} (\mathscr{L}_{\xi_1} \xi_2 - \mathscr{L}_{\xi_2} \xi_1)^M \\ &= \xi_1^P \partial_P \xi_2^M - 5 d^{MNP} d_{STP} \xi_1^S \partial_N \xi_2^T - (\xi_1 \leftrightarrow \xi_2). \end{aligned} \tag{5.5}$$

The kernel of generalised diffeomorphisms is non-trivial as parameters of the form $\xi^M = d^{MNP} \partial_N \eta_P$ does not generate a transformation. To see this consider the transformation due to such a parameter

$$\begin{aligned} \mathscr{L}_\xi V^M = &-d^{MNP} \partial_S \partial_N \eta^P V^S + 10 d^{MNP} d^{QLS} d_{STP} \partial_N \partial_Q \eta_L V^T \\ &+ \underbrace{d^{SNP} \partial_N \eta_P \partial_S V^M}_{=0\,\text{section constraint}} + \underbrace{(\lambda - \frac{1}{3}) \partial_S \partial_N \eta_P d^{SNP} V^M}_{=0\,\text{section constraint}}. \end{aligned} \tag{5.6}$$

The last term can be rewritten as

$$\begin{aligned} d_{STP} d^{P(MN} d^{QL)S} \partial_Q \partial_N X &= \frac{1}{12} d_{STP} (4 d^{SNQ} d^{LMP} + 4 d^{SMN} d^{QLP} + 4 d^{SML} d^{QNP}) \partial_N \partial_Q X \\ &= \frac{2}{3} d_{STP} d^{SMN} d^{QLP} \partial_N \partial_Q X, \end{aligned} \tag{5.7}$$

for some arbitrary field X and the first term vanish due to the section constraint. Using the following identity for the $\mathfrak{e}_6$ cubic invariant

$$d_{STP}d^{P(MN}d^{QL)S} = \frac{2}{15}\delta_T^{(M}d^{NQL)}, \tag{5.8}$$

it is found that

$$\begin{aligned} d_{STP}d^{SMN}d^{QLP}\partial_N\partial_Q X &= \frac{3}{2}\times\frac{2}{15}\delta_T^{(M}d^{NQL)}\partial_N\partial_Q X \\ &= \frac{1}{10}\delta_T^{Q}d^{MNL}\partial_N\partial_Q X. \end{aligned} \tag{5.9}$$

It then follows immediately that $\delta_\xi V^M = 0$ if $\xi^M = d^{MNP}\partial_N\eta_P$.

We can derive other useful properties using (5.8). Consider $U^M = d^{MNK}\partial_N V_K$ with V_K transforming with a weight $\lambda = \frac{2}{3}$. Then U^M transforms as

$$\delta U^M = d^{MNK}\partial_K(\mathscr{L}_\xi V_N) = d^{MNK}\partial_K\left(\xi^S\partial_S V_N + \partial_N\xi^S V_S\partial_S\xi^S V_N - 10d_{NLR}d^{SQR}\partial_Q\xi^L V_S\right). \tag{5.10}$$

Using (5.8) the last term can be rewritten using

$$\begin{aligned} d^{N(MK}d_{NLR}d^{SQ)R}\partial_K(\partial_Q\xi^L V_S) &= \frac{1}{24}\left(8d^{NMK}d_{NLR}d^{SQR} + 8d^{NMQ}d_{NLR}d^{SKR}\right) \\ &\quad + 8\,\underbrace{d^{NQK}d_{NLR}d^{SMR}\partial_K(\partial_Q\xi^L V_S)}_{=0\,\text{section constraint}} \\ &= \frac{2\times 6}{15\times 24}d^{MKN}\partial_K\left(\partial_L\xi^L V_N + \partial_Q\xi^K V_N)\right), \end{aligned} \tag{5.11}$$

and we find

$$\begin{aligned} 10d^{NMK}d_{NLR}d^{SQR}\partial_K(\partial_Q\xi^L V_S) &= -10d^{NMQ}d_{NLR}d^{SKR}\partial_K(\partial_Q\xi^L V_S) \\ &\quad + d^{MKN}\partial_K(\partial_L\xi^L V_N + d^{MNK}\partial_K(\partial_Q\xi^K V_N)). \end{aligned} \tag{5.12}$$

Inserting (5.12) in the transformation of U^M we get

$$\delta U^M = d^{MNK}\xi^S\partial_S\partial_K V_N + 10d^{MNQ}d_{KSN}d^{LSN}d^{SKR}\partial_Q\xi^L\partial_K V^S, \tag{5.13}$$

which shows that U^M of this form transforms as a vector with weight $\lambda = \frac{1}{3}$ which will be useful later on.

As in an ordinary gauge theory any local internal symmetry make bare derivatives non-covariant. The usual solution is to introduce a connection A_μ^M that transforms inhomogeneously such that a covariant derivative can be defined. In this spirit we define a covariant derivative by its action on a vector V

$$\begin{aligned} D_\mu V^M &:= \partial_\mu V^M - \mathscr{L}_{A_\mu}V^M \\ &= \partial_\mu V^M - A_\mu^P\partial_P V^M + P^{MN}{}_{PQ}\,\partial_N A_\mu^P V^Q - \lambda\partial_P A_\mu^P V^M, \end{aligned} \tag{5.14}$$

with A_μ^M being the gauge connection. We then impose that this transforms covariantly for any tensor field X, i.e.

$$\delta_\xi(D_\mu X) \stackrel{!}{=} \mathscr{L}_\xi(D_\mu X). \tag{5.15}$$

Note especially that this is not true for bare derivatives

$$\delta_\xi(\partial_\mu V^N) = \partial_\mu(\delta_\xi V^N) \neq \mathscr{L}_\xi(\partial_\mu V^N), \tag{5.16}$$

since the generalised diffeomorphism parameter is assumed to depend on the external spacetime as well $\xi = \xi(x,z)$. Look at the inhomogeneous transformation $\Delta_\xi D_\mu V = \delta_\xi D_\mu V - \mathscr{L}_\xi D_\mu V$ on a vector field V

$$\begin{aligned}\Delta_\xi D_\mu V^M &= \delta_\xi(\partial_\mu V^M - \mathscr{L}_{A_\mu} V^M) - \mathscr{L}_\xi \partial_\mu V^M + \mathscr{L}_\xi \mathscr{L}_{A_\mu} V^M \\ &= \mathscr{L}_{\partial_\mu \xi} V^M - \mathscr{L}_{\delta A_\mu} V^M + \mathscr{L}_{[\xi, A_\mu]_{\mathrm{E}}} V^M .\end{aligned} \tag{5.17}$$

We thus find that if the gauge connection transforms as $\delta A_\mu = \partial_\mu \xi^M + [\xi, A_\mu]^M_{\mathrm{E}}$, then $D_\mu V^M$ transforms covariantly. However, this is only defined up to a trivial transformation and we use this to slightly rewrite the transformation of A^M_μ by noting that

$$\begin{aligned}[][\xi, A_\mu]^M_{\mathrm{E}} &= \xi^P \partial_P A^M_\mu - A^P_\mu \partial_P \xi^M - 5 d^{MNP} d_{STP} \xi^S \partial_N A^T_\mu + 5 d^{MNP} d_{STP} A^S_\mu \partial_N \xi^T \\ &= \xi^P \partial_P A^M_\mu - A^P_\mu \partial_P \xi^M - 10 d^{MNP} d_{STP} \xi^S \partial_N A^T_\mu + 5 d^{MNP} d_{STP} \partial_N (\xi^S A^T_\mu) \\ &:= D_\mu \xi^M + d^{MNP} \partial_N \zeta_P ,\end{aligned} \tag{5.18}$$

with $d^{MNP}\partial_N \zeta_P$ being a parameter in the kernel of generalised diffeomorphisms and, moreover, defining ξ^M to transform with a weight $w = \frac{1}{3}$. We thus find that the covariant derivative is indeed covariant if

$$\delta A^M_\mu = D_\mu \xi^M . \tag{5.19}$$

In analogy with Yang-Mills theory we would then want to construct a field strength tensor $\tilde{F}^M_{\mu\nu}$ that transforms covariantly under the duality group, i.e. $\delta_\xi \tilde{F}^M_{\mu\nu} = \mathscr{L}_\xi \tilde{F}^M_{\mu\nu}$. The obvious guess would be to define $\tilde{F}^M_{\mu\nu}$ as

$$\begin{aligned}\tilde{F}^M_{\mu\nu} :&= 2\partial_{[\mu} A^M_{\nu]} - [A_\mu, A_\nu]^M_{\mathrm{E}} \\ &= 2\partial_{[\mu} A^M_{\nu]} - 2 A^P_{[\mu} \partial_P A^M_{\nu]} - 10 d^{MNP} d_{STP} A^S_{[\mu} \partial_N A^T_{\nu]}\end{aligned} \tag{5.20}$$

Varying the gauge potential we find

$$\begin{aligned}\delta \tilde{F}^M_{\mu\nu} &= 2\partial_{[\mu} \delta A^M_{\nu]} + 2 \delta A^K_{[\mu} \partial_K A^M_{\nu]} - 2 A^K_{[\mu} \partial_K \delta A^M_{\nu]} + \\ &\quad + 10 d^{MNP} d_{STP} \left(\delta A^S_{[\mu} \partial_N A^T_{\nu]} + A^S_{[\mu} \partial_N \delta A^T_{\nu]} \right) ,\end{aligned} \tag{5.21}$$

which by comparison with the expression

$$2 D_{[\mu} \delta A^M_{\nu]} = 2\partial_{[\mu} \delta A^M_{\nu]} - 2 A^P_{[\mu} \partial_P \delta A^M_{\nu} + 2 \partial_P A^M_{[\mu} \delta A^P_{\nu]} - 20 d^{MNP} d_{STP} \partial_N A^S_{[\mu} \delta A^T_{\nu]} , \tag{5.22}$$

where we used that A^M_μ transforms with weight $\frac{1}{3}$, can be written as

$$\delta \tilde{F}^M_{\mu\nu} = 2 D_{[\mu} \delta A^M_{\nu]} + 10 d^{MNP} d_{STP} \partial_N \left(A^S_{[\mu} \delta A^T_{\nu]} \right) . \tag{5.23}$$

The last term in the transformation of $\tilde{F}^M_{\mu\nu}$ is non-covariant and in order to define a covariant field strength tensor we will introduce a 2-form $B_{\mu\nu M}$ that transforms as

$$\Delta B_{\mu\nu M} = \delta B_{\mu\nu N} + d_{MNS} A^N_{[\mu} \delta A^S_{\nu]} . \tag{5.24}$$

This 2-form also possess a gauge symmetry parameterised by a 1-form parameter $\Sigma_{\mu N}$ with weight $\frac{2}{3}$. We can then define a covariant field strength tensor of A^M_μ as

$$F^M_{\mu\nu} = 2\partial_{[\mu} A^M_{\nu]} - [A_\mu, A_\nu]^M_E + 10 d^{MNS} \partial_N B_{\mu\nu S} . \tag{5.25}$$

To see that this is covariant we first define the gauge transformation of A_μ^M and $B_{\mu\nu N}$ as

$$\begin{aligned} \delta A_\mu^M &= D_\mu \xi^M - 10 d^{MNK} \partial_N \Sigma_{\mu K}, \\ \Delta B_{\mu\nu M} &= 2 D_{[\mu} \Sigma_{\nu]M} + d_{MNK} \xi^N F_{\mu\nu}^K. \end{aligned} \tag{5.26}$$

Note that $B_{\mu\nu M}$ appeared only in the combination $d^{MNK}\partial_K B_{\mu\nu N}$ so that the gauge variation $\Delta B_{\mu\nu M}$ is only defined up to a term that vanish when contracted with $d^{MNK}\partial_K$. The gauge variation of $F_{\mu\nu}^M$ is then found to be

$$\begin{aligned} \delta F_{\mu\nu}^M = {} & 2 D_{[\mu} D_{\nu]} \xi^M - 20 d^{MNK} D_{[\mu} \partial_K \Sigma_{\nu]N} \\ & + 20 d^{MNK} \partial_K D_{[\mu} \Sigma_{\nu]N} + 10 d^{MNK} d_{NSP} \partial_K \left(\xi^S F_{\mu\nu}^P \right). \end{aligned} \tag{5.27}$$

In order to see that $F_{\mu\nu}^M$ transforms as a vector we use that

$$2 D_{[\mu} D_{\nu]} V^M = -2 \mathscr{L}_{A_{[\mu}} \partial_{\nu]} V^M - 2 \partial_{[\mu} \mathscr{L}_{A_{\nu]}} V^M + \mathscr{L}_{[A_\mu, A_\nu]_{\mathrm{E}}} V^M, \tag{5.28}$$

and it is straightforward to expand the first two terms and find

$$\begin{aligned} 2 D_{[\mu} D_{\nu]} V^M &= -2 \partial_{[\mu} A_{\nu]}^K \partial_K V^M + 2 \partial_K \partial_{[\mu} A_\nu^M + 2(\lambda - \frac{1}{3}) \partial_P \partial_{[\mu} A_{\nu]} P V^M \\ & \quad - 20 d^{MNL} d_{SPL} \partial_N \partial_{[\mu} A_{\nu]}^P V^S + \mathscr{L}_{[A_\mu, A_\nu]_{\mathrm{E}}} V^M \\ &= -\mathscr{L}_{F_{\mu\nu}} V^M. \end{aligned} \tag{5.29}$$

Moreover, in the gauge variation (5.27) there is a term $d^{MNK} D_\mu \partial_K \Sigma_{\nu N}$ and we note that the covariant derivative and the partial derivative does in general not commute. However, noting that $\Sigma_{\mu N}$ carries a weight $\frac{2}{3}$ we can use that $d^{MNK}\partial_K \Sigma_{\nu N}$ transforms as a vector with weight $\frac{1}{3}$. We then find

$$\begin{aligned} d^{MNK} D_\mu \partial_K V_N = {} & d^{MNK} \partial_K \partial_\mu V_N - d^{MNK} A_\mu^S \partial_S \partial_K V_N \\ & + \underbrace{d^{SNK} \partial_S A_\mu^M \partial_K V_N}_{=0 \text{ section constraint}} - 10 d^{MLR} d_{LST} d^{TNK} \partial_R A_\mu^S \partial_K V_N. \end{aligned} \tag{5.30}$$

The last term can be rewritten using the same calculation as in (5.12) and we thus get

$$\begin{aligned} d^{MNK} D_\mu \partial_K V_N = {} & d^{MNK} \partial_K \partial_\mu V_N - d^{MNK} A_\mu^S \partial_S \partial_K V_N \\ & + 10 d^{MLK} d_{LST} d^{TNR} \partial_R A_\mu^S \partial_K V_N - d^{MNK} \partial_S A_\mu^S \partial_K V_N - d^{MNK} \partial_K A_\mu^S \partial_S V_N. \end{aligned} \tag{5.31}$$

On the other hand, consider

$$\begin{aligned} d^{MNK} \partial_K D_\mu V_N = {} & d^{MNK} \partial_K \left(\partial_\mu V_N - A_\mu^S \partial_S V_N - \partial_S A_\mu^S V_N \right) \\ & + 10 d^{MLK} d_{LST} d^{TNR} \partial_R A_\mu^T V_N - \underbrace{d^{MNK} \partial_K (\partial_N A_\mu^S V_S)}_{=0 \text{ section constraint}}, \end{aligned} \tag{5.32}$$

from which it is easily found that

$$d^{MNK} \partial_K D_\mu V_N = d^{MNK} D_\mu \partial_K V_N, \tag{5.33}$$

if V_N transforms with a weight $\frac{2}{3}$. Using this together with (5.29) it is straightforward to find the variation of the field strength tensor $F_{\mu\nu}^M$ in (5.27) as

$$\begin{aligned} \delta F_{\mu\nu}^M &= \xi^K \partial_K F_{\mu\nu}^M - \partial_K \xi^M F_{\mu\nu}^M + 10 d_{NLR} d^{MKR} \partial_K \xi^L F_{\mu\nu}^N \\ &= \mathscr{L}_\xi F_{\mu\nu}^M, \end{aligned} \tag{5.34}$$

and $F^M_{\mu\nu}$ thus transforms as a covariant vector with weight $\frac{1}{3}$.

Having to introduce higher order form fields in order to ensure covariance for gauge connections might seem a bit strange. However, this is a typical feature of so-called gauged supergravities [22]. These are maximal supergravities obtained by compactifaction on a torus T^d in which a subgroup G_0 of the global duality group is promoted to a local symmetry. As above, one then has to introduce gauge fields in order to define covariant derivatives and the corresponding field strength tensors will also transform non-covariantly. One solves this problem by successively adding higher form fields as we have done above but however, these have to transform in particular representations R_p of the duality group G in order to get covariant field strength tensors. This leads to the notion of a "tensor hierarchy" of p-forms that have to be introduced in order to define gauged supergravities. Moreover, the tensor hierarchy algebra constructed in [8] predicts precisely this series of representations.

In [37] an action for the external metric $g_{\mu\nu}$, the gauge field A^M_μ, the two-form $B_{\mu\nu M}$ and the scalar fields $\mathcal{M}_{MN}$ is found that is invariant under external diffeomorphisms as well as internal generalised diffeomorphisms. This action is given by

$$\begin{aligned} S_{E_{6(6)}} =& \int \mathrm{d}x^5 \mathrm{d}z^{27} \sqrt{|g|} \left(\tilde{R} + \frac{1}{24} g^{\mu\nu} D_\mu \mathcal{M}_{MN} D_\nu \mathcal{M}^{MN} - \frac{1}{4} \mathcal{M}_{MN} F^M_{\mu\nu} F^N_{\mu\nu} - V(\mathcal{M}_{MN}, g_{\mu\nu}) \right) \\ &+ \int \mathrm{d}x^5 \mathrm{d}z^{27} \mathcal{L}_{\text{top}}, \end{aligned} \tag{5.35}$$

where $\mathcal{L}_{\text{top}}$ is a topological term that couples A^M_μ with $B_{\mu\nu M}$ and, especially, the scalar potential is given by

$$\begin{aligned} V = -\frac{1}{24} \mathcal{M}^{MN} \partial_M \mathcal{M}^{KL} \partial_N \mathcal{M}_{KL} + \frac{1}{2} \mathcal{M}^{MN} \partial_M \mathcal{M}^{KL} \partial_L \mathcal{M}_{NK} \\ -\frac{1}{2} g^{-1} \partial_M g \partial_N \mathcal{M}^{MN} - \frac{1}{4} \mathcal{M}^{MN} g^{-1} \partial_M g g^{-1} \partial_N g - \frac{1}{4} \mathcal{M}^{MN} \partial_M g^{\mu\nu} \partial_N g_{\mu\nu}. \end{aligned} \tag{5.36}$$

In the extended geometries introduced in Chapter 6 it is the potential V that we will derive a general expression for.

5.1.1 Solving the section constraint

In order to embed ordinary supergravity in the exceptional field theory discussed above we need to solve the section constraint

$$d^{MNS} \partial_N \otimes \partial_S = 0. \tag{5.37}$$

A trivial solution is to drop the dependence on the internal coordinates completely by setting $\partial_M = 0$. In this case the action (5.35) reduces to that of supergravity compactified on T^6 and especially the lagrangian for the scalar sector is given by the non-linear sigma model introduced in Section 3.2.1. Moreover, it is in this case that the global "hidden" symmetry group $E_{6(6)}$ supergravity compactified on T^6 becomes a manifest symmetry which was the main motivation for introducing extended geometries. Note that in this case we clearly have a truncated theory in comparison to the full $E_{6(6)}$ exceptional field theory which is not truncated.

By looking at the affine extended Dynkin diagram for $\mathfrak{e}_{6(6)}$, shown in Figure 5.1, we can find at least two other interesting solutions corresponding to different embeddings of a $\mathfrak{gl}$

subalgebra. In order to see how this solves the section constraint begin with decomposing **27** and the adjoint **78** under $\mathfrak{sl}(6)\oplus\mathbb{R}$ embedded as $\mathfrak{sl}(6)\oplus\mathbb{R}\subset\mathfrak{sl}(6)\oplus\mathfrak{sl}(2)\subset\mathfrak{e}_{6(6)}$

$$\begin{cases}\mathbf{27}\to\mathbf{6}_{-1}\oplus\mathbf{15}\oplus\mathbf{6}_1,\\ \mathbf{78}\to\mathbf{1}_{-2}\oplus\mathbf{20}_{-1}\oplus[\mathbf{35}+\mathbf{1}]_0\oplus\mathbf{20}_1\oplus 1_2,\end{cases} \tag{5.38}$$

with the subscript denoting the eigenvalue of $\mathbb{R}$. In components the vector field is then given by $z^M=(z^i,z_{ij},z^{\hat{i}})$ and the section constraint can be solved by setting

$$\partial^{ij}=0,\qquad \partial_{\hat{i}}=0, \tag{5.39}$$

with derivatives acting on arbitrary fields. This solves the section constraint since the only possibly non-trivial term is $d^{ijk}\partial_j\otimes\partial_k$ which vanishes as well. The reason for this is that d^{MNP} is an invariant tensor and any non-trivial term of d^{MNP} should carry weight 0 under $\mathbb{R}$. Before looking at the second solution to the section constraint consider an element in the coset space $E_{6(6)}/USp(8)$ which we in the beginning argued to parameterise the scalar fields and the corresponding coset element of the $GL(6)$ subalgebra

$$\frac{E_{6(6)}}{USp(8)}\to\frac{GL(6)}{SO(6)}, \tag{5.40}$$

and we can identify $g_{mn}\in\frac{GL(6)}{SO(6)}$. The remaining $\dim Usp(8)-\dim SO(6)=21$ compact generators of $Usp(8)$ can be used to gauge away $\mathbf{20}_{-1}$ and $\mathbf{1}_{-2}$ in (5.38) such that we are left with $A_{mnp}\in\mathbf{20}_1$ and $\chi\in\mathbf{1}_2$, with χ coming from a dualised three-form in five dimensions. We have thus seen that this correspond to the M-theory solution of the section constraint. The complete dictionary between $E_{6(6)}$ exceptional field theory with this section constraint and M-theory is given in [37].

A similar story follows for the embedding $\mathfrak{sl}(5)\times\mathfrak{sl}(2)\times\mathbb{R}\subset\mathfrak{sl}(6)\oplus\mathfrak{sl}(2)\oplus\mathbb{R}$ given in Figure 5.1

$$\begin{cases}\mathbf{27}\quad\to(\mathbf{5},\mathbf{1})_4\oplus(\bar{\mathbf{5}},\mathbf{2})_1\oplus(\mathbf{10},\mathbf{1})_{-2}\oplus(\mathbf{1},\mathbf{2})_{-5},\\ \mathbf{78}\quad\to(\mathbf{5},\mathbf{1})_{-6}\oplus(\overline{\mathbf{10}},\mathbf{2})_{-3}\oplus[(\mathbf{24},\mathbf{1})\oplus(\mathbf{1},\mathbf{3})\oplus(\mathbf{1},\mathbf{1})]_0\oplus(\mathbf{10},\mathbf{2})_3\oplus(\bar{\mathbf{5}},\mathbf{1})_6.\end{cases} \tag{5.41}$$

In components the vector is given by $z^M=(z^i,z_{i\alpha},z^{ij},z_\alpha)$ and we can solve the section constraint by setting

$$\partial^{i\alpha}=0,\qquad \partial_{ij}=0,\qquad \partial^\alpha=0, \tag{5.42}$$

where again the derivatives act implicitly on any fields. This again solves the section constraint since d^{ijk} vanish due to the grading. Considering again an element in the coset $E_{6(6)}$ and the corresponding coset element in $GL(5)\times SL(2)$

$$\frac{E_{6(6)}}{USp(8)}\to\frac{GL(5)}{SO(5)}\times\frac{SL(2)}{SO(2)} \tag{5.43}$$

We can again use the remaining $\dim USp(8)-\dim SO(5)-\dim SO(2)=25$ compact generators of $USp(8)$ to gauge away $(\bar{\mathbf{5}},1)_{+6}\oplus(\mathbf{10},2)_3$. This corresponds to the type IIB supergravity solution of the section constraint, to see this remember that the bosonic sector of type IIB consist of a metric g_{ij}, a two-form B_2, the dilaton ϕ and form fields C_0, C_2 and C_4. We then identify $g_{ij}\in GL(5)/SL(5)$, $(\phi,C_0)\in SL(2)/U(1)$, $(B_2,C_2)\in(\overline{\mathbf{10}},\mathbf{2})_{-3}$ and $C_4\in(\mathbf{5},\mathbf{1})_{-6}$. The complete dictionary between $E_{6(6)}$ exceptional field theory with this section constraint and type IIB supergravity is given in [37].

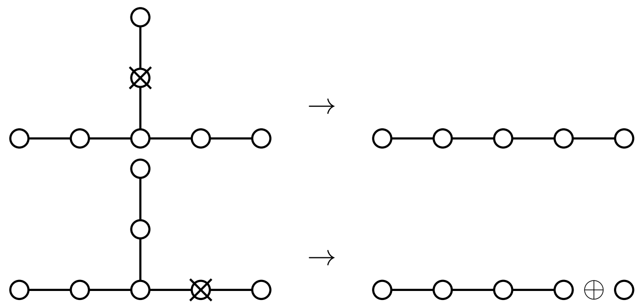

Figure 5.1: Two different solutions to the section constraint with the upper corresponding to $\mathfrak{e}_{6(6)} \to \mathfrak{sl}(6) \oplus \mathbb{R}$ and the lower to $\mathfrak{e}_{6(6)} \to \mathfrak{sl}(5) \oplus \mathfrak{sl}(2) \oplus \mathbb{R}$. Note that the factor of $\mathbb{R}$ has been excluded from the diagram and that we have drawn the affine extension of $E_{6(6)}$ which by deletion of any node gives a subalgebra of $E_{6(6)}$.

6

Extended Geometries

In this chapter the general construction of extended geometries constructed in [5] are described. These are generalisations of double geometry and exceptional geometry to a geometry based on a Kac-Moody structure algebra $\mathfrak{g} \times \mathbb{R}^+$ and an irreducible highest weight module $R(\lambda)$. The construction is built around the generalised diffeomorphisms and closure of their algebra. Moreover, the so-called section constraint is crucial for many reasons, in particular closure.

When the structure algebra is that of a hidden duality group appearing when compactifying, say, a supergravity theory the extended geometries describe the purely internal (scalar) degrees of freedom of that theory. However, the geometries introduced in this chapter are described without reference to any specific physical theory and compactification ansatz.

In Sections 6.1 – 6.3 we follow the construction of extended geometries presented in [5] by especially focusing on finding constraints to ensure closure of generalised diffeomorphisms. Moreover, a pseudo-action which is invariant under generalised diffeomorphisms encoding the dynamics of the generalised metric is introduced. In Section 6.4 we introduce a covariant derivative following [40] in order to derive purely geometric objects such as torsion and curvature.

6.1 Extended spacetime and setup

Extended geometries will be based on the split real-form of a Kac-Moody algebra $\mathfrak{g} \times \mathbb{R}$ exponentiated to a group $G \times \mathbb{R}^+$ and an irreducible integrable highest weight coordinate module $R(\lambda)$ with derivatives in the dual module $\partial_M \in \overline{R(\lambda)}$.

Ordinary geometry is formulated in a coordinate independent manner with transition functions valued in $GL(d)$; in extended geometries on the other hand this rôle is played by $G \times \mathbb{R}^+$. However, we will only consider local features since global transformations in general are still difficult to deal with. In the specific example of double field theory global transformations are well behaved and discussed in [41, 42].

As introduced in Section 2.3 a Kac-Moody algebra is described in terms of its Cartan matrix a_{ij} which we assume to be symmetrisable, i.e. there exists a diagonal matrix D with non-zero entries D_j such that Da is a symmetric matrix. Note that the Kac-Moody algebra $\mathfrak{g}(a)$ is not the bosonic extended algebra $\mathscr{A}(a)$ introduced in Section 2.3. However, we will use both the bosonic extension $\mathscr{A}(\mathfrak{g})$ and the fermionic extension $\mathscr{B}(a)$ in Section 6.2.2 to show closure of generalised diffeomorphisms. If the Cartan matrix a_{ij} associated to a rank r algebra $\mathfrak{g}$ is invertible we let $i,j=1,2,\dots r$. Otherwise if the corank $k>0$ we extend the Cartan matrix such that a_{ij} is invertible with $i,j=1,2,\dots r+k$. Since a is assumed to be symmetrisable there exists an inner product on the root space that we define as

$$(\alpha_i,\alpha_j) := D_i a_{ij}, \tag{6.1}$$

where $\alpha_{i,j}$ are simple roots which we normalise as $D_i = \frac{(\alpha_i,\alpha_i)}{2}$. We also define coroots $\alpha_i^\vee$ as $a_{ij} = \alpha_j(\alpha_i^\vee) = \frac{2}{(\alpha_i,\alpha_i)}(\alpha_i,\alpha_j)$. Symmetrising a from the right by D^{-1} we get a symmetric inner product on the coroots as

$$(\alpha_i^\vee,\alpha_j^\vee) := a_{ij}D_j^{-1}. \tag{6.2}$$

Introducing fundamental weights Λ_i dual to the coroots $\Lambda_i(\alpha_j^\vee)=\delta_{ij}$ induces a metric on the weight space as

$$(\Lambda_i,\Lambda_j) := D_i a_{ij}^{-1}. \tag{6.3}$$

6.2 Generalised diffeomorphisms

Generalised diffeomorphisms play a central rôle in the construction of extended geometries, the general form of which is

$$\delta_\xi V^M = \mathscr{L}_\xi V^M = \xi^N\partial_N V^M + Z^{MN}{}_{PQ}\partial_N\xi^P V^Q, \tag{6.4}$$

where Z is an invariant tensor of $\mathfrak{g}$. The first term can be viewed as a transport term coming from a Taylor expansion of the argument and the second term is a projection operator which projects the ${}^N{}_P$ indices on the adjoint module, $\mathfrak{g}\oplus\mathbb{R}^+$. The invariant tensor Z is hence given by

$$Z^{MN}{}_{PQ} = -k\eta_{\alpha\beta}T^{\alpha M}{}_Q T^{\beta N}{}_P + \beta\delta^M_Q\delta^N_P, \tag{6.5}$$

where $\eta_{\alpha\beta}$ is the inverse of the invariant bilinear form on $\mathfrak{g}$, T^α are the generators in the representation $R(\lambda)$ and k,β constants to be determined. This can be written in an equivalent way using the tensor

$$Y^{MP}{}_{NQ} = Z^{MP}{}_{NQ} + \delta^M_N\delta^P_Q = -k\eta_{\alpha\beta}T^{\alpha M}{}_Q T^{\beta N}{}_P + \beta\delta^M_Q\delta^N_P + \delta^M_N\delta^P_Q \tag{6.6}$$

as

$$\begin{aligned}\mathscr{L}_\xi V^M &= \xi^N\partial_N V^M - \partial_N\xi^M V^N + Y^{MN}{}_{PQ}\partial_N\xi^P V^Q\\ &= L_\xi V^M + Y^{MN}{}_{PQ}\partial_N\xi^P V^Q.\end{aligned} \tag{6.7}$$

Here L_ξ denotes the ordinary Lie derivative which shows that the Y-tensor can be interpreted as the departure of the generalised Lie derivative in the structure algebra $\mathfrak{g}\oplus\mathbb{R}$ compared to that of ordinary geometry. This is precisely the Y-tensor introduced and derived in Section 2.3.3 which we will see below. Especially, by comparison with (2.113) we find $\beta = k(\lambda,\lambda)-1$.

A crucial consistency condition is whether the algebra closes or not, to see this examine

$$\left([\mathscr{L}_\xi,\mathscr{L}_\eta] - \mathscr{L}_{\frac{1}{2}(\mathscr{L}_\xi\eta-\mathscr{L}_\eta\xi)}\right)V^M \stackrel{?}{=} 0. \tag{6.8}$$

After a good deal of algebra one finds that

$$\left([\mathscr{L}_\xi, \mathscr{L}_\eta] - \mathscr{L}_{\frac{1}{2}(\mathscr{L}_\xi \eta - \mathscr{L}_\eta \xi)}\right) V^M = \frac{1}{2} Z^{MP}{}_{QN} Y^{QR}{}_{ST}\, \xi^T \partial_P \partial_R \eta^S V^N - (\xi \leftrightarrow \eta) \tag{6.9}$$

if the Y-tensor satisfies certain constraints. The first is the (strong) section constraint

$$Y^{MN}{}_{PQ}\, \partial_M \otimes \partial_N = 0, \tag{6.10}$$

where $\partial_M \otimes \partial_N$ indicates that each derivative act on an arbitrary field. To get some intuition for this constraint look at the case for DFT with

$$Y^{MN}{}_{PQ} = -2P^{MN}{}_{PQ} + \delta^M_P \delta^N_Q, \tag{6.11}$$

where $P^{MN}{}_{PQ} = -\frac{1}{4}\eta^{IK}\eta^{JL}(T_{IJ})^M_Q (T_{KL})^N_P$ denotes the projection operator onto the adjoint. The generators in the fundamental of $O(D, D)$ [34] are $(T_{IJ})^M_P = \delta^M_I \eta_{JP} - \delta^M_J \eta_{IP}$ and a straightforward calculation shows that

$$Y^{MN}{}_{PQ}\, \partial_M \otimes \partial_N = \eta^{MN} \partial_M \otimes \partial_N, \tag{6.12}$$

which indeed is the section constraint for DFT introduced in Chapter 4. Moreover, the Y-tensor also needs to satisfy the following constraints for the algebra to close

$$\begin{aligned}\Big(Y^{MN}{}_{TQ}\, Y^{TP}{}_{[SR]} + 2Y^{MN}{}_{[R|T|}\, Y^{TP}{}_{S]Q} \\ -Y^{MN}{}_{[RS]}\, \delta^P_Q - 2Y^{MN}{}_{[S|Q|}\, \delta^P_{R]}\Big)\partial_{(N} \otimes \partial_{P)} = 0,\end{aligned} \tag{6.13a}$$

$$\begin{aligned}\Big(Y^{MN}{}_{TQ}\, Y^{TP}{}_{(SR)} + 2Y^{MN}{}_{(R|T|}\, Y^{TP}{}_{S)Q} \\ -Y^{MN}{}_{(RS)}\, \delta^P_Q - 2Y^{MN}{}_{(S|Q|}\, \delta^P_{R)}\Big)\partial_{[N} \otimes \partial_{P]} = 0.\end{aligned} \tag{6.13b}$$

These identities are derived in Appendix A. Whether the remaining term in (6.9) needed for closure vanish or not depends on the specific algebra together with the choice of coordinate module $R(\lambda)$, a simple criterion for this will be derived below using certain extensions of $\mathfrak{g}$. As of yet neither the particular form nor any symmetry properties of Y have been used. However, inserting the explicit form (6.6) in (6.13a)-(6.13b) one finds that they are satisfied if the section constraint is [6].

6.2.1 Section constraint

Up to the remaining term in (6.9) closure of the algebra is entirely dependent on solving the section constraint (6.12). To this end examine the tensor product of the coordinate module with itself. This can trivially be decomposed as

$$\begin{aligned} R(\lambda) \otimes R(\lambda) &= \vee^2 R(\lambda) \oplus \wedge^2 R(\lambda) \\ &= (R^s_h \oplus R_2) \oplus \left(R^a_h \oplus \tilde{R}_2\right), \end{aligned} \tag{6.14}$$

with $R^{s,a}_h$ being the highest weight representation in the symmetrised and anti-symmetrised product respectively and

$$R_2 = \vee^2 R(\lambda) \ominus R^s_h, \tag{6.15}$$

$$\tilde{R}_2 = \wedge^2 R(\lambda) \ominus R^a_h. \tag{6.16}$$

It is easily seen that the highest weight representation in the symmetrised product is given by $R_h^s = R(2\lambda)$. Solving the section constraint can be done in two steps, first impose the weak section stating that any momenta $|p\rangle \in \overline{R(\lambda)}$ lie in a minimal orbit of G, which is equivalent to $|p\rangle \otimes |p\rangle \in \overline{R(2\lambda)}$ [43, 44]. Secondly we demand that the product of two arbitrary momenta $|p\rangle, |q\rangle \in \overline{R(\lambda)}$ contain only the dual of the highest modules in the symmetric R_h^s and anti-symmetric R_h^a product of $R(\lambda)$ respectively. In other words, the second step means that we set

$$(\partial \otimes \partial)\,|_{\overline{R_2}} = 0, \qquad (\partial \otimes \partial)\,|_{\overline{\tilde{R}_2}} = 0, \tag{6.17}$$

where $\otimes$ indicate that the derivatives act on arbitrary fields.

The section constraint plays an important rôle in extended geometries as this specifies the embedding $GL(d) \hookrightarrow G \times \mathbb{R}^+$ that defines ordinary geometry. However, crucially the section constraint is solved in a G covariant manner. Moreover, any solution to the section constraint can be reached by a G transformation of another solution while the d-dimensional subspace that span the vector representation of $GL(d)$ is stabilised by the $GL(d)$ subgroup of G.

In order to make progress note that the highest weight representations in the anti-symmetrised product R_s^a are given by representations $R(2\lambda - \alpha_i)$ with $\lambda(\alpha_i^\vee) \neq 0$. The highest weight state of such a representation is given by

$$|2\lambda - \alpha_i\rangle = |\lambda\rangle \otimes |\lambda - \alpha_i\rangle - |\lambda - \alpha_i\rangle \otimes |\lambda\rangle, \tag{6.18}$$

which indeed has the weight $2\lambda - \alpha_i$. However, as will be shown below not all such highest weight representations can be kept.

Consider the quadratic Casimir invariant evaluated on $R(\lambda)$, $R(2\lambda)$ and $R(2\lambda - \alpha_i)$ using (2.70)

$$\begin{aligned} C_2(R(\lambda)) &= \frac{1}{2}(\lambda, \lambda + 2\rho), \\ C_2(R(2\lambda)) &= 2C_2(R(\lambda)) + (\lambda, \lambda), \\ C_2(R(2\lambda - \alpha_i)) &= C_2(R(2\lambda)) - 2(\alpha_i, \lambda) + \frac{1}{2}(\alpha_i, \alpha_i) - (\alpha_i, \rho) \\ &= C_2(R(2\lambda)) - \lambda_i(\alpha_i, \alpha_i), \end{aligned} \tag{6.19}$$

where we in the second step in the last equation used the definition of coroots and the expression of the Weyl vector given in (2.38). Evaluate the Casimir on a state in $R(2\lambda)$, which without loss of generality we take to be $|\lambda\rangle \otimes |\lambda\rangle$, using the explicit form in (6.19)

$$\begin{aligned} \frac{1}{2}\eta_{\alpha\beta}T^\alpha T^\beta \left(|\lambda\rangle \otimes |\lambda\rangle\right) = \left(\frac{1}{2}\eta_{\alpha\beta}T^\alpha T^\beta|\lambda\rangle\right) \otimes |\lambda\rangle + |\lambda\rangle \otimes \left(\frac{1}{2}\eta_{\alpha\beta}T^\alpha T^\beta|\lambda\rangle\right) \\ + \eta_{\alpha\beta}T^\alpha \otimes T^\beta|\lambda\rangle \otimes |\lambda\rangle. \end{aligned} \tag{6.20}$$

On the other hand evaluate the right hand side of (6.19) on the same state

$$\left(2C_2(R(\lambda)) + (\lambda, \lambda)\right)|\lambda\rangle \otimes |\lambda\rangle = \left(\frac{1}{2}T^\alpha T^\beta|\lambda\right) \otimes |\lambda\rangle + |\lambda\rangle \otimes \left(\frac{1}{2}T^\alpha T^\beta|\lambda\right) + (\lambda, \lambda)|\lambda\rangle \otimes |\lambda\rangle. \tag{6.21}$$

We thus find the algebraic condition

$$\left(\eta_{\alpha\beta}T^\alpha \otimes T^\beta - (\lambda, \lambda)\right)|\lambda\rangle \otimes |\lambda\rangle = 0 \tag{6.22}$$

when imposing (6.19). A similar condition is derived for states in $R(2\lambda-\alpha_i)$ by evaluating the Casimir operator on the highest weight state $|2\lambda-\alpha_i\rangle$

$$\begin{aligned}&[C_2(R(2\lambda-\alpha_i))-2C_2(R(\lambda))-(\lambda,\lambda)+\lambda_i(\alpha_i,\alpha_i)]\,|2\lambda-\alpha_i\rangle\\&\quad=\Big(\eta_{\alpha\beta}T^\alpha\otimes T^\beta-(\lambda,\lambda)+\lambda_i(\alpha_i,\alpha_i)\Big)\,|2\lambda-\alpha_i\rangle=0.\end{aligned}\tag{6.23}$$

Consider the highest weight vector $|\lambda\rangle$ in a section together with some other vector $|q\rangle$ also in the section. The symmetrised and anti-symmetrised product of these two vectors should satisfy the constraints derived above, i.e.

$$\Big(\eta_{\alpha\beta}T^\alpha\otimes T^\beta-(\lambda,\lambda)\Big)\,(|\lambda\rangle\otimes|q\rangle+|q\rangle\otimes|\lambda\rangle)=0,\tag{6.24}$$

and

$$\Big(\eta_{\alpha\beta}T^\alpha\otimes T^\beta-(\lambda,\lambda)+\lambda_i(\alpha_i,\alpha_i)\Big)\,(|\lambda\rangle\otimes|q\rangle-|q\rangle\otimes|\lambda\rangle)=0.\tag{6.25}$$

This is trivially satisfied by the $|q\rangle=|\lambda\rangle$. Consider a state $|q\rangle=e_{-\alpha_i}|\lambda\rangle$ with α_i a positive simple root with $\lambda_i\neq0$. The weak section condition on $|q\rangle\otimes|q\rangle$ then gives

$$\begin{aligned}\Big(\eta_{\alpha\beta}T^\alpha\otimes T^\beta-(\lambda,\lambda)\Big)\,e_{-\alpha_i}|\lambda\rangle\otimes e_{-\alpha_i}|\lambda\rangle&=(\lambda-\alpha_i,\lambda-\alpha_i)e_{-\alpha_i}|\lambda\rangle\otimes e_{-\alpha_i}|\lambda\rangle\\&\quad+(1+\sigma)\sum_{\alpha\in\Delta^+}e_\alpha e_{-\alpha_i}|\lambda\rangle\otimes e_{-\alpha}e_{-\alpha_i}|\lambda\rangle.\end{aligned}\tag{6.26}$$

The first term has to vanish separately from which we get $\lambda_i=1$. The second term then also vanish since $(\alpha_i-\alpha)$ with $\alpha_i\neq\alpha$ can not be written as a sum of positive roots and the term with $\alpha=\alpha_i$ vanish for $\lambda_i=1$. Consider then the state $(1+\sigma)|\lambda\rangle\otimes e_{-\alpha_i}|\lambda\rangle$, with σ the permutation operator, which also should satisfy the symmetric constraint. For this state we find

$$\begin{aligned}\Big(\eta_{\alpha\beta}T^\alpha\otimes T^\beta-(\lambda,\lambda)\Big)\,|\lambda\rangle\otimes e_{-\alpha_i}|\lambda\rangle&=(\lambda,\lambda-\alpha_i)(1+\sigma)|\lambda\rangle\otimes e_{-\alpha_i}|\lambda\rangle\\&\quad+(1+\sigma)\sum_{\alpha\in\Delta^+}e_{-\alpha}|\lambda\rangle\otimes e_\alpha e_{-\alpha_i}|\lambda\rangle.\end{aligned}\tag{6.27}$$

The only contribution from the sum comes from $\alpha=\alpha_i$, since otherwise $(\alpha_i-\alpha)$ can not be written as a sum of positive roots, which cancels the term proportional to λ_i in the first term. We have thus found that $|\lambda\rangle$, $|q\rangle$, and $|\lambda\rangle+|q\rangle$ all satisfy the weak section constraint.

Consider now another state $|p\rangle=e_{-\beta-\alpha_i}|\lambda\rangle$ which also lies in the section. From the weak section constraint on $(1+\sigma)|\lambda\rangle\otimes|p\rangle$ we find

$$\begin{aligned}\Big(\eta_{\alpha\beta}T^\alpha\otimes T^\beta-(\lambda,\lambda)\Big)\,e_{-\alpha_i}|\lambda\rangle\otimes e_{-\beta-\alpha_i}|\lambda\rangle&=(1+\sigma)\,[(\lambda,\lambda-\alpha_i-\beta)-(\lambda,\lambda)]\,|\lambda\rangle\otimes e_{-\beta-\alpha_i}|\lambda\rangle\\&\quad+(1+\sigma)\sum_{\alpha\in\Delta^+}e_{-\alpha}|\lambda\rangle\otimes e_\alpha e_{-\beta-\alpha_i}|\lambda\rangle.\end{aligned}\tag{6.28}$$

The term in the sum with $\alpha=\beta+\alpha_i$ contributes a term

$$(1+\sigma)(\lambda(\alpha_i^\vee)+\lambda(\beta^\vee))e_{-\beta-\alpha_i}|\lambda\rangle\otimes|\lambda\rangle,\tag{6.29}$$

which cancels the remaining term from the first line in (6.28). The remaining terms from the sum consists of α such that $\alpha\neq\beta+\alpha_i$ and $(\alpha,\lambda)\neq0$ since otherwise if $(\alpha,\lambda)=0$, then $e_{-\alpha}|\lambda\rangle$ vanish. The contribution from $\alpha = \alpha_i$ and $\alpha = \beta$ vanish only if $(\beta, \lambda) = 0$. The remaining states carry a weight $\lambda - (\alpha_i + \beta - \alpha)$, with α_i not included in α and $(\alpha, \lambda) \neq 0$, from which it follows that $\alpha_i + \beta - \alpha$ is not a sum of positive roots and any such state thus vanish.

Lastly consider the weak section constraint on $|p\rangle \otimes |p\rangle$

$$\begin{aligned}\Big(\eta_{\alpha\beta}T^\alpha \otimes T^\beta - (\lambda,\lambda)\Big)\, e_{-\alpha_i-\beta}|\lambda\rangle \otimes e_{-\alpha_i-\beta}|\lambda\rangle = [(\lambda-\alpha_i-\beta, \lambda-\alpha_i-\beta)]e_{-\alpha_i-\beta}|\lambda\rangle \otimes e_{-\alpha_i-\beta}|\lambda\rangle \\ + (1+\sigma)\sum_{\alpha\in\Delta^+} e_\alpha e_{-\alpha_i-\beta}|\lambda\rangle \otimes e_{-\alpha}e_{-\alpha_i-\beta}e|\lambda\rangle.\end{aligned} \tag{6.30}$$

Any element α that contains a simple root $\alpha_j \neq \alpha_i$ such that α_j is not included in β will not contribute to the sum since then $(\alpha_i + \beta - \alpha)$ can not be written as a sum of positive roots. We then choose $\beta = \beta_j$ according to

$$\beta_j = \sum_{k=1}^{j} \alpha_{i+k}, \tag{6.31}$$

such that $a_{mn} = -1$ for $|m-n| = 1$ and $a_{mn} = 0$ for $|m-n| > 0$ for $m, n = i, i+1, \ldots, i+(d-2)$ for some value $d-2$. With this choice the set of roots $\{\alpha_i, \beta_j\}_{j=1,2,\ldots d-2}$ correspond to simple positive roots of a $\mathfrak{sl}(d) \subset \mathfrak{g}$ subalgebra. Moreover, the states $e_{-\beta_j-\alpha_i}|\lambda\rangle$ span the vector representation under this algebra. In this representation e_α, with α a positive root of $\mathfrak{sl}$, is an upper-triangular matrix and $e_{-\alpha}$ its transpose, from which it follows that either $e_\alpha e_{-\beta_j-\alpha_i}|\lambda\rangle$ or $e_{-\alpha}e_{-\beta_j-\alpha_i}|\lambda\rangle$ vanish. Moreover, we find that

$$-2(\alpha_i, \beta) + (\beta, \beta) = 0, \tag{6.32}$$

if $\beta = \beta_j$ and which ensures that (6.30) vanish. We have thus found that also $|p\rangle$ and $|p\rangle \otimes |\lambda\rangle$ satisfy the weak section constraint. Moreover, there is an anti-symmetric condition from (6.22) which is seen to be fulfilled by looking at the following

$$\begin{aligned}\Big(\eta_{\alpha\beta}T^\alpha \otimes T^\beta - (\lambda,\lambda)\Big)\, |\lambda\rangle \otimes e_{-\alpha-\beta}|\lambda\rangle &= [(\lambda, \lambda-\alpha_i-\beta) - (\lambda,\lambda)]|\lambda\rangle \otimes e_{-\alpha_i-\beta}|\lambda\rangle \\ &\quad + \sum_{\alpha\in\Delta^+} e_{-\alpha}|\lambda\rangle \otimes e_\alpha e_{-\alpha_i-\beta}|\lambda\rangle \\ &= -|\lambda\rangle \otimes |p\rangle + |p\rangle \otimes |\lambda\rangle,\end{aligned} \tag{6.33}$$

using similar arguments as above to find that the only non-vanishing contribution from the sum comes from $\alpha = \alpha_i + \beta$. It is immediate that $e_{-\alpha_i-\beta_j}|\lambda\rangle + e_{-\alpha_i-\beta_k}|\lambda\rangle$ for any j, k also satisfy the section constraint. For $\beta_j = \beta_k$ we have already seen that is true. For $\beta_j \neq \beta_k$ one finds that $(h, h) - (\lambda, \lambda)$ acts as -1. In the sum over roots only roots α contained in the $\mathfrak{sl}$ subalgebra is possibly non-zero using the same argument as above. It is, moreover, seen that there is precisely one choice of root $\alpha \propto \beta_j - \beta_k$, which acts as the permutation operator on $e_{-\alpha_i-\beta_j}|\lambda\rangle \otimes e_{-\alpha_i-\beta_k}|\lambda\rangle$. This precisely cancels the contribution from $(h, h) - (\lambda, \lambda)$.

Any two vectors $|p\rangle$ and $|q\rangle$ in a section thus satisfy

$$\Big(\eta_{\alpha\beta}T^\alpha \otimes T^\beta - (\lambda,\lambda) + (1-\sigma)\Big)\, |p\rangle \otimes |q\rangle = 0, \tag{6.34}$$

with the last term only contributing to the anti-symmetric part. Comparing with the invariant Y tensor introduced in (6.6) we find for any vector $|p\rangle$ and $|q\rangle$ in a section

$$Y|p\rangle \otimes |q\rangle = 0, \tag{6.35}$$

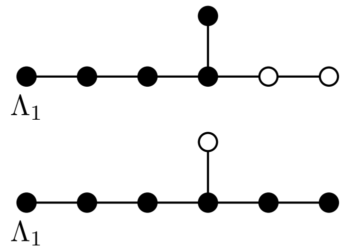

Figure 6.1: Dynkin diagram of $\mathfrak{e}_7$ with black nodes corresponding to two different sections.

with $k = 1$ and $\beta = (\lambda, \lambda) - 1$.

A section can thus be constructed as follows

1. Pick a highest weight state $|\lambda\rangle$.
2. Enlarge the section by adding the state $e_{-\alpha_i}|\lambda\rangle$ with $\lambda_i = 1$.
3. Add another state $e_{-\alpha_{i+1}} e_{-\alpha_i}|\lambda\rangle$ if $\lambda_i = 0$.
4. Continue and add more states $e_{-\alpha_{i+l}} e_{-\alpha_{i+l-1}} \dots e_{-\alpha_i}|\lambda\rangle$ as long as $\lambda_{i+l} \neq 0$ only for $l = 0$.

If one reaches a branching in the diagram one choose one direction and should stop when reaching a node with multiple connections. By construction these states span a d-dimensional representation under the "gravity-line" corresponding to an embedding of $GL(d)$ in G. As an example of this consider an extended geometry based on $E_{7(7)}$ and the coordinate module $R(\lambda) = R(\Lambda_1)$. Two possible sections are then shown in Figure 6.1 where the black nodes denote the set of simple roots defining the $\mathfrak{sl}$ subalgebra[1].

6.2.2 Ancillary transformation and closure of algebra

The obstruction to closure of generalised diffeomorphisms was shown in (6.9) to be

$$\Big([\mathscr{L}_\xi, \mathscr{L}_\eta] - \mathscr{L}_{\frac{1}{2}(\mathscr{L}_\xi \eta - \mathscr{L}_\eta \xi)}\Big) V^M = \frac{1}{2} Z^{MP}{}_{QN} Y^{QR}{}_{ST}\, \xi^T \partial_P \partial_R \eta^S V^N - (\xi \leftrightarrow \eta)\,, \qquad (6.36)$$

given that the section constraint is fulfilled. The goal is to rewrite this in a form that makes it easy to see whether it vanish or not. Consider therefore

$$\begin{aligned} Z^{MP}{}_{QN} Y^{QR}{}_{ST}\, \xi^T \partial_P \partial_R \eta^S V^N &= \Big(-k\eta_{\alpha\beta} T^{\alpha M}{}_N T^{\beta P}{}_Q Y^{QR}{}_{ST} + \beta \delta^M_N Y^{PR}{}_{ST}\Big) \xi^T \partial_P \partial_R \eta^S V^N, \\ &= (k^2 \eta_{\alpha\beta} T^{\alpha M}{}_N T^{\beta P}{}_Q \eta_{\gamma\rho} T^{\gamma Q}{}_T T^{\rho R}{}_S \\ &\quad - k\beta \eta_{\alpha\beta} T^{\alpha M}{}_N T^{\beta P}{}_T \delta^R_S - k\eta_{\alpha\beta} T^{\alpha M}{}_N T^{\beta P}{}_S \delta^R_T) \xi^T \partial_P \partial_R \eta^S V^N, \end{aligned} \qquad (6.37)$$

where we expanded the Z-tensor in the first term on the RHS and then used that the second term vanish due to the section constraint. The strategy is then to commute $T^{\beta P}{}_Q$ and $T^{\gamma Q}{}_T$ in the term proportional to k^2 in the second line in order to again apply the

[1] This should not be confused with the notation for short/long roots.

section constraint

$$\begin{aligned}(6.37)|_{k^2} &= k^2\eta_{\alpha\beta}T^{\alpha M}{}_N T^{\beta P}{}_Q \eta_{\gamma\rho}T^{\gamma Q}{}_T T^{\rho R}{}_S \xi^T\partial_P\partial_R\eta^S V^N \\ &= k^2\eta_{\alpha\beta}T^{\alpha M}{}_N \eta_{\gamma\rho}T^{\rho R}{}_S \left(f^{\alpha\gamma}{}_\sigma T^{\sigma P}{}_T + T^{\gamma P}{}_Q T^{\beta Q}{}_T\right)\xi^T\partial_P\partial_R\eta^S V^N \\ &= k^2\left(\eta_{\gamma\rho}\eta_{\alpha\beta}f^{\alpha\gamma}{}_\sigma T^{\alpha M}{}_N T^{\rho R}{}_S T^{\sigma P}{}_T\right)\xi^T\partial_P\partial_R\eta^S V^N \\ &+ k\eta_{\alpha\beta}T^{\alpha M}{}_N T^{\beta Q}{}_T \left(\beta\delta^R_S\delta^P_Q + \delta^R_Q\delta^P_S\right)\xi^T\partial_P\partial_R\eta^S V^N.\end{aligned} \tag{6.38}$$

Inserting (6.38) in (6.37) the two terms proportional to β cancel and we find

$$\begin{aligned}Z^{MP}{}_{QN}Y^{QR}{}_{ST}\,\xi^T\partial_P\partial_R\eta^S V^N = &\left(k^2\eta_{\gamma\rho}\eta_{\alpha\beta}f^{\alpha\gamma}{}_\sigma T^{\alpha M}{}_N T^{\rho R}{}_S T^{\sigma P}{}_T\right)\xi^T\partial_P\partial_R\eta^S V^N \\ &+ k\eta_{\alpha\beta}T^{\alpha M}{}_N T^{\beta Q}{}_T\left(\delta^R_Q\delta^P_S - \delta^P_Q\delta^R_S\right)\xi^T\partial_P\partial_R\eta^S V^N.\end{aligned} \tag{6.39}$$

In index-free notation the failure of generalised diffeomorphisms to close can thus be written as

$$\left([\mathscr{L}_\xi,\mathscr{L}_\eta] - \mathscr{L}_{\frac{1}{2}(\mathscr{L}_\xi\eta - \mathscr{L}_\eta\xi)}\right)V^M = \Sigma_\alpha T^\alpha|V\rangle, \tag{6.40}$$

with

$$\Sigma^\alpha = \frac{k}{2}\langle\partial_\eta|\otimes\langle\partial_\eta|S^\alpha|\xi\rangle\otimes|\eta\rangle - (\xi\leftrightarrow\eta) \tag{6.41}$$

and

$$S^\alpha = -kf^\alpha{}_{\beta\gamma}T^\beta\otimes T^\gamma + T^\alpha\otimes\mathbf{1} - \mathbf{1}\otimes T^\alpha. \tag{6.42}$$

The tensor S^α is easily seen to satisfy $\sigma S^\alpha = -S^\alpha\sigma$ which implies that it can be decomposed as

$$S^{\alpha MN}{}_{PQ} = S^{\alpha(MN)}{}_{[PQ]} + S^{\alpha[MN]}{}_{(PQ)}. \tag{6.43}$$

Moreover, in (6.40) only the part $S^{\alpha(MN)}{}_{[PQ]}$ contributes and hence the expression for Σ^α simplifies

$$\Sigma^\alpha \to \Sigma^\alpha = -\frac{k}{2}\langle\partial_\eta|\otimes\langle\partial_\eta|S^\alpha\frac{1-\sigma}{2}|\xi\rangle\otimes|\eta\rangle + (\xi\leftrightarrow\eta)\,. \tag{6.44}$$

The relevant part $\frac{1+\sigma}{2}S^\alpha$ can be conveniently rewritten in index-free notation as

$$\begin{aligned}\frac{1+\sigma}{2}S^\alpha &= [\mathbf{1}\otimes T^\alpha, -k\eta_{\beta\gamma}T^\beta\otimes T^\gamma + \beta\mathbf{1} + \sigma] \\ &= [\mathbf{1}\otimes T^\alpha, Y].\end{aligned} \tag{6.45}$$

Inserting (6.45) in (6.44) and using the section constraint this further reduce to

$$\Sigma^\alpha \to \Sigma^\alpha = -\frac{k}{2}\langle\partial_\eta|\langle\partial_\eta|\mathbf{1}\otimes T^\alpha Y_-|\xi\rangle|\eta\rangle + (\xi\leftrightarrow\eta), \tag{6.46}$$

with $Y_- = Y\frac{1-\sigma}{2}$.

This simplification of Σ^α together with the formalism developed in Section 2.3.3 allows us to find a simple condition on the presence of ancillary transformations. Remember that $Y^{MN}{}_{PQ} = Y^{(MN)}{}_{(PQ)} + Y^{[MN]}{}_{[PQ]}$ and

$$Y^{MN}{}_{PQ} = \frac{k}{2}\left(g^{MN}{}_{QP} - \tilde{g}^{MN}{}_{QP}\right), \tag{6.47}$$

and due to the symmetry of g and $\tilde{g}$ only the latter contributes in Σ^α. Explicitly if the following expression is satisfied the algebra of generalised diffeomorphisms closes

$$\delta^M_K T^{\alpha N}{}_S \tilde{g}^{SK}{}_{PQ} \partial_{(M} \otimes \partial_{N)} = 0, \tag{6.48}$$

which by the definition of the invariant tensor $\tilde{g}$ is equivalent to

$$([\![[\![T^\alpha, \tilde{E}^M]\!], \tilde{E}^N]\!], [\![\tilde{F}_P, \tilde{F}_Q]\!])\partial_{(M} \otimes \partial_{N)} = 0. \tag{6.49}$$

In order to determine if this holds note that $[\![[\![T^\alpha, \tilde{E}^{(M}]\!], \tilde{E}^{N)}]\!] \in \vee^2 R(\lambda)$. Due to the section constraint however the only possibly non-trivial such element lies in $R(2\lambda)$. Assuming that the LHS of (6.49) does not vanish is equivalent to finding some element in the structure algebra $x \in \mathfrak{g}$ (the contribution from the center of $\mathscr{A}_0$ is trivially zero) and a symmetric tensor Λ_{MN} such that

$$\Lambda_{MN}[\![\tilde{E}^M, [\![\tilde{E}^N, x]\!]]\!] \neq 0. \tag{6.50}$$

Moreover, since $\mathscr{B}_{-2} = \vee^2 R(2\lambda) \ominus R(2\lambda)$ the tensor also satisfies $\Lambda_{MN}[\![E^M, E^N]\!] = 0$ which in particular is satisfied for $[\![e_0, e_0]\!]$. This implies that the expression (6.49) is non-zero if we can find an $x \in \mathfrak{g}$ such that

$$[\![\tilde{e}_0, [\![\tilde{e}_0, x]\!]]\!] \neq 0. \tag{6.51}$$

Note that we have used the isomorphism between $\mathscr{A}_{\pm 1}$ and $\mathscr{B}_{\pm 1}$. Consider an element $x = e_\alpha$ with an associated positive root α of $\mathfrak{g}$ such that $\alpha(\alpha_0^\vee) < -1$. With this assumption it follows that

$$[\![\tilde{e}_0, e_\alpha]\!] \neq 0 \qquad \text{since} \qquad [\![\tilde{f}_0, [\![\tilde{e}_0, e_\alpha]\!]]\!] = -\alpha(\alpha_0^\vee)e_\alpha \neq 0, \tag{6.52}$$

using the Jacobi identity. In the same spirit we examine if the element $[\![\tilde{e}_0, [\![\tilde{e}_0, e_\alpha]\!]$ vanish or not. Using again the Jacobi identity we find

$$\begin{aligned} [\![\tilde{f}_0, [\![\tilde{e}_0, [\![\tilde{e}_0, e_\alpha]\!]]\!]]\!] &= [\![\tilde{e}_0, \underbrace{[\![\tilde{f}_0, [\![\tilde{e}_0, e_\alpha]\!]]\!]}_{\propto -\alpha(\alpha_0^\vee)}]\!] - [\![\underbrace{[\![\tilde{e}_0, \tilde{f}_0]\!]}_{h_0}, [\![\tilde{e}_0, e_\alpha]\!]]\!] \\ &= -2(1 + \alpha(\alpha_0^\vee))[\![\tilde{e}_0, e_\alpha]\!], \end{aligned} \tag{6.53}$$

which by the assumption $\alpha(\alpha_0^\vee) < -1$ is non-vanishing. Thus if we can find such a positive root α so-called ancillary transformations will be present. Since α is assumed to be positive we can expand it as $\sum_i c_i \alpha_i$, with $c_i \geq 0$ and α_i a simple root of $\mathfrak{g}$ such that

$$\begin{aligned} \alpha(\alpha_0^\vee) &= \sum_i c_i \alpha_i(\alpha_0^\vee) = \sum_i c_i A_{0i} \\ &= -\frac{k}{2} \sum_i c_i (\alpha_i, \alpha_i)\lambda_i \overset{!}{<} -1. \end{aligned} \tag{6.54}$$

If λ is not a fundamental weight it is always possible to find a root α such that the inequality is satisfied and ancillary transformations are thus present in this case.

For sufficiency assume that $\lambda = \Lambda_j$ is a fundamental weight. In this case with $k = 2/(\alpha_j, \alpha_j)$ we have

$$\alpha(\alpha_0^\vee) = -c_j. \tag{6.55}$$

Thus for any root α with $c_j \geq 2$ ancillary transformations are present. Take the root vector $x = e_\theta$ associated to the highest root θ of $\mathfrak{g}$. In (2.37) the highest root θ was expanded

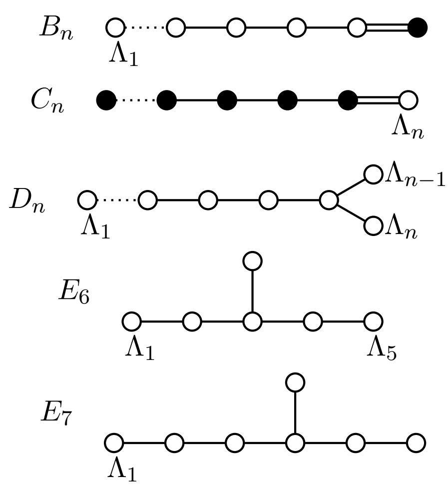

Figure 6.2: All structure algebras with possible choice of coordinate module for which no ancillary transformation are shown. The $\mathfrak{a}_n$ family is not included in the figure since any choice of fundamental weight leads to an algebra of diffeomorphisms without ancillary transformations.

in simple roots and the coefficients were the Coxeter labels. It thus follows immediately that for no ancillary transformation being present $\lambda = \Lambda_j$ is a fundamental weight and, moreover, that the corresponding Coxeter label c_j is equal to one. In fact this is not only a necessary but also a sufficient condition. This can be seen by noting that by applying any positive simple root e_i to $[\![\tilde{e}_0, [\![\tilde{e}_0, e_\theta]\!]]\!]$ and using the Jacobi identity as well as $[\![e_i, e_\theta]\!] = 0$, we find

$$[\![e_i, [\![\tilde{e}_0, [\![\tilde{e}_0, e_\theta]\!]]\!]]\!] = [\![\tilde{X}^M, [\![\tilde{e}_0, e_\theta]\!]]\!] + [\![\tilde{e}_0, [\![\tilde{X}^M, e_\theta]\!]]\!], \tag{6.56}$$

with $\tilde{X}^M := [\![e_i, \tilde{e}_0]\!] \in R(\lambda)$. In other words we can build any object $\Lambda_{MN}[\![\tilde{X}^M, [\![\tilde{X}^N, e_\theta]\!]]\!] = 0$ starting from $[\![\tilde{e}_0, [\![\tilde{e}_0, e_\theta]\!]]\!]$. Note by construction that $\Lambda_{MN} \in \overline{R(2\lambda)}$ and hence it follows directly that $\Lambda_{MN}[\![E^M, E^N]\!] = 0$ as well. On the other hand, applying a root vector associated to a simple negative root to the same expression we find

$$\begin{aligned} [\![f_i, [\![\tilde{e}_0, [\![\tilde{e}_0, e_\theta]\!]]\!]]\!] &= [\![\tilde{e}_0, [\![\tilde{e}_0, [\![f_i, e_\theta]\!]]\!]]\!] \\ &= [\![\tilde{e}_0, [\![\tilde{e}_0, e_{\theta-\alpha_i}]\!]]\!], \end{aligned} \tag{6.57}$$

where we used that $[\![f_i, \tilde{e}_0]\!] = 0$. We thus see that the expression (6.56) is true for any element $x \in \mathfrak{g}$.

We have thus shown that ancillary transformations (6.40) vanish if and only if the highest weight of the coordinate module is a fundamental weight and the corresponding Coxeter label is equal to one. Such modules are easily found for finite-dimensional simple Lie algebras and can be found in e.g. [17]. For $\mathfrak{a}_n$ this includes any choice Λ_i while all other examples are given in Figure 6.2. Moreover, infinite-dimensional structure algebras always have ancillary transformations [5].

6.3 Dynamics

Having developed the structure of generalised diffeomorphisms to describe geometry based on a general structure algebra we want to describe the dynamics of some fields on this geometry. Especially, we will write down a pseudo-action that determines the dynamics for a generalised metric $G_{MN} \in G \times \mathbb{R}^+/K$, where G is the structure group and K the maximal compact subalgebra of G. In the case of compactifying 11-dimensional supergravity on a torus T^d the generalised metric will contain the purely internal degrees of freedom of the metric as well as those of the three-form (dualised to obtain the maximal amount of scalars) and the structure group is given by the $E_{d(d)}$ series.

Note that in the theory below we only consider an extended geometry based on a generalised structure group with some coordinate module $R(\lambda)$. In other words there is no external spacetime nor any compactification ansatz. In order to include external degrees of freedom one should look at DFT or EFT. In this case there would be purely external fields as well as gauge potentials, or "graviphotons", that can be considered as gauge fields for the, in this case, internal generalised diffeomorphisms. Typically the corresponding field strength tensor will however not transform covariantly but fails by a trivial transformation. To account for this one successively has to add higher rank form fields that transforms in particular representations of the structure group. This is the procedure that leads to the notion of a "tensor hierarchy". The exact spectrum of representations of the tensor hierarchies can be derived from tensor hierarchy algebras developed in [8]. The construction and rôle of these tensor hierarchy algebras in extended geometries are further discussed in [9, 10] and references therein.

In order to describe the dynamics the action should be quadratic in derivatives of the metric and it should transform as a scalar density of weight 1. The action will only transform properly after the section constraint is applied and in that sense it is a pseudo-action. This means that we manually have to impose the section constraint and only look at the classical theory. In order to e.g. quantize the theory by performing a path integral one would most likely have to find a way to dynamically generate the section constraint as a constraint.

Analogously to the metric in Riemannian geometry being an element in $GL(d)/SO(d)$ the generalised metric is an element in $G \times \mathbb{R}^+/H$. Moreover, we take the weight of the metric to be $-2w$ instead of the canonical weight -2β of a rank two tensor. This as we will see is necessary to ensure that the lagrangian transforms properly.

Crucially the metric determines a local embedding of the maximal compact subalgebra K. To see this remember that the maximal compact subalgebra $\mathfrak{k}$ is the subset of $\mathfrak{g}$ that has eigenvalue 1 under an involution. The metric determines such a local involution on the generators T^α of $\mathfrak{g}$ given by

$$\begin{aligned} \star : \qquad & \mathfrak{g} \to \mathfrak{g} \\ & T^\alpha \mapsto T^{\alpha\star} = (G T^\alpha G^{-1})^T, \end{aligned} \tag{6.58}$$

with ^{T} being the ordinary transpose. In index notation with G_{MN} being the metric, G^{MN} its inverse and the generators $T^{\alpha M}{}_N$ this corresponds to

$$(T^{\alpha\star})^M{}_N = G_{NP} T^{\alpha P}{}_S G^{SM}. \tag{6.59}$$

It is then easily seen that linear combinations $T+T^\star$ and $T-T^\star$ span $\mathfrak{k}$ and $\mathfrak{g}\ominus\mathfrak{k}$ respectively. Note that $\eta_{\alpha\beta}$ is invariant under this evolution and hence $\eta_{\alpha\beta} T^{\alpha\star} \otimes T^{\beta\star} = \eta_{\alpha\beta} T^\alpha \otimes T^\beta$.

As in the discussion of non-linear sigma models in Section 3.2.1 we will differentiate the dynamical field and contract with its inverse to obtain (similar to the Maurer-Cartan form) an element in the corresponding coset algebra $\mathfrak{g} \times \mathbb{R}/\mathfrak{k}$

$$(G^{-1}\partial_M G)^N{}_P = T^{\alpha N}{}_P \Pi_{M\alpha} + \delta^P_N \Theta_M. \tag{6.60}$$

Note that since the metric is valued in the coset we have $\Pi_{M\alpha}T^{\alpha\star} = \Pi_{M\alpha}T^\alpha$. Due to the non-covariant derivative in the definition of $\Pi_{M\alpha}$ and Θ_M these will not transform covariantly under generalised diffeomorphisms and to find an appropriate action we should determine the inhomogeneous transformation of these fields.

The inhomogeneous transformation of a field is given by the difference $\Delta_\xi = \delta_\xi - \mathscr{L}_\xi$ and we therefore look at

$$\delta_\xi\left(\partial_M G_{NP}\right) = \partial_M\left(\xi^K\partial_K G_{NP} - 2Z^{SK}{}_{T(N|}\partial_K\xi^T G_{|P)S}\right), \tag{6.61}$$

$$\begin{aligned}\mathscr{L}_\xi\left(\partial_M G_{NP}\right) &= \xi^K\partial_K\partial_M G_{NP} - 2Z^{SK}{}_{T(N|}\partial_K\xi^T\partial_M G_{|P)S}\\ &\quad - Z^{SK}{}_{TM}\partial_K\xi^T\partial_S G_{NP}.\end{aligned} \tag{6.62}$$

Using the section constraint in the last term in (6.62) we find the inhomogeneous transformation

$$\Delta_\xi\left(G^{-1}\partial_M G\right)^N{}_P = -2Z^{SK}{}_{T(N}\partial_{K)}\partial_M\xi^T. \tag{6.63}$$

In terms of the fields $\Pi_{M\alpha}$ and Θ_M this translates to

$$\Delta\Pi_{M\alpha} = \eta_{\alpha\beta}\left(T^\beta + T^{\star\beta}\right)^S{}_T\partial_M\partial_S\xi^T, \tag{6.64}$$

$$\Delta\Theta_M = -2w\partial_M\partial_K\xi^K. \tag{6.65}$$

The ansatz for the lagrangian will then be

$$\mathcal{L} = c_1\mathcal{L}_1(\Pi_{M\alpha}) + c_2\mathcal{L}_2(\Pi_{M\alpha}) + c_3\mathcal{L}_3(\Theta_M) + c_4\mathcal{L}_4(\Pi_{M\alpha},\Theta_M), \tag{6.66}$$

with c_i constant coefficients and $\mathcal{L}_i$ field dependent terms given by

$$\begin{cases}\mathcal{L}_1 = G^{MN}\eta^{\alpha\beta}\Pi_{M\alpha}\Pi_{N\beta},\\ \mathcal{L}_2 = G^{KL}T^{\alpha M}{}_K T^{\beta N}{}_L\Pi_{N\alpha}\Pi_{M\beta},\\ \mathcal{L}_3 = G^{MN}\Theta_M\Theta_N,\\ \mathcal{L}_4 = G^{MK}T^{\alpha N}{}_K\Pi_{M\alpha}\Theta_N.\end{cases} \tag{6.67}$$

The inhomogeneous transformation of $\mathcal{L}_1$ is straightforward to compute

$$\begin{aligned}\Delta_\xi\mathcal{L}_1 &= 2G^{MN}\eta^{\alpha\beta}\Pi_{M\alpha}(T^\gamma + T^{\star\gamma})^S{}_T\eta_{\gamma\beta}\partial_N\partial_S\xi^T\\ &= 4G^{MN}\Pi_{M\alpha}T^{\alpha S}{}_T\partial_N\partial_S\xi^T,\end{aligned} \tag{6.68}$$

where we used that $\Pi_{M\alpha} \in \mathfrak{g} \ominus \mathfrak{k}$. It is likewise found that $\mathcal{L}_3$ transforms as

$$\Delta\mathcal{L}_3 = -4wG^{MN}\Theta_M\partial_N(\partial\cdot\xi). \tag{6.69}$$

In a similar fashion we find for $\mathcal{L}_4$

$$\Delta_\xi\mathcal{L}_4 = G^{MK}T^{\alpha N}{}_K\left(T^\beta + T^{\star\beta}\right)^S{}_T\partial_M\partial_S\xi^T\Theta_N - 2wG^{MK}T^{\alpha N}{}_K\Pi_{M\alpha}\partial_N(\partial\cdot\xi). \tag{6.70}$$

Using that $\eta_{\alpha\beta}T^\alpha\otimes T^{\star\beta} = \eta_{\alpha\beta}T^{\star\alpha}\otimes T^\beta$ and the definition of the involution $\star$ it is found that

$$\begin{aligned}\Delta_\xi\mathcal{L}_4 &= G^{MK}T^{\alpha N}{}_K T^{\alpha S}{}_T\partial_M\partial_S\xi^T\Theta_N + G^{NL}T^{\alpha M}{}_L T^{\alpha S}{}_T\partial_M\partial_S\xi^T\Theta_N\\ &\quad - 2wG^{MK}T^{\alpha N}{}_K\Pi_{M\alpha}\partial_N(\partial\cdot\xi).\end{aligned} \tag{6.71}$$

By the definition of Θ_N there is a derivative in ${}_N$, hence the section constraint can be used to simplify both terms in the first line. We thus get

$$\begin{aligned}\Delta_\xi \mathcal{L}_4 = {} & (2\beta + 2w + 1) G^{MN} \Theta_M \partial_N (\partial \cdot \xi) + G^{MN} \partial_M \partial_N \xi^T \Theta_T + \\ & - 2w G^{MN} \partial_M \partial_N (\partial \cdot \xi),\end{aligned} \tag{6.72}$$

where we also integrated by parts and discarded total derivatives to obtain the term containing $\partial^3 \xi$.

The remaining term $\mathcal{L}_2$ requires a bit more work, to begin with we find (below we make no distinction between upper and lower adjoint indices and leave $\eta_{\alpha\beta}$ implicit)

$$\begin{aligned}\Delta \mathcal{L}_2 &= 2G^{KL} T^{\alpha M}{}_K T^{\beta N}{}_L \left(T^\alpha + T^{\star\alpha}\right)^S{}_T \partial_n \partial_S \xi^T \Pi_{M\beta} \\ &= 2G^{KL} \left(T^{\alpha M}{}_K T^{\alpha S}{}_T + G_{KQ} T^{\alpha Q}{}_R G^{RM} T^{\alpha S}{}_T\right) \partial_N \partial_S \xi^T T^{\beta N}{}_L \Pi_{M\beta}.\end{aligned} \tag{6.73}$$

In the first term on the second line use the section constraint and in the second term commute $(T^\beta T^\alpha)^N{}_R = (T^\alpha T^\beta)^N{}_R + f^{\beta\alpha}{}_\gamma T^{\gamma N}{}_R$ to find

$$\begin{aligned}\Delta \mathcal{L}_2 = {} & 2G^{KL} \left(\beta \delta^M_K \delta^S_T + \delta^M_T \delta^S_K\right) T^{\beta N}{}_L \partial_N \partial_S \xi^T \Pi_{M\beta} \\ & + 2G^{RM} \left(T^{\alpha N}{}_L T^{\beta L}{}_R + f^{\beta\alpha}{}_\gamma T^{\gamma N}{}_R\right) T^{\alpha S}{}_T \partial_N \partial_S \xi^T \Pi_{M\beta}.\end{aligned} \tag{6.74}$$

Use the section constraint on the first term in the second line to find

$$\begin{aligned}\Delta_\xi \mathcal{L}_2 = {} & 2(2\beta + 1) G^{RM} T^{\beta N}{}_R \partial_N (\partial \cdot \xi) \Pi_{M\beta} + 2G^{SL} T^{\alpha N}{}_L \partial_S \partial_N \xi^M \Pi_{M\alpha} \\ & + 2G^{RM} f^{\beta\alpha}{}_\gamma T^{\gamma N}{}_R T^{\alpha S}{}_T \partial_N \partial_S \xi^T \Pi_{M\beta}\end{aligned} \tag{6.75}$$

From the definition of the S^α tensor in (6.42)

$$S^{\alpha SN}{}_{RT} = -f^{\alpha\beta\gamma} T^{\beta N}{}_R T^{\gamma S}{}_T + T^{\alpha N}{}_R \delta^S_T - T^{\alpha S}{}_T \delta^N_R, \tag{6.76}$$

we can rewrite the term containing the structure constants as

$$\begin{aligned}2G^{RM} f^{\beta\alpha}{}_\gamma T^{\gamma N}{}_R T^{\alpha S}{}_T \partial_N \partial_S \xi^T \Pi_{M\beta} = {} & 2G^{RM} S^{\alpha SN}{}_{RT} \partial_N \partial_S \xi^T \Pi_{M\alpha} \\ & - 2G^{RM} T^{\alpha N}{}_R \partial_N (\partial \cdot \xi) \Pi_{M\alpha} + 2G^{RM} T^{\alpha S}{}_T \partial_R \partial_S \xi^T \Pi_{M\alpha}.\end{aligned} \tag{6.77}$$

Inserting (6.77) in (6.75) we find

$$\begin{aligned}\Delta_\xi \mathcal{L}_2 = {} & 2G^{RM} S^{\alpha SN}{}_{RT} \partial_N \partial_S \xi^T \Pi_{M\alpha} + 4\beta G^{RM} T^{\beta N}{}_R \partial_N (\partial \cdot \xi) \Pi_{M\beta} \\ & + 2G^{SL} T^{\alpha N}{}_L \partial_S \partial_N \xi^M \Pi_{M\alpha} + 2G^{RM} T^{\alpha S}{}_T \partial_R \partial_S \xi^T \Pi_{M\alpha}.\end{aligned} \tag{6.78}$$

Note that the last term in the second line equals $\frac{1}{2}\Delta_\xi \mathcal{L}_1$. Using the definition of $\Pi_{M\alpha}$ rewrite the second term in the first line as

$$4\beta G^{RM} T^{\beta N}{}_R \partial_N (\partial \cdot \xi) \Pi_{M\beta} = 4\beta G^{RM} G^{NS} \partial_M G_{SR} \partial_N (\partial \cdot \xi) - 4\beta G^{MN} \Theta_M \partial_N (\partial \cdot \xi), \tag{6.79}$$

which by partial integration and neglecting total derivatives can in turn be written as

$$4\beta G^{RM} T^{\beta N}{}_R \partial_N (\partial \cdot \xi) \Pi_{M\beta} = 4\beta G^{MN} \partial_M \partial_N (\partial \cdot \xi) - 4\beta G^{MN} \Theta_M \partial_N (\partial \cdot \xi). \tag{6.80}$$

An analogous calculation for the first term in the second line in (6.78) gives

$$2G^{SL} T^{\alpha N}{}_L \partial_S \partial_N \xi^M \Pi_{M\alpha} = 2G^{MN} \partial_M \partial_N (\partial \cdot \xi) - 2G^{MN} \partial_M \partial_N \xi^T \Theta_T. \tag{6.81}$$

The final transformation is then given by

$$\begin{aligned}\Delta\mathcal{L}_2 = 2G^{RM}S^{\alpha SN}{}_{RT}\partial_N\partial_S\xi^T\Pi_{M\alpha} + 2(2\beta+1)G^{MN}\partial_M\partial_N(\partial\cdot\xi) \\ - 4\beta G^{MN}\Theta_M\partial_N(\partial\cdot\xi) - 2G^{MN}\partial_M\partial_N\xi^T\Theta_T + \frac{1}{2}\Delta_\xi\mathcal{L}_1\end{aligned} \tag{6.82}$$

In order for the lagrangian $\mathcal{L}$ to transform covariantly, up to the ancillary transformations and boundary terms, we thus find

$$\begin{cases} 2c_1 + c_2 = 0, \\ c_4 - 2c_2 = 0, \\ -2wc_4 + 2(2\beta+1)c_2 = 0, \\ -4wc_3 + (2\beta+2w+1)c_4 - 4\beta c_2 = 0. \end{cases} \tag{6.83}$$

Setting $c_2 = -1$ to fix the overall scale the following expression

$$\mathcal{L} = \frac{1}{2}\mathcal{L}_1(\Pi_{M\alpha}) - \mathcal{L}_2(\Pi_{M\alpha}) - \frac{(\lambda,\lambda)}{(\lambda,\lambda)-\frac{1}{2}}\mathcal{L}_3(\Theta_M) - 2\mathcal{L}_4(\Pi_{M\alpha},\Theta_M), \tag{6.84}$$

transforms as

$$\Delta_\xi\mathcal{L} = -2G^{RM}S^{\alpha SN}{}_{RT}\partial_N\partial_S\xi^T\Pi_{M\alpha} \tag{6.85}$$

if the metric has a weight $w = (\lambda,\lambda) - 1/2$. Moreover, the weight of $\mathcal{L}$ is then 1 as it should since

$$2[(\lambda,\lambda) - 1/2] - 2[(\lambda,\lambda) - 1] = 1. \tag{6.86}$$

We have thus found a lagrangian that transforms like a scalar density of weight 1 up to ancillary and boundary terms. However, covariance is not manifest and one would like to derive the same expression with manifest transformation properties. In ordinary geometry this is done by introducing an affine connection and from this the Ricci tensor and Ricci scalar are derived which roughly correspond to the equations of motion and the Einstein-Hilbert lagrangian respectively. An attempt to do this will be presented below.

6.4 Covariant formalism

As in ordinary geometry fields containing bare partial derivatives typically transforms non-covariantly since diffeomorphisms act locally. This is again true in extended geometries and the fact that the lagrangian (6.84) transformed covariantly were due to some delicate cancellations. We here follow [40] which studied exceptional geometry with the structure algebra $\mathfrak{e}_{d(d)} \times \mathbb{R}$ for $d \leq 8$. The discussion in [40] were however in many cases general and can be directly transferred to the formalism of extended geometries presented above and in [5].

Introduce a covariant derivative $\mathscr{D}_M$ that acts on a vector field V^N as

$$\mathscr{D}_M V^N = \partial_M V^N - \Gamma_{MP}{}^N V^P, \tag{6.87}$$

with $\Gamma_{MP}{}^N$ being the connection. Just as a connection in Einstein gravity or the Yang-Mills connection in a non-abelian gauge theory this connection is a adjoint valued "one-form", i.e. $\Gamma_{MP}{}^N \in \overline{R(\lambda)} \otimes [\mathfrak{g} \oplus \mathbb{R}]$. Especially this exclude any specific symmetry properties between the lower indices. Imposing that $D_M V^N$ transforms covariantly one finds that the connection has to transform inhomogeneously as

$$\Delta_\xi \Gamma_{MN}{}^P = Z^{PQ}{}_{RN} \partial_M \partial_Q \xi^R. \tag{6.88}$$

Since the tensor Z manifestly projects ${}^P{}_N$ and ${}^Q{}_R$ onto the adjoint representation it is easily seen that the inhomogeneous term is a derivative of an element taking values in the structure algebra $\mathfrak{g} \times \mathbb{R}$. Moreover, imposing Leibniz property and that the covariant derivative reduces to a partial derivative on functions the action on arbitrary tensor fields is well-defined.

The covariant derivative introduced above is valid for tensors, i.e. an element in $R(\lambda)^p \otimes \overline{R(\lambda)}^q$ which transforms under $\mathbb{R}$ with the canonical weight $(p-q)\beta$. For a tensor density with weight w there is an extra contribution to the covariant derivative as

$$D_P X^{M_1 M_2 \ldots M_p}{}_{N_1 N_2 \ldots N_q} = \mathscr{D}_P X^{M_1 M_2 \ldots M_p}{}_{N_1 N_2 \ldots N_q} - \frac{1}{\beta |R(\lambda)|} (w - (p-q)\beta) \Gamma_{MS}{}^S, \tag{6.89}$$

with $|R(\lambda)| = \dim R(\lambda)$ and we note that D_P reduce to $\mathscr{D}_P$ when acting on a tensor.

In ordinary geometry one usually continues by demanding that the connection is metric-compatible and torsion-free which uniquely specifies the Levi-Civita connection. However, this will turn out to be rather complicated in extended geometries and in fact a unique choice of such a connection is generally not possible to find.

A metric compatible connection is one that satisfies

$$D_M G_{NP} = 0, \tag{6.90}$$

or explicitly

$$\partial_M G_{NS} + 2\Gamma_{M(N}{}^T G_{S)T} - \frac{1}{\beta |R(\lambda)|} \Gamma_{MK}{}^K G_{NS} = 0. \tag{6.91}$$

Multiplying (6.91) with G^{PS} this further implies

$$\left(G^{-1} \partial_M G\right){}^P{}_N = \Gamma_{MN}{}^P + G_{NS} \Gamma_{MT}{}^S G^{TP} - \frac{1}{\beta |R|} \Gamma_{MK}{}^K \delta_N^P. \tag{6.92}$$

We thus see that metric compatibility uniquely determines the part in $\mathfrak{g} \ominus \mathfrak{k}$. Note especially that the RHS of (6.92) is projected onto $T^\alpha + T^{\star\alpha}$ by the definition of the locally embedded compact subalgebra. Moreover, (6.92) shows that the building block of the lagrangian (6.84) is in fact a connection projected onto $\mathfrak{g}/\mathfrak{k}$.

6.4.1 Torsion

Torsion of a connection can be defined in at least two equivalent ways (at least when ancillary transformations are absent). Following [40] we define torsion of a connection as the part that transforms homogeneously, i.e. the part of the connection which is not contained in (6.88). In fact, only the part in $\left(\vee^2 \overline{R(\lambda)} \ominus \overline{R_h^s}\right) \otimes R(\lambda)$ of the connection transforms inhomogeneously due to (6.88), with R_s^h defined in (6.14). The non-torsion parts of the connection are thus given by the modules in the overlap

$$\overline{R(2\lambda)} \otimes R(\lambda) \cap \left[\overline{R(\lambda)} \otimes (\mathfrak{g} \oplus \mathbb{R})\right], \tag{6.93}$$

with the remaining modules of the connection being torsion.

Consider the following combination [40]

$$T_{MN}{}^P = \Gamma_{MN}{}^P + Z^{PQ}{}_{RN}\Gamma_{QM}{}^R, \tag{6.94}$$

and we wish to determine whether this transforms covariantly under generalised diffeomorphisms

$$\begin{aligned}\Delta_\xi T_{MN}{}^P &= Z^{PS}{}_{TN}\partial_M\partial_S\xi^T + Z^{PQ}{}_{RN}Z^{RS}{}_{TM}\partial_Q\partial_S\xi^T \\ &= \left(Z^{PS}{}_{TN}\delta^Q_M + Z^{PQ}{}_{RN}Z^{RS}{}_{TM}\right)\partial_Q\partial_S\xi^T.\end{aligned} \tag{6.95}$$

Continuing with this expression we find

$$\begin{aligned}\text{RHS of (6.95)} &= Z^{PQ}{}_{RN}Y^{RS}{}_{TM}\,\partial_Q\partial_S\xi^T \\ &= -T^{\alpha P}{}_N T^{\alpha Q}{}_R Y^{RS}{}_{TM}\,\partial_Q\partial_S\xi^T,\end{aligned} \tag{6.96}$$

where in the second line we have expanded Z and used the section constraint on the term in Z that contains a factor of β. By expanding the Y-tensor we further find

$$\begin{aligned}\text{RHS of (6.96)} &= T^{\alpha P}{}_N\left((T^\alpha T^\beta)^Q{}_M T^{\beta S}{}_T - \beta\delta^S_T T^{\alpha Q}{}_M - \delta^S_M T^{\alpha Q}{}_T\right)\partial_Q\partial_S\xi^T \\ &= T^{\alpha P}{}_N\left(-f^{\alpha\beta\gamma}T^{\beta Q}{}_M T^{\gamma S}{}_T + T^{\alpha Q}{}_M\,\delta^S_T - T^{\alpha Q}{}_T\,\delta^S_M\right)\partial_Q\partial_S\xi^T,\end{aligned} \tag{6.97}$$

where going to the second line we have expressed the first term as a commutator, used the section constraint and that $\partial_Q\partial_S$ is symmetric. We thus see that the expression (6.94) transforms covariantly if ancillary transformations are absent since $T_{MN}{}^P$ transforms as

$$\Delta_\xi T_{MN}{}^P = T^{\alpha P}{}_N\,S^{\alpha SQ}{}_{MT}\partial_Q\partial_S\xi^T. \tag{6.98}$$

However, although this expression for torsion transforms homogeneously the operator

$$\mathcal{O}_{MN,Q}{}^{P,ST} = \delta^S_M\delta^T_N\delta^P_Q + Z^{PS}{}_{QN}\delta^T_M \tag{6.99}$$

does not satisfy $\mathcal{O}^2 = \mathcal{O}$ and hence is not a projection operator. Instead each module of torsion has different eigenvalues under $\mathcal{O}$. An explicit projection operator would be convenient when setting torsion to zero, however, we were not able to find such an expression in the general case. In the construction of tensor hierarchy algebras in [9] it was shown that torsion sits at level -1 with respect to a $\mathbb{Z}$-grading. It would therefore be interesting to see if it is possible to derive a projection operator from this algebra, similar to the way that the Y-tensor was derived from extensions of the structure algebra in Section 2.3.3.

Setting torsion to zero in (6.94) imposes the following constraint

$$\begin{aligned}0 &= \Gamma_{MN}{}^P + Z^{PQ}{}_{RN}\Gamma_{QM}{}^R \\ &= 2\Gamma_{[MN]}{}^P + Y^{PQ}{}_{RN}\,\Gamma_{QM}{}^R,\end{aligned} \tag{6.100}$$

where the second line follows from the definition of Z and Y. It is convenient to make explicit that the connection is valued in $\overline{R(\lambda)}\otimes(\mathfrak{g}\oplus\mathbb{R})$

$$\Gamma_{MN}{}^P = \Gamma^\alpha_M T^{\alpha P}{}_N + \Gamma_M\delta^P_N. \tag{6.101}$$

Inserting (6.101) in (6.100) we find

$$\begin{aligned}0 = T^{\alpha P}{}_N&\left(\Gamma^\alpha_M - \Gamma^\beta_Q(T^\alpha T^\beta)^Q{}_M - \Gamma_Q T^{\alpha Q}{}_M\right) \\ &+ \delta^P_N\left(\Gamma_M + \beta\Gamma^\alpha T^{\alpha R}{}_M + \beta\Gamma_M\right).\end{aligned} \tag{6.102}$$

Multiplying with ${T^{\gamma N}}_P$ and δ^N_P respectively each line in (6.102) should vanish separately. As an example consider an extended geometry with structure algebra $\mathfrak{e}_{7(7)} \times \mathbb{R}$, maximal compact subalgebra $\mathfrak{su}(8)/\mathbb{Z}_2{}^2$ and the coordinate module $R(\lambda) = \mathbf{56}$ which is self-dual. The connection then contains the following modules

$$\mathbf{56} \otimes (\mathbf{133} \oplus \mathbf{1}) = 2 \cdot \mathbf{56} \oplus \mathbf{912} \oplus \mathbf{6480}. \tag{6.103}$$

To determine the parts that are not torsion we note that $\overline{R(2\Lambda_1)} = \overline{\mathbf{1463}}$, with Λ_1 being the highest weight of $\mathbf{56}$, is the highest weight module of $\vee^2 \mathbf{56}$ and we thus find

$$\overline{\mathbf{1463}} \otimes \mathbf{56} = \mathbf{56} \oplus \mathbf{6480} \oplus \ldots, \tag{6.104}$$

where ... denote terms not present in the connection. We thus see that the non-torsion part of the connection contains $\mathbf{56} \oplus \mathbf{6480}$ while the remaining $\mathbf{56} \oplus \mathbf{912}$ thus are torsion. We continue with this example to determine which parts of the connection are specified by imposing metric compatibility and vanishing torsion. Metric compatibility fixes the part in $\mathbf{56} \otimes (\mathfrak{g} \times \mathbb{R}/\mathfrak{k})$ as argued above. Decomposing the remaining terms under the maximal compact subalgebra we find

$$(\mathbf{28} \oplus \overline{\mathbf{28}}) \otimes \mathbf{63} = \mathbf{28} \oplus \mathbf{36} \oplus \mathbf{420} \oplus \mathbf{1280} \oplus \text{conj.} \tag{6.105}$$

Moreover, decomposing torsion and non-torsion respectively we find

$$\begin{aligned} \mathbf{56} \oplus \mathbf{912} &\to \mathbf{28} \oplus \mathbf{36} \oplus \mathbf{420} \oplus \text{conj.}, \\ \mathbf{56} \oplus \mathbf{6480} &\to \mathbf{28} \oplus \mathbf{420} \oplus \oplus \mathbf{1280} \oplus \mathbf{1512} \oplus \text{conj}, \end{aligned} \tag{6.106}$$

which by comparing with (6.105) shows that the modules $\mathbf{1280} \oplus \overline{\mathbf{1280}}$ of $\mathfrak{su}(8)$ are not determined by imposing metric compatibility and vanishing torsion. For the relevant modules for other algebras in the $\mathfrak{e}_d$-series see [40].

The problem with having non-determined parts in the connection is that these would lead to extra d.o.f. besides the ones that we are trying to describe, i.e. those of the generalised metric. Thus, in order to write down a meaningful theory one should only acquire expressions that are independent of these modules.

6.5 Generalised Ricci tensor

In Section 6.3 we derived an action that is invariant under generalised diffeomorphisms which further can be used to derive the equations of motion for the generalised metric. However, as stated above the invariance is not manifest. Below we will work towards a manifest formulation following [40] by constructing a generalised Ricci tensor that transforms covariantly under generalised diffeomorphisms. However, there are several intrinsic problems due to the fact that the affine connection is not uniquely determined by metric compatibility and vanishing torsion.

The most natural thing to start with is to derive curvature by taking commutators of covariant derivatives (here the commutators act on an element in $\overline{R(\lambda)}$ which we have skipped)

$$[D_M, D_N]^P{}_Q = 2\partial_{[M}\Gamma_{N]Q}{}^P + 2\Gamma_{[M|S|}{}^P\Gamma_{N]Q}{}^S. \tag{6.107}$$

[2] We ignore the discrete $\mathbb{Z}_2$ factor and consider only the double cover $\mathfrak{su}(8)$ below.

Expressing the connection with indices in the adjoint as in (6.101) the abelian part of the second factor above vanish and we are left with

$$\begin{aligned}[D_M, D_N]^P{}_Q = {}& 2T^{\alpha P}{}_Q\,\partial_{[M}\Gamma^\alpha_{N]} + 2\delta^P_Q\partial_{[M}\Gamma_{N]} \\ & + f^{\alpha\beta\gamma}T^{\gamma P}{}_Q\,\Gamma^\alpha_M\Gamma^\beta_N + 2T^{\alpha P}{}_Q\,\Gamma^\alpha_{[M}\Gamma_{N]}.\end{aligned} \tag{6.108}$$

Consider the inhomogeneous transformation of $\partial_M\Gamma_{NQ}{}^P$ which by a straightforward calculation at this point is given by

$$\begin{aligned}\Delta_\xi\partial_M\Gamma_{NQ}{}^P = {}& Z^{PR}{}_{SP}\,\partial_M\partial_N\partial_R\xi^S - \Delta_\xi\Gamma_{MN}{}^R\Gamma_{RQ}{}^P \\ & + \Delta_\xi\Gamma_{MR}{}^P\Gamma_{NQ}{}^R - \Delta_\xi\Gamma_{MQ}{}^R\Gamma_{NR}{}^P.\end{aligned} \tag{6.109}$$

Anti-symmetrising in ${}_{[MN]}$ we get

$$2\Delta_\xi\partial_{[M}\Gamma_{N]Q}{}^P = Y^{RS}{}_{TN}\,\Delta_\xi\Gamma_{SM}{}^T\Gamma_{RQ}{}^P - T^{\alpha R}{}_N\,S^{\alpha SL}{}_{MT}\partial_L\partial_S\xi^T\Gamma_{RQ}{}^P - 2\Delta_\xi\left(\Gamma_{[M|Q|}{}^R\Gamma_{N]R}{}^P\right), \tag{6.110}$$

where the first two terms come from the transformation of torsion, the last term from the second line in (6.109) and the term containing $\partial^3\xi$ vanish due to symmetry. The last term in (6.110) cancels the contribution from $[\Gamma, \Gamma]$ in $[D, D]$ and we thus get

$$\Delta_\xi[D_M, D_N]^P{}_Q = Y^{RS}{}_{TN}\,\Delta_\xi\Gamma_{SM}{}^T\Gamma_{RQ}{}^P - T^{\alpha R}{}_N\,S^{\alpha SL}{}_{MT}\partial_L\partial_S\xi^T\Gamma_{RQ}{}^P. \tag{6.111}$$

It is worth to note that due to the symmetry properties of S^α only the part anti-symmetric in ${}_{MT}$ contributes in this expression. In double geometry with $\mathfrak{g} = \mathfrak{o}(d, d)$ it is possible to continue and construct a four-index object that indeed transforms covariantly. In the general case this has not been done and instead we will use (6.111) to construct a two-index tensor that transform like a tensor, this will then correspond to a generalised Ricci tensor.

Contract (6.111) with δ^N_P and symmetrise in ${}_{(MQ)}$ to find

$$\Delta_\xi[D_{(M}, D_{|P|}]^P{}_{Q)} = \Delta_\xi\left(\frac{1}{2}Y^{RS}{}_{TP}\,\Gamma_{SM}{}^T\Gamma_{RQ}{}^P\right) - T^{\alpha R}{}_P\,S^{\alpha SL}{}_{(M|T}\partial_L\partial_S\xi^T\Gamma_{R|Q)}{}^P. \tag{6.112}$$

The first term in the expression above is easily canceled and we thus have constructed a symmetric two-index tensor R_{MN} that transforms covariantly up to ancillary transformations. Explicitly R_{MN} is given by

$$\begin{aligned}R_{MN} = {}& \partial_{(M}\Gamma_{|P|N)}{}^P - \partial_P\Gamma_{(MN)}{}^P + \Gamma_{(MN)}{}^S\Gamma_{PS}{}^P \\ & - \Gamma_{P(M}{}^S\Gamma_{N)S}{}^P - \frac{1}{2}Y^{RS}{}_{TP}\,\Gamma_{SM}{}^T\Gamma_{RN}{}^P,\end{aligned} \tag{6.113}$$

and transforms as

$$\Delta_\xi R_{MN} = -T^{\alpha R}{}_P\,S^{\alpha SL}{}_{(M|T}\partial_L\partial_S\xi^T\Gamma_{R|Q)}{}^P. \tag{6.114}$$

Note that the transformation of R_{MN} is independent of whether the torsion vanishes or not. At this point this is indeed a two-index tensor but the modules of the connection which are not determined by metric compatibility and vanishing torsion may still appear in R_{MN}. We can further project R_{MN} on to $\mathfrak{g}/\mathfrak{k}$ as follows

$$R^\alpha := (G^{-1}T^\alpha)^{MN}R_{MN}. \tag{6.115}$$

It is the projection R^α that is a possible candidate for the equations of motion for the generalised metric. It would therefore be interesting to find a metric-compatible torsion-free connection specified in terms of the generalised metric and compare R^α with the equations of motion derived from (6.84). This procedure would correspond to imposing that the undetermined modules in the connection vanish. For consistency one would then have to make sure that the variation of R^α does not contain any of these undetermined modules as well. This is done for the exceptional case in [40].

7

Flux compactifications and (exceptional) generalised geometry

In this chapter we study (exceptional) generalised geometries. These are geometries based on double geometries and exceptional geometries with a global solution to the section constraint. Especially this implies that fields are acted on naturally by the relevant duality group and are patched together by the geometric subgroup of the duality group. Importantly this allows for the presence of a non-trivial form field as a part of the geometry. Especially geometrical tools used for fluxless compactifications can be generalised and applied also in the presence of fluxes in the generalised geometric formulation. This merger of gravitational degrees of freedom and non-gravitiational degrees of freedom is behind the effectivness of generalised geometry in the study of flux compactifications.

In the first two sections we discuss shortly some phenomenology of compactification as well as introducing geometrical tools to study supersymmetric compactifications without fluxes. In Section 7.3 we consider supersymmetric backgrounds with fluxes and introduce some basics of (exceptional) generalised geometry. The generalised geometrical tools are then used in Section 7.4 to re-express the conditions of supersymmetric flux compactifications as integrability conditions of generalised structures. Since this topic by itself could be a whole thesis we will not be able to give all the details, for more on compactifications in general as well as Calabi-Yau compatictifications we refer to [12, 28]. For the formulation of generalised geometry we refer to [45, 46, 47] and to [13, 16] for its application to study flux compactifications.

7.1 Phenomenology and moduli stabilisation

As is well-known string theory/M-theory lives in a 10/11-dimensional spacetime, yet our everyday experience tells us otherwise. Hence, in order for string theory/M-theory to be able to describe the real world it has to have a consistent low-energy description that matches our observations. Firstly, such a theory should reproduce the standard model of particle physics which has been confirmed to high precision, i.e. among other things it should be described by a gauge theory with gauge group $SU(3) \times SU(2) \times U(1)$, together with the correct particle spectrum. Another important notion is that of supersymmetry, notably the standard model is not supersymmetric in contrast to string theory/M-theory. The latter two are maximally supersymmetric theories and thus most, perhaps all, supersymmetries needs to be spontaneously broken.

The idea of compactification, where a compact internal manifold is "small" compared to the typical size of say strings, is one way to obtain an effective theory in lower dimensions. A generic feature of such compactifications is the presence of scalar fields typically due to internal legs of some tensor field. Especially important are massless such scalar fields, called *moduli*. The vacuum expection values of moduli are not fixed and appear as free parameters in the theory. For a given compactification ansatz there are possibly hundreds or thousands of such moduli which determine physical properties in the external theory, e.g. gauge couplings, Yukawa couplings as well as the size and structure of the internal manifold. However, if all these properties are free parameters the theory loses most of its predictive power. Moreover, massless scalar fields would give rise to a long-ranged force which would invalidate Einstein's equivalence principle. Of course the coupling could be extremely small such that these forces are not experimentally observable. However, this approach comes with a lot of problems and instead one typically tries to generate mass to these scalar fields, this is called *moduli stabilisation*. Generating a potential for the scalar fields would imply that the moduli fields are no longer free parameters but rather fixed by the procedure that stabilises the moduli, e.g. a complicated internal geometry or presence of gauge fluxes.

The simplest internal manifold to compactify on is a torus but such a compactification preserves all the supersymmetry, i.e. $\mathcal{N} = 8$ in four external dimensions. In order to break some supersymmetry we could abandon the torus and compactify on something more complicated, e.g. a Calabi-Yau three-fold which we will discuss further below. Another possibility is to introduce a non-zero flux for some $(p+1)$-form field

$$\int_{\Sigma_{p+1}} C_{p+1} \neq 0, \tag{7.1}$$

corresponding to a p-brane winding some non-trivial cycle Σ_{p+1} in the internal manifold. Such a configuration schematically generates a potential for the moduli through the kinetic term of such a form field

$$V \sim \int_{\mathcal{M}_{\text{int}}} \mathrm{d}C_{p+1} \wedge \star_g \mathrm{d}C_{p+1}, \tag{7.2}$$

where $\mathcal{M}_{\text{int}}$ denotes the internal manifold, and the presence of the metric moduli in the hodge dual $\star$ is the reason for a potential being generated. This can be seen by expanding C_{p+1} around its vacuum expectation value. This is one mechanism for generating mass to a modulus (which by definition no longer is a modulus).

Phenomenology of flux compactifications could fill out a thesis of its own and for this reason we refer the reader to the literature for further discussion about this. See e.g. [12, 28] and references therein.

7.2 Zero flux compactification

To begin with we will start by reviewing compactifications on backgrounds without fluxes. The idea is to find vacua (solutions to the equations of motion and the Bianchi identities) such that spacetime separates into an external spacetime $\mathcal{M}_{\text{ext}}$ and an internal spacetime $\mathcal{M}_{\text{int}}$ as

$$\mathcal{M} = \mathcal{M}_{\text{ext}} \times \mathcal{M}_{\text{int}}. \tag{7.3}$$

However, finding such solutions by solving the equations of motion are generally rather difficult and we will take a different route. Supergravity in $D = 11$ has $\mathcal{N} = 1$ local supersymmetry, this is the maximal amount of supersymmetry with a total of 32 supercharges and a generic vacuum solution will (spontaneuously) break all of this supersymmetry. By restricting to solutions that preserve a certain amount of supersymmetry, say $\mathcal{N} = 1$ in $D = 4$, we will find constraints that give a more tractable way of finding vacua. Moreover, we will assume that the external spacetime is Minkowski with possible generalisations, especially to AdS.

Supersymmetry is generated by a supercharge $\mathcal{Q}$ and a parameter ϵ, both being spinors of $SO(1, D-1)$. A vacuum $|0\rangle$ is supersymmetric if it is annihilated by the supersymmetry transformation

$$\bar{\epsilon}\mathcal{Q}|0\rangle = 0. \tag{7.4}$$

By taking an arbitrary field Θ and looking at its variation under the supersymmetry transformation we find

$$\langle\delta_\epsilon\Theta\rangle = \langle 0|\,[\bar{\epsilon}\mathcal{Q}, \Theta]\,|0\rangle = 0. \tag{7.5}$$

Schematically we have (from now variations should be understood implicitly to be valid as expectation values)

$$\delta_\epsilon\Theta_{\text{boson}} \sim \bar{\epsilon}\Theta_{\text{fermion}}, \qquad \delta_\epsilon\Theta_{\text{fermion}} \sim \epsilon\Theta_{\text{boson}}. \tag{7.6}$$

By the assumption that $\dim \mathcal{M}_{\text{ext}} = d$ being a d-dimensional Minkowski space we have to decompose the spinor representations under $SO(1, d-1) \times SO(D-d)$, where D is the dimension of the full spacetime. Typically these will transform in a non-trivial representation under $SO(1, d-1)$ and hence have to vanish in order to preserve Poincaré symmetry of the vacuum. Thus in order to make sure that the vacuum preserve some amount of supersymmetry we need to find non-trivial solutions of $\delta_\epsilon\Theta_{\text{fermions}} = 0$. Note that if such a solution is found we also have to make sure that the equations of motion and Bianchi identities are fulfilled.

With a constant dilaton and vanishing fluxes the supersymmetry variation of the gravitino ψ_M (which is always present in supergravity) [28] is given by

$$\delta_\epsilon\psi_M = \nabla_M\epsilon = 0 \quad \Longleftrightarrow \quad \nabla_\mu\epsilon = 0, \qquad \nabla_m\epsilon = 0, \tag{7.7}$$

where we have used that there is no mixing between greek external indices and latin internal indices in the connection due to the metric being of a product form. A spinor satisfying these conditions is called a *Killing spinor*. From this one can easily show by taking a commutator of covariant derivatives that a necessary condition for the existence of a covariantly constant spinor is that the internal manifold is Ricci-flat (obviously also the external Minkowski space is Ricci-flat)

$$R_{mn} = 0. \tag{7.8}$$

Curvature of a manifold is closely connected to the way an object transforms upon parallel transport. In this spirit take some object v and parallel transport it around a closed loop on the manifold. Generally the object transforms in some representation of the holonomy group $\mathscr{H} \subseteq O(d)$ under parallel transport for a d-dimensional Riemannian manifold, since the norm does not change assuming a metric compatible connection. Likewise, a spinor parallel transported in a loop transforms in a representation of $\mathscr{H}$ as well and hence, especially, also the Killing spinor. But since the Killing spinor is covariantly constant it does not transform when parallel transported and needs to be a singlet under the holonomy group. Hence, a necessary condition for the existence of a covariantly constant spinor is related to the restriction of the holonomy group, which in turn put constraints on the metric. Such features as reduction of holonomy and structure group are conveniently described in terms of fibre bundles briefly explained below.

7.2.1 Mathematical interlude – Fibre bundles

A short introduction to the notion of fibre bundles are introduced in order to set the terminology. This will be convenient for more complicated compactifications later on as well for the discussion of generalised geometry. Just as a manifold is a topological space that looks locally like $\mathbb{R}^n$, i.e. there exists some coordinate chart on $U_i \subseteq \mathcal{M}$ such that $\phi : \mathcal{M} \to \mathbb{R}^n$, a bundle is a topological space that "looks locally" like a product of two topological spaces. Moreover, in the cases we are interested in these topological spaces will furthermore be differentiable manifolds.

A bundle is a triple $(P, \mathcal{M}, \pi)$, typically also denoted $P \xrightarrow{\pi} \mathcal{M}$, where P is the total space, $\mathcal{M}$ the base space and $\pi : P \to \mathcal{M}$ the projection, which is a surjection onto the base space. The notion of a bundle is a rather general concept and we will instead only be interested in (differentiable) fibre bundles. A (differentiable) fibre bundle $(P, \mathcal{M}, \pi, F, G)$ is defined by a bundle $(P, \mathcal{M}, \pi)$, a differentiable manifold F called the fibre and a Lie group G called the structure group. In order for a fibre bundle to "look locally" like a product manifold there exists patches on $U_i \subseteq \mathcal{M}$ diffeomorphisms and linear maps $\phi_i : U_i \times F \to \pi^{-1}(U_i)$ such that $\pi \circ \phi_i(u, f) = u$, where $u \in U_i$, $f \in F$ and $\pi^{-1}(U_i) = F_p \cong F$ is the fibre at $u \in U_i$. The maps ϕ_i are usually called local trivialisations since the map $\phi^{-1} \circ \pi^{-1}$ maps any U_i to $U_i \times F$ which "looks like a product manifold". In order for this to make sense there needs to be some compatibility on overlapping patches $U_i \cap U_j \neq \emptyset$. Take a point $p \in P$ such that $\pi(p) = u$ with $u \in U_i \cap U_j$. Consider then mapping $\phi_{u,i}$

$$
\begin{aligned}
\phi_{u,i} : F &\to F_u, \\
f &\mapsto f_u = \phi_i(u, f),
\end{aligned}
\tag{7.9}
$$

which assigns to each point in F an element in the fibre F_u at u. On the overlap we can look at

$$
\phi_j(u, f) = \phi_i(u, t_{ij}(f)),
\tag{7.10}
$$

where $t_{ij} : f \to f$ defined on $U_i \cap U_j$ are called transition functions. Composing with ϕ_i^{-1} we find

$$
\phi_i^{-1} \circ \phi_j(u, f) = (u, \phi_{i,u}^{-1} \circ \phi_{j,u} f),
\tag{7.11}
$$

and by the definition of the transition functions we find $t_{ij,u} = \phi_{i,u}^{-1} \circ \phi_{j,u}$. The compatibility requirement is then to demand that $t_{ij} : U_i \cap U_j \to R(G)$, with $R(G)$ some representation of G, such that fibres are patched together by the structure group G. This moreover defines a left action of G on the bundle.

A section of a bundle $P \xrightarrow{\pi} \mathcal{M}$ is defined as a map $\sigma : \mathcal{M} \to P$ such that $\pi \circ \sigma = \mathrm{id}_{\mathcal{M}}$. A well known example is a section of the tangent bundle, $T\mathcal{M}$, which is a bundle defined as

$$
TM = \bigcup_{x \in \mathcal{M}} \{(x, y) | x \in \mathcal{M}, y \in T_x\mathcal{M}\},
\tag{7.12}
$$

with $T_x\mathcal{M}$ being the tangent space of $\mathcal{M}$ at a point $x \in \mathcal{M}$. A section then associates a vector in the fibre $T_x\mathcal{M}$ to each point $x \in \mathcal{M}$

$$\begin{aligned} \sigma: \quad \mathcal{M} &\to \mathcal{M} \times T_x\mathcal{M} \\ x &\mapsto (x, X), \qquad X \in T_x\mathcal{M}, \end{aligned} \tag{7.13}$$

which is just an ordinary vector field on $\mathcal{M}$. Moreover, the space of sections of a bundle P is often denoted $\Gamma(P)$.

An important class of fibre bundles are principal fibre bundles. A principal fibre bundle (P,$\mathcal{M}$,π,G) is a fibre bundle with a typical fibre being the structure group G itself, i.e. for some $u \in U \subseteq \mathcal{M}$ we have $\pi^{-1}(u) \cong G$. Moreover, there is a right action of G on P as

$$\triangleleft : P \times G \to P, \tag{7.14}$$

which is free. An important example of a principle fibre bundle is the frame bundle. The frame bundle is a fibre bundle with a typical fibre consisting of all ordered basis of the tangent space. In any patch U_i on a d-dimensional manifold $\mathcal{M}$ there exist local coordinates which spans a basis $\{\partial_\mu\}_{\mu=1,2,\ldots d}$, and any other basis can be written as

$$e_\alpha = e_\alpha{}^\mu \partial_\mu, \tag{7.15}$$

with $\alpha = 1, 2, \ldots d$. The elements $e_\alpha{}^\mu$ can be seen as $d \times d$-dimensional matrices and thus the fibre is isomorphic to $GL(d)$.

Importantly we also want to consider tensor fields on a manifold $\mathcal{M}$ which is formally introduced by looking at associated vector bundles to a principle fibre bundle. Suppose that $(P, \pi, \mathcal{M}, G)$ is a principle fibre bundle and $(\rho, \mathbb{V})$ a G-representation. Then an associated vector bundle $P_\mathbb{V}$ is given by

$$P_\mathbb{V} = (P \times \mathbb{V})/ \sim_G, \tag{7.16}$$

with $\sim_G$ defined by the natural action of G on both P and $\mathbb{V}$. By construction elements of the associated vector bundle is thus invariant under the action of the structure group G, and the action of G on the first factor P can in some sense be considered as the inverse action of G on $\mathbb{V}$. As an example consider an element $v = v^\alpha e_\alpha$ in tangent bundle $v \in T\mathcal{M}$ which is an associated vector bundle, on which an element $\Lambda \in G$ acts as

$$v \mapsto \rho(\Lambda)^\alpha{}_\beta v^\beta e_\gamma \Lambda^\gamma{}_\alpha. \tag{7.17}$$

From invariance of v we find that $\rho(\Lambda)^\alpha{}_\beta = (\Lambda)^{-1\alpha}{}_\beta$.

What we mainly will be interested in is the possibility of finding so-called reduced structures or H-structures for some some closed subgroup $H \subset G$. A principal bundle $(P, \pi, \mathcal{M}, G)$ has an H-structure if it is possible to choose frames in P such that the structure group reduces to H. A typical example of this is an $O(d) \subset GL(d)$ structure on Riemannian manifolds. Due to the presence of a metric we can consider orthonormal frames $\tilde{e}_a$ such that

$$g(\tilde{e}_a, \tilde{e}_b) = \delta_{ab}, \tag{7.18}$$

with g being the metric. This defines an $O(d)$ structure since on overlapping patches orthonormal frames are related by $O(d)$ transformations. The importance of this in what will follow is the fact that finding a reduced structure is equivalent to the presence of a globally defined non-vanishing tensor on $\mathcal{M}$. Especially this tensor is invariant under some subgroup $H \subset G$. Since it is globally defined we can choose a frame $\{e_\alpha\}_{a=1,2,\dots d}$ such that the tensor is constant on overlapping patches which effectively reduce $GL(d)$ to H.

The notion of reduced structure group is a vast subject and we will not go into more details. Moreover, connections and the notion of holonomy is also best described in the language of fibre bundles. We refer the reader to [48] for further discussion about these topics.

7.2.2 Compactification on Calabi-Yau

Having reviewed the notion of reduced structure group and special holonomy a Calabi-Yau n-fold, CY_n, can be defined as $2n$-dimensional compact Riemannian manifolds with holonomy contained in $SU(n)$[28]. We will be particularly interested in CY_3 such that starting with $D = 10$ supergravity, or string theory, the external spacetime is four-dimensional.

Since there is a reduced holonomy group we know that for a CY_3 there exists some p-forms, or equivalently a Killing spinor, invariant under the holonomy group $\mathscr{H} = SU(3)$. This can be found by looking at the decomposition of one-, two- and three-forms of $Spin(6) \cong SU(4) \to SU(3)$

$$\begin{aligned}
\mathbf{6} &\to \mathbf{3} + \bar{\mathbf{3}}, \\
\mathbf{15} &\to \mathbf{8} + \mathbf{3} + \bar{\mathbf{3}} + \mathbf{1}, \\
\mathbf{20} &\to \mathbf{6} + \bar{\mathbf{6}} + \mathbf{3} + \bar{\mathbf{3}} + 2 \cdot \mathbf{1}.
\end{aligned}$$

It thus exist a one-form ω and two three-forms (these actually combine into a complex three-form) Ω that are singlets under $\mathscr{H}$. This pair of nowhere vanishing tensors defines a *structure*. Note that we already know that such a structure is equally well defined by a nowhere vanishing globally defined spinor $\mathbf{4}$ of $SU(4)$ which decomposes under $\mathscr{H}$ as

$$\mathbf{4} \to \mathbf{3} + \mathbf{1}. \tag{7.19}$$

Explicitly, a Weyl-spinor ψ of $Spin(1,9)$ decomposed under $Spin(1,3) \times Spin(6)$ can be parameterised as

$$\psi = \eta_+ \otimes \epsilon_+ + \eta_- \otimes \epsilon_-, \tag{7.20}$$

where $\eta_\pm$ and $\epsilon_\pm$ are Weyl-spinors with chirality ± 1 under $Spin(1,3)$ and $Spin(6)$ respectively. Morever, imposing that ψ is a Majorana-Weyl spinor in ten dimensions further imply that η_+ is the charge conjugate of η_- and likewise for $\epsilon_\pm$.

In what follows we will mainly consider structures defined by some bosonic forms, such as (ω, Ω), but from the observation above we know that this corresponds to the existence of a globally defined nowhere vanishing spinor. To preserve supersymmetry such a spinor also satisfies a differential condition and, hence, also the structure (ω, Ω) needs to satisfy certain differential conditions. In fact, the structure (ω, Ω) is related to the Killing spinor as

$$\omega_{mn} = \mp \mathrm{i} \epsilon_\pm^\dagger \gamma_{mn} \epsilon_\pm, \tag{7.21}$$

$$\Omega_{mnp} = -\mathrm{i} \epsilon_-^\dagger \gamma_{mnp} \epsilon_+. \tag{7.22}$$

Moreover, in order for the structures to be compatible they should satisfy the compatibility conditions

$$\omega \wedge \Omega = 0 \qquad \omega \wedge \omega \wedge \omega = \frac{3\mathrm{i}}{4}\Omega \wedge \bar{\Omega}. \tag{7.23}$$

Taking the covariant derivative of these definitions we find that solving the Killing spinor equations in terms of the structure (ω, Ω) is simply the statement that they are both closed forms

$$\mathrm{d}\omega = 0, \qquad \mathrm{d}\Omega = 0. \tag{7.24}$$

7.3 Flux compactifications and generalised geometry

In the case of vanishing flux we have seen that the Killing spinor equation led to special holonomy, which in turn put restrictions on the connection of the internal manifold. In the presence of fluxes the Killing spinor is no longer covariantly constant, however, the existence of globally defined nowhere vanishing spinors still imply a reduced structure group. This is summarised by

$$\begin{aligned} \text{vanishing flux} &\Longleftrightarrow \text{special holonomy}, \\ \text{non-zero flux} &\Longleftrightarrow \text{reduced structure group}. \end{aligned}$$

In the presence of fluxes the Killing spinor equations of type II supergravity are given by [28]

$$\begin{aligned} \delta_\epsilon \psi_M &= \left(\nabla_M + \frac{1}{8}\Gamma^{NP} H_{MNP} \mathscr{P} + \frac{1}{16}\mathrm{e}^{\Phi} \sum_n \not{F}_n \Gamma_M \mathscr{P}_n \right) \epsilon = 0, \\ \delta_\epsilon \lambda &= \left(\not{\partial}\Phi + \frac{1}{12}\Gamma^{MNP} H_{MNP} \mathscr{P} + \frac{1}{8}\mathrm{e}^{\Phi} \sum_n (-1)^n (5-n) \not{F}_n \mathscr{P}_n \right) \epsilon = 0, \end{aligned} \tag{7.25}$$

where $\mathscr{P}$ and $\mathscr{P}_n$ are 2×2-matrices, $\not{F}_n = \frac{1}{n!}\Gamma^{M_1 M_2 \ldots M_n} F_{M_1 M_2 \ldots M_n}$ and $n \in \{0, 1, \ldots, 10\}$ are even(odd) for type IIA(B). In the case of vanishing flux and a constant dilation this reduces to a covariantly constant spinor ϵ as discussed above. We thus see that the presence of gauge fluxes are according to (7.25) obstructions to the Killing spinor being covariantly constant with respect to the Levi-Civita connection.

The obstruction to the Killing spinor being covariantly constant is conveniently analysed in terms of *intrinsic torsion*. It is always possible to construct a connection ∇ with torsion, such that the Killing spinor is covariantly constant with respect to that connection. Consider therefore a connection $\nabla = \nabla' - \kappa$ such that $\nabla_n V_m = \partial_n V_m - {\Gamma'}_{nm}{}^p V_p - \kappa_{nm}{}^p V_p$, where ∇' is the Levi-Civita connection and κ is the so-called contorsion tensor. Generically the connection coefficents $\Gamma_{mn}{}^p$ take values in $\Lambda^1 \otimes \mathfrak{g}$, where Λ^1 is the one-form representation of $G = GL(d)$ and $\mathfrak{g} = \mathfrak{gl}(d)$ is the corresponding Lie algebra. Assuming metric compatibility fixes the part in $\mathfrak{gl}(d)/\mathfrak{so}(d)$ as we saw in Section 6.4 of the pair ${}_m{}^p$. This is contained in the Levi-connection but the Levi-Civita connection contain further terms that ensures vanishing torsion, $\Gamma_{[mn]}{}^p = 0$. This implies that the contorsion take values in

$$\kappa_{mn}{}^p \in \Lambda_1 \otimes \mathfrak{h}, \tag{7.26}$$

where $\mathfrak{h} = \mathfrak{so}(d)$ is the maximal compact subalgebra of $\mathfrak{gl}(d)$. Assuming a reduced structure group $G' \subseteq SO(d)$ implies that we can separate the contorsion into two parts

$$\kappa = \kappa_{\text{int}} + \kappa_0, \qquad \text{where} \quad \kappa_0 \in \Lambda_1 \otimes \mathfrak{g}' \quad \text{and} \quad \kappa_{\text{int}} \in \Lambda_1 \otimes \mathfrak{g}'^{\perp}. \tag{7.27}$$

The point now being that since the Killing spinor is a singlet under $\mathfrak{g}'$, the failure of the Killing spinor to be covariantly constant *with respect to the Levi-Civita connection* is given by κ_{int}, i.e. the intrinsic torsion. The part κ_{int} can further be decomposed into irreducible representations of $\mathfrak{g}'$ called torsion classes. As an example take $G = GL(6)$ and $G' = SU(3)$. Under G' we have the following decompositions $\Lambda_1 \to \mathbf{3} \oplus \bar{\mathbf{3}}$, $\mathfrak{g}' = \mathbf{8}$ and $\mathfrak{g}'^{\perp} = \mathbf{1} \oplus \mathbf{3} \oplus \bar{\mathbf{3}}$ and

$$\kappa_{\text{int}} \in \bigoplus_{i=1}^{5} \mathscr{W}_i, \tag{7.28}$$

where $\mathscr{W}_i$ are the five torsion classes given by

$$\begin{aligned}
\mathscr{W}_1 &= \mathbf{1} \oplus \mathbf{1}, \\
\mathscr{W}_2 &= \mathbf{8} \oplus \mathbf{8}, \\
\mathscr{W}_3 &= \mathbf{6} \oplus \bar{\mathbf{6}}, \\
\mathscr{W}_4 &= \mathbf{3} \oplus \bar{\mathbf{3}}, \\
\mathscr{W}_5 &= \mathbf{3} \oplus \bar{\mathbf{3}}.
\end{aligned}$$

Using complex geometry and cohomology one can show, see [28] for details, that the exterior derivative of the structure (ω, Ω) can be written in terms of torsion classes as

$$\mathrm{d}\omega = \frac{3}{2}\mathrm{Im}(\overline{\mathscr{W}}_1\Omega) + \mathscr{W}_4 \wedge \omega + \mathscr{W}_3 \tag{7.29}$$

and

$$\mathrm{d}\Omega = \mathscr{W}_1\omega \wedge \omega + \mathscr{W}_2 \wedge \omega + \overline{\mathscr{W}}_5 \wedge \Omega. \tag{7.30}$$

We thus see that all five torsion classes vanish for CY_3 in order for (ω, Ω) to be closed. Moreover, we have seen explicitly that the presence of intrinsic torsion corresponds to compactifications with fluxes, which in turn is an obstruction to finding a covariantly constant Killing spinor.

7.3.1 Generalised geometry

We have seen above that demanding some preserved supersymmetry in fluxless compactifications led to two constraints:

- the existence of globally defined spinors led to a reduced structure group characterised by (ω, Ω),
- imposing the Killing spinor equation led to a reduced holonomy.

Including fluxes alters the Killing spinor equation and the holonomy group need no longer be reduced. This effectively weakens the constraints of the internal manifold which at the same time make things more complicated. In the spirit of extended geometries we will try to generalise the notion of geometry in order to accommodate flux degree of freedoms in terms of what is called (exceptional) *generalised geometry*. The reason for doing this being that it is then possible to define an appropriate generalisation of the holonomy, or rather of the structure, together with some integrability conditions. This will imply that a supersymmetric compactification should have vanishing so-called generalised intrinsic torsion. Especially, since gauge degrees of freedom are included in the geometry, this holds true even when non-trivial fluxes are present. Below we will briefly introduce (exceptional) generalised geometry needed for discussing flux compactifications and generalised structures, for a complete introduction to this field we refer to the literature [45, 46, 47, 49].

In extended geometries we enlarged spacetime by introducing coordinates in some module of the underlying structure group G, and imposed a section constraint specifying a $GL(d) \hookrightarrow G$ embedding. Generalised geometry can be considered as an extended geometry with a global solution to the section constraint and a structure group given by the "geometric" subgroup of G as we will see.

Therefore consider a manifold $\mathcal{M}$ and its associated tangent bundle $T\mathcal{M}$. A basis of the tangent space in local coordinates is simply given by $\{\partial/\partial x^i\}_{i=1,\ldots d}$, where $d = \dim \mathcal{M}$. Now recall that in DFT space-time is enlarged to $2d$-dimensions by letting $x^i \to (x^i, \tilde{x}_i)$, where the latter is the vector representation of $O(d,d)$. In generalised geometry one instead considers only a generalised tangent bundle on which, in the DFT case, $O(d,d) \times \mathbb{R}^+$ acts naturally and, in the exceptional case, $E_{d(d)} \times \mathbb{R}^+$ acts naturally. In the former case the generalised tangent bundle is isomorphic to a sum of the tangent bundle and the cotangent bundle

$$E \cong T\mathcal{M} \oplus T^*\mathcal{M}, \tag{7.31}$$

i.e. at a point $x \in \mathcal{M}$ the typical fibre is isomorphic to the formal sum of a vector and a one-form. These are precisely the parameters for ordinary diffeomorphisms and the NS-NS sector two-form gauge transformations.

There exists a natural inner product on sections $X^I = \xi + \lambda$, $Y^I = \eta + \rho$, where $X, Y \in \Gamma(TM \oplus T^*\mathcal{M})$, given by

$$(X, Y) = \frac{1}{2}\left(\rho(x) + \lambda(y)\right), \tag{7.32}$$

which also can be written as $\eta_{MN} X^M Y^N$ with X^M, Y^N being the components of X, Y respectively and

$$\eta_{MN} = \frac{1}{2}\begin{pmatrix} 0 & \mathbf{1} \\ \mathbf{1} & 0 \end{pmatrix}. \tag{7.33}$$

This inner product is manifestly invariant under $SO(d,d)$ transformations parameterised by elements in the algebra

$$O = \begin{pmatrix} A & \beta \\ B & -A^T \end{pmatrix}, \tag{7.34}$$

with $A \in \mathfrak{gl}(d)$, an anti-symmetric two-vector $\beta = \beta^{ij}$ and B a two-form.

In order to include the presence of the NS-NS two-form B we will look at "twisted" elements. These are sections of E that provide explicitly the isomorphism stated between an element $\tilde{X} \in \Gamma(E)$ and $X = v + \lambda \in \Gamma(TM \oplus T^*\mathcal{M})$, and is given by

$$\tilde{X} := \mathrm{e}^B X = v + (\lambda + \iota_v B), \tag{7.35}$$

with $\iota_v B(y) = B(v,y)$ denoting the interior product. The two-form B is not globally defined, instead on overlapping patches $U_i \cap U_j \neq 0$ it is possibly shifted by an exact two-form

$$B_{(i)} - B_{(j)} = \mathrm{d}\lambda_{(ij)}, \tag{7.36}$$

with $\lambda_{(ij)}$ a one-form. This means that the twisted vectors are patched together as

$$\tilde{X}_{(i)} - \tilde{X}_{(j)} = \iota_v \mathrm{d}\lambda_{(ij)}. \tag{7.37}$$

Patching of twisted vectors are therefore done by elements in the geometric subgroup $GL(d) \ltimes \Lambda^2_{\text{cl}}(\mathcal{M})$, with $\Lambda^2_{\text{cl}}(\mathcal{M})$ being closed two-forms on $\mathcal{M}$. There are some further consistency conditions which implies that this specifies a "gerbe structure" [50], but this is outside the scope of the thesis and will not be needed. The infinitesimal action of a generalised diffeomorphisms, with the structure group $O(d,d) \times \mathbb{R}^+$, are parameterised by elements in $Y = w + \rho \in \Gamma(E)$ using the generalised Lie derivative on $X = x + \lambda \in \Gamma(E)$ as

$$\mathscr{L}_Y X^M = V^N \partial_N X^M - (\partial \times_{\text{ad}} V)^M{}_N X^N. \tag{7.38}$$

Here we embedded the ordinary derivative in ∂_M as $\partial_M = (\partial_i, 0)$, and we denoted $(V \times D)_{\text{ad}}$ the projection on the adjoint of $\overline{E} \times E$ as $(D \times_{\text{ad}} V)^M{}_N = D_N V^M - \eta_{NP}\eta^{MS} D_S V^P$. This agrees with the definition of the generalised Lie derivative introduced in DFT.

7.3.2 Exceptional generalised geometry

In the same spirit we discuss exceptional generalised geometry ($d \leq 7$) based on $E_{d(d)} \times \mathbb{R}^+$ and a generalised tangent bundle

$$E \cong T\mathcal{M} \oplus T^*\mathcal{M} \oplus \ldots, \tag{7.39}$$

where $\ldots$ denote extra terms coming from the decomposition of the coordinate module of $E_{d(d)}$ under the $GL(d,\mathbb{R})$ subgroup. As an example consider $\dim \mathcal{M} = 7$, then the coordinate module is the fundamental $\mathbf{56}$ module of $E_{7(7)}$ which decompose under $GL(7,\mathbb{R})$ as

$$\mathbf{56} \to \mathbf{7} + \overline{\mathbf{7}} + \mathbf{21} + \overline{\mathbf{21}}, \tag{7.40}$$

corresponding to a vector, a one-form, a two-form and a five-form. The generalised tangent bundle E is then given by

$$E \cong T\mathcal{M} \oplus \Lambda^2 T^*\mathcal{M} \oplus \Lambda^5 T^*\mathcal{M} \oplus \left(T^*\mathcal{M} \otimes \Lambda^7 T^*\mathcal{M}\right). \tag{7.41}$$

This is the general form for E for $d \leq 7$ with the last term absent if $d < 7$. A section $X \in \Gamma(E)$ on a patch U_i is given by

$$X = v + \omega + \sigma + \tau, \tag{7.42}$$

with $v \in \Gamma(TU_i)$, $\omega \in \Gamma(\Lambda^2 T^* U_i)$, $\sigma \in \Gamma(\Lambda^5 T^* U_i)$ and $\tau \in \Gamma(T^* U_i \otimes \Lambda^7 T^* U_i)$. Likewise the adjoint of $\mathfrak{e}_{7(7)} \times \mathbb{R}$ decompose as

$$\mathbf{133} \to \mathbf{1} \oplus \mathbf{48} \oplus \mathbf{7} \oplus \overline{\mathbf{7}} \oplus \mathbf{35} \oplus \overline{\mathbf{35}}. \tag{7.43}$$

The corresponding adjoint tangent bundle is therefore given by

$$\mathrm{ad}\, F \cong \mathbb{R} \oplus (T^*\mathcal{M} \otimes T\mathcal{M}) \oplus \Lambda^3 T^*\mathcal{M} \oplus \Lambda^6 T^*\mathcal{M} \oplus \Lambda^3 T\mathcal{M} \oplus \Lambda^6 T\mathcal{M}, \tag{7.44}$$

with a section $W \in \Gamma(\operatorname{ad} F)$ on a patch U_i given by

$$W = l + r + a + \tilde{a} + \alpha + \tilde{\alpha}, \tag{7.45}$$

with $l \in \mathbb{R}$, $r \in (T^*U_i \otimes TU_i)$, $a \in \Lambda^3 T^*U_i$, $\tilde{a} \in \Lambda^6 T^*U_i$, $\alpha \in \Lambda^3 TU_i$ and $\tilde{\alpha} \in \Lambda^6 TU_i$. Note especially the presence of a three-form and a six-form which corresponds to the three-form and the dualised six-form of M-theory. Moreover, a section of $\Gamma(E)$ contains a vector, a one-form, a two-form and a five-form which parameterise diffeomorphisms and gauge transformations of these form fields.

As in the introduction to $O(d,d)$ generalised geometry above, sections of E and $\operatorname{ad} F$ are twisted by the form fields of the adjoint bundle, providing the isomorphism stated above. Explicitly, a section $\tilde{X} \in \Gamma(E)$ of the generalised tangent bundle is twisted by the form fields in the adjoint bundle as

$$\tilde{X} = \mathrm{e}^{a+\tilde{a}} X, \tag{7.46}$$

with an action of $E_{7(7)} \times \mathbb{R}^+$ defined below. Elements in the generalised tangent bundle are therefore patched together on $U_{(i)} \cap U_{(j)} \neq 0$ as

$$X_{(i)} = \mathrm{e}^{\mathrm{d}\Lambda_{(ij)} + \mathrm{d}\tilde{\Lambda}_{(ij)}} X_{(j)}, \tag{7.47}$$

with a three-form $\mathrm{d}\Lambda_{(ij)}$ and a six-form $\mathrm{d}\tilde{\Lambda}_{(ij)}$ corresponding to the gauge symmetry of the three-form and six-form potential in M-theory respectively.

There is a natural action of $E_{d(d)} \times \mathbb{R}^+$, given in e.g. [16], on a section $X \in \Gamma(E)$ as $E' = W \cdot E$ which in components is given by

$$\begin{aligned}
v' &= lv + r \cdot v + \alpha \lrcorner \omega - \tilde{\alpha} \lrcorner \sigma, \\
\omega' &= lv + r \cdot \omega + v \lrcorner a + \alpha \lrcorner \sigma + \tilde{\alpha} \lrcorner \tau, \\
\sigma' &= l\sigma + r \cdot \sigma + v \lrcorner \tilde{a} + a \wedge \omega + \alpha \lrcorner \tau, \\
\tau' &= l\tau + r \cdot \tau - j\tilde{a} \wedge \omega + ja \wedge \sigma,
\end{aligned} \tag{7.48}$$

with the definitions

$$\begin{aligned}
(v \wedge v')^{i_1 i_2 \ldots i_{n+n'}} &:= \frac{(n+n')!}{n!n'!} v^{[i_1 i_2 \ldots} v'^{i_{n+1} \ldots i_{n'}]}, \\
(\omega \wedge \omega')_{i_1 \ldots i_{n+n'}} &:= \frac{(n+n')!}{n!n'!} \omega_{[i_1 i_2 \ldots} \omega'_{i_{n+1} \ldots i_{n+n'}]}, \\
(v \lrcorner \omega)_{i_1 \ldots i_{n'-n}} &:= \frac{1}{n!} v^{j_1 j_2 \ldots j_n} \omega_{j_1 j_2 \ldots j_n i_1 \ldots i_{n'-n}} \qquad \text{if } n < n', \\
(v \lrcorner \omega)^{i_1 \ldots i_{n'-n}} &:= \frac{1}{n'!} v^{j_1 j_2 \ldots j_{n'} i_{n'+1} \ldots i_{n-n'}} \omega_{j_1 j_2 \ldots j_{n'}} \qquad \text{if } n' < n, \\
(jv \lrcorner j\omega)^i{}_j &:= \frac{1}{(n-1)!} v^{ik_1 k_2 \ldots k_{n-1}} \omega_{jk_1 k_2 \ldots k_{n-1}}, \\
(j\omega \wedge \omega')_{i, i_1 i_2 \ldots i_d} &:= \frac{d!}{(n-1)!(d+1-n)!} \omega_{i[i_1 i_2 \ldots i_{n-1}} \omega'_{i_n i_{n+1} \ldots i_d]}.
\end{aligned} \tag{7.49}$$

Moreover, the $\mathfrak{gl}(d)$ subalgebra parameterised by an element r acts naturally as $r \cdot v^i = r^i{}_j v^i$, $r \cdot \omega_{ij} = -r^k{}_i \omega_{kj} - r^k{}_j \omega_{ik}$ with obvious extensions to other fields. The adjoint action

$W'' = [W, W']$ with elements $W \in \Gamma(\mathrm{ad}\, F)$ and $W' \in \Gamma(\mathrm{ad}\, F)$ is further given by

$$\begin{aligned}
l'' &= \frac{1}{3}(\alpha \lrcorner a' - \alpha' \lrcorner a) + \frac{2}{3}(\tilde{\alpha}' \lrcorner \tilde{a} - \tilde{\alpha} \lrcorner \tilde{a}'), \\
r'' &= r \cdot r' + j\alpha \lrcorner j a' - j\alpha' \lrcorner j a - \frac{1}{3}\mathbf{1}(\alpha \lrcorner a' - \alpha' \lrcorner a) \\
&\quad + j\tilde{\alpha}' \lrcorner j\tilde{a} - j\tilde{\alpha} \lrcorner j\tilde{a}' - \frac{2}{3}\mathbf{1}(\tilde{\alpha}' \lrcorner \tilde{a} - \tilde{\alpha} \lrcorner \tilde{a}'), \\
a'' &= r \cdot a' - r' \cdot a + \alpha' \cdot \tilde{a} - \alpha \lrcorner \tilde{a}' \\
\tilde{a}'' &= r \cdot \alpha' - r' \cdot \alpha - a \wedge a', \\
\alpha'' &= r \cdot \alpha' - r' \cdot \alpha + \tilde{\alpha}' \lrcorner a - \tilde{\alpha} \lrcorner a', \\
\tilde{\alpha}'' &= r \cdot \tilde{\alpha}' - r' \cdot \tilde{\alpha} - \alpha \wedge \alpha'.
\end{aligned} \tag{7.50}$$

Just as in the $O(d,d)$ generalised case, as well as extended geometries, we introduce a generalised Lie derivative by the now well-known structure

$$\mathscr{L}_Y X^M := Y^N \partial_N X^M - (\partial_N \times_{\mathrm{ad}} Y)^M{}_N X^N, \tag{7.51}$$

where we again embedded the ordinary partial derivative as $\partial_M = (\partial_i, 0, 0, 0)$ in obvious notation. Moreover, $\times_{\mathrm{ad}}$ denotes the projection of $\overline{E} \otimes E$ to the adjoint. Using $(\partial \times_{\mathrm{ad}} X) = \partial \times v + \mathrm{d}\omega + \mathrm{d}\sigma$ it is easily found that

$$\begin{aligned}
\mathscr{L}_X X' =& L_v v' + (L_v \omega - v' \lrcorner \mathrm{d}\omega) + (L_v \sigma' - v' \lrcorner \mathrm{d}\sigma - \omega' \wedge \mathrm{d}\omega) \\
&+ (L_v \tau' - j\omega' \wedge \mathrm{d}\sigma - j\sigma' \wedge \mathrm{d}\omega) \\
=& L_v X' - W \cdot X',
\end{aligned} \tag{7.52}$$

with L_v being the ordinary Lie derivative with respect to the $GL(7)$ vector component v and $W = \mathrm{d}\omega + \mathrm{d}\sigma \in \Gamma(\mathrm{ad}\, F)$.

7.4 Generalised structure

The bosonic sector of M-theory is given by the metric g_{MN} and the three-form form field A, and its corresponding four-form field strength $F = \mathrm{d}A$. We consider a warped product ansatz for the metric

$$\mathrm{d}s^2 = \mathrm{e}^{2A(y)} \mathrm{d}s^2(\mathbb{R}^{1,D-1}) + \mathrm{d}s^2(\mathcal{M}), \tag{7.53}$$

where x, y are coordinates on $\mathbb{R}^{1,D-1}$ and $\mathcal{M}$ respectively, and A depends only on the internal manifold to preserve external Poincaré invariance. The non-trivial fluxes will be chosen to lie entirely in $\mathcal{M}$, again to preserve the symmetries in the external spacetime. For simplicity we will consider the specific example of $D = 4$, such that the internal manifold is seven-dimensional following the work of [16]. The generalised tangent space with respect to an $E_{7(7)} \times \mathbb{R}^+$ exceptional generalised geometry is then given by

$$E \cong T\mathcal{M} \oplus \Lambda^2 T^*\mathcal{M} \oplus \Lambda^5 T^*\mathcal{M} \oplus \left(T^*\mathcal{M} \otimes \Lambda^7 T^*\mathcal{M}\right). \tag{7.54}$$

In ordinary geometry the reduction of the structure group to some G-structure, where G is a subgroup of the structure group, proved fruitful to characterise geometries preserving some amount of supersymmetry. The goal is therefore to again find "reduced structures", this time for some $G \subset E_{7(7)}$ subgroup. This is done by looking for tensors invariant under the action of G, or in other words, finding singlets when decomposing modules of $E_{7(7)}$ under G.

To this end decompose the adjoint representation **133** of $E_{7(7)}$ under $SU(2)\times Spin^*(12) \subset E_{7(7)}$, with $Spin^*(12)$ a specific real form of the double cover of $SO(12)$,

$$\mathbf{133} \to (\mathbf{1}, \mathbf{66}) + (\mathbf{2}, \mathbf{32}) + (\mathbf{3}, \mathbf{1}). \tag{7.55}$$

We thus find a triplet of $E_{7(7)}$ tensors which are invariant under $Spin^*(12)$. Such a triplet, J_α, of invariant tensors define a structure, called the hyper-multiplet structure, and are sections of the adjoint bundle

$$J_\alpha \in \Gamma(\mathrm{ad}\, F \otimes (T^*\mathcal{M})^{1/2}). \tag{7.56}$$

This hyper-multiplet structure then satisfies the $\mathfrak{su}(2)$ algebra

$$[J_\alpha, J_\beta] = 2\rho\epsilon_{\alpha\beta\gamma}J_\gamma, \tag{7.57}$$

with $\rho \in \det T^*\mathcal{M}$. Note that, as in the case of vanishing flux, the hyper-multiplet structure J_α can be written in terms of the Killing spinors. Before moving on to the integrability condition that the hyper-multiplet structure should satisfy we look for another structure. Decomposing the fundamental **56** of $E_{7(7)}$ under $E_{6(2)}$ we find two more singlets,

$$\mathbf{56} \to \mathbf{27} + \overline{\mathbf{27}} + 2 \cdot \mathbf{1}. \tag{7.58}$$

The two singlets in this decomposition V and $\hat{V}$ defines the vector-multiplet structure, and hence also an $E_{6(2)}$ sub-bundle. There is a further requirement that the singlet V is chosen such that

$$q(V) > 0, \tag{7.59}$$

where q is the quartic invariant of E_7. This ensures that the stabiliser group is precisely the real form $E_{6(2)}$ [51]. The structure $\hat{V}$ is further defined in terms of V as

$$s(Y, \hat{V}) = \frac{2}{\sqrt{q(V)}} q(Y, V, V, V), \tag{7.60}$$

with $s(\cdot, \cdot)$ being the symplectic invariant of $E_{7(7)}$ and this should be valid for any section $Y \in \Gamma(E)$. It is also convenient to combine these into a complex object as $X := V + \mathrm{i}\hat{V}$.

Just as in ordinary Calabi-Yau there are certain consistency conditions between the invariant tensors that defines the structure. Specifically, this further restricts the structure group to their overlap defining a $SU(6) = E_{6(2)} \cap Spin^*(12)$ structure [13]. The compatibility condition between the hyper-multiplet J_α and the vector-multiplet $X = V + \mathrm{i}\hat{V}$ is given by

$$\begin{aligned} J_+ \cdot X = J_- \cdot X = 0 \qquad &\Longleftrightarrow \qquad J_\alpha \cdot V = 0 \\ \frac{1}{2}\mathrm{i}s(X, \overline{X}) = \rho^2 \qquad &\Longleftrightarrow \qquad \kappa(J_\alpha, J_\beta) = -2\sqrt{q(V)}\delta_{\alpha\beta} \end{aligned} \tag{7.61}$$

with $\rho \in \det T^*\mathcal{M}^{\frac{1}{2}}$ and $J_\pm = J_1 \pm \mathrm{i}J_2$.

7.4.1 Integrability of structures

Integrability of the hyper-multiplet structure and of the vector-multiplet structure are in analogy to the fluxless case given by differential conditions on the corresponding invariant tensors. In other words, these differential conditions are necessary for the Killing spinor satisfying the Killing spinor equation, which in turn is the condition for preserving a certain amount of supersymmetry. In the case of vanishing flux this was related to vanishing intrinsic torsion, i.e. given a G-structure the existence of a G-compatible torsion-free connection implied that the structure was integrable. Motivated by this analogy we will look at the conditions of vanishing generalised intrinsic torsion.

7.4.1.1 Generalised intrinsic torsion

To define generalised intrinsic torsion we first have to define a covariant derivative, D, as follows

$$D_M \cdot V^N = \partial_M V_N + \Gamma_{MP}{}^N V^P, \tag{7.62}$$

where $V \in \Gamma(E)$, $\Gamma_{MP}{}^N \in \Gamma(E^* \otimes \mathrm{ad}\, F)$ and $\partial_M = (\partial_m, 0, 0, 0)$. Imposing Leibniz rule

$$D_M(fV) = (D_M f) \otimes V + f(D_M V), \tag{7.63}$$

the action of a covariant derivative is easily extended to sections of any associated vector bundle, i.e. to any other generalised tensor field, analogous to the covariant derivative in Section 6.4. The torsion of such a connection is then defined by

$$\mathscr{T}(V) \cdot X = L^D_V X - L_V X, \tag{7.64}$$

where X is an arbitrary generalised tensor field, L^D_V is the generalised Lie derivative with the replacement $\partial \to D$ and $\cdot$ denotes the adjoint action. Note that this definition agrees with the one presented in extended geometries, i.e. torsion is the part of the connection that transforms homogeneously under generalised diffeomorphisms, at least when ancillary transformations are absent. In the $E_{7(7)}$ case we already know from Section 6.4.1, that the torsion $\mathscr{T}$ of the $E_{7(7)}$ covariant derivative are given by

$$\mathscr{T} \in \mathbf{56} \oplus \mathbf{912}. \tag{7.65}$$

We define a G-compatible connection as one that preserves G-invariant tensors defining the structure. As an example an $E_{6(2)}$ structure were defined by an $E_{6(2)}$ invariant tensor $V \in \Gamma(E)$, the connection D is then $E_{6(2)}$ compatible if $DV = 0$. Assuming a connection D is compatible with the G-structure any other compatible connection is given by $D \to D + \Sigma$, with

$$\Sigma \in \Gamma(E^* \otimes \mathrm{ad}\, P_G), \tag{7.66}$$

where $\mathrm{ad}\, P_G$ has a typical fibre $\mathrm{Lie}\, G = \mathfrak{g}$. Given a G-compatible connection with torsion we define the intrinsic torsion as the part of torsion that can not be removed by adding a term Σ. With this definition it is clear that the presence of intrinsic torsion is an obstruction to finding a G-compatible torsion-free connection. This is analogous to the appearence of intrinsic torsion introduced for ordinary geometry.

Another way to put it is to define a map ϕ that projects Σ onto its torsion part. The intrinsic torsion is then given by

$$W_{\text{int}} = W/\operatorname{im}\phi, \tag{7.67}$$

with W_{int} and W being the space of intrinsic torsion and the space of torsion respectively. For more discussion about this see [16].

To clarify this take our main example with $E_{7(7)} \times R^+$ and $G = Spin^*(12) \times SU(2)$. To see what parts are intrinsic torsion decompose the torsion $\mathbf{56} \oplus \mathbf{912}$ under G to find

$$\mathbf{56} \oplus \mathbf{912} \to 2 \cdot (\mathbf{12}, \mathbf{2}) + (\mathbf{32}, \mathbf{1}) + (\mathbf{32}, \mathbf{3}) + (\mathbf{220}, \mathbf{2}) + (\mathbf{352}, \mathbf{1}). \tag{7.68}$$

Likewise we decompose Σ under G using $\mathbf{56} \to (\mathbf{12}, \mathbf{2}) + (\mathbf{32}, \mathbf{1})$

$$\begin{aligned}((\mathbf{12}, \mathbf{2}) + (\mathbf{32}, \mathbf{1})) \times (\mathbf{66}, \mathbf{1}) =& (\mathbf{220}, \mathbf{2}) + (\mathbf{12}, \mathbf{2}) + (\mathbf{560}, \mathbf{2}) + (\mathbf{22}, \mathbf{1}) \\ &+ (\mathbf{1728}, \mathbf{1}) + (\mathbf{352}, \mathbf{1}).\end{aligned} \tag{7.69}$$

The two big modules containing $\mathbf{1728}$ and $\mathbf{560}$ are not torsion and, hence, we find that the intrinsic torsion are given by

$$W_{\text{int}} = (\mathbf{12}, \mathbf{2}) + (\mathbf{32}, \mathbf{3}). \tag{7.70}$$

Performing the same analysis for $G = E_{6(2)}$ is straightforward, especially, the torsion decompose as

$$\mathbf{56} + \mathbf{912} \to \mathbf{1} + 2 \cdot \mathbf{27} + \mathbf{78} + \mathbf{351} + \text{c.c.} \tag{7.71}$$

We also have

$$(\mathbf{1} + \mathbf{27} + \text{c.c.})) \times \mathbf{78} = \mathbf{78} + \mathbf{27} + \mathbf{351} + \mathbf{1728} + \text{c.c.} \tag{7.72}$$

We thus find the intrinsic torsion in the $E_{6(2)}$ case to be

$$W_{\text{int}}^{E_{6(2)}} = \mathbf{1} + \mathbf{27} + \text{c.c.} \tag{7.73}$$

Consider the definition of an $E_{6(2)}$ compatible covariant derivative $DV = 0$. From the definition of torsion it is easily found that

$$\begin{aligned}\mathscr{L}_V V^M &= \mathscr{L}_V^D V^M - \mathscr{T} \cdot V^M \\ &= \underbrace{V^N D_N V^M}_{=0} - \underbrace{D_N V^M V^N}_{=0} - \mathscr{T} \cdot V^M.\end{aligned} \tag{7.74}$$

By definition of intrinsic torsion we can split $\mathscr{T} = \mathscr{T}_{\text{int}} + \mathscr{T}_0$, with $\mathscr{T}_{\text{int}} \in \mathbf{56} \otimes (\mathfrak{e}_{7(7)} \times \mathbb{R}/\mathfrak{e}_{6(2)})$ and $\mathscr{T}_0 \in \mathbf{56} \otimes \mathfrak{e}_{6(2)}$, and since V is invariant under $\mathfrak{e}_{6(2)}$ we find

$$\mathscr{L}_V V = -\mathscr{T}_{\text{int}} \cdot V. \tag{7.75}$$

Demanding that the intrinsic torsion vanish thus gives the integrability condition on the vector-multiplet structure

$$\mathscr{L}_V V = 0. \tag{7.76}$$

This is further equivalent to $\mathscr{L}_X \overline{X} = 0$.

It was further shown in [16] that the vanishing of intrinsic torsion and, hence integrability of the generalised structure, is equivalent to vanishing of so-called *momentum maps*. For the hyper-multiplet structure these are given by a triplet of momentum maps, μ_α, acting on any element $Y \in \Gamma(E)$ as

$$\mu_\alpha(Y) := -\frac{1}{2}\epsilon_{\alpha\beta\gamma}\int_{\mathcal{M}} \kappa(J_\beta, \mathscr{L}_Y J_\gamma), \tag{7.77}$$

with κ being the $\mathfrak{e}_{7(7)}$ Killing form. The integrability is then the requirement that the triplet of momentum maps vanish for any element in the generalised tangent bundle,

$$\mu_\alpha(Y) = 0 \qquad \text{for any } \; Y \in \Gamma(E). \tag{7.78}$$

Likewise the integrability condition in (7.76) can be equivalently written as a momentum map μ given by

$$\mu(Y) := -\frac{1}{2}\int_{\mathcal{M}} s(V, \mathscr{L}_Y V), \tag{7.79}$$

and an integrable vector-multiplet structure V satisfies

$$\mu(Y) = 0, \qquad \text{for any } \; Y \in \Gamma(E). \tag{7.80}$$

If the hyper-multiplet structure and the vector-multiplet structure separately satisfies the integrability condition they define a reduced structure. To define a $SU(6)$ structure they also need to satisfy the compatiblity condition given in (7.61), as well as the condition

$$\mathscr{L}_X J_\alpha = 0, \tag{7.81}$$

with $X = V + \mathrm{i}\hat{V}$. This is needed in order for the intrinsic torsion with respect to the $SU(6)$ structure to vanish [16].

In [16, 52] it was shown that vanishing intrinsic torsion, and hence vanishing of the momentum maps, with respect to a $Spin^*(12) \cap E_{6(2)} = SU(6)$ structure is equivalent to preserving $\mathcal{N} = 2$ supersymmetry. The idea is that vanishing supersymmetry variations of fermionic fields are naturally expressed as Killing spinors being *generalised covariantly constant*. One then shows that this demands vanishing intrinsic torsion with respect to a $SU(6)$ structure. Moreover, since the gauge degrees of freedom are naturally included in the geometry, these also appear naturally through the covariant derivative; this should be compared to the ordinary formulation discussed above, were they instead were obstructions to finding covariantly constant Killing spinors. Using exceptional generalised geometry we have thus seen how to use generalised geometrical tools to study supersymmetric flux compactifications, especially this is similar to the way geometrical tools, introduced in Section 7.2, are used to study supersymmetric compactifications with vanishing flux. The appearance of $SU(6)$ can be further understood by noting that the Killing spinor transforms in $\mathbf{8}$ under (the double cover) of the maximal compact subgroup $SU(8)$ of $E_{7(7)}$. Demanding $\mathcal{N} = 2$ preserved supersymmetry is then equivalent to the Killing spinor being stabilised by $SU(6) \subset SU(8)$.

There are further interesting models to study using the tools introduced above. In [14, 53] compactifications with an external AdS spacetime are examined. This is further applied to study the AdS/CFT correspondence in [54], in which they compare deformations of the CFT to deformations of the hyper-multiplet and vector-multiplet in generalised geometry. In [55] it was further shown that any supersymmetric flux compactification is equivalent to some integrable G structure.

7.4.2 Example Calabi-Yau

To clarify some of the concepts introduced above we consider a specific example of compactification on $CY_3 \times S^1$ following [16]. Especially, we would like to reproduce the known results obtained from ordinary geometric tools. However, in order to continue with a generalised geometry based on $E_{7(7)} \times \mathbb{R}^+$ we consider a slight extension by adding a circle S^1. Just as in the Calabi-Yau case discussed above, there exists a closed symplectic two-form ω, a closed complex three-form Ω and then there also exists a closed one-form $\zeta = \mathrm{d}y$, with y being the coordinate on S^1. These structures satisfy the following compatibility conditions

$$\begin{cases} \omega \wedge \Omega = 0, \\ \omega \wedge \omega \wedge \omega = \frac{3\mathrm{i}}{4}\Omega \wedge \bar{\Omega}, \\ \iota_{\zeta^\#}\omega = \iota_{\zeta^\#}\Omega = 0, \end{cases} \tag{7.82}$$

where we have defined an isomorphism $^\#$ on any form field as $(\rho^\#)^{i_1 i_2 \dots i_p} = g^{i_1 j_1} g^{i_2 j_2} \dots \rho_{j_1 j_2 \dots j_p}$, especially $\zeta^\# = \partial_y$, with y being the coordinate along S^1. Moreover, the integrability conditions state that ω, Ω and ζ are all closed and the volume form is given by

$$\rho^2 := \mathrm{vol}_7 = \frac{\mathrm{i}}{8}\Omega \wedge \bar{\Omega} \wedge \zeta. \tag{7.83}$$

These invariant tensors can be written explicitly in a basis $\{e^a\}_{a=1,2,\dots 7}$, with $e_7 = \zeta = \mathrm{d}y$, as

$$\begin{aligned} \omega &= e^{12} + e^{34} + e^{56}, \\ \Omega &= (e^1 + \mathrm{i}e^2) \wedge (e^3 + \mathrm{i}e^4) \wedge (e^5 + \mathrm{i}e^6), \end{aligned} \tag{7.84}$$

with $e^{ij} = e^i \wedge^j$. From this it follows immediately that the compatibility conditions are satisfied. Especially, in this basis we define the hodge dual, $\star$, as

$$\star\, e^{i_1 i_2 \dots i_p} = \frac{1}{q!}\epsilon^{i_1 i_2 \dots i_p}{}_{j_1 j_2 \dots j_q} e^{j_1 j_2 \dots j_q}, \tag{7.85}$$

and $\epsilon^{12\dots d} = \epsilon^{12\dots d} = 1$. Importantly the inner product of forms with the same degree λ, ρ is then given by

$$\mathrm{vol}_7(\rho^\# \lrcorner \rho) = \lambda \wedge \star\rho. \tag{7.86}$$

Moreover, it is easily found that $\star\Omega = \mathrm{i}\Omega \wedge \zeta$ which will be useful below.

We then want to show that there exists an integrable hyper-multiplet structure, J_α, and an integrable vector-multiplet structure, V, given in terms of (ω, Ω, ζ) which satisfies the generalised compatibility and integrability conditions. We will assume that the forms (ω, Ω, ζ) are given by the expressions above and satisfy the same compatibility and integrability conditions. In [16] these conditions are on the other hand derived from the generalised conditions.

The hyper-multiplet structure is given by

$$\begin{aligned} J_+ &= \underbrace{\frac{1}{2}\rho\Omega}_{a} \underbrace{-\frac{1}{2}\rho\Omega^\#}_{\alpha}, \\ J_- &= \underbrace{\frac{1}{2}\rho\bar{\Omega}}_{a} \underbrace{-\frac{1}{2}\rho\bar{\Omega}^\#}_{\alpha}, \\ J_3 &= \underbrace{\frac{1}{2}\rho I}_{r} \underbrace{-\mathrm{i}\frac{1}{16}\rho\Omega \wedge \bar{\Omega}}_{\tilde{a}} \underbrace{-\frac{1}{16}\mathrm{i}\rho\Omega^\# \wedge \bar{\Omega}^\#}_{\tilde{\alpha}}, \end{aligned} \tag{7.87}$$

with $I^m{}_n = -\omega^m{}_n = \frac{1}{8}\mathrm{i}(\bar{\Omega}^{mpq}\omega_{npq} - \Omega^{mpq}\bar{\Omega}_{npq})$ being the complex structure, and we have for convenience identified the elements with the general decomposition given in (7.45). The vector-multiplet structure on the other hand is given by

$$X = \underbrace{\zeta^\#}_{v} + \underbrace{\mathrm{i}\omega}_{\omega} \underbrace{- \frac{1}{2}\zeta \wedge \omega \wedge \omega}_{\sigma} \underbrace{- \mathrm{i}\zeta \otimes \mathrm{vol}_7}_{\tau}. \tag{7.88}$$

First we need to check that J_α indeed is a $\mathfrak{su}(2)$ triplet using the action defined in (7.50). Denoting $W'' = [J_+, J_-]$ we find

$$\begin{aligned}
l'' &= \frac{\rho^2}{12}(-\Omega^{ijk}\overline{\Omega_{ijk}} + \bar{\Omega}^{ijk}\Omega_{ijk}) = 0, \\
r'' &= \rho^2 j(-\frac{1}{2}\Omega^\#)\lrcorner j(\frac{1}{2}\bar{\Omega}) - \rho^2 j(-\frac{1}{2}\bar{\Omega}^\#)\lrcorner j(\frac{1}{2}\Omega) \\
&\quad - \frac{\rho^2}{3}\mathbf{1}(-\frac{1}{2}\Omega^\#\lrcorner\frac{1}{2}\bar{\Omega} + \frac{1}{2}\bar{\Omega}^\#\lrcorner\frac{-1}{2}\Omega) \\
&= -2\mathrm{i}I\rho^2, \\
a'' &= 0, \\
\tilde{a}'' &= -\frac{\rho^2}{4}\Omega \wedge \bar{\Omega}, \\
\alpha'' &= 0, \\
\tilde{\alpha}'' &= -\frac{\rho^2}{4}\Omega^\# \wedge \bar{\Omega}^\#.
\end{aligned} \tag{7.89}$$

It is then seen that indeed $W'' = -4\mathrm{i}\rho J_3$. Now instead let $W'' = [J_+, J_3]$ and we find

$$\begin{aligned}
l'' &= 0, \\
r'' &= 0, \\
a'' &= -\frac{1}{8}\rho^2 I \cdot \Omega + \frac{\mathrm{i}}{32}\rho^2\Omega^\#\lrcorner(\Omega \wedge \bar{\Omega}), \\
\tilde{a}'' &= 0, \\
\alpha'' &= \frac{1}{8}\rho^2 I \cdot \Omega^\# - \frac{\mathrm{i}}{16}\rho^2(\Omega^\# \wedge \bar{\Omega})\lrcorner\Omega, \\
\tilde{\alpha}'' &= 0,
\end{aligned} \tag{7.90}$$

which, using $I \cdot \Omega = -3\mathrm{i}\Omega$ and $I \cdot \Omega^\# = -3\mathrm{i}\Omega^\#$, gives $[J_+, J_3] = 2\mathrm{i}\rho J_+$ and indeed $\{J_\pm, J_3\}$ is a $\mathfrak{su}(2)$ triplet.

In order to check the compatibility condition $J_+ \cdot X = J_- \cdot X = 0$ we repeat the action of an element $W = r + a + \tilde{a} + \alpha + \tilde{\alpha} \in \mathrm{ad}\, F.$ on an element $Y = v + \omega + \sigma + \tau$,

$$\begin{aligned}
v' &= r \cdot v + \alpha\lrcorner\omega - \tilde{\alpha}\lrcorner\sigma, \\
\omega' &= r \cdot \omega + v\lrcorner a + \alpha\lrcorner\sigma + \tilde{\alpha}\lrcorner\tau, \\
\sigma' &= r \cdot \sigma + v\lrcorner\tilde{a} + a \wedge \omega + \alpha\lrcorner\tau, \\
\tau' &= r \cdot \tau - j\tilde{a} \wedge \omega + ja \wedge \sigma.
\end{aligned} \tag{7.91}$$

Consider first $X' = J_+ \cdot X = 0$, then by the action (7.91) we find

$$\begin{aligned}
v' &= -\frac{\mathrm{i}}{2}\rho\Omega^{\#}\lrcorner\omega, \\
\omega' &= \frac{1}{2}\rho\zeta^{\#}\lrcorner\Omega + \frac{1}{4}\rho\Omega^{\#}\lrcorner(\zeta\wedge\omega\wedge\omega), \\
\sigma' &= \frac{\mathrm{i}}{2}\rho\Omega\wedge\omega + \frac{\mathrm{i}}{2}\rho\Omega^{\#}\lrcorner(\zeta\otimes\mathrm{vol}_7), \\
\tau' &= -\frac{1}{4}\rho\Omega\wedge\zeta\wedge\omega\wedge\omega.
\end{aligned} \tag{7.92}$$

Using the explicit form of Ω and ω it is easily found that $v' = \Omega^{\#}\lrcorner\omega = 0$, which also sets $\omega' = 0$ using that $\iota_{\zeta^{\#}}\Omega = \iota_{\zeta^{\#}}\omega = 0$. The second term in σ' also vanish since $\mathrm{vol}_7 \propto \omega\wedge\omega\wedge\omega$ and $\Omega\wedge\omega = 0$. Moreover, since $\Omega\wedge\omega\wedge\omega = 0$ it follows that τ' vanish as well. Likewise it is found that $J_- \cdot X = 0$. We also need to check the normalisation condition $\frac{1}{2}\mathrm{i}s(X, \overline{X}) = \rho^2$, with the symplectic invariant given for any vector $Y, Y' \in \Gamma(E)$

$$s(Y, Y') = -\frac{1}{4}(\iota_v\tau - \iota_{v'}\tau + \sigma\wedge\omega' - \sigma'\wedge\omega). \tag{7.93}$$

We thus find

$$\begin{aligned}
\frac{1}{2}\mathrm{i}s(X, \overline{X}) :&= -\frac{1}{8}\mathrm{i}\left(-2\mathrm{i}\iota_{\zeta^{\#}}\zeta\mathrm{vol}_7 + \mathrm{i}\zeta\wedge\omega\wedge\omega\wedge\omega\right) \\
&= \rho^2,
\end{aligned} \tag{7.94}$$

where we used $\rho^2 = \mathrm{vol}_7 = \frac{1}{8}\mathrm{i}\Omega\wedge\overline{\Omega}\wedge\zeta$ and the compatibility condition $\omega\wedge\omega\wedge\omega = \frac{3}{4}\mathrm{i}\Omega\wedge\overline{\Omega}$. This shows that J_α and X indeed defines a compatible $SU(6)$ structure. Moreover, this reproduces the known compatibility condition from ordinary Calabi-Yau compactification introduced in Section 7.2.2.

We also need to check if this structure is integrable, corresponding to satisfying the differential conditions on the underlying Killing spinor. As we will see this reproduce the known integrability condition on the Calabi-Yau structure. Consider therefore the triplet of momentum maps $\mu_\alpha(V) = 0$, which determines the integrability of the hyper-multiplet structure. These can be rewritten in a $J_\pm$ basis as

$$\begin{aligned}
\mu_3(Y) &= -\frac{\mathrm{i}}{2}\int_{\mathcal{M}}\kappa(J_-, \mathscr{L}_Y J_+) \\
&= -\frac{\mathrm{i}}{2}\int_{\mathcal{M}}\kappa(J_-, L_v J_+) + \frac{\mathrm{i}}{2}\int_{\mathcal{M}}\kappa(J_-, (\mathrm{d}\omega + \mathrm{d}\sigma)\cdot J_+) \\
&= -\frac{\mathrm{i}}{2}\int_{\mathcal{M}}\kappa(J_-, L_v J_+) + 2\mathrm{i}\int_{\mathcal{M}}\rho\kappa(\mathrm{d}\omega + \mathrm{d}\sigma, J_3),
\end{aligned} \tag{7.95}$$

where going to the second line we have used the observation in (7.52), and going to the last line we used the invariance of the Killing form together with the fact that the hyper-multiplet is a $\mathfrak{su}(2)$ triplet. Likewise we can define the plus component of the momentum map as

$$\begin{aligned}
\mu_+(Y) &= -\mathrm{i}\int_{\mathcal{M}}\kappa(J_3, \mathscr{L}_Y J_+) \\
&= -\mathrm{i}\int_{\mathcal{M}}\kappa(J_3, L_v J_+) + 2\int_{\mathcal{M}}\rho\kappa(\mathrm{d}\omega + \mathrm{d}\sigma, J_+),
\end{aligned} \tag{7.96}$$

with similar steps as above and it straightforward to define an analogous expression for μ_-. These momentum maps $\mu_3, \mu_\pm$ should vanish for any generalised vector $Y = v + \tilde{\omega} + \sigma + \tau \in \Gamma(E)$ and since the momentum maps are linear we can consider each $GL(7)$ component separately. Note that we have denoted the two-form $\tilde{\omega}$ to not cause confusion with the two-form ω present in the structure. Note also that the component τ of Y drops out of the momentum maps trivally. To continue we need the $\mathfrak{e}_{7(7)}$ Killing form given by

$$\kappa(W, W') = \frac{1}{2}\mathrm{tr}(r)\mathrm{tr}(r') + \mathrm{tr}(rr') + \alpha \lrcorner a' + \alpha' \lrcorner a - \tilde{\alpha} \lrcorner \tilde{a}' - \tilde{\alpha}' \lrcorner \tilde{a}. \tag{7.97}$$

Consider then

$$\mu_+(\sigma) = 2\int_{\mathcal{M}} \rho\kappa(\mathrm{d}\sigma, J_+) = 0, \tag{7.98}$$

since J_+ does not contain any six-form. Likewise $L_v J+$ contains a three-form and an anti-symmetric three-vector, but since J_3 does not we find using the Killing-form

$$\mu_+(v) = 0. \tag{7.99}$$

The only non-zero contribution from μ_+ thus comes from the $\tilde{\omega}$ component, $\kappa(\mathrm{d}\tilde{\omega}, \frac{1}{2}\rho\Omega^\#) = \frac{1}{2}\rho\Omega^\# \lrcorner \mathrm{d}\tilde{\omega}$, and we thus get the momentum map

$$\mu_+(\tilde{\omega}) \propto \int_{\mathcal{M}} \rho^2 \Omega^\# \lrcorner \mathrm{d}\tilde{\omega}. \tag{7.100}$$

Using that

$$\begin{aligned} \int_{\mathcal{M}} \mathrm{vol}_7 \Omega^\# \lrcorner \mathrm{d}\tilde{\omega} &= \int_{\mathcal{M}} \mathrm{d}\tilde{\omega} \wedge \star\Omega \\ &= \mathrm{i}\int_{\mathcal{M}} \mathrm{d}\tilde{\omega} \wedge \Omega \wedge \zeta, \end{aligned} \tag{7.101}$$

and integration by parts this reduces to

$$\mu_+(\tilde{w}) \propto \int_{\mathcal{M}} \mathrm{d}(\Omega \wedge \zeta) \wedge \tilde{\omega} = 0, \tag{7.102}$$

which indeed is fulfilled since Ω and ζ are closed. Considering the μ_3 momentum map we find

$$\mu_3(\tilde{\omega}) = 2\mathrm{i}\int_{\mathcal{M}} \rho\kappa(\mathrm{d}\tilde{\omega}, J_3) = 0, \tag{7.103}$$

since J_3 does not contain any anti-symmetric three-vector. For the action of the six-form we find

$$\begin{aligned} \mu_3(\sigma) &= 2\mathrm{i}\int_{\mathcal{M}} \rho\kappa(\mathrm{d}\sigma, J_3) \\ &= -\frac{1}{8}\int_{\mathcal{M}} \rho^2 (\Omega^\# \wedge \Omega^\#) \lrcorner \mathrm{d}\tilde{\sigma}. \end{aligned} \tag{7.104}$$

Using the same procedure as above we find

$$\int_{\mathcal{M}} \mathrm{vol}_7 (\Omega^\# \wedge \bar{\Omega}^\#) \lrcorner \mathrm{d}\sigma = \int_{\mathcal{M}} \mathrm{d}\sigma \wedge \star(\Omega \wedge \bar{\Omega}) \propto \int_{\mathcal{M}} \mathrm{d}\zeta \wedge \sigma = 0, \tag{7.105}$$

which indeed is fulfilled since ζ is closed. The last condition from the triplet of momentum maps is the following

$$\begin{aligned} \mu_3(v) &= -\frac{\mathrm{i}}{2}\int_{\mathcal{M}} \kappa(J_-, L_v J_+) \\ &= \frac{\mathrm{i}}{8}\int_{\mathcal{M}} \rho\left(\bar{\Omega}^\# \lrcorner L_v(\rho\Omega) + L_v(\rho\Omega^\#) \lrcorner \bar{\Omega}\right). \end{aligned} \tag{7.106}$$

Since Ω is closed it is also covariantly constant and $\overline{\Omega}^{\#} \lrcorner \Omega$ is constant. We can then use $\int_{\mathcal{M}} \overline{\Omega}^{\#} \lrcorner \Omega L_v \rho^2 = 0$, assuming that the boundary term vanishes, to move ρ outside of the Lie derivative. Using $\rho^2 \overline{\Omega}^{\#} \lrcorner L_v \Omega = L_v \Omega \wedge \star \overline{\Omega}$ we then find

$$\begin{aligned} \mu_3(v) &\propto \int_{\mathcal{M}} L_v \Omega \wedge \star \overline{\Omega} \\ &= -\mathrm{i} \int_{\mathcal{M}} (\mathrm{d}\iota_v \Omega + \iota_v \mathrm{d}\Omega) \wedge \overline{\Omega} \wedge \zeta = 0, \end{aligned} \tag{7.107}$$

which follows from integration by parts and that Ω and $\overline{\Omega}$ are closed. This shows that the hyper-multiplet is integrable.

The integrability condition for the vector-multiplet structure is given by $\mathscr{L}_X \overline{X} = 0$, i.e.

$$\mathscr{L}_X \overline{X} = L_{\zeta^{\#}} \overline{X} - \mathrm{i}\mathrm{d}\omega \cdot \overline{X} - (\zeta \wedge \omega \wedge \mathrm{d}\omega) \cdot \overline{X} = 0. \tag{7.108}$$

This can be seen to vanish immediately by noting that $\zeta^{\#}$ is a Killing vector and using that $\mathrm{d}\omega$ is a closed form. This shows that X defines an integrable vector-multiplet structure.

In order to define an exceptional Calabi-Yau the structures, i.e. to set to zero the intrinsic torsion of the $SU(6)$ structure, must also satisfy $\mathscr{L}_X J_+ = \mathscr{L}_X J_- = 0$. This is given by

$$\mathscr{L}_X J_+ = L_{\zeta^{\#}} J_+ - \mathrm{i}\mathrm{d}\omega \cdot J_+ + \frac{1}{2}\mathrm{d}(\zeta \wedge \omega \wedge \omega) \cdot J_+, \tag{7.109}$$

which again is easily seen to be fulfilled by the same reasoning as above.

We thus see that the hyper-multiplet structure in (7.87) and the vector-multiplet structure in (7.88) together with the compatibility conditions and integrability conditions reproduce the known conditions obtained via ordinary results. Although fluxes were not included in this example the formalism applies equally well when they are, for examples of this see [16, 53]. The main difference when including fluxes is that one has to look at elements that are "twisted" by the adjoint action of flux fields, the twisting in the case above were trivial.

8

Conclusions

String theory/M-theory are the most prominent frameworks of quantum gravity, but there is still much about these theories that we do not know. Especially the notion of describing extended objects has led to interesting dualities such as T-duality and U-duality. As we have argued the corresponding low-energy effective theories, supergravity theories, have related "hidden" symmetry groups when compactified on torii. The rôle of these symmetries in M-theory is the motivation for this thesis and for developing extended geometries.

Gravitational degrees of freedom and non-gravitational degrees of freedom are typically mixed under these dualities and hence a covariant formalism suggest a merger of the underlying fields leading to the notion of extended geometries. This extension is done by introducing generalised diffeomorphisms and demanding that fields transform covariantly under such transformations. The latter is equivalent to the closure of the algebra of diffeomorphisms which in turn demands the section constraint, i.e. a local embedding of ordinary geometry. One can then construct an action invariant under generalised diffeomorphisms that encodes the dynamics of a generalised metric. Moreover, it is also possible to construct geometrical objects such as a covariant derivative, torsion and a generalised Ricci tensor which characterise the extended geometry.

With a global solution to the section constraint and a "twisting" by form fields an extended geometry reduces to what is called generalised geometry. This formalism is potent in describing, among other things, compactifications of supergravity with fluxes. Including fluxes typically invalidates the geometrical tools that works well to describe compactifications without fluxes. Since extended geometries merge gravitational and non-gravitational degrees of freedom appropriate generalisations appear naturally and compactifications can be dealt with using generalised geometrical tools.

In Chapter 6 a covariant formalism was introduced as well as a generalised Ricci tensor. The latter is a candidate for the equations of motion for the generalised metric and it would be interesting to compare this with the equations of motion derived from the action introduced in the same chapter. In order to do this one should find a metric-compatible and torsion-free connection in terms of the generalised metric. However, these two conditions do not fully specify the connection and we were not able to find such an expression for a general structure algebra. This would be an interesting direction for future studies.

Another pressing issue that would be interesting to solve is that of the global structure and global transformations. As of yet the extended geometries are generally restricted to the local behaviour, exceptions are for example double geometries. Another interesting question is that of the section constraint which we have seen play a crucial part in defining a consistent theory. A truly different theory would be obtained if the section constraint could be dropped completely.

Extended geometries ultimately presents an interesting way to merge gravitational and non-gravitational degrees of freedom in order to exploit a larger part of the underlying symmetry of M-theory. The full rôle of extended geometries in a complete formulation of M-theory remains to be seen, but unification and symmetries have been incredibly successful guiding tools to a better understanding of the fundamental laws of nature more than once before.

A

Closure of generalised diffeomorphisms

In this appendix we provide the explicit calculation of the closure of generalised diffeomorphisms. We want to look at $[\mathscr{L}_\xi, \mathscr{L}_\eta]V^M$ and start with one term

$$\begin{aligned}\mathscr{L}_\xi(\mathscr{L}_\eta V^M) &= \mathscr{L}_\xi(\eta^P\partial_P V^M - \partial_P\eta^M V^P + Y^{MR}{}_{ST}\,\partial_R\eta^S V^T)\\ &= (\mathscr{L}_\xi\eta^P)\partial_P V^M + \eta^P\mathscr{L}_\xi(\partial_P V^M) - (\mathscr{L}_\xi\partial_P\eta^M)V^P - \partial_P\eta^M\mathscr{L}_\xi V^P\\ &+ Y^{MR}{}_{ST}\,(\mathscr{L}_\xi\partial_R\eta^S)V^T + Y^{MR}{}_{ST}\,\partial_R\eta^S\mathscr{L}_\xi V^T.\end{aligned} \tag{A.1}$$

The first term gives

$$\begin{aligned}(\mathscr{L}_\xi\eta^P)\partial_P V^M &= \xi^K\partial_K\eta^P\partial_P V^M - \partial_K\xi^M\partial_P V^K + \partial_K\xi^K\partial_K V^M\\ &+ Y^{MR}{}_{ST}\,(\partial_R\xi^S)\partial_P V^T - \underbrace{Y^{TR}{}_{SP}\,(\partial_R\xi^S)\partial_T V^M}_{=0\,\text{section constraint}}.\end{aligned} \tag{A.2}$$

The second term gives

$$\begin{aligned}\eta^P(\mathscr{L}_\xi\partial_P V^M) &= \eta^P\xi^K\partial_K\partial_P V^M - \eta^P\partial_K\xi^M\partial_P V^K + \eta^P\partial_P\xi^K\partial_K V^M\\ &+ Y^{MR}{}_{ST}\,(\partial_R\xi^S)\partial_P V^T - \underbrace{Y^{TR}{}_{SP}\,(\partial_R\xi^S)\partial_T V^M}_{=0\,\text{section constraint}},\end{aligned} \tag{A.3}$$

where we note that the first terms is symmetric in $\xi \leftrightarrow \eta$ and will vanish in the commutator. The third term gives

$$\begin{aligned}-(\mathscr{L}_\xi\partial_P V^M)V^P &= -\xi^K\partial_K\partial_P\eta^M V^P + \partial_K\xi^M\partial_P\eta^K V^P - \partial_P\xi^K\partial_K\eta^M V^P\\ &- Y^{MR}{}_{ST}\,(\partial_R\xi^S)\partial_\eta^T V^P + \underbrace{Y^{TR}{}_{SP}\,(\partial_R\xi^S)\partial_T V^P}_{=0\,\text{section constraint}}.\end{aligned} \tag{A.4}$$

The fourth term is given by

$$-\partial_P\eta^M\mathscr{L}_\xi V^P = -\partial_\eta^M\xi^K\partial_K V^P + \partial_P\eta^M\partial_K\xi^P V^K - \underbrace{\partial_P\eta^M Y^{PR}{}_{ST}\,(\partial_R\xi^S)\partial_P\eta^M}_{=0\,\text{section constraint}}. \tag{A.5}$$

The fifth term is given by

$$\begin{aligned}Y^{MR}{}_{ST}\,(\mathscr{L}_\xi\partial_R\eta^S)V^T &= Y^{MR}{}_{ST}\,V^T\left(\xi^K\partial_K\partial_R\eta^S - \partial_K\xi^S\partial_R\eta^K + \partial_R\xi^K\partial_K\eta^S\right)\\ &+ Y^{MR}{}_{ST}\,V^T(Y^{SK}{}_{LQ}\,(\partial_K\xi^L)\partial_R\eta^Q - \underbrace{Y^{QK}{}_{LR}\,(\partial_K\xi^L)\partial_Q\eta^S}_{=0\,\text{section constraint}}).\end{aligned} \tag{A.6}$$

And finally the last term is given by

$$Y^{MR}{}_{ST}\,(\partial_R\eta^S)\mathscr{L}_\xi V^T = Y^{MR}{}_{ST}\,\left(\partial_R\eta^S\xi^K\partial_K V^T - \partial_K\xi^T V^K + Y^{TK}{}_{QP}\,(\partial_K\xi^Q)V^P\right). \tag{A.7}$$

We also want to calculate $\frac{1}{2}(\mathscr{L}_{\mathscr{L}_\xi\eta} - \mathscr{L}_{\mathscr{L}_\eta\xi})V^M$ to this end calculate

$$\mathscr{L}_{\mathscr{L}_\xi\eta}V^M = \mathscr{L}_{\xi^K\partial_K\eta}V^M - \mathscr{L}_{\partial_K\xi\eta^K}V^M + \mathscr{L}_{Y^{\circ R}{}_{ST}\,\partial_R\xi^S\eta^T}V^M. \tag{A.8}$$

The first term gives

$$\begin{aligned}\mathscr{L}_{\xi^K\partial_K\eta}V^M &= \xi^K\partial_K\eta^P\partial_P V^M - \left(\partial_P\xi^K\partial_K\eta^M + \xi^K\partial_K\partial_P\eta^M\right)V^P\\ &\quad + Y^{MR}{}_{ST}\,\left(\partial_\xi^K\partial_K\eta^S + \xi^K\partial_K\partial_R\eta^S\right)V^T.\end{aligned} \tag{A.9}$$

The second term gives

$$\begin{aligned}-\mathscr{L}_{\partial_K\xi\eta^K}V^M &= -\partial_K\xi^P\eta^K\partial_P V^M + \left(\eta^K\partial_K\partial_P\xi^M + \partial_K\xi^M\partial_P\eta^K\right)V^P\\ &\quad - Y^{MR}{}_{ST}\,\left(\partial_R\partial_K\xi^S\eta^K + \partial_K\xi^S\partial_R\eta^K\right)V^T.\end{aligned} \tag{A.10}$$

And the last term gives

$$\begin{aligned}\mathscr{L}_{Y^{\circ R}{}_{ST}\,\partial_R\xi^S\eta^T}V^M &= \underbrace{Y^{KR}{}_{ST}\,(\partial_R\xi^S)\eta^T\partial_K V^M}_{=0\,\text{section constraint}} - Y^{MR}{}_{ST}\,V^K\left(\partial_K\partial_R\xi^S\eta^T + \partial_R\xi^S\partial_K\eta^T\right)\\ &\quad + Y^{MQ}{}_{LK}\,Y^{LR}{}_{ST}\,V^K\left(\partial_Q\partial_R\xi^S\eta^T + \partial_R\xi^S\partial_Q\eta^T\right).\end{aligned} \tag{A.11}$$

It is then straightforward but tedious to calculate $[\mathscr{L}_\xi, \mathscr{L}_\eta]V^M - \frac{1}{2}(\mathscr{L}_{\mathscr{L}_\xi\eta} - \mathscr{L}_{\mathscr{L}_\eta\xi})V^M$ and one finds that the contribution from the terms proportional to $\partial^2\xi$ is given by

$$\frac{1}{2}\left(Y^{MR}{}_{ST}\,\partial_K\partial_R\xi^S - Y^{MQ}{}_{LK}\,Y^{LR}{}_{ST}\,\partial_Q\partial_R\xi^S\right)V^K\eta^T = -\frac{1}{2}Z^{MQ}{}_{LK}Y^{LR}{}_{ST}\,\partial_Q\partial_R\xi^S V^T\eta^K, \tag{A.12}$$

which is nothing but an ancillary transformation. The remaining terms are proportional to $(\partial\xi\partial\eta)$ and given by

$$\begin{aligned}&\frac{1}{2}Y^{MR}{}_{ST}\,Y^{SK}{}_{LQ}\,\partial_K\xi^L\partial_R\eta^Q V^T + Y^{MR}{}_{ST}\,Y^{TK}{}_{QP}\,\partial_R\eta^S\partial_K\xi^Q V^P\\ &-\frac{1}{2}Y^{MR}{}_{ST}\,\partial_K\xi^T\partial_R\eta^S V^K - Y^{MR}{}_{ST}\,\partial_K\xi^S\partial_R\eta^K V^T - (\xi\leftrightarrow\eta) \stackrel{!}{=} 0.\end{aligned} \tag{A.13}$$

The identities that the Y tensor need to satisfy given in (6.13a)-(6.13b) are then easily derived from this expression.

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