Thesis for the Degree of Licentiate of Philosophy

Computational Aspects of Lévy-Driven SPDE Approximations

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Abstract

In order to simulate solutions to stochastic partial differential equations (SPDE) they must be approximated in space and time. In this thesis such fully discrete approximations are considered, with an emphasis on finite element methods combined with rational semigroup approximations. There are several notions of the error resulting from this. One of them is the weak error, measured in terms of the mean of a functional applied to the solution. To approximate the mean, one typically employs Monte Carlo and multilevel Monte Carlo methods that are based on generating a large number of realizations of the approximate solution to the SPDE.

The thesis consists of two papers. In Paper 1 the additional error caused by Monte Carlo and multilevel Monte Carlo methods when one attempts to simulate the weak error is analysed Upper and lower bounds are derived for the different methods and simulations illustrate the results.

When using multilevel Monte Carlo methods to estimate the weak error, along with other properties of the SPDE, it is important that the discretizations used are sufficiently stable in a mean square sense. In Paper 2 a framework for the analysis of the asymptotic mean square stability of a general stochastic recursion scheme is set up. This framework is then applied to several discretizations of an SPDE, which results in a series of sufficient conditions for stability. Some of these results are found to be sharp in simulations.

In addition to the two papers, a deterministic scheme for the simulation of weak errors in the context of finite elements is presented.

Keywords: Stochastic partial differential equations, numerical approximation of stochastic differential equations, finite element method, Monte Carlo, multilevel Monte Carlo, variance reduction techniques, weak convergence, asymptotic mean square stability, multiplicative noise, Lévy processes

List of included papers

Paper 1 Annika Lang and Andreas Petersson. Monte Carlo versus multilevel Monte Carlo in weak error simulations of SPDE approximations. *Math. Comput. Simulation*, 143:99 – 113, 2018. doi: `10.1016/j.matcom.2017.05.002`

Paper 2 Annika Lang, Andreas Petersson, and Andreas Thalhammer. Mean-square stability analysis of approximations of stochastic differential equations in infinite dimensions. *BIT Numer. Math.*, 2017. doi: `10.1007/s10543-017-0684-7`

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Contents

1 Introduction

The topic of this thesis is the study of some computational aspects of approximations to stochastic partial differential equations (SPDE) of the form

$$\begin{aligned} \mathrm{d}X(t) &= (AX(t) + F(X(t)))\,\mathrm{d}t + G(X(t))\,\mathrm{d}L(t), \\ X(0) &= X_0, \end{aligned} \tag{1}$$

where $t \in [0, T]$. Here the process X takes values in a Hilbert space H, the stochastic Lévy process L takes values in another Hilbert space U, F maps elements from H to H and G maps elements of H onto a space of operators from U to H. The operator A on H is the generator of a strongly continuous semigroup of bounded linear operators in H. An example of such an operator is the Laplacian Δ on the space of square integrable functions on some domain $D \subset \mathbb{R}^d$, $d \in \{1, 2, 3\}$, so that (1) becomes a stochastic heat equation, which one can interpret as describing heat flow perturbed by some noise, perhaps due to measurement errors. Here the noise is said to be *multiplicative*, since the operator G depends on X. SPDE have many other uses in fields such as biology, engineering and finance, see, e.g., [18, 19] for an overview of such applications.

As the solution X to (1) is a stochastic process, a natural quantity of interest is the mean value $\mathbb{E}[X(t)]$ of it at some time $t \in [0, T]$, or perhaps the mean value $\mathbb{E}[\phi(X(t))]$ of some functional $\phi : H \to \mathbb{R}$ of the solution. Since analytical solutions to (1) are hardly ever available, an approximation $\hat{X}(t)$ is used instead and the quantity $|\,\mathbb{E}[\phi(\hat{X}(t))] - \mathbb{E}[\phi(X(t))]|$ is referred to as a weak error. The topic of weak error analysis has been met with increasing interest in the SPDE community during recent years.

In order to approximate quantities like $\mathbb{E}[\phi(X(t))]$, Monte Carlo methods are often employed, which are based on generating a large number of realizations of X. That is to say, an approximation of the solution to (1) has to be computed many times, something that is computationally very expensive. This may explain why simulations that illustrate theoretical results on convergence with respect to the weak error are rarely available, as one employs Monte Carlo methods to approximate the weak error $|\,\mathbb{E}[\phi(\hat{X}(t))] - \mathbb{E}[\phi(X(t))]|$ itself. The first paper of this thesis analyses the additional error caused by approximating the weak error using various Monte Carlo approaches.

One of the methods considered in Paper 1 is the multilevel Monte Carlo method, which is based on approximating $\mathbb{E}[\phi(X(t))]$ by applying the Monte Carlo method to a sequence $\hat{X}_0, \hat{X}_1, \dots, \hat{X}_\ell, \dots$ of approximations of X indexed by a level ℓ. Typically the accuracy of $\hat{X}_\ell$ increases as $\ell \to \infty$ but so does the computational cost. The main idea of the multilevel Monte Carlo method is to compute a different number of realizations for each level, from a few when ℓ is big to many when ℓ is small. By choosing the right balance between the accuracy of the approximation and the number of realizations at each level, the multilevel Monte Carlo method can be more efficient than standard Monte Carlo methods, while retaining the same degree of accuracy. To ensure that it is more efficient, the approximation should be sufficiently stable at all levels. More precisely, this refers to the asymptotic mean square stability of $\hat{X}$, the property that $\mathbb{E}[\|\hat{X}(t)\|_H^2] \to 0$ as $t \to \infty$. The second paper of this thesis sets up a framework for analysing the asymptotic mean square stability of approximations to (1) as well as more general finite-dimensional recursion schemes.

The following sections provide the theoretical background for the papers, along with summaries of them. In Section 2 we set up the notation we use and review basic results from the fields of functional analysis and probability theory, with an emphasis on random fields and stochastic integration. Section 3 contains results on SPDE and finite element approximations and Section 4 introduces Monte Carlo and multilevel Monte Carlo methods. In Sections 5 and 7 we summarize Papers 1 and 2 while in Section 6, inspired by the results of Paper 2, we present a way of computing weak errors that avoids the use of Monte Carlo methods.

2 Preliminaries

In this chapter we introduce the concepts and notation needed for the construction of solutions and approximations of SPDE. For proofs and more details on the standard claims made in this part of the thesis, the reader is referred to [7, 13, 20].

2.1 Functional analysis

Let $(U, \langle\cdot,\cdot\rangle_U)$ and $(H, \langle\cdot,\cdot\rangle_H)$ be real separable Hilbert spaces and let $(B, \|\cdot\|_B)$ and $(E, \|\cdot\|_B)$ be real Banach spaces. We write $L(B;E)$ for the Banach space of linear and bounded operators from B to E, or $L(B)$ if $E = B$. Given an orthonormal basis $(e_i)_{i\in\mathbb{N}}$ of H we denote by $L_{\mathrm{HS}}(H;U) \subseteq L(H;U)$ the Hilbert space of *Hilbert–Schmidt operators* with inner product

$$\langle F, G\rangle_{L_{\mathrm{HS}}(H;U)} = \sum_{i=1}^{\infty} \langle Fe_i, Ge_i\rangle_U ,$$

and whenever $U = H$ we write $L_{\mathrm{HS}}(H)$ for $L_{\mathrm{HS}}(H;H)$. It is not hard to see that this inner product is independent of the chosen orthonormal basis $(e_i)_{i\in\mathbb{N}}$. The embedding $L_{\mathrm{HS}}(H;U) \subseteq L(H;U)$ is continuous with embedding constant 1, i.e., for $F \in L_{\mathrm{HS}}(H;U)$, $\|F\|_{L(H;U)} \leq \|F\|_{L_{\mathrm{HS}}(H;U)}$. Another notion that we will use is the *trace* of a self-adjoint and positive semidefinite operator $Q \in L(H)$ which is for any orthonormal basis $(e_i)_{i\in\mathbb{N}}$ of H defined by

$$\mathrm{Tr}(Q) = \sum_{i=1}^{\infty} \langle Qe_i, e_i\rangle_H = \|Q^{1/2}\|^2_{L_{\mathrm{HS}}(H)},$$

where $Q^{1/2}$, which exists if $\mathrm{Tr}(Q) < \infty$, is the unique self-adjoint and positive semidefinite operator for which $Q^{1/2}Q^{1/2} = Q$. Next, let us assume that $-A : \mathrm{dom}(A) \subseteq H \to H$ is a densely defined, linear, self-adjoint and positive definite operator with compact inverse $(-A)^{-1}$. By the spectral theorem applied to $(-A)^{-1}$ we get an orthonormal eigenbasis

$(e_i)_{i\in\mathbb{N}}$ of H and a positive sequence $(\lambda_i)_{i\in\mathbb{N}}$ of eigenvalues of $-A$ that is increasing and for which $\lim_i \lambda_i = \infty$. For $r \geq 0$ we define fractional powers of $-A$ by

$$(-A)^{\frac{r}{2}} f = \sum_{i=1}^{\infty} \lambda_i^{\frac{r}{2}} \langle f, e_i \rangle_H e_i$$

for $f \in \dot{H}^r$, where

$$\dot{H}^r = \mathrm{dom}((-A)^{\frac{r}{2}}) = \left\{ f \in H : \|f\|_r^2 < \infty \right\}$$

is a separable Hilbert space when equipped with the inner product

$$\langle \cdot, \cdot \rangle_r = \left\langle (-A)^{\frac{r}{2}} \cdot, (-A)^{\frac{r}{2}} \cdot \right\rangle_H .$$

With this construction it holds for $s \leq r$ that $\dot{H}^r \subseteq \dot{H}^s$. The operator A is the generator of a semigroup, which we define below.

Definition 2.1. Let B be a Banach space. A family $(E(t))_{t\in[0,\infty)}$ with $E(t) \in L(B)$ for all $t \geq 0$ is called a *semigroup of operators on B* if

(i) $E(0) = I$, where I is the identity operator and

(ii) $E(t + s) = E(t)E(s)$ for all $s, t \geq 0$.

If in addition to this

(iii) $\lim_{t\searrow 0} E(t)f = f$ for all $f \in B$,

it is said to be *strongly continuous* or a C_0*-semigroup*. If it also satisfies

(iv) $\|E(t)\|_{L(B)} \leq 1$ for all $t \geq 0$,

then it is called a C_0*-semigroup of contractions*.

The linear operator A defined by

$$Af = \lim_{t\searrow 0} \frac{E(t)f - f}{t},$$

with $\mathrm{dom}(A)$ being the space of all $f \in B$ such that the limit exists, is called the *infinitesimal generator* of the semigroup.

In our setting, an analytic C_0-semigroup of contractions $(E(t))_{t\in[0,\infty)}$ is, for $t \geq 0$ and $f \in H$, given by $E(t)f = \sum_{i=1}^{\infty} e^{-\lambda_i t} \langle f, e_i \rangle_H e_i$. It can be seen that A is the generator of this semigroup. Next, we give a concrete standard example of the notions given so far.

Example 2.2. Let $H = L^2(D;\mathbb{R})$ be the space of square integrable functions on a bounded convex domain $D \subset \mathbb{R}^d$, $d \in \mathbb{N}$, with polygonal boundary. Let for a function f on D the operator A be given by

$$Af = \nabla \cdot (a(x)\nabla f) - c(x)f$$

with Dirichlet boundary conditions, where $a, c : D \to \mathbb{R}$ are sufficiently smooth functions with $c(x) \geq 0$ and $a(x) \geq a_0 > 0$ for all $x \in D$. Then $-A$ fulfils the assumptions above and its fractional powers give rise to the sequence of Hilbert spaces $(\dot{H}^r)_{r\geq 0}$. One can show that $\dot{H}^1 = H_0^1(D)$ and $\dot{H}^2 = H^2(D) \cap H_0^1(D)$ where $H^k(D)$, $k \geq 0$, denotes the Sobolev space of order k and $H_0^1(D)$ is the subspace of $H^1(D)$ containing the functions that are zero at the boundary of D.

We end this section with a brief review of tensor products of Hilbert spaces, which are used in Paper II. For Hilbert spaces H and U the *algebraic tensor product* $H \otimes_0 U$ is the vector space of finite sums $\sum_{i=1}^n f_i \otimes g_i$, where $f_i \in H$ and $g_i \in U$ for $i = 1, \ldots, n$, along with the equivalence relations

$$\begin{aligned}(f_1 + f_2) \otimes g_1 &= f_1 \otimes g_1 + f_2 \otimes g_1 \\ f_1 \otimes (g_1 + g_2) &= f_1 \otimes g_1 + f_1 \otimes g_2 \\ (\lambda f_1) \otimes g_1 &= f_1 \otimes (\lambda g_1) = \lambda(f_1 \otimes g_1),\end{aligned}$$

where $\lambda \in \mathbb{R}$. The *Hilbert tensor product* $H \otimes U$, or just $H^{(2)}$ when $U = H$, is defined as the completion of the algebraic tensor product with respect to the norm induced by the inner product

$$\langle f_1 \otimes g_1, f_2 \otimes g_2 \rangle_{H \otimes U} = \langle f_1, f_2 \rangle_H \langle g_1, g_2 \rangle_U .$$

If $(e_{1,i})_{i\in\mathbb{N}}$ and $(e_{2,i})_{i\in\mathbb{N}}$ are orthonormal bases of H and U respectively then $(e_{1,i} \otimes e_{2,j})_{i,j\in\mathbb{N}}$ is an orthonormal basis of $H \otimes U$. The same statement holds when we drop the orthonormality requirement if the spaces involved are finite-dimensional.

The Hilbert tensor product can also be constructed by identifying $H \otimes U \cong L_{\mathrm{HS}}(U; H)$ where the element $f_1 \otimes g_1$, $f_1 \in H$, $g_1 \in U$, is interpreted as the mapping

$$g_2 \mapsto \langle g_1, g_2 \rangle_U f_1.$$

Here $\cong$ denotes the existence of an isometric isomorphism.

2.2 Probability theory

To be able to speak of stochastic processes in Hilbert spaces, we must first introduce the concept of a Hilbert space-valued random variable. Let $(\Omega, \mathcal{A}, (\mathcal{F}_t)_{t\in[0,T]}, P)$ be a complete filtered probability space satisfying the usual conditions, which is to say that $\mathcal{F}_0$ contains all P-null sets and $\mathcal{F}_t = \cap_{s>t}\mathcal{F}_s$ for all $t \in [0, T]$. For a Hilbert space H, an *H-valued random variable*, or just a *random variable* if $H = \mathbb{R}$, is an $(\mathcal{A}, \mathcal{B}(H))$-measurable function

$X : \Omega \mapsto H$. Here $\mathcal{B}(H)$ refers to the Borel σ-algebra on H. The *expectation* of an H-valued random variable is defined by the Bochner integral

$$\mathbb{E}[X] = \int_\Omega X(\omega)\,\mathrm{d}P(\omega)$$

whenever $\|X\|_{L^1(\Omega;H)} = \mathbb{E}[\|X\|_H] < \infty$. The *covariance* of $X \in L^2(\Omega;H)$ is defined by

$$\mathsf{Cov}(X) = \mathbb{E}\left[(X - \mathbb{E}[X]) \otimes (X - \mathbb{E}[X])\right]$$

and with the identification $H \otimes H \cong L_{\mathrm{HS}}(H)$ its counterpart, the unique self-adjoint positive semidefinite operator $Q \in L_{\mathrm{HS}}(H)$ for which, with $f \in H$,

$$Qf = \mathbb{E}\left[\langle X - \mathbb{E}[X], f\rangle_H (X - \mathbb{E}[X])\right]$$

is called the *covariance operator* of X although when there is no risk of confusion we also refer to this operator as the *covariance* of X. From this definition one can see that

$$\mathrm{Tr}(Q) = \|X - \mathbb{E}[X]\|^2_{L^2(\Omega;H)} < \infty$$

and Q is said to be of *trace class*. The identity

$$\mathbb{E}[\langle X - \mathbb{E}[X], f\rangle_H \langle X - \mathbb{E}[X], g\rangle_H] = \langle Qf, g\rangle_H \tag{2}$$

for any $f, g \in H$ is a straightforward consequence of the definition of Q.

A probability measure μ on H is called *Gaussian* if for some $m \in H$ and $Q \in L(H)$ with $\mathrm{Tr}(Q) < \infty$, its characteristic functional is given by

$$\hat{\mu}(f) = \int_H \exp\left(i\langle g, f\rangle_H\right)\mathrm{d}\mu(g) = \exp\big(i\langle f, m\rangle_H - \tfrac{1}{2}\langle Qf, f\rangle_H\big).$$

It can be shown that for each $m \in H$ and self-adjoint positive semidefinite $Q \in L(H)$ with $\mathrm{Tr}(Q) < \infty$ such a measure exists. A random variable X is said to be *Gaussian* if its image measure $P \circ X^{-1}$ is a Gaussian probability measure and we write $X \sim \mathcal{N}(m, Q)$. In this case $m = \mathbb{E}[X]$ and Q is the covariance operator of X.

For an interval $[0,T]$, with $T < \infty$, an *H-valued stochastic process* $(X(t))_{t\in[0,T]}$ is a family of H-valued random variables. It is said to be *adapted to the filtration* $(\mathcal{F}_t)_{t\in[0,T]}$ if for each $t \in [0,T]$, $X(t)$ is $\mathcal{F}_t$-measurable. Two H-valued stochastic processes X, Y are said to be *modifications* of one another if for all $t \in [0,T]$, $X(t) = Y(t)$ P-almost surely.

The class of stochastic processes most important in this thesis is that of the so called Lévy processes. Since we will only have use for Lévy processes that are square integrable and have zero mean, as this entails that they are square integrable martingales, these are the ones that we define next. For a trace class self-adjoint positive semidefinite operator $Q \in L(H)$, an H-valued stochastic process $(L(t))_{t\in[0,T])}$ is said to be a *mean zero square integrable Q-Lévy process with respect to the filtration* $(\mathcal{F}_t)_{t\in[0,T]}$ if

- $L(0) = 0$ P-almost surely,
- L is continuous in probability, i.e., for any $\epsilon > 0$ and $t \in [0, T]$,

$$\lim_{\substack{s \to t \\ s \geq 0}} P(\|L(t) - L(s)\|_H > \epsilon) = 0,$$

- L has independent and stationary increments,
- L is adapted to $(\mathcal{F}_t)_{t \in [0,T]}$,
- $L(t) - L(s)$ is independent of $\mathcal{F}_s$ for all $0 \leq s < t \leq T$,
- L is square integrable, i.e., $\|L(t)\|_{L^2(\Omega;H)} < \infty$ for all $t \geq 0$ and
- $L(t) - L(s)$, $0 \leq s < t \leq T$, has zero mean and covariance $(t - s)Q$.

With these assumptions it holds that L is an H-valued square integrable martingale. If in addition to this for all $0 \leq s < t \leq T$ the increment $L(t) - L(s) \sim \mathcal{N}(0, (t - s)Q)$, L is said to be a *Q-Wiener process* (or a *standard Brownian motion* in the case of $H = \mathbb{R}$ and $Q = 1$) with respect to the filtration $(\mathcal{F}_t)_{t \in [0,T]}$, and we then denote it by W. An expansion of a mean zero square integrable Q-Lévy process $(L(t))_{t \in [0,T])}$ on the orthonormal eigenbasis $(p_i)_{i \in \mathbb{N}}$ of Q is called the *Karhunen–Loève expansion*

$$L = \sum_{i=1}^{\infty} \sqrt{\mu_i} L_i p_i, \tag{3}$$

where $(\mu_i)_{i \in \mathbb{N}}$ is the decreasing sequence of positive eigenvalues of Q in H, where we note that such eigenpairs exist since the assumption of a finite trace ensures that Q is compact. Furthermore, $(L_i)_{i \in \mathbb{N}}$ is a sequence of uncorrelated identically distributed real-valued Q-Lévy processes with $Q = 1$. If L is a Q-Wiener process then for any $i \in \mathbb{N}$, L_i is a real-valued standard Brownian motion.

2.3 Random fields

Let us now consider the setting of Example 2.2 where $D \subset \mathbb{R}^d$ is a bounded convex domain with polygonal boundary and $H = L^2(D; \mathbb{R})$. We introduce random fields on D and give a condition under which such fields are elements of $L^2(\Omega; H)$.

Definition 2.3. A *random field* is a collection of random variables $(X(x))_{x \in D}$ such that the mapping $x \times \omega \mapsto X(x)[\omega]$ is measurable with respect to the product σ-algebra $\mathcal{B}(D) \otimes \mathcal{A}$.

Definition 2.4. A *second order random field* X is a random field with $X(x) \in L^2(\Omega; \mathbb{R})$ for all $x \in D$. Its *covariance function* C is given by $C(x, y) = \mathsf{Cov}(X(x), X(y))$ for $x, y \in D$ and its *mean function* m by $m(x) = \mathbb{E}[X(x)]$.

If we now assume that the pair of functions C and m fulfils

$$\int_D C(x,x) + m(x)^2 \,\mathrm{d}x < \infty,$$

then, as a consequence of the joint measurability and Fubini's theorem, the mapping given by $\omega \mapsto X(\cdot)[\omega]$, which we denote by X, is in $L^2(\Omega; H)$ and the mean function m is equal to the expectation of X. For the same reason, the covariance operator of X is for $f \in H$ and $x \in D$ given by

$$\begin{aligned} Qf(x) &= \mathbb{E}\left[\langle X - \mathbb{E}[X], f\rangle_H \left(X(x) - \mathbb{E}[X(x)]\right)\right] \\ &= \mathbb{E}\left[\int_D \left(X(y) - m(y)\right)\left(X(x) - m(x)\right) f(y) \,\mathrm{d}y\right] = \int_D C(x,y) f(y) \,\mathrm{d}y, \end{aligned}$$

a fact that can be exploited to numerically approximate the eigenpairs of Q from knowing only the covariance function. This can then in turn be used to generate samples of, e.g., a Lévy process by using the Karhunen–Loève expansion (3). We say that C is a *kernel* of the covariance operator, or just a *covariance kernel* for short. With the parameters $\sigma^2, \kappa \in \mathbb{R}$, an example of a common covariance kernel is the *exponential kernel*

$$C(x,y) = \frac{\sigma^2}{(2\pi)^{d/2}\kappa(d-1)!!} \exp(-\kappa|x-y|)$$

which one obtains as a special case of the *Matérn covariance kernel*

$$C(x,y) = \frac{2^{1-\nu}\sigma^2}{(4\pi)^{d/2}\Gamma(\nu + d/2)\kappa^{2\nu}} (\kappa|x-y|)^\nu K_\nu(\kappa|x-y|)$$

by setting $\nu = 1/2$. Here K_ν denotes the modified Bessel function of the second kind. In Figure 1 we see a realization of an approximation of a Q-Wiener process taking values in $H = L^2(D; \mathbb{R})$, where $D = [0,1]^2$. Here Q is the covariance operator corresponding to a Matérn kernel with parameters $\nu = 3$, $\kappa = 25$ and σ^2 chosen so that $C(x,x) = 5$. The approximation was generated with FEniCS (see [2]) using the approach of [17].

2.4 Stochastic integration

To make sense of solutions to (1) we need a theory for the integration of stochastic processes with respect to square integrable martingales. That is to say, in our setting, for $t \in [0,T]$ and an $L(U;H)$-valued stochastic process Ψ on $[0,T]$, we want to make sense of the H-valued *stochastic Itô integral*

$$I_t^L(\Psi) = \int_0^t \Psi(s) \,\mathrm{d}L(s)$$

with respect to a mean zero square integrable U-valued Q-Lévy process L. We briefly reiterate the results of [20, Chapter 8] for our simpler setting of integration with respect

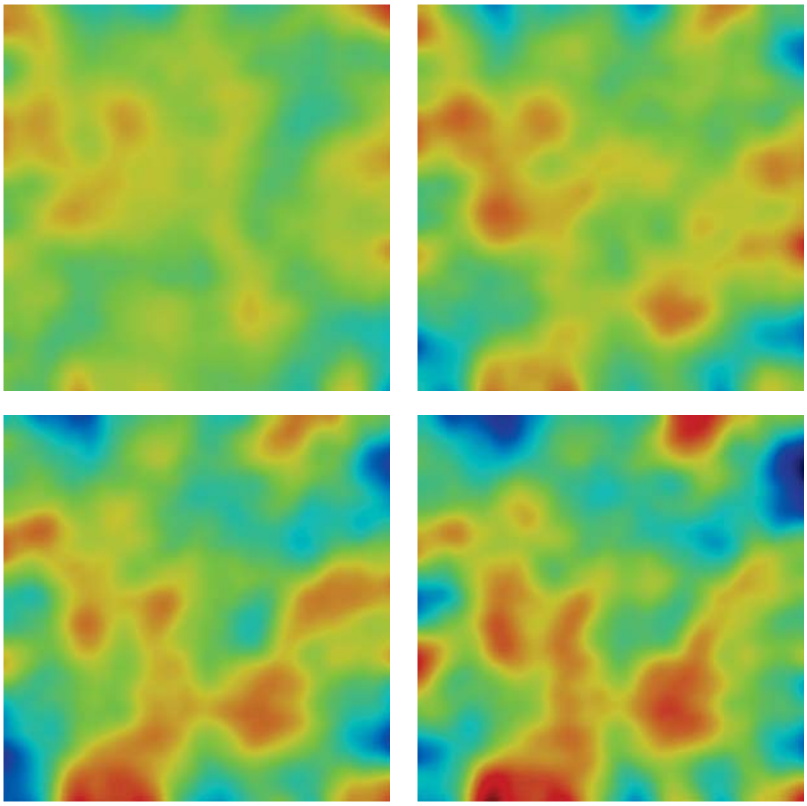


Figure 1: Realization of a Q-Wiener process in $H = L^2([0,1]^2)$, sampled at times $t = 0.25, 0.50, 0.75$ and 1.00.

to such processes. The integral is first defined in terms of so called *simple* integrands, which are those $L(U;H)$-valued stochastic processes Ψ for which there, with $m \in \mathbb{N}$, exist a sequence of times $0 = t_0 < t_1 < \ldots < t_i < t_{i+1} < \ldots < t_m = T$, a sequence $(\Psi_i)_{i=1}^{m-1}$ of $L(U;H)$-valued operators and a sequence $(A_i)_{i=1}^{m-1}$ of events in $\mathcal{F}_{t_i}$ such that

$$\Psi(s) = \sum_{i=0}^{m-1} 1_{A_i} 1_{(t_i,t_{i+1}]}(s)\Psi_i$$

for $s \in [0,T]$, where 1_{A_i} and $1_{(t_i,t_{i+1}]}$ are *indicator functions*, i.e., for a set A

$$1_A(x) = \begin{cases} 1 & \text{if } x \in A, \\ 0 & \text{otherwise.} \end{cases}$$

For these processes one sets

$$I_t^L(\psi) = \sum_{i=0}^{m-1} 1_{A_i} \Psi_i (L(t_{i+1} \wedge t) - L(t_i \wedge t))$$

for which the so called *Itô isometry*

$$\mathbb{E}\left[\|I_t^L(\Psi)\|_H^2\right] = \mathbb{E}\left[\int_0^t \|\Psi(s)\|^2_{L_{\mathrm{HS}}(U_0;H)} \,\mathrm{d}s\right] \tag{4}$$

holds true, where $U_0 = Q^{1/2}(U)$ is the Hilbert space equipped with the inner product

$$\langle \cdot, \cdot \rangle_{U_0} = \left\langle Q^{-\frac{1}{2}} \cdot, Q^{-\frac{1}{2}} \cdot \right\rangle_U,$$

$Q^{-1/2}$ denoting the pseudo-inverse of $Q^{1/2}$. The space $\mathcal{N}_T^L(H)$ of admissible integrands is now defined as the completion of the space of simple processes with respect to the norm

$$\|\Psi\|_T = \left(\mathbb{E}\left[\int_0^T \|\Psi(s)\|^2_{L_{\mathrm{HS}}(U_0;H)} \,\mathrm{d}s\right]\right)^{\frac{1}{2}},$$

and $I_t^L : \mathcal{N}_T^L(H) \to L^2(\Omega; H)$ is well defined as a continuous extension. Since $\|\Psi\|_t \leq \|\Psi\|_T$, the Itô isometry (4) holds true for any $t \in [0,T]$ and any admissible integrand.

One can also construct $\mathcal{N}_T^L(H)$ by

$$\mathcal{N}_T^L(H) = L^2(([0,T] \times \Omega, \mathcal{P}_T, \,\mathrm{d}t \otimes P)\,; L_{\mathrm{HS}}(U_0; H))$$

where $\mathcal{P}_T$ denotes the predictable σ-algebra, i.e., the σ-algebra generated by the set

$$\{(s,t] \times A \subseteq [0,T] \times \Omega \,|\, 0 \leq s < t, A \in \mathcal{F}_s\}.$$

A stochastic process which is measurable with respect to this σ-algebra is said to be *predictable*.

3 Stochastic partial differential equations and approximations

Let us now return to (1), the SPDE of the introduction, to discuss what we mean by a solution to it. Recall that the equation is given by

$$\begin{aligned} \mathrm{d}X(t) &= (AX(t) + F(X(t)))\,\mathrm{d}t + G(X(t))\,\mathrm{d}L(t), \\ X(0) &= X_0, \end{aligned} \tag{5}$$

where $t \in [0,T]$, $X_0 \in L^2(\Omega;H)$, $F : H \to H$, $G : H \to L_{\mathrm{HS}}(U_0;H)$ and L is a U-valued Q-Lévy process. This is to be understood as the integral equation

$$X(t) = X_0 + \int_0^t AX(s) + F(X(s))\,\mathrm{d}s + \int_0^t G(X(s))\,\mathrm{d}L(s), \tag{6}$$

where the first integral is of Bochner type and the second is the stochastic integral introduced in Section 2.4. In order to make sense of this process, there are several notions of solutions in the literature. Solutions formulated in terms of the integral equation (6) are referred to as *strong solutions*. We will, however, be concerned with the weaker concept of *mild solutions* which are formulated in terms of the semigroup $(E(t))_{t\in[0,T]}$ generated by A.

Definition 3.1. Let $X_0 \in L^2(\Omega;H)$. A predictable process $X = (X(t))_{t\in[0,T]}$ is called a *mild solution to Equation* (5) if

$$\sup_{t\in[0,T]} \|X(t)\|_{L^2(\Omega;H)} < \infty$$

and for all $t \in [0,T]$

$$X(t) = E(t)X_0 + \int_0^t E(t-s)F(X(s))\,\mathrm{d}s + \int_0^t E(t-s)G(X(s))\,\mathrm{d}L(s),\ P\text{-a.s.}$$

Under the following set of assumptions, [20, Theorem 9.15] and [20, Theorem 9.29] ensure the existence of an up to modification unique mild solution X of Equation (5). Since in our setting $(E(t))_{t\in[0,T]}$ is a semigroup of contractions, [20, Theorem 9.29] also guarantees the existence of a càdlàg modification of X.

Assumption 3.2. The parameters F, G and L of Equation (5) fulfil the following.

- For $F : H \to H$ there exist a constant $C > 0$ such that for all $f, g \in H$

$$\|F(f) - F(g)\|_H \le C\|f - g\|_H.$$

- For $G : H \to L_{\mathrm{HS}}(U_0;H)$ there exist a constant $C > 0$ such that for all $f, g \in H$

$$\|G(f) - G(g)\|_{L_{\mathrm{HS}}(U_0;H)} \le C\|f - g\|_H.$$

- L is a U-valued mean zero square integrable Q-Lévy process.

In general, these are stronger than what is needed for the existence of a mild solution. See e.g., [20, Chapter 9] and [13, Chapter 2] for examples of relaxed assumptions.

3.1 Spatial discretization

Since an analytical solution to (5) is rarely available, one has to *discretize* the equation in space and in time if one want to simulate it on a computer. We speak of a *fully discrete approximation* of the mild solution X if it is discretized in both space and time. To arrive at such an approximation is the goal of this section and the next. In this first part, which mirrors the setting of [13], we consider spatial discretizations. The main idea is to seek solutions to (5) in some finite-dimensional subspace of H, where the operators involved are replaced with finite-dimensional counterparts.

Let $(V_h)_{h\in(0,1]}$ be a family of subspaces of $\dot{H}^1$ such that $\dim(V_h) = N_h < \infty$, $h \in (0,1]$. By $P_h : H \to V_h$ and $R_h : \dot{H}^1 \to V_h$ we denote orthogonal projectors onto V_h with respect to the inner products of H and $\dot{H}^1$ respectively, which is to say that for all $f \in H$, $g \in \dot{H}^1$ and $f_h \in V_h$ we have $\langle P_h f, f_h\rangle_H = \langle f, f_h\rangle_H$ and $\langle R_h g, f_h\rangle_1 = \langle g, f_h\rangle_1$. In order to guarantee the convergence of a fully discrete approximation to the mild solution, the following assumption on the sequence $(V_h)_{h\in(0,1]}$ formulated in terms of these projectors is given in [13].

Assumption 3.3. There exists a constant $C > 0$ such that the following statements are true.

- For all $f \in \dot{H}^1$, $\|P_h f\|_1 \leq C\|f\|_1$.
- For all $f \in \dot{H}^1$, $\|R_h f - f\|_H \leq Ch\|f\|_1$.
- For all $f \in \dot{H}^2$, $\|R_h f - f\|_H \leq Ch^2\|f\|_2$.

From [13, Section 3.2] we cite two examples of subspace sequences that fulfil Assumption 3.3.

Example 3.4 (Standard finite element method)**.** Consider the setting of Example 2.2, where $H = L^2(D;\mathbb{R})$, $D \subseteq \mathbb{R}^d$ with $d \in \{1,2,3\}$. Let $(T_h)_{h\in(0,1]}$ be a regular quasi-uniform family of triangulations of D with h being the maximal mesh size. We let V_h be the space of all functions that are continuous and piecewise linear on T_h and zero at the boundary of D.

Example 3.5 (Spectral Galerkin method)**.** Let us restrict Example 3.4 to the case that $D = [0,1]$ and $A = \Delta$ with Dirichlet boundary conditions. In this setting the orthonormal eigenbasis $(e_i)_{i\in\mathbb{N}}$ and the sequence of eigenvalues $(\lambda_i)_{i\in\mathbb{N}}$ to $-A$ are explicitly known to be $e_i = \sqrt{2}\sin(i\pi\cdot)$ and $\lambda_i = i^2\pi^2$ for all $i \in \mathbb{N}$. If we now, for $N \in \mathbb{N}$, set $h = \lambda_{N+1}^{-1/2}$ and $V_h = \operatorname{span}(e_1, e_2, \ldots, e_N)$ we get a sequence $(V_h)_{h\in(0,1]}$ that fulfils Assumption 3.3.

Given a fixed subspace V_h, $-A_h : V_h \to V_h$, the discrete version of $-A$, is now defined by the relationship

$$\langle -A_h f_h, g_h\rangle_H = \langle f_h, g_h\rangle_1 = \left\langle (-A)^{\frac{1}{2}} f_h, (-A)^{\frac{1}{2}} g_h \right\rangle_H$$

for all $f_h, g_h \in V_h$. It is straightforward to see that this operator is self-adjoint and positive definite (hence invertible) on V_h which, in the same way as before, entails the existence of an orthonormal eigenbasis $(e_{h,i})_{i=1,\dots,N_h}$ of V_h and an increasing positive sequence $(\lambda_{h,i})_{i=1,\dots,N_h}$ of eigenvalues of $-A_h$. For the same reason, A_h generates a C_0-semigroup of contractions $(E_h(t))_{t\in[0,T]}$. Therefore, replacing F and G by P_hF and P_hG respectively, Assumption 3.2 also guarantees the existence of a *semidiscrete mild solution* X_h to the equation

$$\begin{aligned} \mathrm{d}X_h(t) &= (A_hX_h(t) + P_hFX_h(t))\,\mathrm{d}t + P_hG(X_h(t))\,\mathrm{d}L(t), \\ X_h(0) &= P_hX_0. \end{aligned}$$

3.2 Spatio-temporal discretization

In this section we further discretize the semidiscrete solution by considering *rational approximations* of $(E_h(t))_{t\in[0,T]}$, where we follow the approach of [21]. Let us therefore for convenience consider a uniform time grid given by $t_j = j\Delta t$ for $j = 0,\dots,N_{\Delta t}$, where $N_{\Delta t} \in \mathbb{N}$ and $\Delta t = TN_{\Delta t}^{-1}$.

A *rational approximation of order* p of the exponential function is a rational function $R : \mathbb{C} \to \mathbb{C}$ satisfying that there exist constants $C, \delta > 0$ such that for all $z \in \mathbb{C}$ with $|z| < \delta$

$$|R(z) - \exp(z)| \leq C|z|^{p+1}.$$

Since R is rational there exist polynomials r_{n} and r_{d} such that $R = r_{\mathrm{d}}^{-1}r_{\mathrm{n}}$. With the introduced notation, $R(\Delta tA_h) \in L(V_h)$ is for all $f_h \in V_h$ given by

$$R(\Delta tA_h)f_h = r_{\mathrm{d}}^{-1}(\Delta tA_h)r_{\mathrm{n}}(\Delta tA_h)f_h = \sum_{k=1}^{N_h} \frac{r_{\mathrm{n}}(-\Delta t\lambda_{h,k})}{r_{\mathrm{d}}(-\Delta t\lambda_{h,k})} \langle f_h, e_{h,k}\rangle_H\, e_{h,k}.$$

The family of $L(V_h)$-valued operators $(E_{h,\Delta t}(t))_{t\in[0,T]}$ defined by $E_{h,\Delta t}(t) = R(\Delta tA_h)^j$ for $t \in [t_{j-1}, t_j)$, $j = 1,\dots,N_{\Delta t}$, is the rational approximation of the semigroup $(E_h(t))_{t\in[0,T]}$.

Let us now, for $f_h \in V_h$, define a *deterministic approximation operator* on V_h by

$$D^{\mathrm{det}}_{\Delta t,h}(f_h) = R(\Delta tA_h)f_h + r_{\mathrm{d}}^{-1}(\Delta tA_h)\Delta tP_hF(f_h), \tag{7}$$

and, for $j = 1,\dots,N_{\Delta t}$, a *stochastic approximation operator* by

$$D^{\mathrm{stoch},j}_{\Delta t,h}(f_h) = r_{\mathrm{d}}^{-1}(\Delta tA_h)P_hG(f_h)\Delta L^j, \tag{8}$$

where the Lévy increment $\Delta L^j = L(t_{j+1}) - L(t_j)$. Then, the fully discrete approximation $X_{h,\Delta t} = (X^{t_j}_{h,\Delta t})_{j=0,\dots,N_{\Delta t}}$ of the mild solution to Equation (5) is, for $j = 0,\dots,N_{\Delta t} - 1$, given by the recursion scheme

$$\begin{aligned} X^{t_{j+1}}_{h,\Delta t} &= D^{\mathrm{det}}_{\Delta t,h}X^{t_j}_{h,\Delta t} + D^{\mathrm{stoch},j}_{\Delta t,h}X^{t_j}_{h,\Delta t}, \\ X^0_{h,\Delta t} &= P_hX_0. \end{aligned} \tag{9}$$

Example 3.6. An important example of a rational approximation of $(E_h(t))_{t\in[0,T]}$ is the *backward Euler scheme*, where $R(\Delta t A_h)$ is defined through $r_{\mathrm{d}}(x) = 1 - x$ and $r_{\mathrm{n}}(x) = 1$ for all $x \in \mathbb{R}$, $x \neq 1$. One can then rewrite Scheme (9) as

$$X^{t_{j+1}}_{h,\Delta t} - X^{t_j}_{h,\Delta t} = \Big(A_h X^{t_{j+1}}_{h,\Delta t} + P_h F(X^{t_j}_{h,\Delta t})\Big)\,\Delta t + P_h G(X^{t_j}_{h,\Delta t})\Delta L^j, \tag{10}$$

where $j = 0, \ldots, N_{\Delta t} - 1$. Another example is the *forward Euler scheme* defined through $r_{\mathrm{d}}(x) = 1$ and $r_{\mathrm{n}}(x) = 1 + x$ for all $x \in \mathbb{R}$, which can similarly be rewritten as

$$X^{t_{j+1}}_{h,\Delta t} - X^{t_j}_{h,\Delta t} = \Big(A_h X^{t_j}_{h,\Delta t} + P_h F(X^{t_j}_{h,\Delta t})\Big)\,\Delta t + P_h G(X^{t_j}_{h,\Delta t})\Delta L^j,$$

for $j = 0, \ldots, N_{\Delta t} - 1$.

3.3 Strong and weak convergence

To finish this introduction to approximations of SPDE we briefly review results for two different notions of convergence of the family $\hat{X} = (X_{h,\Delta t}, h \in (0,1], N_{\Delta t} \in \mathbb{N})$ to the mild solution X of Equation (5). The approximation is said to *strongly converge* to X if

$$\sup_{j\in\{1,\ldots,N_{\Delta t}\}} \|X^{t_j}_{h,\Delta t} - X(t_j)\|_{L^2(\Omega;H)} \to 0$$

as $h, \Delta t \to 0$. However, one might not always be interested in approximating X in a mean square sense but only in the mean value of a functional of the solution. We say that $\hat{X}$ *converges weakly* to X if

$$\sup_{j\in\{1,\ldots,N_{\Delta t}\}} \left|\mathbb{E}\left[\phi\big(X^{t_j}_{h,\Delta t}\big) - \phi\big(X(t_j)\big)\right]\right| \to 0$$

as $h, \Delta t \to 0$ for all $\phi \in C^2_{\mathrm{p}}(H;\mathbb{R})$, the space of all twice Fréchet differentiable mappings $\phi : H \to \mathbb{R}$ such that the derivatives of ϕ have at most polynomial growth. One can show that for such mappings, strong convergence implies weak convergence.

Let us now introduce a set of assumptions that together with the previous ones ensures the strong convergence of $\hat{X}$ to X.

Assumption 3.7. The rational approximation of $(E_h(t))_{t\in[0,T]}$ is given by the backward Euler scheme, i.e., R is defined through $r_{\mathrm{d}}(x) = 1 - x$ and $r_{\mathrm{n}}(x) = 1$ for all $x \in \mathbb{R}$, $x \neq 1$. Furthermore, the noise is assumed to be Gaussian, i.e., $L = W$ is a U-valued Q-Wiener process, and the initial value $X_0 \in \dot{H}^1$ is deterministic.

An important consequence of this assumption is, by [13, Theorem 2.25], the uniform bound

$$\sup_{t\in[0,T]} \|X(t)\|_{L^p(\Omega;H)} < \infty, \tag{11}$$

which holds for any $p \geq 1$. The following result on the strong convergence is a direct consequence of [13, Theorem 3.14].

Theorem 3.8. *Under Assumptions 3.2, 3.3 and 3.7, for any $p \geq 1$, there exists a constant C such that, for any fully discrete approximation $X_{h,\Delta t}$,*

$$\sup_{j\in\{1,\dots,N_{\Delta t}\}} \|X_{h,\Delta t}^{t_j} - X(t_j)\|_{L^p(\Omega;H)} \leq C\left(h + (\Delta t)^{\frac{1}{2}}\right).$$

We say that the strong convergence is of rate 1 in space and 1/2 in time. As a rule of thumb the weak convergence rate is twice that of the strong rate (see e.g. [3, 9, 12] for fully discrete approximations with additive noise and [4, 6, 8, 10, 11] for semi-discrete approximations with multiplicative noise). Proving this in this specific setting is work in progress, so in the mean time we formulate the following conjecture.

Conjecture 3.9. *Under Assumptions 3.2, 3.3 and 3.7, for all $\phi \in C_{\mathrm{p}}^2(H;\mathbb{R})$, there exists a constant C such that, for any fully discrete approximation $X_{h,\Delta t}$,*

$$\sup_{j\in\{1,\dots,N_{\Delta t}\}} \left|\mathbb{E}\left[\phi\big(X_{h,\Delta t}^{t_j}\big) - \phi\big(X(t_j)\big)\right]\right| \leq C\left(h^2 + \Delta t\right).$$

4 Monte Carlo methods

Since the mean of functionals of the mild solution of Equation (5) cannot in general be explicitly evaluated, we have to introduce an approximation of the expectation operator $\mathbb{E}[\cdot]$. We will consider two approximations that are both based on simulating a large number of approximate solutions to (5) and taking the average of a functional applied to it, but we formulate the theory in the general setting of real-valued random variables.

The *Monte Carlo estimator* E_N of a real-valued random variable $Y \in L^2(\Omega;\mathbb{R})$ is given by

$$E_N[Y] = \frac{1}{N}\sum_{i=1}^{N} Y^{(i)},$$

where $(Y^{(i)})_{i=1}^N$ is a sequence of independent, identically distributed random variables that have the same law as Y. The convergence of $E_N[Y]$ to $\mathbb{E}[Y]$ as $N \to \infty$ is ensured by a mean square version of the *law of large numbers*

$$\begin{aligned}
\|\,\mathbb{E}[Y] - E_N[Y]\|_{L^2(\Omega;\mathbb{R})}^2 &= \left\|\frac{1}{N}\sum_{i=1}^{N}\big(\mathbb{E}[Y] - Y^{(i)}\big)\right\|_{L^2(\Omega;\mathbb{R})}^2 \\
&= \frac{1}{N^2}\sum_{i=1}^{N}\big\|\mathbb{E}[Y] - Y^{(i)}\big\|_{L^2(\Omega;\mathbb{R})}^2 \\
&= \frac{1}{N}\mathsf{Var}(Y) \leq \frac{1}{N}\|Y\|_{L^2(\Omega;\mathbb{R})}^2.
\end{aligned} \tag{12}$$

Instead of a single random variable Y we can consider a sequence $(Y_\ell)_{\ell\in\mathbb{N}_0}$ of random variables, where $Y_\ell \in L^2(\Omega;\mathbb{R})$ and the index $\ell \in \mathbb{N}_0$ is referred to as a *level*. The *multilevel*

Monte Carlo estimator E^L of $Y_L \in (Y_\ell)_{\ell\in\mathbb{N}_0}$ is, for $L \in \mathbb{N}$, defined by

$$E^L[Y_L] = E_{N_0}[Y_0] + \sum_{\ell=1}^{L} E_{N_\ell}[Y_\ell - Y_{\ell-1}],$$

where $(N_\ell)_{\ell=0}^L$ consists of level specific numbers of samples in the respective Monte Carlo estimators. A telescoping sum argument shows that as an estimator of $\mathbb{E}[Y_L]$ the multilevel Monte Carlo estimator is unbiased. Under the assumption that $(Y_\ell)_{\ell\in\mathbb{N}_0}$ converges to some random variable Y, a calculation similar to (12) gives the error estimate

$$\|\,\mathbb{E}[Y] - E^L[Y_L]\|_{L^2(\Omega;\mathbb{R})} \le |\,\mathbb{E}[Y - Y_L]\,| + \left(\frac{1}{N_0}\|Y_0\|^2_{L^2(\Omega;\mathbb{R})} + \sum_{l=1}^{L}\frac{1}{N_\ell}\|Y_\ell - Y_{\ell-1}\|^2_{L^2(\Omega;\mathbb{R})}\right)^{\frac{1}{2}}.$$

In this context, the advantage of using a multilevel Monte Carlo estimator compared to a standard Monte Carlo estimator is due to the flexibility allowed by letting the sample sizes $(N_\ell)_{\ell=0}^L$ depend on a bound on $\|Y_\ell - Y_{\ell-1}\|_{L^2(\Omega;\mathbb{R})}$, $\ell = 1,\dots,L$. In the case that sampling Y_ℓ for small ℓ is computationally cheaper then sampling Y_L, we can let the sampling effort be concentrated on the coarser levels $\ell \ll L$. We then need to choose the sample sizes in such a way that the overall error is balanced. The following theorem, which is a special case of [14, Theorem 1], shows how one does this and provides bounds on the overall computational work.

Theorem 4.1. *Let $(a_\ell)_{\ell\in\mathbb{N}_0}$ be a decreasing sequence of positive real numbers that converges to zero and assume that there exist constants C_1, C_2, C_3 and a parameter $\eta \in [0,1]$, such that for all $\ell \in \mathbb{N}$, Y_ℓ and $Y_{\ell-1}$ fulfil*

$$|\mathbb{E}[Y - Y_\ell]| \le C_1 a_\ell,$$

and

$$\|Y_\ell - Y_{\ell-1}\|_{L^2(\Omega;\mathbb{R})} \le C_2 a_\ell^\eta$$

for $\ell \in \mathbb{N}_0$ and that Y_0 fulfils $\|Y_0\|_{L^2(\Omega;\mathbb{R})} = C_3$. For $L \in \mathbb{N}, \ell = 1,\dots,L$, $\epsilon > 0$, set $N_\ell = \lceil a_L^{-2} a_\ell^\eta \ell^{1+\epsilon}\rceil$, where $\lceil\cdot\rceil$ is the ceiling function, and $N_0 = \lceil a_L^{-2}\rceil$. Then

$$\|\,\mathbb{E}[Y] - E^L[Y_L]\,\|_{L^2(\Omega;\mathbb{R})} \le \left(C_1^2 + C_3 + C_2 + \zeta(1+\epsilon)\right)^{\frac{1}{2}} a_L,$$

where ζ denotes the Riemann zeta function. Furthermore, assume that, for some constants C_4, C_5, C_6 and $\kappa, \delta > 0$, the work $\mathcal{W}_\ell^B$ of one calculation of $Y_\ell - Y_{\ell-1}$, $\ell \ge 1$, is bounded by $C_4 a_\ell^{-\kappa}$, that the work of one calculation of Y_0 is bounded by C_5 and that the addition of the Monte Carlo estimators adds $C_6 a_L^{-\delta}$ to the overall work load $\mathcal{W}_L$. Then there exists another constant C_7 such that $\mathcal{W}_L$ is bounded by

$$\mathcal{W}_L \le C_7 a_L^{-2}\left(C_5 + C_4\sum_{\ell=1}^{L} a_\ell^{-(\kappa-2\eta)}\ell^{1+\epsilon}\right) + C_6 a_L^{-\delta}.$$

Furthermore, if there exists $a > 1$ such that $a_\ell = \mathrm{O}(a^{-\ell})$ then the bound on $\mathcal{W}_L$ simplifies to

$$\mathcal{W}_L = \begin{cases} \mathrm{O}(a_L^{-\max\{2,\delta\}}), & \text{if } \kappa < 2\eta \\ \mathrm{O}(\max\{a_L^{-(2+\kappa-2\eta)} L^{2+\epsilon}, a_L^{-\delta}\}), & \text{if } \kappa \geq 2\eta. \end{cases}$$

Example 4.2. Let us consider a concrete example of a Monte Carlo simulation in the context of Examples 2.2 and 3.4 under Assumptions 3.2 and 3.7 and compare the computational costs of the Monte Carlo and multilevel Monte Carlo estimators in this setting. Below we let $C > 0$ be a general constant that may change from line to line.

Recall that we seek to approximate the $L^2(D;\mathbb{R})$-valued solution of Equation (5) with the family $\hat{X} = (X_{h,\Delta t}, h \in (0,1], N_{\Delta t} \in \mathbb{N})$, where the V_h-valued sequence $X_{h,\Delta t}$ is given by the backward Euler scheme (10) and V_h is the space of all piecewise linear functions on T_h, a triangulation with maximal mesh width h. Let us introduce a subsequence of approximations, indexed by levels ℓ, by $\check{X} = (X_\ell = X_{h,\Delta t}, h = 2^{-\ell}, N_{\Delta t} = h^{-2}, \ell \in \mathbb{N}_0)$, and suppose that it is primarily the end time value $X(T)$ that we are interested in. Theorem 3.8 then ensures that there for all $p \geq 1$ exists a constant C such that for all $\ell \in \mathbb{N}_0$

$$\|X(T) - X_\ell^T\|_{L^p(\Omega;H)} \leq C2^{-\ell}.$$

Given a functional $\phi \in C_{\mathrm{p}}^2(H;\mathbb{R})$ we now set $Y = \phi(X(T))$ and let Y_ℓ of $(Y_\ell)_{\ell\in\mathbb{N}_0}$ be given by $Y_\ell = \phi(X_\ell^T)$. By the mean value theorem for Fréchet differentiable mappings we have

$$\begin{aligned} &\|Y - Y_\ell\|^2_{L^2(\Omega;\mathbb{R})} \\ &\quad = \mathbb{E}\left[\left|\phi\big(X(T)\big) - \phi\big(X_\ell^T\big)\right|^2\right] \\ &\quad = \mathbb{E}\left[\left|\int_0^1 \left\langle \phi'\big(X_\ell^T + s\big(X(T) - X_\ell^T\big)\big), X(T) - X_\ell^T\right\rangle_H \,\mathrm{d}s\right|^2\right] \\ &\quad \leq C\|X(T) - X_\ell^T\|^2_{L^{2(p+1)}(\Omega;H)}\left(1 + \|X(T)\|^{2p}_{L^{2(p+1)}(\Omega;H)} + \|X_\ell^T\|^{2p}_{L^{2(p+1)}(\Omega;H)}\right) \\ &\quad \leq C\|X(T) - X_\ell^T\|^2_{L^{2(p+1)}(\Omega;H)} \end{aligned} \tag{13}$$

by using Hölder's inequality and the assumption that there exists $p \geq 2$, $C > 0$ such that $\|\phi'(f)\|_H \leq C(1 + \|f\|_H^p)$ for all $f \in H$ (cf. [13, Chapter 1]). Here the constant of the first inequality depends on p but not on ℓ. In the second inequality, we use the uniform bound (11) on X and the fact that the convergence result of Theorem 3.8 implies a similar bound on X_ℓ to get a constant which depends on X but not on ℓ. By a similar argument one shows that there exists a constant C not depending on ℓ, such that

$$\|Y_\ell\|_{L^2(\Omega;\mathbb{R})} = \mathbb{E}\left[\left|\phi\left(X_\ell^T\right)\right|^2\right]^{\frac{1}{2}} \leq C \tag{14}$$

for all $\ell \in \mathbb{N}_0$.

We use these results to deduce that the error of the standard Monte Carlo estimation of $\mathbb{E}[\phi(X(T))]$ is, for $\ell \in \mathbb{N}_0$, by the triangle inequality, Conjecture 3.9 and (12) bounded by

$$\begin{aligned}
&\left\|\mathbb{E}\left[\phi(X(T))\right] - E_N\left[\phi(X_\ell^T)\right]\right\|_{L^2(\Omega;\mathbb{R})} \\
&\quad \leq \left\|\mathbb{E}\left[\phi(X(T))\right] - \mathbb{E}\left[\phi(X_\ell^T)\right]\right\|_{L^2(\Omega;\mathbb{R})} + \left\|\mathbb{E}\left[\phi(X_\ell^T)\right] - E_N\left[\phi(X_\ell^T)\right]\right\|_{L^2(\Omega;\mathbb{R})} \\
&\quad \leq C2^{-2\ell} + \frac{1}{\sqrt{N}}\|\phi(X_\ell^T)\|_{L^2(\Omega;\mathbb{R})} \leq C\left(2^{-2\ell} + N^{-\frac{1}{2}}\right),
\end{aligned}$$

where the constant of the second inequality is that of Conjecture 3.9. Therefore, to ensure that the Monte Carlo error does not dominate the error of the approximation of $\mathbb{E}[\phi(X(T))]$ one should set the number of samples $N \simeq 2^{4\ell}$. For the multilevel Monte Carlo scheme, Theorem 3.8 along with Equation (13) ensure that

$$\begin{aligned}
\|Y_\ell - Y_{\ell-1}\|_{L^2(\Omega;\mathbb{R})} &\leq \|Y - Y_\ell\|_{L^2(\Omega;\mathbb{R})} + \|Y - Y_{\ell-1}\|_{L^2(\Omega;\mathbb{R})} \\
&\leq C\left(2^{-\ell} + 2^{-\ell-1}\right) \leq C2^{-\ell},
\end{aligned}$$

so that the conditions of Theorem 4.1 are fulfilled with $a_\ell = 2^{-2\ell}$, $\ell \in \mathbb{N}_0$ and $\eta = 1/2$. Therefore there exists a constant C such that for all $L \in \mathbb{N}$

$$\|\,\mathbb{E}\,[Y] - E^L\,[Y_L]\,\|_{L^2(\Omega;\mathbb{R})} \leq C2^{-2\ell}$$

as long as the level dependent sample sizes N_ℓ, $\ell \in \mathbb{N}_0$, are chosen to be $N_\ell = \lceil 2^{4L-2\ell}\ell^{1+\epsilon}\rceil$ for $\ell > 0$ and $N_0 = 2^{4L}$. This means that for a given level L, the majority of samples are taken at a coarse level while retaining the same rate of convergence compared to the standard Monte Carlo method. Assuming that the computational work of solving the backward Euler system (10) at one time step of level ℓ is bounded by $\mathrm{O}(2^{\ell d})$, where d is the dimension of the underlying spatial domain, and that the computational cost of computing $Y_\ell - Y_{\ell-1}$ is roughly equivalent to the cost of Y_ℓ, for $\ell \in \mathbb{N}_0$, the total cost of computing $E^L\,[Y_L]$ is by Theorem 4.1 bounded by $\mathcal{W}_L = \mathrm{O}(2^{(3+d)L}L^{2+\epsilon})$. Thus the computational cost of the multilevel Monte Carlo estimation is significantly cheaper than that of the single level Monte Carlo estimator $E_N\left[\phi(X_L^T)\right]$ with $N = 2^{4L}$ samples, which in comparison is bounded by $\mathcal{W}_L = \mathrm{O}(2^{(6+d)L})$, while retaining the same rate of convergence.

5 Summary of Paper 1

Consider the analysis of weak errors for fully discrete approximations of solutions to SPDEs, that is to say, in the context of Example 4.2, errors of the type $|\,\mathbb{E}[Y - Y_\ell]|$ where ϕ is some given functional, $Y = \phi(X(T))$ and Y_ℓ of $(Y_\ell)_{\ell\in\mathbb{N}_0}$ is given by $Y_\ell = \phi(X_\ell^T)$ for each level $\ell \in \mathbb{N}_0$. This topic has been investigated in the community of numerical analysis of SPDEs for some time. Yet, simulations that illustrate the theoretical results of such investigations are rarely available. Furthermore, while weak convergence for equations driven by additive noise exist, cf. Section 3.3, theoretical results for the case of multiplicative noise are still work in progress, at least for a finite element spatial discretization. In these cases, simulations of weak convergence rates can inform us about the plausibility of claims on the rate, such as the one of Conjecture 3.9.

One reason for the lack of simulations in the literature is the computational expense of simulating a solution to an SPDE, which must be repeated a large number of times when using a Monte Carlo method to approximate the expectation that is part of the weak error. Due to this computational complexity, it is important to carefully consider which Monte Carlo method one chooses in order to accurately simulate weak error rates. In Paper 1 we present four methods of simulating such rates and analyse the additional error caused by the Monte Carlo approximation involved in each of them.

In the paper, the analysis is done for the more general problem of approximating the quantity $|\mathbb{E}[Y - Y_\ell]|$, where $(Y_\ell)_{\ell \in \mathbb{N}_0}$ is a sequence of mean square integrable random variables converging to $Y \in L^2(\Omega; \mathbb{R})$. If one were interested in estimating $\mathbb{E}[Y - Y_\ell]$, the method of *common random numbers* would tell us that when Y and Y_ℓ are positively correlated, which is reasonable to assume in the case that the latter random variable is an approximation of the former, an estimator of the form $\mathbb{E}[Y] - E_N[Y_\ell]$ is outperformed by $E_N[Y - Y_\ell]$, since the former has higher variance and both are unbiased. Now, when estimating $|\mathbb{E}[Y - Y_\ell]|$, the estimators $|\mathbb{E}[Y] - E_N[Y_\ell]|$ and $|E_N[Y - Y_\ell]|$ are in general biased, so a direct comparison cannot be made. Instead we show that the mean squared error of the former estimator is bounded from below by

$$\| |\mathbb{E}[Y - Y_\ell]| - |\mathbb{E}[Y] - E_N[Y_\ell]| \|_{L^2(\Omega;\mathbb{R})} \geq -|\mathbb{E}[Y - Y_\ell]| + \left(|\mathbb{E}[Y - Y_\ell]|^2 + N^{-1} \mathsf{Var}[Y_\ell] \right)^{1/2}$$

and from above by

$$\| |\mathbb{E}[Y - Y_\ell]| - |\mathbb{E}[Y] - E_N[Y_\ell]| \|_{L^2(\Omega;\mathbb{R})} \leq N^{-1/2} (\mathsf{Var}[Y_\ell])^{1/2}.$$

For the latter estimator, the corresponding bounds are shown to be

$$\begin{aligned} &\| |\mathbb{E}[Y - Y_\ell]| - |E_N[Y - Y_\ell]| \|_{L^2(\Omega;\mathbb{R})} \\ &\quad \geq -|\mathbb{E}[Y - Y_\ell]| + \left(|\mathbb{E}[Y - Y_\ell]|^2 + N^{-1} \mathsf{Var}[Y - Y_\ell] \right)^{1/2} \end{aligned}$$

and

$$\| |\mathbb{E}[Y - Y_\ell]| - |E_N[Y - Y_\ell]| \|_{L^2(\Omega;\mathbb{R})} \leq N^{-1/2} (\mathsf{Var}[Y - Y_\ell])^{1/2}.$$

Therefore, under the assumption that the quantity of interest $|\mathbb{E}[Y - Y_n]| \ll N^{-1/2}$ is very small, which is usually the case in the context of weak error simulations of SPDE approximations, the former estimator will behave like $N^{-1/2}(\mathsf{Var}[Y_\ell])^{1/2}$ and the latter like $N^{-1/2}(\mathsf{Var}[Y - Y_\ell])^{1/2}$. So if Y and Y_ℓ have a positive correlation, which they in general do in such simulations, the additional error of the latter estimator will be significantly smaller than that of the former.

In addition to this, the additional error caused by estimating $|\mathbb{E}[Y - Y_L]|$ with a multilevel Monte Carlo estimator $|\mathbb{E}[Y] - E^L[Y_L]|$ is analysed. We find that the mean squared error of this estimator is bounded from below by

$$\begin{aligned}
&\| |\mathbb{E}[Y - Y_L]| - |\mathbb{E}[Y] - E^L[Y_L]| \|_{L^2(\Omega;\mathbb{R})} \\
&\quad \geq -|\mathbb{E}[Y - Y_L]| + \left(|\mathbb{E}[Y - Y_L]|^2 + N_0^{-1} \mathsf{Var}[Y_0] + \sum_{\ell=1}^{L} N_\ell^{-1} \mathsf{Var}[Y_\ell - Y_{\ell-1}] \right)^{1/2}
\end{aligned}$$

and from above by

$$\| |\mathbb{E}[Y - Y_L]| - |\mathbb{E}[Y] - E^L[Y_L]| \|_{L^2(\Omega;\mathbb{R})} \leq \left(N_0^{-1} \mathsf{Var}[Y_0] + \sum_{\ell=1}^{L} N_\ell^{-1} \mathsf{Var}[Y_\ell - Y_{\ell-1}] \right)^{1/2}.$$

When choosing the sample sizes similarly to the choice made in Theorem 4.1, it turns out that

$$\| |\mathbb{E}[Y - Y_L]| - |\mathbb{E}[Y] - E^L[Y_L]| \|_{L^2(\Omega;\mathbb{R})} \simeq |\mathbb{E}[Y - Y_L]|,$$

which is to say that the additional error coming form the Monte Carlo method will asymptotically not affect the observed rate of the weak error simulations. For completeness, the additional error caused by the multilevel Monte Carlo estimator $|E^L[Y - Y_L]|$ was also analysed, although one should note that this estimator is of no practical interest. This is due to the fact that $E_{N_0}[Y - Y_0]$ has to be computed, i.e., many samples of the exact solution must be generated, which destroys the idea of multilevel Monte Carlo methods.

An attempt was then made to simulate weak error rates using these estimators in the context of a fully discrete approximation of the one-dimensional stochastic heat equation driven by multiplicative Wiener noise, using a finite element approximation in space and the backward Euler scheme in time, that is to say Example 4.2 restricted to the case that $A = \Delta$. Due to the large sample sizes involved, this simulation is computationally highly expensive and was therefore performed on a cluster at Chalmers Centre for Computational Science and Engineering (C3SE). It was observed that the estimators $E_N[Y - Y_\ell]$ and $|\mathbb{E}[Y] - E^L[Y_L]|$, where Y is replaced by a reference solution, outperform $\mathbb{E}[Y] - E_N[Y_\ell]$ in the sense that the simulated weak error rate more closely resembles the prediction of Conjecture 3.9. The error rates were also simulated for the simpler case of approximating a geometric Brownian motion. The lower computational costs of this allowed for finer simulations, which illustrated the theoretical bounds in an even clearer way.

6 Deterministic simulation of weak convergence rates

In Paper 1 the functional ϕ used in the actual simulations of the weak error was chosen to be $\phi = \| \cdot \|_H^2$, so that the goal of the computations carried out was nothing but the approximation of the second moment $\mathbb{E}[\|X(T)\|_H^2]$ of the end time value of the solution of Equation (5). During the writing of Paper 2, we noted that in this case there is a simple deterministic method for the simulation of weak convergence rates when the terms of the SPDE are linear. The goal of this section is a brief description of it. To this end, let us again consider the setting of Example 4.2 under the simplifying assumption that $F = 0$. Recall that $H = L^2(D;\mathbb{R})$ is the space of square integrable functions on a bounded domain $D \subset \mathbb{R}^d$, $d \in \mathbb{N}$, and V_h is the space of all piecewise linear functions on some triangulation T_h of D. The backward Euler scheme (10) that gives the approximation $X^T_{h,\Delta t} \in L^2(\Omega; V_h)$ of the end time value $X(T) \in L^2(\Omega; H)$ of the solution to the SPDE (5) can then be rewritten as

$$(I - \Delta t A_h)\, X^{t_{j+1}}_{h,\Delta t} = X^{t_j}_{h,\Delta t} + P_h G(X^{t_j}_{h,\Delta t})\Delta L^j,$$

where $j = 0, \dots, N_{\Delta t} - 1$. Since V_h is finite-dimensional and G is assumed to be linear, we have that $P_h G(\cdot)\Delta L^j$ is an $L(V_h)$-valued random variable. Hence, we may rewrite the backward Euler scheme as

$$(I - \Delta t A_h)\, X^{t_{j+1}}_{h,\Delta t} = \left(I + P_h G(\cdot)\Delta L^j\right) X^{t_j}_{h,\Delta t}$$

and by tensorizing this equation we get

$$\begin{aligned}&((I - \Delta t A_h) \otimes (I - \Delta t A_h)) \left(X^{t_{j+1}}_{h,\Delta t} \otimes X^{t_{j+1}}_{h,\Delta t}\right)\\ &= \left(I \otimes I + I \otimes P_h G(\cdot)\Delta L^j + P_h G(\cdot)\Delta L^j \otimes I + P_h G(\cdot)\Delta L^j \otimes P_h G(\cdot)\Delta L^j\right) \left(X^{t_j}_{h,\Delta t} \otimes X^{t_j}_{h,\Delta t}\right).\end{aligned}$$

Applying the expectation operator $\mathbb{E}[\cdot]$ to both sides of this tensorized equation then yields a simple recursion scheme for finding $\mathbb{E}[X^T_{h,\Delta t} \otimes X^T_{h,\Delta t}] \in V_h^{(2)}$. To see this, note first that for the second operator in the parenthesis on the left hand side of the equation we have, since $X^{t_j}_{h,\Delta t}$ is $\mathcal{F}_{t_j}$-measurable and ΔL^j is independent of $\mathcal{F}_{t_j}$ with zero mean,

$$\begin{aligned}&\mathbb{E}\left[\left(I \otimes P_h G(\cdot)\Delta L^j\right)\left(X^{t_j}_{h,\Delta t} \otimes X^{t_j}_{h,\Delta t}\right)\right]\\ &\quad = \mathbb{E}\left[\left(I \otimes \mathbb{E}[P_h G(\cdot)\Delta L^j | \mathcal{F}_{t_j}]\right)\left(X^{t_j}_{h,\Delta t} \otimes X^{t_j}_{h,\Delta t}\right)\right] = 0,\end{aligned}$$

and for the same reason the mean of the third operator applied to $X^{t_j}_{h,\Delta t} \otimes X^{t_j}_{h,\Delta t}$ is also zero. Therefore, using the facts that $X^{t_j}_{h,\Delta t}$ is $\mathcal{F}_{t_j}$-measurable and that ΔL^j is independent of $\mathcal{F}_{t_j}$ once again along with the fact that deterministic bounded linear operators and the expectation operator commute, we end up with the scheme

$$\begin{aligned}&((I - \Delta t A_h) \otimes (I - \Delta t A_h))\, \mathbb{E}\left[X^{t_{j+1}}_{h,\Delta t} \otimes X^{t_{j+1}}_{h,\Delta t}\right]\\ &\quad = \left(I \otimes I + \mathbb{E}\left[P_h G(\cdot)\Delta L^j \otimes P_h G(\cdot)\Delta L^j\right]\right) \mathbb{E}\left[X^{t_j}_{h,\Delta t} \otimes X^{t_j}_{h,\Delta t}\right].\end{aligned} \tag{15}$$

for $j = 0, \dots, N_{\Delta t} - 1$. Except for the term $\mathbb{E}\left[P_h G(\cdot)\Delta L^j \otimes P_h G(\cdot)\Delta L^j\right]$, this scheme is straightforward to implement on a computer by the use of Kronecker products of standard sparse finite element matrices.

To see how we may represent the remaining term as a matrix, we consider an example of the linear operator G important in applications, namely the linear *Nemytskii* operator which describes pointwise multiplication in the domain D. For this we assume that the noise lives on the same space as the solution, i.e., that $U = H = L^2(D;\mathbb{R})$, and that for each $j = 0,\dots,N_{\Delta t}-1$, ΔL^j is a random field on D with covariance operator $\Delta t Q$. The Nemytskii operator is then, for $f,g \in H$ and $x \in D$, given by

$$G(f)g[x] = f(x)g(x).$$

Then, any entry of the matrix representation of $\mathbb{E}\left[P_h G(\cdot)\Delta L^j \otimes P_h G(\cdot)\Delta L^j\right]$ is given by applying this operator to a basis vector $\psi_k \otimes \psi_\ell$ of $V_h^{(2)}$ and taking the inner product of the result and another basis vector $\psi_m \otimes \psi_n$. In our case, the matrix entry ends up being

$$\begin{aligned}
&\left\langle \mathbb{E}\left[P_h G(\psi_k)\Delta L^j \otimes P_h G(\psi_\ell)\Delta L^j\right], \psi_m \otimes \psi_n \right\rangle_H \\
&\quad = \mathbb{E}\left[\left\langle P_h G(\psi_k)\Delta L^j \otimes P_h G(\psi_\ell)\Delta L^j, \psi_m \otimes \psi_n \right\rangle_H\right] \\
&\quad = \mathbb{E}\left[\left\langle G(\psi_k)\Delta L^j, \psi_m \right\rangle_H \left\langle G(\psi_\ell)\Delta L^j, \psi_n \right\rangle_H\right] \\
&\quad = \mathbb{E}\left[\left\langle \Delta L^j, \psi_k\psi_m \right\rangle_H \left\langle \Delta L^j, \psi_\ell\psi_n \right\rangle_H\right] \\
&\quad = \Delta t \left\langle Q\psi_k\psi_m, \psi_\ell\psi_n \right\rangle_H = \Delta t \int_D \int_D C(x,y)\psi_\ell(x)\psi_n(x)\psi_k(y)\psi_m(y)\,\mathrm{d}x\,\mathrm{d}y,
\end{aligned}$$

where we used the fact that G is the Nemytskii operator in the third equality and Identity (2) in the fourth. Note that the product $\psi_\ell\psi_n \in H$ since it is assumed that $V_h \subset \dot{H}^1$.

Scheme (15) is now ready to be implemented and it results in $\mathbb{E}[X^T_{h,\Delta t} \otimes X^T_{h,\Delta t}] \in V_h^{(2)}$. Then, writing

$$X^T_{h,\Delta t} = \sum_{i=1}^{N_h} x_i \psi_i$$

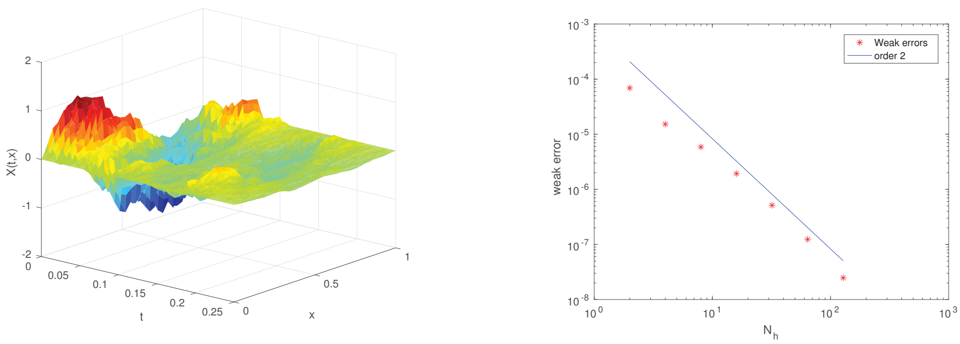
using random coefficients $x_1,\dots,x_{N_h}$, we see that we can easily calculate $\mathbb{E}\left[\|X^T_{h,\Delta t}\|_H^2\right]$ from $\mathbb{E}[X^T_{h,\Delta t} \otimes X^T_{h,\Delta t}]$ since

$$\mathbb{E}\left[\|X^T_{h,\Delta t}\|_H^2\right] = \mathbb{E}\left[\left\langle X^T_{h,\Delta t}, X^T_{h,\Delta t}\right\rangle_H\right] = \sum_{i,j=1}^{N_h} \mathbb{E}\left[x_i x_j\right] \left\langle \psi_i, \psi_j \right\rangle_H$$

and

$$\mathbb{E}[X^T_{h,\Delta t} \otimes X^T_{h,\Delta t}] = \sum_{i,j=1}^{N_h} \mathbb{E}[x_i x_j] \psi_i \otimes \psi_j.$$

Let us now demonstrate this technique in a setting that is very similar to the one of Paper 1, where we analysed the additional errors caused by Monte Carlo approximations when attempting to numerically simulate weak convergence rates. We set $D = [0,1]$, $T = 1$ and $A = \Delta$ with zero boundary conditions, i.e., we are dealing with the homogeneous stochastic heat equation in one dimension. We let G be the linear Nemytskii operator defined above and choose Q to be the covariance operator corresponding to the exponential kernel given by $C(x,y) = 20\exp(-2|x-y|)$, $x,y \in D$. With the notation of Example 4.2 we compute $\mathbb{E}[Y_\ell] = \mathbb{E}[\|X_\ell^T\|_H^2]$ for $\ell = 1,\dots,7$ using $\mathbb{E}[\|X_8^T\|_H^2]$ as a reference solution when calculating the weak error. In Figure 2(a) we see a realization of the process $(X_6^{t_j})_{j=0,\dots,N_{\Delta t}/4}$ when $L = W$ is a Q-Wiener process. In this setting, Conjecture 3.9 predicts a weak convergence rate of order 2 in space and as we can see from Figure 2(b), this is consistent with our numerical computations. Here, the initial value was given by $X_0(x) = \sin(\pi x)$ for $x \in D$ and the computations were done in MATLAB.



(a) A realization of the solution X.

(b) Weak errors for the fully discrete approximation of X, computed with a deterministic method.

Figure 2: The one-dimensional stochastic heat equation with linear multiplicative noise.

7 Summary of Paper 2

Let us assume that the SPDE (5) is fully linear, i.e., that the operators F and G fulfil $F \in L(H)$ and $G \in L(H; L(U;H))$. A property of such SPDE that has gathered some interest in the community in recent years is the qualitative behaviour of the second moment of the solution to (5). This is commonly analysed in terms of the *equilibrium* or *zero solution* $(X_{\mathrm{e}}(t) = 0)_{t \geq 0}$ which is called *mean square stable* if, for every $\varepsilon > 0$, there exists $\delta > 0$ such that $\mathbb{E}[\|X(t)\|_H^2] < \varepsilon$ for all $t \geq 0$ whenever $\mathbb{E}[\|X_0\|_H^2] < \delta$. It is called *asymptotically mean square stable* if it is mean square stable and there exists $\delta > 0$ such that $\mathbb{E}[\|X_0\|_H^2] < \delta$ implies $\lim_{t\to\infty} \mathbb{E}[\|X(t)\|_H^2] = 0$.

While the main focus of the analysis of approximations of solutions to (5) has been on showing strong and weak convergence, cf. Theorem 3.8 and Conjecture 3.9, these properties do not guarantee that the approximation shares the same (asymptotic) mean square stability properties as the analytical solution. The goal of Paper 2 is to generalize the existing theory of asymptotic mean square stability analysis of approximations to the solutions of finite-dimensional stochastic differential equations to more general approximations, such as the ones introduced in Section 3.2. An important application of mean square stability can be found in multilevel Monte Carlo methods. If the solution is mean square unstable on any of the included levels, this is enough for the estimator to not behave as it should, see, e.g., [1].

The goal of the first part of the paper is the analysis of the asymptotic mean square stability of the general linear recursion scheme

$$\begin{aligned} X_h^{j+1} &= D_{\Delta t,h}^{\mathrm{det}} X_h^j + D_{\Delta t,h}^{\mathrm{stoch},j} X_h^j, \\ X_h^0 &= X_h^0, \end{aligned} \tag{16}$$

for $j \in \mathbb{N}_0$, where the $L(V_h)$-valued deterministic and stochastic operators $D_{\Delta t,h}^{\mathrm{det}}$ and $D_{\Delta t,h}^{\mathrm{stoch},j}$ are not yet assumed to be given by (7) and (8). The $\mathcal{F}_0$-measurable initial condition X_h^0 is assumed to be square integrable. For this scheme, an *equilibrium* (solution) is given by the zero solution, which is defined as $X_{h,\mathrm{e}}^j = 0$ for all $j \in \mathbb{N}_0$. It is called *mean square stable* if, for every $\varepsilon > 0$, there exists $\delta > 0$ such that $\mathbb{E}[\|X_h^j\|_H^2] < \varepsilon$ for all $j \in \mathbb{N}_0$ whenever $\mathbb{E}[\|X_h^0\|_H^2] < \delta$ and *asymptotically mean square stable* if it is mean square stable and there exists $\delta > 0$ such that $\mathbb{E}[\|X_h^0\|_H^2] < \delta$ implies $\lim_{j\to\infty} \mathbb{E}[\|X_h^j\|_H^2] = 0$. When there is no risk of confusion, the scheme (16) is itself said to be (asymptotically) mean square stable when this holds.

First, the asymptotic mean square stability is analysed under the general assumption that the family $(D_{\Delta t,h}^{\mathrm{stoch},j}, j \in \mathbb{N}_0)$ is $\mathcal{F}$-compatible in the sense that the random operator $D_{\Delta t,h}^{\mathrm{stoch},j}$ is $\mathcal{F}_{t_{j+1}}$-measurable and $\mathbb{E}[D_{\Delta t,h}^{\mathrm{stoch},j} | \mathcal{F}_{t_j}] = 0$ for all $j \in \mathbb{N}_0$. It is also assumed that, for all $j \in \mathbb{N}_0$,

$$\|D_{\Delta t,h}^{\mathrm{stoch},j}\|_{L^2(\Omega;L(V_h))} = \mathbb{E}[\|D_{\Delta t,h}^{\mathrm{stoch},j}\|_{L(V_h)}^2]^{1/2} < \infty$$

and

$$\mathbb{E}\left[D_{\Delta t,h}^{\mathrm{stoch},j} \otimes D_{\Delta t,h}^{\mathrm{stoch},j} \,\middle|\, \mathcal{F}_{t_j} \right] = \mathbb{E}\left[D_{\Delta t,h}^{\mathrm{stoch},j} \otimes D_{\Delta t,h}^{\mathrm{stoch},j} \right].$$

This is a natural assumption that is true when $D_{\Delta t,h}^{\mathrm{stoch},j}$ is given by (8). Next, under the additional assumption that $(D_{\Delta t,h}^{\mathrm{stoch},j}, j \in \mathbb{N}_0)$ has constant covariance, i.e., that for all $j \in \mathbb{N}_0$,

$$\mathbb{E}\left[D_{\Delta t,h}^{\mathrm{stoch},j} \otimes D_{\Delta t,h}^{\mathrm{stoch},j} \right] = \mathbb{E}\left[D_{\Delta t,h}^{\mathrm{stoch},0} \otimes D_{\Delta t,h}^{\mathrm{stoch},0} \right], \tag{17}$$

it is shown that the zero solution of (16) is asymptotically mean square stable if and only if the stability operator

$$\mathcal{S} = D_{\Delta t,h}^{\mathrm{det}} \otimes D_{\Delta t,h}^{\mathrm{det}} + \mathbb{E}[D_{\Delta t,h}^{\mathrm{stoch},0} \otimes D_{\Delta t,h}^{\mathrm{stoch},0}] \in L(V_h^{(2)})$$

satisfies $\rho(\mathcal{S}) = \max_{i=1,\ldots,N_h^2} |\lambda_i| < 1$, where $\lambda_1, \ldots, \lambda_{N_h^2}$ are the eigenvalues of $\mathcal{S}$.

The second part of the paper treats the asymptotic mean square stability of the scheme of Section 3.2, i.e., it is assumed that (16) approximates the mild solution to the SPDE (5).

It is shown that when $D^{\mathrm{det}}_{\Delta t,h}$ and $D^{\mathrm{stoch},j}_{\Delta t,h}$ are given by (7) and (8), the stability operator simplifies to

$$\mathcal{S} = D^{\mathrm{det}}_{\Delta t,h} \otimes D^{\mathrm{det}}_{\Delta t,h} + \Delta t\, (C \otimes C) q \in L(V_h^{(2)}),$$

where $q = \sum_{k=1}^{\infty} \mu_k p_k \otimes p_k \in U^{(2)}$ and $C \in L(U; L(V_h))$ with

$$Cu = r_{\mathrm{d}}^{-1}(\Delta t A_h) P_h G(\cdot) u,$$

recalling that the eigenvectors of Q are given by $(p_i)_{i\in\mathbb{N}}$ and the eigenvalues by $(\mu_i)_{i\in\mathbb{N}}$. A similar result is shown for the higher order Milstein scheme, the convergence of which was analysed in [5].

In the remainder of the paper we derive sufficient conditions for the asymptotic mean square stability of the scheme of Section 3.2 for different rational approximations of the discrete semigroup $(E_h(t))_{t\in[0,T]}$, among those the backward and forward Euler scheme of Example 3.6. For example, the first of these is seen to be asymptotically mean square stable if

$$\frac{(1 + \Delta t \|F\|_{L(H)})^2 + \Delta t\, \mathrm{Tr}(Q) \|G\|^2_{L(H;L(U;H))}}{(1 + \Delta t \lambda_{h,1})^2} < 1,$$

where $\lambda_{h,1}$ is the smallest eigenvalue of the discrete operator $-A_h$. These conditions are based on the observation that $\rho(\mathcal{S}) \leq \|\mathcal{S}\|_{L(V_h)}$. Using these results, a condition that ensures the asymptotic mean square stability of both the zero solution to (5) and its approximation with the backward Euler scheme is derived under Assumption 3.7. A similar result is again shown for the Milstein scheme. Simulations using both spectral and finite element Galerkin methods illustrate the theoretical results.

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