Thesis for the Degree of Licentiate of Philosophy

Compact Relativistic Matter Shells with Massless and Charged Particles

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Abstract

The Vlasov model is a matter model that is widely used in physics. In the context of astrophysics and cosmology it describes an ensemble of self gravitating, collisionless particles. These particles can be interpreted as for example ions, stars, or galaxies. If the gravity that the particles create collectively is described by General Relativity the Vlasov equation couples to Einstein's field equations and the Einstein-Vlasov system is obtained. If it is furthermore assumed that the particles carry a charge, then an electromagnetic field is created collectively which in addition to the space-time curvature governs the particles' trajectories and which satisfies Maxwell's equations.

This thesis focuses on static solutions, in spherical symmetry. In the introduction the Einstein-Vlasov-Maxwell system is introduced formally and the equations in spherical symmetry are derived. Further, the results of this thesis are set into relation to the current sate of research. Finally, a numerical section characterizing the solutions constructed analytically in the papers concludes the introduction.

In Paper I spherically symmetric, static shell solutions for massless particles are constructed. The difficulty lies in showing that the matter quantities of these solutions are of compact support. A different approach than in the massive case is necessary. Furthermore the obtained solutions are compared to the concept of geons, introduced by John Wheeler in 1955. Geons are solutions of the Einstein-Maxwell system that are spherically symmetric on a time average and have long lifetimes. They were originally intended to serve as models for individual particles.

In paper II charged particles are considered. In the first part local existence of spherically symmetric, static solutions of the Einstein-Vlasov-Maxwell system around the center of symmetry is established. Then, based on that, the existence of compactly supported, static solutions with finite mass is shown for small particle charges by a perturbation argument. In the last part the existence of thin shell solutions is proven for arbitrary values of the particle charge. The proof yields sequences of shell solutions approaching an infinitesimally thin shell. Some properties of this limit are discussed.

Keywords: General Relativity, Vlasov matter, Static Solutions, Spherical Symmetry, massless Einstein-Vlasov system, Einstein-Vlasov-Maxwell system

List of appended papers

Paper I Håkan Andréasson, David Fajman & Maximilian Thaller, *Models for Self-Gravitating Photon Shells and Geons*, Ann. Henri Poincaré **18** (2017), 681-705

Paper II Maximilian Thaller, *Existence of Static Solutions of the Einstein-Vlasov-Maxwell System and the Thin Shell Limit*, Preprint

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Maximilian Thaller
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List of abbreviations and symbols

$\mathscr{B}(\mathscr{M})$	Space of vector fields in the manifold M
EV-system	Einstein-Vlasov system
EVM-system	Einstein-Vlasov-Maxwell system
$\star$	Hodge star operator, $\star : \Lambda^k(V) \to \Lambda^{n-k}(V)$, $\dim(V) = n$
i_X	Interior derivative with respect to the vector field X
$\Lambda^k(V)$	Space of differential forms on the vector space V
$\mathscr{L}_X$	Lie derivative along the vector field X
∇_X	Covariant derivative along the vector field X
TOV equation	Tolman-Oppenheimer-Volkov equation

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Part I

Introduction

1 Physical Interpretation of the Results

The centerpiece of this thesis are two papers with results on static solutions of the Einstein-Vlasov system. The Vlasov equation describes the motion of an ensemble of particles moving through space without collisions. Thereby the particles' trajectories are determined by forces or space-time curvature due to gravity, electro-magnetic interaction or other effects. The gravitational field can be described either classically by a gravitational potential satisfying a Poisson equation or with General Relativity via curvature of space time. The curvature is encoded in the metric tensor on the space-time manifold which satisfies Einstein's field equations. In the former case one deals with the Vlasov-Poisson system, in the latter case with the Einstein-Vlasov system. Are the particles charged interaction via a Lorentz force is part of the model. The electromagnetic field contributes a term to the Vlasov equation. At the same time the electro-magnetic field fulfills Maxwell's equations. For the model of an ensemble of collisionless particles with general relativistic gravitation and interaction via a collectively created electro-magnetic field the Einstein-Vlasov-Maxwell system (EVM-system) is obtained. Depending on the assumed properties of the particles the Einstein-Vlasov system can be given different physical interpretations. Some examples are galaxies, globular clusters, and galaxy clusters.

There are various applications of the Vlasov matter model in physics. In plasma physics particles with repulsive forces are interpreted as ions. In astrophysics the particles represent stars or galaxies to provide models for galaxies or galaxy clusters, see e.g. the book [14]. A classical description of gravity, i.e. with the Newtonian theory often yields realistic descriptions. This means the Vlasov-Poisson system is studied. The Vlasov-Poisson system is the non-relativistic limit of the Einstein-Vlasov system. It is currently better understood than the Einstein-Vlasov system and it has an easier structure. However it only provides physically reasonable models if the conditions are not extreme. Examples for such situations are a plasma of ions, or dilute regions of a galaxy if the stars move not to fast. In other situations, for example regions of extreme gravity like in the center of a galaxy, situations with extremely fast moving bodies, or cosmological models for a matter filled universe as a whole, the Einstein-Vlasov system has to be used. The Einstein-Vlasov system covers the mentioned situations whereas the Vlasov-Poisson system cannot cover all physical effects that we observe.

In the first paper, Paper I, the static Einstein-Vlasov system in spherical symmetry is considered for massless particles. The existence of solutions of this system with a particle distribution function which has compact support in space is proven. In this model the trajectories of the massless particles are null geodesics, the same kind of trajectories that photons are modeled by in the framework of General Relativity. In contrast to Newtonian gravity, in the description by general relativity, light is interacting with gravity, too. The following effects can be modeled. If a light ray passes a massive object it will be bent by its gravitational field. This phenomenon is generally referred to as *gravitational lensing*. Bright structures behind a massive object appear distorted as if the light would pass through a giant lens. Figure 1 shows a photo of the gravitational lensing of a luminous red galaxy. This effect can be used to

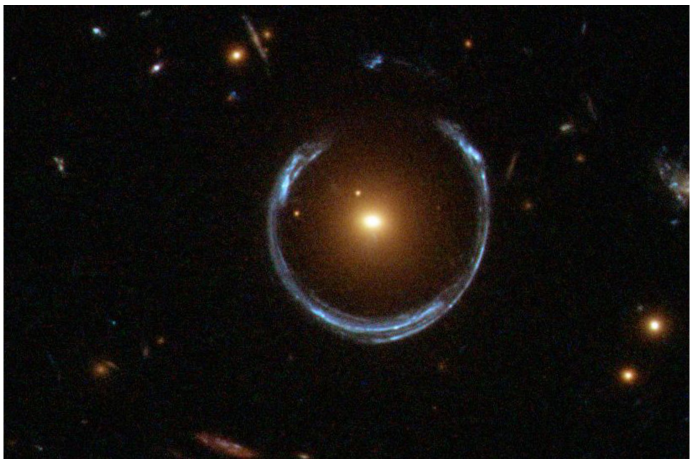

Fig. 1: Gravitational lensing caused by the mass of a luminous red galaxy, photographed by the Hubble space telescope

encounter heavy but dark objects and to measure their mass [17]. If a light ray passes a very dense object very closely it can even be trapped on a circular or at least closed orbit.

In General Relativity objects with non-zero rest mass create a gravitational field - like in the classical picture. However, massless particles bend space-time, too, through the other forms of energy that they carry. A static solution of the massless Einstein-Vlasov system with matter quantities of compact support describes a matter configuration where the involved particles do not fill the whole universe but are confined to a finite part of space-time. Thus they create a gravitational field sufficiently strong to keep themselves together. In Paper I the problem of existence of such a static solution is considered and it turns out that static solutions with matter quantities of bounded support are possible, but they occur only in quite extreme situations where the space time curvature is very strong and the particles are very dense. Situations whose properties are very far from the regime where the Newtonian description of gravity applies.

In the second paper, Paper II, charged particles are considered. In the charged case the particles create an electric field in addition to gravitational effects. It is intuitively plausible that the gravitational and the electro-magnetic effects act in an oppositional manner. Whereas the gravity tends to keep the particles together the electric force pushes the particles away from one another. In the paper existence of spherically symmetric static solutions is proved in two scenarios. First for small values of the particle charge. Second for particles of arbitrary particle charge but they have to be densely packed so that strong gravitational fields are present.

The study of the EVM-system, too, is physically motivated. It provides a candidate for a model of galactic nebulae. These nebulae consist of large clouds of ionized matter. One example are planetary nebulae - the remnants of a certain kind of supernova [33]. Planetary nebulae consist of a cloud of ionized hydrogen distributed around a white dwarf. However, it has to be said that these nebulae often are not well described by static solutions and that they are relatively dilute, i.e. not in the realm of strong gravity that makes General Relativity as a model necessary.

A different physical motivation is that solutions of the EVM-system with charged particles might share some properties with axially symmetric, rotating solutions of the Einstein-Vlasov system. Rotating solutions are not static but merely stationary. Charge could serve as technically simpler substitute for angular momentum to obtain first insights into the properties of such solutions. For example, as discussed in the introduction of Paper II, the quantity $\frac{2m(r)}{r}$, where r is the areal radius and $m(r)$ is the Hawking mass, gives a measure for how relativistic a solution is. A sequence of spherically symmetric static solutions where $\sup_r \frac{2m(r)}{r} \to 1$ would approach a solution with a black hole region. The existence of such a sequence is however ruled out. For a broad variety of matter models it is known that in spherical symmetry $\sup_r \frac{2m(r)}{r} \leq \frac{8}{9}$, cf. [5]. If the solution has a total angular momentum however, the situation is different. In the numerical work [2] the authors construct highly relativistic, axially symmetric stationary solutions of the Einstein-Vlasov system that contain ergo regions. An adiabatic transition, i.e. a sequence of stationary solutions, from a regular solution to a black hole seems possible. The spherically symmetric EVM-system seems to exhibit this behavior, too. In the numerical study [9] the authors construct sequences of static solutions where $\sup_r \frac{2m(r)}{r} \to 1$.

2 The Mass Shell

2.1 Definition

Now the Einstein-Vlasov-Maxwell system is introduced formally. The described particles are characterized by a particle mass m_0 and a particle charge q_0. The particles are assumed to be identical. Of course, if one sets $q_0 = 0$, one obtains just the Einstein-Vlasov system. First we define the mass shell $\mathscr{P}_{m_0}$ of a Lorentzian manifold $(\mathscr{M}, g)$ of dimension $n = 4$ corresponding to particles of mass m_0. The signature of the Lorentzian metric is $(-,+,+,+)$ and we denote by ∇ the Levi-Civita connection on $\mathscr{M}$. Assume that on the tangent bundle $T\mathscr{M}$ of $\mathscr{M}$ we have on a neighborhood U (which can be the whole of $T\mathscr{M}$) the coordinates $(x^\mu, p^\nu) \in \mathbb{R}^8$. In this thesis Greek indices run from 0 to 3 and Latin indices from 1 to 3. Furthermore the Einstein summation convention is used if the same index shows up upstairs and downstairs in the same formula. The trajectory $(X(s), P(s))$ of a freely falling particle, an affinely parameterized curve in the eight dimensional manifold $T\mathscr{M}$, fulfills the system of equations

$$\frac{\mathrm{d}X^\mu}{\mathrm{d}s}(s) = P^\mu(s), \tag{2.1}$$

$$\frac{\mathrm{d}P^\mu}{\mathrm{d}s}(s) = -\Gamma^\mu_{\alpha\beta}(X(s))\, P^\alpha(s) P^\beta(s) + q_0 F^\mu{}_\nu(X(s))\, P^\nu(s), \tag{2.2}$$

where the Christoffel symbols $\Gamma^\mu_{\alpha\beta} = \mathrm{d}x^\mu\,(\nabla_\alpha \partial_\beta)$ are given by the formula

$$\Gamma^\mu_{\alpha\beta}(x) = -\frac{1}{2} g^{\mu\delta}(x) \left(\frac{\partial g_{\beta\delta}}{\partial x^\alpha}(x) + \frac{\partial g_{\delta\alpha}}{\partial x^\beta}(x) - \frac{\partial g_{\alpha\beta}}{\partial x^\delta}(x) \right), \tag{2.3}$$

and $F \in \Lambda^2(T\mathscr{M})$ is the electromagnetic field tensor which is obtained as exterior derivative of the four potential $A \in \Lambda^1(T\mathscr{M})$. With $\Lambda^k(T\mathscr{M})$ the space of differential k-forms on $\mathscr{M}$ is denoted.

The quantity $-g_{X(s)}(P(s), P(s))$ can be interpreted as rest mass of the particle whose trajectory $(X^\mu(s), P^\mu(s))$ fulfills (2.1)–(2.2). One would expect that the rest mass is conserved along the trajectory (2.1)–(2.2). To see that this is indeed the case one can introduce and use a symplectic structure on $T\mathscr{M}$, following [29]. First we define the Hamilton function

$$H : T\mathscr{M} \to \mathbb{R}, \qquad H(x,p) = \frac{1}{2} g_x(p,p). \tag{2.4}$$

The index x means that we take $g_x(p,p) = g_{\mu\nu}(x) p^\mu p^\nu$. Furthermore with the metric g on $\mathscr{M}$ we define the Poincaré one form given by

$$\Theta_{(x,p)} := (g_{\mu\nu}(x) p^\mu + q_0 A_\nu(x)) \mathrm{d}x^\nu. \tag{2.5}$$

Then the symplectic form Ω_s can be defined via

$$\Omega_s := \mathrm{d}\Theta = g_{\mu\nu}\mathrm{d}p^\mu \wedge \mathrm{d}x^\nu + \left(\frac{\partial g_{\alpha\nu}}{\partial x^\mu}p^\alpha + \frac{q_0}{2}F_{\mu\nu}\right)\mathrm{d}x^\mu \wedge \mathrm{d}x^\nu. \tag{2.6}$$

The Hamilton function (2.4) defines a Hamiltonian vector field, that we call L, via

$$\mathrm{d}H = -i_L\Omega_s. \tag{2.7}$$

The operator $i : \mathscr{B}(T\mathscr{M}) \times \Lambda^k(T(T\mathscr{M})) \to \Lambda^{k-1}(T(T\mathscr{M}))$ is called interior derivative and is defined by $i_X\alpha(\cdot,\cdot,\dots,\cdot) = \alpha(X,\cdot,\dots,\cdot)$. With $\mathscr{B}(T\mathscr{M})$ we denote the space of vector fields on $T\mathscr{M}$. The name L is inspired by the fact that Liouville's theorem applies, i.e. the phase space volume is conserved along the characteristic curves of L. We check that the integral curves of L are exactly the curves fulfilling (2.1), i.e. that L is given by

$$L = p^\mu\frac{\partial}{\partial x^\mu} + \left(q_0 F^\mu{}_\nu p^\nu - \Gamma^\mu_{\alpha\beta}p^\alpha p^\beta\right)\frac{\partial}{\partial p^\mu}. \tag{2.8}$$

To this end we split L formally into two parts:

$$L = L_1^\mu\frac{\partial}{\partial x^\mu} + L_2^\nu\frac{\partial}{\partial p^\nu}. \tag{2.9}$$

We have

$$\mathrm{d}H = \frac{1}{2}\frac{\partial g_{\mu\nu}}{\partial x^\alpha}p^\mu p^\nu \mathrm{d}x^\alpha + g_{\mu\nu}p^\mu\mathrm{d}p^\nu \tag{2.10}$$

and

$$i_L\Omega_s = \left[g_{\mu\nu}L_2^\mu + \left(\frac{\partial g_{\mu\nu}}{\partial x^\alpha} - \frac{\partial g_{\mu\alpha}}{\partial x^\nu}\right)p^\mu L_1^\alpha + q_0 F_{\alpha\nu}L_1^\alpha\right]\mathrm{d}x^\nu - g_{\mu\nu}L_1^\mu\mathrm{d}p^\nu. \tag{2.11}$$

Comparing coefficients we obtain

$$L_1^\mu = p^\mu, \qquad L_2^\alpha = -\Gamma^\alpha_{\mu\nu}p^\mu p^\nu + q_0 F^\alpha{}_\mu p^\mu \tag{2.12}$$

which shows that (2.8) holds.

By virtue of the Cartan identity, we see that H is conserved along integral curves of L by the following short calculation,

$$\mathscr{L}_L H = \mathrm{d}\left(i_L H\right) + i_L\mathrm{d}H = -i_L^2\Omega_s = 0. \tag{2.13}$$

Here $\mathscr{L}_X$ denotes the Lie derivative with respect to the vector field X. In the second equality sign it has been used that the interior derivative of a scalar function is always zero and in the third equality sign antisymmetry of Ω_s has been exploited. This motivates the definition of the mass shell $\mathscr{P}_{m_0}$ as

$$\mathscr{P}_{m_0} = \{(x^\alpha, p^\alpha) \in T\mathscr{M} \,:\, g_{\alpha\beta}(x)p^\alpha p^\beta = -m_0^2,\, p \neq 0, p\,\text{future dir.}\}. \tag{2.14}$$

Recall that a vector X is called future directed if $g(\partial_t, X) < 0$. The mass shell is a seven dimensional submanifold of $T\mathscr{M}$ which can be equipped with the coordinates $(x^\mu, p^i) \in \mathbb{R}^7$. The mass shell can be interpreted as the region of $T\mathscr{M}$ that can be accessed by particles of the rest mass m_0.

In the chargeless case $q_0 = 0$ the integral curves of the Hamiltonian vector field L are the lifts of geodesics to the tangent bundle. In this case L is called the *geodesic spray* and denoted by $\mathscr{X}$.

The condition $g_x(p,p) = -m_0^2$ in the definition (2.14) of the mass shell yields a submanifold of $T\mathscr{M}$ with two components, given by

$$p^0 = \frac{1}{-g_{00}} \left(g_{a0}p^a \pm \sqrt{g_{a0}^2 \left(p^a\right)^2 - g_{00}\left(g_{ab}p^a p^b + m_0^2\right)} \right). \tag{2.15}$$

If $m_0 > 0$ these components are disconnected. By imposing that p is future directed we choose the plus sign in (2.15). That the characteristic curves of L stay in $\mathscr{P}_{m_0}$ means that L is tangent to $\mathscr{P}_{m_0}$. Thus we can regard L as a vector field on $\mathscr{P}_{m_0}$. We can construct a basis $(b_1, \dots, b_7)$ of $T\mathscr{P}_{m_0}$ in therms of the standard basis $(\partial_{x^\mu}, \partial_{p^\nu})$ of $T(T\mathscr{M})$ by

$$b_{1+\mu} = \frac{\partial}{\partial x^\mu} + \frac{\partial p^0}{\partial x^\mu}\frac{\partial}{\partial p^0}, \quad b_{4+j} = \frac{\partial}{\partial p^j} + \frac{\partial p^0}{\partial p^j}\frac{\partial}{\partial p^0}. \tag{2.16}$$

Applying ∂_{x^μ} or ∂_{p^j} respectively to the identity $-m_0^2 = g_{\mu\nu}p^\mu p^\nu$ we obtain

$$\frac{\partial p^0}{\partial x^\mu} = -\frac{1}{2g_{0\alpha}p^\alpha}\frac{\partial g_{\beta\gamma}}{\partial x^\mu}p^\beta p^\gamma, \tag{2.17}$$

$$\frac{\partial p^0}{\partial p^j} = -\frac{1}{g_{0\alpha}p^\alpha}g_{\beta j}p^\beta. \tag{2.18}$$

A straight forward calculation then gives

$$p^\mu \frac{\partial p^0}{\partial x^\mu} - \Gamma^j_{\alpha\beta}p^\alpha p^\beta \frac{\partial p^0}{\partial p^j} = -\Gamma^0_{\alpha\beta}p^\alpha p^\beta \tag{2.19}$$

and by antisymmetry of $F_{\mu\nu}$ we have

$$q_0 F^j{}_\mu p^\mu \frac{\partial p^0}{\partial p^j} = q_0 F^0{}_\mu p^\mu \tag{2.20}$$

and we can write the Hamiltonian vector field L as

$$L = p^\mu b_{\mu+1} + \left(q_0 F^j{}_\mu p^\mu - \Gamma^j_{\alpha\beta}p^\alpha p^\beta\right) b_{j+4}. \tag{2.21}$$

2.2 Volume element

One can construct a volume element on the mass shell by the following procedure which follows [28]. First we construct a volume element on the tangent bundle $T\mathscr{M}$ from the symplectic form Ω_s, defined in (2.6). We have $\dim(T\mathscr{M}) = 2n$ and we consider the $(2n)$-form

$$\Lambda_{(x,p)} := \frac{(-1)^{\frac{n(n-1)}{2}}}{n!} \underbrace{\Omega_s \wedge \cdots \wedge \Omega_s}_{n-\text{times}}. \tag{2.22}$$

There exists an orthonormal basis of $T(T\mathscr{M})$ such that the form Λ is a volume element with respect to this basis if $\Lambda_{(x,p)}(\partial_{p^0}, \ldots, \partial_{p^3}, \partial_{x^0}, \ldots, \partial_{x^3}) > 0$ for all $(x,p) \in T\mathscr{M}$. (We say $\Lambda_{(x,p)}$ is a volume form on $T\mathscr{M}$ with respect to the basis $(e_1(x,p), \ldots, e_8(x,p))$ if $\Lambda_{(x,p)}(e_1(x,p), \ldots, e_8(x,p)) = 1$ for all points $(x,p) \in T\mathscr{M}$.) Indeed we have

$$\begin{aligned}
\Lambda_{(x,p)} &= \frac{(-1)^{\frac{n(n-1)}{2}}}{n!} g_{\mu_1\nu_1} \cdots g_{\mu_n\nu_n} \mathrm{d}p^{\mu_1} \wedge \mathrm{d}x^{\nu_1} \wedge \cdots \wedge \mathrm{d}p^{\mu_n} \wedge \mathrm{d}x^{\nu_n} \\
&= \frac{1}{n!} g_{\mu_1\nu_1} \cdots g_{\mu_n\nu_n} \mathrm{d}p^{\mu_1} \wedge \cdots \wedge \mathrm{d}p^{\mu_n} \wedge \mathrm{d}x^{\nu_1} \wedge \cdots \wedge \mathrm{d}x^{\nu_n} \\
&= \frac{1}{n!} g_{\mu_1\nu_1} \cdots g_{\mu_n\nu_n} \varepsilon^{\mu_1\ldots\mu_n} \varepsilon^{\nu_1\ldots\nu_2} \mathrm{d}p^0 \wedge \cdots \wedge \mathrm{d}p^{n-1} \wedge \mathrm{d}x^0 \wedge \cdots \wedge \mathrm{d}x^{n-1} \\
&= \det(g_{\mu\nu}) \mathrm{d}p^0 \wedge \cdots \wedge \mathrm{d}p^{n-1} \wedge \mathrm{d}x^0 \wedge \cdots \wedge \mathrm{d}x^{n-1}.
\end{aligned} \tag{2.23}$$

Next we observe that the vector field

$$N = \frac{1}{p_0} \frac{\partial}{\partial p^0} \tag{2.24}$$

fulfills $\mathrm{d}H(N) = 1$, so it has a component normal to $\mathscr{P}_{m_0}$. This means in particular that we can extend N to a basis $(N, X_2, \ldots, X_{2n})$ of $T(T\mathscr{M})$ where the vectors $X_2, \ldots, X_{2n}$ are tangent to the mass shell $\mathscr{P}_{m_0}$. Define now

$$\sigma := i_N \Lambda. \tag{2.25}$$

We have

$$(\mathrm{d}H \wedge \sigma)(N, X_2, \ldots, X_{2n}) = \sigma(X_2, \ldots, X_{2n}) = \Lambda(N, X_2, \ldots, X_{2n}), \tag{2.26}$$

so $\mathrm{d}H \wedge \sigma = \Lambda$. Using the expression (2.23) we find

$$\sigma = \frac{|\det(g_{\mu\nu})|}{-p_0} \mathrm{d}p^1 \wedge \cdots \wedge \mathrm{d}p^3 \wedge \mathrm{d}x^0 \wedge \cdots \wedge \mathrm{d}x^3. \tag{2.27}$$

Furthermore, $\iota^*\sigma$ is a volume form on $\mathscr{P}_{m_0}$ since

$$(\iota^*\sigma)(X_2,\ldots,X_{2n}) = \sigma(X_2,\ldots,X_{2n}) = \Lambda(X_1,X_2,\ldots,X_{2n}) > 0. \tag{2.28}$$

The right hand side is always larger than zero since Λ is a volume form. Observe that the pullback $(\iota^*\sigma)$ is uniquely determined by the volume form Λ on $T\mathscr{M}$.

The mass shell is a fiber bundle over the space time manifold $\mathscr{M}$. Since $\mathscr{M}$ has the natural volume element $\sqrt{|\det(g_{\mu\nu})|}\mathrm{d}x^0\wedge\mathrm{d}x^1\wedge\mathrm{d}x^2\wedge\mathrm{d}x^3$ the volume form $\mu_{\mathscr{P}_x}$ of a fiber $\mathscr{P}_x$ of the mass shell $\mathscr{P}_m$ is then given by

$$\mu_{\mathscr{P}_x} = \frac{\sqrt{|\det(g_{\mu\nu}(x))|}}{-p_0}\mathrm{d}p^1\wedge\mathrm{d}p^2\wedge\mathrm{d}p^3. \tag{2.29}$$

An alternative way of obtaining the form (2.29) of the volume element on a fiber of the mass shell works by considering the restriction of the metric $g_{T_x\mathscr{M}} = \eta_{\mu\nu}\mathrm{d}p^\mu\otimes\mathrm{d}p^\nu$ on the tangent space $T_x\mathscr{M}$ to $\mathscr{P}_x$. To this end one replaces

$$\mathrm{d}p^0 = \frac{\partial p^0}{\partial p^j}\mathrm{d}p^j \tag{2.30}$$

and calculates the determinant of the remaining matrix. The volume element is then given by $\mu_{\mathscr{P}_x} = \sqrt{|\det g_{\mathscr{P}_x}|}\mathrm{d}p^1\wedge\mathrm{d}p^2\wedge\mathrm{d}p^3$ which gives the same formula (2.29).

3 The Einstein-Vlasov-Maxwell System

Now the Einstein-Vlasov-Maxwell system (EVM-system) can be presented. A solution of the EVM system for particles with mass $m_0 \geq 0$ and charge $q_0 \geq 0$ is a four dimensional Lorentzian manifold $(\mathscr{M}, g)$, a particle distribution function $f \in C(\mathscr{P}_{m_0};\mathbb{R}_+)$, and an electro-magnetic field tensor $F \in \Lambda^2(T\mathscr{M})$ such that the EVM-system,

$$R_{\mu\nu}[g] - \frac{1}{2}R[g]g_{\mu\nu} = 8\pi\left(T_{\mu\nu}[f] + \tau_{\mu\nu}[f]\right), \tag{3.1}$$

$$T_{\mu\nu}[f] = g_{\mu\alpha}g_{\nu\beta}\int_{\mathscr{P}_x} f(x,p)p^\alpha p^\beta \mu_{\mathscr{P}_x}, \tag{3.2}$$

$$\tau_{\mu\nu}[f] = \frac{1}{4\pi}\left(-\frac{1}{4}g_{\mu\nu}F_{\alpha\beta}F^{\alpha\beta} + F_{\nu\alpha}F_\mu{}^\alpha\right), \tag{3.3}$$

$$\mathscr{L}_L f = 0, \tag{3.4}$$

$$\mathrm{d}F = 0, \tag{3.5}$$

$$\nabla_\alpha F^{\alpha\beta} = -4\pi q_0\int_{\mathscr{P}_x} f(x,p)p^\beta\mu_{\mathscr{P}_x} \tag{3.6}$$

is satisfied.

A few words to explain this system and the involved quantities are in order. Equation (3.1) is Einstein's field equation. It is a second order non-linear partial integro-differential equation. The unknowns are the components $g_{\mu\nu}$ of the metric tensor g, as well as the particle distribution function f entering the energy momentum tensor. On the left hand side we haven the Ricci tensor $R_{\mu\nu}[g]$ and the Ricci scalar $R[g]$, both curvature quantities that can be calculated in terms of the metric and its partial derivatives up to second order. Indeed, the Christoffel symbols are defined by (2.3) and the components of the Riemann tensor can be calculated by

$$R^{\alpha}{}_{\beta\gamma\delta} = \partial_\gamma \Gamma^{\alpha}_{\beta\delta} + \Gamma^{\alpha}_{\nu\gamma}\Gamma^{\nu}_{\beta\delta} - \partial_\delta \Gamma^{\alpha}_{\beta\gamma} - \Gamma^{\alpha}_{\nu\delta}\Gamma^{\nu}_{\beta\gamma}. \tag{3.7}$$

Then we have

$$R_{\alpha\beta}[g] = R^{\mu}{}_{\alpha\mu\beta}, \quad R[g] = g^{\alpha\beta} R_{\alpha\beta}[g]. \tag{3.8}$$

On the right hand side of equation (3.1) we have the energy momentum tensor $T_{\mu\nu}[f]$ which is defined in equation (3.2) and which is determined by the particle distribution function f. The integral goes over a fiber $\mathscr{P}_x$ of the mass shell $\mathscr{P}_{m_0}$, defined in (2.14). Furthermore we have the electromagnetic contribution $\tau_{\mu\nu}[F]$ to the energy momentum tensor given in (3.3). The electromagnetic field tensor $F_{\mu\nu}$ satisfies the Maxwell equations (3.5) and (3.6). The Vlasov equation (2.7) states that the particle distribution function f stays constant along the integral curves of the Hamiltonian vector field L, given in (2.21).

4 The Results in the Context of Research

The Einstein-Vlasov system and the Einstein-Vlasov-Maxwell system admit solutions of various different types. These solutions describe space-times with Vlasov matter which can have fundamentally different features. Among the scenarios that have been studied are dynamical solutions describing black hole formation or dispersion, solutions with various types of symmetry, space-times with compact and non-compact hypersurfaces of constant time coordinate, and also solutions with static space-times. The review article [7] or the book [26] give a good overview over the state of research. This thesis is about static solutions. A solution is called static if the space-time manifold $\mathscr{M}$ possesses a timelike Killing field K which is irrotational. This is true if there are coordinates (t, x, y, z) such that the vector field $K = \partial_t$ is a Killing vector field, i.e. $\mathscr{L}_K g = 0$, and $g_{0a} = 0$, $a = 1, \ldots, 3$. In these coordinates the components of the metric tensor and the particle distribution function f are independent of t. Physically, static solutions can be interpreted as configurations of stellar matter that do not change their shape over time even though the individual particles in the matter formation are moving. In a disc galaxy for example all stars are orbiting around the center, thus they are at move, but the macroscopic shape of the galaxy remains unchanged. Of course, it is not an easy question for what astrophysical objects static solutions are a good description. We refer to the book [14] for some background on the mathematical modeling of stellar matter.

A rigorous analytic study of static solutions of the Einstein-Vlasov system commenced in 1993. Alan Rendall and Gerhard Rein proved in [27] the existence of spherically symmetric static solutions with massive particles. However this result covers only isotropic solutions, i.e. solutions where the pressure is the same in each spatial direction x, y, z. The non-vanishing components of the energy momentum tensor can be interpreted as matter quantities like energy density and pressure. Cf. Section 6.2 for more details. In later works the assumption of isotropy has been removed and the existence of different classes of static solutions has been established for massive particles [23, 24, 11, 12]. It is an important difference between the matter model of a perfect fluid and Vlasov matter that the latter one can have anisotropic pressure. Wolansky proved the existence of spherically symmetric, static solutions with massive particles in [32] by a variational method. Furthermore it was shown that the particle distribution function of these solutions is of bounded support [22].

Perturbation arguments have proved to be a fruitful tool in the analysis of static solutions of the Einstein-Vlasov system. In [24] Gerhard Rein uses a perturbation argument to prove the existence of shell-like static solutions. This means that the spatial support of the particle distribution functions is a shell around the center of symmetry. In [10] the authors use a perturbation argument to show the existence of static solutions with massive particles in the presence of a positive cosmological constant. In the proof of existence of static solution with massive particles of small charge, which is part of Paper II in this thesis, perturbation techniques have been employed, too. In each case the implementation of this idea is of course different and has to be adapted to the structure of the perturbation which can be quite complex.

It is noticeable that so far existence results on static solutions of the Einstein-Vlasov system have been achieved almost exclusively in spherical symmetry. Important exceptions are [11] where the authors use a technique based on the implicit function theorem to prove that non spherically symmetric deviations of spherically symmetric static solutions also are static solutions of the Einstein-Vlasov system. In [12] the authors generalize this result to rotating solutions, i.e. particle distributions with non-vanishing overall angular momentum. Rotating solutions are not static any more but merely stationary. The numerical work [13] gives an overview over the different classes of static solutions in spherical symmetry.

The matter of static, spherically symmetric solutions cannot be arbitrarily dense. More precisely, there cannot exist a static solution where the ratio $2m(r)/r$ exceeds the value $\frac{8}{9}$ at any radius r. The quantity $m(r)$, called Hawking mass, is basically given by the integral over the energy density over the ball of area radius r around the center of symmetry, see (2.15). All quantities are defined in Section 5.2 below. This upper bound on the stability radius for objects of a certain mass can be seen as generalization of the well-known Buchdahl inequality. This bound has been established for the Einstein-Vlasov system in [4]. It has been generalized to a large class of different matter models (that Vlasov matter is a special case of) [5], to solutions with a cosmological constant [8] and to solutions with charged matter [6]. In this context a technique has been developed that exploits very well the mechanisms of the Einstein-Vlasov system in spherical symmetry and that has been employed for two of the results in this thesis, too. This technique basically establishes that by choosing boundary conditions at the center of symmetry one can construct a sequence of static solutions which approaches a solution with an infinitesimally thin shell of Vlasov matter. In this limit the inequality

$$\sup_{r\in[0,\infty)} \frac{2m(r)}{r} \leq \frac{8}{9} \tag{4.1}$$

becomes sharp [3].

As a matter of fact the bound (4.1) yields global existence of static solution almost directly if no additional mechanism such as an electric field or a cosmological constant is present. Existence of a static solution in spherical symmetry can be proven by the following scheme. By virtue of the symmetry assumptions the Einstein-Vlasov system can be transformed into a system of ordinary differential equations in r with boundary conditions at $r = 0$. One wishes to integrate these equations from the center of symmetry outwards. To this end first a local existence result is established. This works usually by a contraction method. A differential operator is constructed from the Einstein-Vlasov equations with the property that a static solution is a fixed point and that acts as a contraction on a suitable set of function over an interval $[0, \delta]$ in the radial axis. In the second step a continuation criterion is formulated. Practically this means that one identifies which terms in the equations can diverge as one moves outwards along the r-axis. To finish the proof of existence one eventually shows that these terms stay bounded for all r. This boundedness follows from the bound (4.1).

In order to obtain physically meaningful solutions it is vital to show that the matter quantities are of bounded support or at least that the mass of the solution is finite. So although the existence of static, spherically symmetric solutions of the Einstein-Vlasov system with massless particles follows easily from the bounds on $2m(r)/r$ the question if there exist solutions with matter quantities of compact support remains interesting. This cannot be shown with the same techniques as in the massive case. On the other hand, compactly supported massless static solutions have been observed numerically [1]. However, an analytical understanding of solutions with a particle distribution function of compact support has been missing. A contribution to this analytical understanding is made in the first paper of this thesis where existence of thin shells of massless Vlasov matter is proven. To this end the method described in the preceding paragraph is used. In particular the solutions constructed in this paper are thin shells of Vlasov matter where $\sup_r 2m(r)/r$ is close to its upper bound $\frac{8}{9}$. These solutions are highly relativistic. One should point out that in contrast to the massive case there are no indications that solutions with a dilute particle distribution and compact support exist.

For the case of charged particles some results on the evolution problem can be found in the literature. Existence of initial data satisfying the constraint equations has been proven in [19]. In [21, 18], generalizing [25] in the uncharged case, global in time existence is proven for solutions with small initial data.

As already indicated, if the particles are charged existence of a static solution of the EVM-system does not follow trivially any more from bounds on $\sup_r 2m(r)/r$. In the charged case an analogous inequality to (4.1), the inequality

$$\sqrt{\frac{m_g(r)}{r}} \leq \frac{1}{3} + \sqrt{\frac{1}{9} + \frac{q(r)^2}{3r^2}} \tag{4.2}$$

holds. Here $m_g(r)$ is a mass parameter including some charge terms, introduced e.g. in Paper II, and $q(r)$ is the charge contained in a ball of radius r. However, (4.2) admits values for $2m(r)/r$ up to 1 which would be a singular point of the solution. In Paper II of this thesis existence of spherically symmetric static solutions of the EVM-system is shown by other techniques. It contributes to the analytical understanding of some classes of static solutions that have already been observed numerically in [9]. Furthermore it complements other work that assume the existence of these solutions, as e.g. [6].

Moreover it provides analytical understanding of sequences of spherically symmetric static solutions of the EVM-system that approach an infinitesimally thin shell. In this limit the inequality (4.2) becomes sharp. So by constructing such a sequence, sharpness of (4.2) is shown. However, it turns out that the quantity $q(r)/r$ in (4.2) tends to zero in this limit. So the sequence of charged shells approaches the uncharged case.

In the uncharged case, thin shells are the unique types of solutions where the inequality (4.1) becomes sharp. In the charged case however, numerical observations in [9] give rise to the conjecture that there is another class of solutions saturating (4.2). These solutions seem to be characterized by $M = R = Q$ where M is the ADM mass of the solution, Q the total charge, and R the bound of the support of the matter quantities of the solution. The analytical understanding of this class of solutions would complement the present picture of steady states with charged Vlasov matter.

5 Charged Particles in Spherical Symmetry

In this thesis results on static solutions of the Einstein-Vlasov system in spherical symmetry are presented. In spherical symmetry the system of equations (3.1) – (2.7) simplifies considerably and reduces to a system of ordinary differential equations. To incorporate spherical symmetry into the system we take the ansatz

$$g = -e^{2\mu(t,r)}\mathrm{d}t^2 + e^{2\lambda(t,r)}\mathrm{d}r^2 + r^2\mathrm{d}\vartheta^2 + r^2\sin^2\vartheta\mathrm{d}\varphi^2 \tag{5.1}$$

for the metric tensor. Here Schwarzschild coordinates $t \in \mathbb{R}$, $r \in [0,\infty)$, $\vartheta \in [0,\pi)$, $\varphi \in [0,2\pi)$ are used. In the remainder of this section we use t and x^0 interchangeably.

5.1 The electric field

This paragraph follows [15]. The electromagnetic field tensor $F \in \Lambda^2(T\mathscr{M})$ is defined to be

$$F = \mathrm{d}A, \tag{5.2}$$

where $A \in \Lambda^1(T\mathscr{M})$ is the electromagnetic four potential. As a direct consequence F is antisymmetric and closed. It fulfills the Maxwell equations (3.5) – (3.6).

Since F is a two form on the four dimensional manifold $\mathscr{M}$, we can split it up into the components E and B as follows,

$$F = E\wedge\mathrm{d}t+B, \quad \text{where} \quad B = B_{23}\mathrm{d}x^2\wedge\mathrm{d}x^3+B_{31}\mathrm{d}x^3\wedge\mathrm{d}x^1+B_{12}\mathrm{d}x^1\wedge\mathrm{d}x^2. \tag{5.3}$$

Now we recall the definition of the Hodge star operator $\star$. Let α be a p-form on an n dimensional manifold equipped with the Lorentzian metric g. Then

$$\star\alpha := \alpha^\star_{j_1\dots j_{n-p}}\mathrm{d}x^{j_1}\wedge\cdots\wedge\mathrm{d}x^{j_{n-p}}, \tag{5.4}$$

$$\alpha^\star_{j_1\dots j_{n-p}} = \sqrt{|\det g|}\alpha^{l_1\dots l_p}\varepsilon_{l_1\dots l_p j_1\dots j_{n-p}}. \tag{5.5}$$

Here the permutation symbol $\varepsilon_{i_1\dots i_k}$ is defined as usual. The Hodge star has the property that for any two p forms α and β we have

$$\alpha\wedge\star\beta = \langle\alpha,\beta\rangle\mathrm{dvol} = \alpha_{i_1\dots i_p}\beta^{i_1\dots i_p}\sqrt{|\det g|}\mathrm{d}x^1\wedge\cdots\wedge\mathrm{d}x^n. \tag{5.6}$$

Like F also the two form $\star F$ can be decomposed. We define $\mathscr{E}$ and $\mathscr{B}$ by the decomposition

$$\star F = \mathscr{E} - \mathscr{B} \wedge \mathrm{d}t. \tag{5.7}$$

In the special case of a flat space time $\mathscr{E}$ and $\mathscr{B}$ are closely related to $\star E$ and $\star B$. Indeed we have

$$\star B = -\mathscr{B} \wedge \mathrm{d}t, \qquad \star(E \wedge \mathrm{d}t) = \mathscr{E}. \tag{5.8}$$

For a more complicated metric however these relations do not hold. The one form E describes the electric field and the one form $\mathscr{B}$ describes the magnetic field.

In a spherically symmetric, static setting the Maxwell equations can be simplified. To this end we first observe

$$\mathrm{d}F = 0 \quad \Leftrightarrow \quad \partial_\alpha F_{\beta\gamma} \mathrm{d}x^\alpha \wedge (\mathrm{d}x^\beta \otimes \mathrm{d}x^\gamma) = 0 \quad \Leftrightarrow \quad \partial_{[\alpha} F_{\beta\gamma]} = 0 \tag{5.9}$$

$$\Leftrightarrow \quad \nabla_{[\alpha} F_{\beta\gamma]} = 0. \tag{5.10}$$

The brackets $[\dots]$ denote the antisymmetrization. Since F is antisymmetric this yields a cyclic sum. To check the last equivalence one verifies that in the cyclic sum the Christoffel symbols cancel out. We consider this equation for $\alpha, \beta, \gamma \neq 0$. We have

$$\mathscr{B}_j = \sqrt{|\det g|} F^{\mu\nu} \varepsilon_{\mu\nu 0 j} \tag{5.11}$$

$$\Leftrightarrow \quad \varepsilon^{ab0j} \mathscr{B}_j = \sqrt{|\det g|} F^{ab} \tag{5.12}$$

$$\Leftrightarrow \quad F_{kl} = g_{k\alpha} g_{l\beta} F^{\alpha\beta} = \frac{1}{\sqrt{|\det g|}} g_{ka} g_{lb} g_{jp} \varepsilon^{ab0j} \mathscr{B}^p, \tag{5.13}$$

where we used $\mathscr{B}_0 = 0$ and $g_{0j} = 0$. For symmetry reasons it follows

$$\partial_b \left(\frac{\sqrt{|\det g|}}{\alpha} \mathscr{B}^b \right) = 0 \tag{5.14}$$

with the lapse function $\alpha = \sqrt{-g_{00}}$. This implies that in spherical symmetry all components of the magnetic field $\mathscr{B}$ are zero. To see this we take the ansatz $\mathscr{B}^j = b(t,r) x^j$. Define the shorthand $C(r) = \sqrt{|\det g|} b(t,r)/\alpha$. Then (5.14) implies

$$\frac{\mathrm{d}}{\mathrm{d}r} \left(r^3 C(r) \right) = 0 \tag{5.15}$$

which is only possible if $b(t,r) = 0$ if one requires a regular solution.

Consider now the electric field E in a spherically symmetric setting. We introduce the reduced field ϵ by $E^j = \epsilon(t,r)\frac{x^j}{r}$. To this end we recall the second Maxwell equation $\nabla_\alpha F^{\alpha\beta} = 4\pi j^\beta$, where we define

$$j^\beta := -q_0 \int_{\mathscr{P}_x} f(x,p) p^\beta \mu_{\mathscr{P}_x}. \tag{5.16}$$

First we consider $\beta = 0$. We have

$$\begin{aligned} \nabla_\alpha F^{\alpha 0} &= \frac{1}{\sqrt{|\det g|}} \partial_\alpha \left(\sqrt{|\det g|} F^{\alpha 0} \right) \\ &= \frac{1}{\sqrt{|\det g|}} \partial_j \left(\sqrt{|\det g|} g^{00} \epsilon(r) \frac{x^j}{r} \right) \end{aligned} \tag{5.17}$$

since we have

$$F^{a0} = g^{ab} g^{00} F_{b0} = g^{ab} g^{00} E_b = g^{00} E^a. \tag{5.18}$$

To be completely concrete here, we plug in the expression (5.1) for the metric and obtain eventually

$$-\frac{e^{-(\mu+\lambda)(t,r)}}{r^2} \frac{\mathrm{d}}{\mathrm{d}r} \left(\epsilon(t,r) r^2 e^{(\lambda-\mu)(t,r)} \right) = 4\pi j^0(t,r). \tag{5.19}$$

It is convenient to define the charge density ϱ_q by

$$\varrho_q(t,r) = -e^{(\mu+\lambda)(t,r)} j^0(t,r) \tag{5.20}$$

and the charge $q(t,r)$ contained in a ball with radius r, given by

$$q(t,r) = e^{(\lambda-\mu)(t,r)} \epsilon(t,r) r^2. \tag{5.21}$$

So in terms of q the Maxwell equation (5.19) reads

$$q'(t,r) = 4\pi r^2 \varrho_q(t,r). \tag{5.22}$$

We consider also the other components of $\nabla_\alpha F^{\alpha\beta}(t,r) = j^\beta(t,r)$. We write

$$\nabla_\alpha F^{\alpha b} = \frac{1}{\sqrt{|\det g|}} \left[\partial_0 \left(\sqrt{|\det g|} F^{0b} \right) + \partial_j \left(\sqrt{|\det g|} F^{jb} \right) \right]. \tag{5.23}$$

The first term is zero if one assumes that both, the metric and the electromagnetic tensor, are independent of time. Since $g_{0j} = 0$ the second terms only contains components of the magnetic field $\mathscr{B}$ which is zero in spherical symmetry. So in total the equations reduce to $j^b(t,r) = 0$ for a time independent solution.

5.2 The Einstein equations

In this section the tt-, the rr- and the $\vartheta\vartheta$- component of the Einstein equations in spherical symmetry are derived. First we calculate the components tt, rr, and $\vartheta\vartheta$ of the tensor τ with the assumed symmetry. To this end we first calculate the components of τ in Cartesian coordinates in terms of ϵ and perform then a coordinate transformation to Schwarzschild coordinates. In Cartesian coordinates the electro magnetic field tensor has the matrix representation

$$(F_{\alpha\beta}) = \begin{pmatrix} 0 & -E_1 & -E_2 & -E_3 \\ E_1 & 0 & 0 & 0 \\ E_2 & 0 & 0 & 0 \\ E_3 & 0 & 0 & 0 \end{pmatrix}, \qquad E_j = g_{jk}E^k = g_{jk}\epsilon\frac{x^k}{r}. \tag{5.24}$$

Note that we use the condition $g_{0i} = 0$ for $i \in \{1, 2, 3\}$, i.e. that the space-time is static. So one obtains

$$\tau_{00} = \frac{1}{8\pi}\epsilon^2 e^{2\lambda}, \tag{5.25}$$

$$\tau_{0m} = 0, \tag{5.26}$$

$$\tau_{mn} = \frac{1}{8\pi}\epsilon^2 e^{2(\lambda-\mu)}\left(\delta_{mn} - \frac{x_m x_n}{r^2} - e^{2\lambda}\frac{x_m x_n}{r^2}\right). \tag{5.27}$$

To simplify notation for the coordinate transformation let $y^0 = t$, $y^1 = r$, $y^2 = \vartheta$, $y^3 = \varphi$. We write for the components $\tilde{\tau}_{\mu\nu}$ of the energy momentum tensor

$$\tilde{\tau}_{\mu\nu} = \tau_{00}\frac{\partial x^0}{\partial y^\mu}\frac{\partial x^0}{\partial y^\nu} + \tau_{ab}\frac{\partial x^a}{\partial y^\mu}\frac{\partial x^b}{\partial y^\nu}. \tag{5.28}$$

A straight forward calculation then yields that in Schwarzschild coordinates τ is in diagonal form and we have

$$\tau = \frac{1}{8\pi}\epsilon^2 e^{2(\lambda-\mu)}\left(e^{2\mu}\mathrm{d}t^2 - e^{2\lambda}\mathrm{d}r^2 + r^2\mathrm{d}\vartheta^2 + r^2\sin^2\vartheta\mathrm{d}\varphi^2\right). \tag{5.29}$$

We also have to consider the Vlasov part (3.2) of the energy momentum tensor. The particle distribution function is assumed to have the symmetries

$$f(t, x, p) = f(st, Ax, Ap), \qquad \text{for all} \quad s \in \mathbb{R}, A \in SO(3). \tag{5.30}$$

Then, in Schwarzschild coordinates $(t, r, \vartheta, \varphi)$ the energy momentum tensor T has diagonal form. Moreover, we can define matter quantities, the energy density ϱ, the radial pressure p, and the tangential pressure p_T, by writing

$$T = \varrho e^{2\mu}\mathrm{d}t^2 + pe^{2\lambda}\mathrm{d}r^2 + p_T r^2\mathrm{d}\vartheta^2 + p_T r^2\sin^2\vartheta\mathrm{d}\varphi^2. \tag{5.31}$$

The energy condition

$$\varrho \geq p + 2p_T \tag{5.32}$$

holds. Using the ansatz (5.1) for the metric one obtains for the components of the Einstein tensor the formulas

$$G_{tt} = e^{2(\mu-\lambda)} \left(\frac{2\lambda'}{r} - \frac{1}{r^2} \right) + \frac{e^{2\mu}}{r^2}, \tag{5.33}$$

$$G_{rr} = \frac{1}{r^2} \left(2r\mu' + 1 - e^{2\lambda} \right), \tag{5.34}$$

$$G_{\vartheta\vartheta} = re^{-2\lambda} \left((\mu' - \lambda') \left(1 + r\mu' \right) + r\mu'' \right) \tag{5.35}$$

as derived in [31]. A prime denotes the derivative with respect to r, We finally derive

$$e^{-2\lambda} \left(2r\lambda' - 1 \right) + 1 = 8\pi r^2 \varrho + \frac{q^2}{r^2}, \tag{5.36}$$

$$e^{-2\lambda} \left(2r\mu' + 1 \right) - 1 = 8\pi r^2 p - \frac{q^2}{r^2}, \tag{5.37}$$

$$e^{-2\lambda} \left((\mu' - \lambda') \left(\frac{1}{r} + \mu' \right) + \mu'' \right) = 8\pi p_T + \frac{q^2}{r^4}. \tag{5.38}$$

These are not all the non trivial Einstein equations. However, one can show that if a solution of the equations (5.36), (5.37) is found, all Einstein equations are solved.

5.3 The Tolman-Oppenheimer-Volkov equation

Both sides of the Einstein equations (5.36)–(5.38) are divergence free. In particular we have $\nabla_\nu \left(T^{r\nu} + \tau^{r\nu} \right) = 0$. This is the so called Tolmann-Oppenheimer-Volkov equation (TOV equation) which shall be derived in a simpler form now. Using the ansatz (5.1) for the metric one can calculate

$$\begin{aligned} \nabla_\nu T^{r\nu} &= \partial_r T^{rr} + \Gamma^r_{\nu\nu} T^{\nu\nu} + \Gamma^\nu_{rr} T^{rr} \\ &= p' e^{-2\lambda} + \mu' e^{-2\lambda} (\varrho + p) + \frac{2}{r} e^{-2\lambda} (p - p_T) \end{aligned} \tag{5.39}$$

and analogously

$$\nabla_\nu \tau^{r\nu} = -e^{-2\lambda} \frac{qq'}{4\pi r^4}. \tag{5.40}$$

A prime denotes the partial derivative with respect to r. So combined we have

$$p' = -\mu' (\varrho + p) - \frac{2}{r} (p - p_T) + \frac{qq'}{4\pi r^4}. \tag{5.41}$$

5.4 Connection to the Reissner-Nordström metric

The Einstein equations (5.36) and (5.37) imply

$$\mu'(r) = \left(1 - \frac{2m(r)}{r} - \frac{F(r)}{r}\right)^{-1} \left(4\pi r p(r) + \frac{m(r)}{r^2} - \frac{q(r)^2}{2r^3} + \frac{F(r)}{2r^2}\right) \quad (5.42)$$

with

$$m(r) = 4\pi \int_0^r s^2 \varrho(s) \mathrm{d}s, \quad (5.43)$$

$$F(r) = \int_0^r \frac{q^2(s)}{s^2} \mathrm{d}s. \quad (5.44)$$

If the matter quantities are of compact support a solution of the EVM system describes a spacetime containing a body of charged Vlasov matter which is surrounded by vacuum. In this vacuum region the metric is given by the Reissner-Nordström metric. To conclude this section, we want to see the connection between the formula (5.42) and the mass and charge parameter, M_{RN} and Q_{RN}, showing up in the line element of the Reissner-Nordström solution. Therefore we assume that there exists a radius R such that there is a vacuum region on $[R, \infty)$. We consider (5.42) for these radii. Let

$$Q = q(R), \quad M = 4\pi \int_0^R s^2 \varrho(s) \mathrm{d}s, \quad P = \int_0^R \frac{q^2(s)}{s^2} \mathrm{d}s. \quad (5.45)$$

Then we have for $r \geq R$ the relation

$$\mu'(r) = \frac{1}{1 - \frac{2M+P+Q^2/R}{r} + \frac{Q^2}{r^2}} \left(\frac{M + P/2 + Q^2/2R}{r^2} - \frac{Q^2}{r^3}\right). \quad (5.46)$$

We observe that the mass parameter M_{RN} in the standard formula for the Reissner-Nordström metric is given by the expression

$$M_{\mathrm{RN}} = M + \frac{P}{2} + \frac{Q^2}{2R}, \quad (5.47)$$

whereas the charge parameter Q_{RN} is just given by Q.

6 The Method of Characteristics

6.1 Conserved quantities

The Vlasov equation can be treated with the method of characteristics. A particle distribution $f \in C(\mathscr{P}_m; \mathbb{R}_+)$ is a solution of the Vlasov equation (2.7) if and only if it is conserved along characteristics, i.e. curves $(X(s), P(s))$ satisfying the system (2.1) – (2.2). The aim now is to find physically meaningful quantities that are conserved along these characteristics. Any function of these conserved quantities will then again be conserved along characteristics and thus be a solution of the Vlasov equation.

Let $p^{(t)}$, $p^{(r)}$, $p^{(\vartheta)}$, $p^{(\varphi)}$ be the canonical momenta to the Schwarzschild coordinates t, r, ϑ, φ. The symmetries of the setting suggest to parameterize the characteristics by the variables

$$r, \quad w = e^{\lambda(r)} p^{(r)}, \quad L = r^4 \left(\left(p^{(\vartheta)}\right)^2 + \sin^2(\vartheta) \left(p^{(\varphi)}\right)^2 \right). \tag{6.1}$$

The variable r is the area radius, w the radial momentum, and L the square of the angular momentum. Due to the symmetry assumptions (5.30) we can, by abuse of notation, consider the particle distribution function as a function of merely r, w, and L, i.e. $f = f(r, w, L)$. We express the characteristic system (2.1) – (2.2) in terms of the variables r, w, L. To this end we first calculate the Christoffel symbols in Cartesian coordinates. The non zero ones are

$$\Gamma^i_{00} = e^{2(\mu-\lambda)} \mu' \frac{x^i}{r}, \tag{6.2}$$

$$\Gamma^i_{jk} = \delta_{jk} \frac{1 - e^{-2\lambda}}{r^2} x^i + \frac{r\lambda' - 1 + e^{-2\lambda}}{r^4} x^i x^j x^k. \tag{6.3}$$

This yields the system

$$\frac{\partial r(s)}{\partial s} = w(s), \tag{6.4}$$

$$\begin{aligned} \frac{\partial w(s)}{\partial s} &= \frac{L(s)}{r(s)^3} + q_0 \frac{q(r(s))}{r(s)^2} e^{\mu(r(s))+\lambda(r(s))} \\ &\quad - \mu'(r(s)) \left(m^2 + w(s)^2 + \frac{L(s)}{r(s)^2} \right), \end{aligned} \tag{6.5}$$

$$\frac{\partial L(s)}{\partial s} = 0. \tag{6.6}$$

We see immediately that the angular momentum variable L is already a conserved quantity along the characteristic curves.

In the uncharged case, $q_0 = 0$, the Hamiltonian vector field L is given by the geodesic spray $\mathscr{X}$ and the characteristic curves are the lifts of geodesics to the tangent bundle $T\mathscr{M}$ of $\mathscr{M}$. This implies that the conserved quantities can be obtained in an elegant way. Let ξ be a Killing vector field and $\gamma^\mu(s)$ a geodesic. Then $g(\dot{\gamma}, \xi)$ is a conserved quantity. The particle energy E and the z-component L_z of the angular momentum can be obtained by taking the Killing fields ∂_t and ∂_φ respectively.

If the particles are charged, their trajectories will not be geodesics any longer. The conserved quantities must be altered according to the structure of the characteristic system (6.4) – (6.6). These quantities read

$$L \quad \text{and} \quad E = e^{\mu(r)}\sqrt{m_0^2 + w^2 + \frac{L}{r^2}} - I_q(r), \tag{6.7}$$

where the term $I_q(r)$ can be read off equation (6.5). We see that

$$\frac{\mathrm{d}}{\mathrm{d}r} I_q(r) = q_0 e^{(\mu+\lambda)(r)} \frac{q(r)}{r^2}. \tag{6.8}$$

6.2 The matter quantities

In the preceding section we have seen that any function $f(r, w, L) = \Phi(E(r, w, L), L)$ which depends on r and w only indirectly via the conserved quantities E and L is a solution of the Vlasov equation.

It is a legit question whether for all static, spherically symmetric solutions of the EVM-system the particle distribution function f is a function of E and L, and not of r. In the case of the Vlasov-Poisson system this is the case. This fact is generally referred to as Jeans' Theorem. For the Einstein-Vlasov system however counterexamples have been found [30].

Using a specific ansatz function Φ for f we will in this section derive integral expressions for the matter quantities ϱ, p, p_T (defined in (5.31), and ϱ_q (defined in (5.20)). These integral expressions are an important step in the method of characteristics because they allow to see the matter quantities as explicitly given functions of the metric functions μ, λ, and the integral term I_q, given in (6.8) as part of the particle energy. Thus

$$\varrho(r) = g_\Phi(r, \mu(r), I_q(r)), \tag{6.9}$$
$$p(r) = h_\Phi(r, \mu(r), I_q(r)), \tag{6.10}$$
$$\varrho_q(r) = q_0 e^\lambda(r) k_\Phi(r, \mu(r), I_q(r)). \tag{6.11}$$

In Paper II these functions are defined and discussed. The important point is that the Vlasov equation has been eliminated from the system.

In this thesis we consider *polytropic* ansatz functions of the form

$$f = \Phi(E, L) = c\left[1 - \frac{E}{E_0}\right]_+^k [L - L_0]_+^\ell . \tag{6.12}$$

The constants are chosen such that

$$c \geq 0, \quad k > -1, \quad L_0 \geq 0, \quad \ell > -\frac{1}{2}, \tag{6.13}$$

and $[x]_+ = x$ if $x \geq 0$ and $[x]_+ = 0$ if $x < 0$. Note that E_0 and L_0 are cut-off parameters for the particle energy and the angular momentum, respectively. They give an upper bound for the energy and a lower bound for the angular momentum.

According to (5.31) we have

$$\varrho(r) = T(e^{-\mu}\partial_t, e^{-\mu}\partial_t), \quad p(r) = T(e^{-\lambda}\partial_r, e^{-\lambda}\partial_r). \tag{6.14}$$

Note that $e^{-\mu}\partial_t$ is the unit vector pointing in t-direction and $e^{-\mu}\partial_t$ is the unit vector pointing in r-direction. The tangential pressure p_T can be calculated by contracting T with any unit vector lying in the plane spanned by ∂_ϑ and ∂_φ. It turns out however that for the calculation it is best to take

$$p_T(r) = \frac{1}{2}\left(g_{\vartheta\vartheta}T^{\vartheta\vartheta} + g_{\varphi\varphi}T_{\varphi\varphi}\right). \tag{6.15}$$

Using (6.1), (6.14), and (6.15) on calculates

$$\varrho(r) = \frac{\pi}{r^2}\int_0^\infty \int_{-\infty}^\infty f(r,w,L)\sqrt{m_0^2 + w^2 + \frac{L}{r^2}}\,\mathrm{d}w\,\mathrm{d}L, \tag{6.16}$$

$$p(r) = \frac{\pi}{r^2}\int_0^\infty \int_{-\infty}^\infty f(r,w,L)\frac{w^2}{\sqrt{m_0^2 + w^2 + \frac{L}{r^2}}}\,\mathrm{d}w\,\mathrm{d}L, \tag{6.17}$$

$$p_T(r) = \frac{\pi}{2r^4}\int_0^\infty \int_{-\infty}^\infty f(r,w,L)\frac{L}{\sqrt{m_0^2 + w^2 + \frac{L}{r^2}}}\,\mathrm{d}w\,\mathrm{d}L, \tag{6.18}$$

$$\varrho_q(r) = q_0 e^{\lambda(r)}\frac{\pi}{r^2}\int_0^\infty \int_{-\infty}^\infty f(r,w,L)\,\mathrm{d}w\,\mathrm{d}L \tag{6.19}$$

Recall that $m_0 \geq 0$ is the particle mass. In Paper II the quantity $z = \varrho - p - 2p_T$ plays a role. Using the formulas (6.16) – (6.18) we see that

$$z(r) = \frac{\pi}{r^2}\int_0^\infty \int_{-\infty}^\infty f(r,w,L)\frac{m_0^2}{\sqrt{m_0^2 + w^2 + \frac{L}{r^2}}}\,\mathrm{d}w\,\mathrm{d}L. \tag{6.20}$$

Now we apply the ansatz (6.12). The integration with respect to L can be carried out explicitly, yielding a constant c_ℓ given in (6.25) below. Moreover it is convenient to express the integrals in terms of the variable

$$\varepsilon = \sqrt{m_0^2 + w^2 + \frac{L}{r^2}}. \tag{6.21}$$

We eventually obtain

$$\varrho(r) = c_\ell r^{2\ell} \xi_{2,\ell+\frac{1}{2}} \left(\sqrt{m_0^2 + \frac{L_0}{r^2}} \right), \tag{6.22}$$

$$p(r) = \frac{c_\ell}{2\ell+3} r^{2\ell} \xi_{0,\ell+\frac{3}{2}} \left(\sqrt{m_0^2 + \frac{L_0}{r^2}} \right), \tag{6.23}$$

$$\varrho_q(r) = c_\ell q_0 e^{\lambda} r^{2\ell} \xi_{1,\ell+\frac{1}{2}} \left(\sqrt{m_0^2 + \frac{L_0}{r^2}} \right), \tag{6.24}$$

with the constant

$$c_\ell = 2\pi \int_0^1 \frac{s^\ell}{\sqrt{1-s}} \mathrm{d}s \tag{6.25}$$

and the shorthand

$$\xi_{\alpha\beta}(t) = \int_t^{e^{-\mu}(E_0+I_q)} \left(1 - \frac{\varepsilon e^{\mu} - I_q}{E_0} \right)^k \varepsilon^\alpha \left(\varepsilon^2 - t^2 \right)^\beta \mathrm{d}\varepsilon. \tag{6.26}$$

One immediately observes the relations

$$\varrho(r) \geq (2\ell+3) p(r) \tag{6.27}$$

and

$$e^{-\mu(r)} \left(E_0 + I_q(r) \right) \varrho_q(r) \geq q_0 e^{\lambda(r)} \varrho(r) \geq \sqrt{m_0^2 + \frac{L_0}{r^2}} \varrho_q(r). \tag{6.28}$$

7 Characterization of Massless Shells

In Paper I of this thesis the existence of spherically symmetric, static solutions of the Einstein-Vlasov system with massless particles is proven. In the proof a class of static shell solutions is constructed. These shell solutions share the property that

$$\Gamma := \sup_{r \in [0,\infty)} \frac{2m(r)}{r} \geq \frac{4}{5}. \tag{7.1}$$

The quantity Γ that we just defined is called concentration parameter. More generally one can conjecture that a large value of Γ is a necessary condition for the support of the matter quantities of a static solution of the Einstein-Vlasov system with massless particles to be finite. This conjecture is motivated by several observations. First, the property (7.1) is an important mechanism for the construction of massless shell solutions in Paper I of this thesis. Second, static solutions of the massless Einstein-Vlasov system with compact support have been observed in numerical work by other authors, for example in [1] where critical collapse has been investigated for massless Vlasov matter, or in [16]. All static solutions in these works have the property that $0.7 \leq \Gamma \leq \frac{8}{9}$. Finally, we made our own numerical calculations to get an impression what values of Γ occur with a polytropic ansatz function of the form (6.12). We were able to construct solutions where Γ takes values down to approximately 0.8. At this place it should be stressed that the conjecture, as formulated in Paper I of this thesis, should be understood as stating that the solutions have to be highly relativistic. We believe that the exact value of Γ is not essential. It seems to be lower than 0.8.

It is instructive to set the observed values of Γ into context. The spacetime surrounding a spherically symmetric object of mass M is described by the Schwarzschild metric. Taking into account the right physical units the line element in Schwarzschild coordinates contains the expression $2MG/(c^2 r)$ where M is the mass of the body, G is the gravitational constant, c the speed of light, and r the radial coordinate. If this quantity is small, spacetime curvature is small. If it is close to one spacetime curvature is extremely large. The same is true for Γ.

At the surface of the sun this quantity takes the value $4.239 \cdot 10^{-6}$. At the surface of the earth it is only $1.39 \cdot 10^{-9}$. Both values are very small compared to 0.8 in (7.1). However, in the universe there exist objects with much stronger gravitational fields. Neutron stars for example are very dense objects. It is not easy to determine the mass and the radius of a neutron star, the review article [20] suggests however that one can take $M = 3 \cdot 10^{30}\,\mathrm{kg}$ and $R = 10\,\mathrm{km}$ in order to make good estimates. This yields $2MG/(c^2 R) \approx 0.4424$. In the vicinity of a black hole the quantity $2MG/(c^2 R)$ becomes even larger and takes values up to 1. On the other hand, if $2MG/(c^2 R) = 0.8$ was achieved by a body of the mass of the earth, its matter would have to be confined to a sphere with a radius of merely $1.1\,\mathrm{cm}$.

Next we discuss a sequence of shell solutions of the massless Einstein-Vlasov system approaching an infinitesimally thin shell. This means if R_1 is the inner radius of the support of the matter quantities and R_2 is the outer radius, then $R_2/R_1 \to 1$. For the particular sequence of shell solutions that we construct we also have $R_1 \to 0$. Such a sequence is interesting because the limit of thin shells is the unique situation in which the inequality $\Gamma \leq \frac{8}{9}$ becomes sharp, if the particles do not carry charge (cf. [4]). Recall that the concentration parameter Γ was defined in (7.1). In this paragraph an example of such a sequence is constructed by numerical means. The Einstein equations are treated as in [13].

In particular the variable $y := e^\mu/E_0$ is introduced and the system

$$y'(r) = \frac{y(r)}{1 - \frac{8\pi}{r}\int_0^r s^2 g_\Phi(s,\mu(s))\mathrm{d}s}\left(4\pi r h_\Phi(r) + \frac{4\pi}{r}\int_0^r s^2 g_\Phi(s)\mathrm{d}s\right), \tag{7.2}$$

$$y(0) = y_c \tag{7.3}$$

is solved using an Euler algorithm. The functions g_Φ and h_Φ are as in (6.9) – (6.10). We keep track of the concentration parameter Γ. Furthermore we consider the trajectories of particles in the calculated space-times, i.e. geodesics, and check how the matter quantities ϱ, p, and p_T behave in relation to one another.

In Paper I of this thesis different shell solutions of the massless Einstein-Vlasov system are constructed numerically. $\Gamma \approx 0.8$ is the smallest value of the concentration parameter that was calculated using a polytropic ansatz function of the type (6.12). We choose the model parameters k, ℓ, L_0, y_c of this solution to calculate the first element of a sequence of solution. Then we decrease y_c until the corresponding solution is a thin shell with $\Gamma \approx \frac{8}{9}$.

Figure 2 shows some elements of the sequence of shells. We observe that the

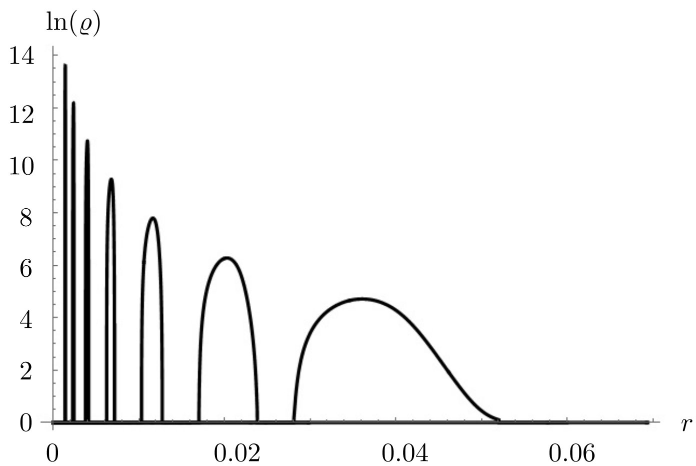

Fig. 2: Profile of the energy density for different central values y_c. The parameters chosen are $k = 0$, $L_0 = 0.01$, $\ell = 0$, $c_0 = 1$, and the initial values y_c are $0.0281, 0.170, 0.103, 0.063, 0.038, 0.023, 0.014$

shells become thinner and move closer to the center of symmetry. Moreover we observe that the maximum value of the energy density ϱ grows rapidly. Figure 3 shows the value of Γ depending on the ratio R_2/R_1 of the outer radius to the inner radius of the matter shell. As proven in [4] the value 8/9 is approached. On the right side in Figure 3 we see the ratio p/p_T and p_T/ϱ. To be precise, for

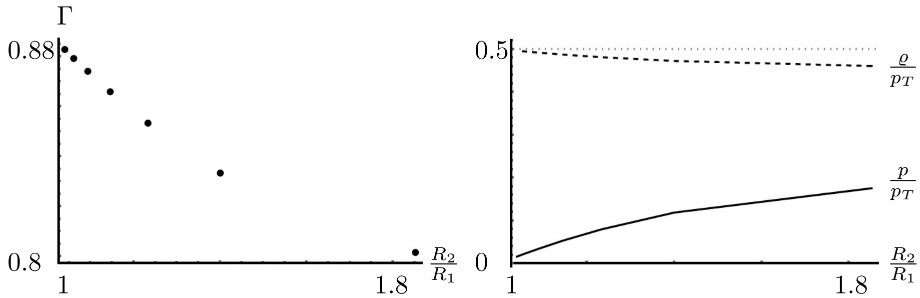

Fig. 3: On the left: Γ for different values of R_2/R_1, the relative thickness of the shell. On the right: the ratio p/p_T and p_T/ϱ (dashed) for different values of R_2/R_1

each solution, characterized by the ratio R_2/R_1 of outer and inner radius of the shell, we plot the following quantity. We evaluate $p(r)/p_T(r)$ at $r = r_{\max}$ where $r_{\max}$ is defined to be the maximum for the numerator, i.e. $p(r)$. Since the limit approaches infinitesimally thin shells the exact radius where p/p_T is evaluated is not essential. We observe that in the limit the maximum of all matter quantities grows rapidly. However, the tangential pressure p_T grows much faster than the radial pressure p and in the limit we approach $\varrho = 2p_T$.

Finally we consider some examples for possible geodesics in the calculated space-times. The geodesics are calculated in the following way. We assume that a static, spherically symmetric space-time has been calculated, as described in the prior paragraph. In particular the functions $\mu(r)$ and $\lambda(r)$ are at disposal. In the setting of massless, chargeless particles the characteristic system of the Vlasov equation in the coordinates (r, w, L), introduced in (6.1) reads

$$\dot{r}(s) = w(s), \tag{7.4}$$

$$\dot{w}(s) = \frac{L(s)}{r(s)^3} - \mu'(r(s))\left(w(s)^2 + \frac{L(s)}{r(s)^2}\right), \tag{7.5}$$

$$\dot{L}(s) = 0, \tag{7.6}$$

where the dot denotes the derivative with respect to the parameter s. In the realm where the shell solution is already close to a thin shell the outer radius R_2 is smaller than $3M$. Nevertheless circular geodesics are possible. Inspecting (7.4)–(7.6) we see that choosing the initial data $w(0) = 0$ and $r(0)$ such that

$$\mu'(\mathring{r}) = \frac{1}{\mathring{r}} \tag{7.7}$$

yields a circular trajectory. If the equation (7.7) has a solution it fulfills $\mathring{r} > R_1$ since $\mu'(R_1) = 0$. On the other hand, we see that (7.7) is equivalent to

$$1 = e^{2\lambda(\mathring{r})} \left(4\pi \mathring{r}^2 p(\mathring{r}) + \frac{m(\mathring{r})}{\mathring{r}} \right). \tag{7.8}$$

The right hand side will become larger than 1 for r contained in the support of the matter quantities, e.g. in view of $e^{2\lambda(r)} \frac{m(r)}{r} \to 4$, as the thin shell limit is approached.

Furthermore, all matter quantities diverge as the shell becomes thinner. This means in particular that there is radial pressure which will be manifested in radial movement.

A solution to this system is calculated with the method of finite differences. We parameterize the particles trajectory bu the coordinates $(r(s), \varphi(s))$. Let $s_0, s_1, s_2, \ldots$ be the discrete steps of the parameter s and Δs be the step width. Write $r_n = r(s_n)$, $\varphi_n = \varphi(s_n)$, $n = 0, 1, 2, \ldots$. These coordinates are evolved by

$$r_{n+1} = r_n + w\Delta s, \quad \varphi_{n+1} = \varphi_n + \frac{\sqrt{L}}{r}\Delta s. \tag{7.9}$$

If we want to consider a massless particle moving inside a shell of massless Vlasov matter only certain initial values for $\mathring{r}$, $\mathring{w}$, $\mathring{L}$ for $r(s)$, $w(s)$, and $L(s)$ are admissible. First, $\mathring{r}$ has to be chosen such that the particle initially is in the shell, i.e. $\mathring{r} \in (R_1, R_2)$. Further the lower bound L_0 on the angular momentum and the upper bound E_0 on the energy give restriction on the choices of $\mathring{w}$ and $\mathring{L}$. We have

$$\begin{aligned}(\mathring{r}, \mathring{L}, \mathring{w}) \in (R_1, R_2) \times \left(L_0, \mathring{r}^2 e^{-2\mu(\mathring{r})} E_0^2 \right) \\ \times \left(-\sqrt{e^{-2\mu(\mathring{r})} E_0^2 - \frac{L}{r^2}}, \sqrt{e^{-2\mu(\mathring{r})} E_0^2 - \frac{L}{r^2}} \right).\end{aligned} \tag{7.10}$$

Two examples are plotted in Figure 4 and 5. As initial data we always choose the maximal possible radial momentum and minimal angular momentum. We observe that in the smaller shell the geodesics turns around much more often than in the broad shell. This is a manifestation of the increasing radial momentum.

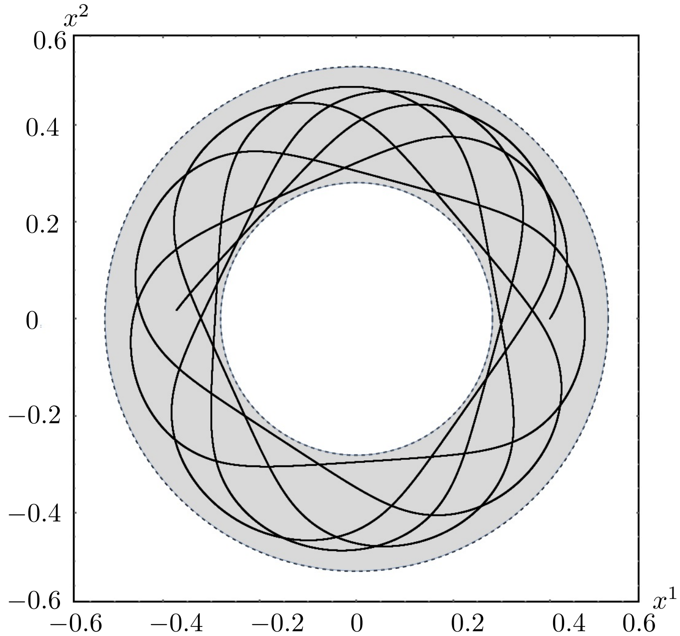

Fig. 4: The spatial support of the matter distribution function f is shaded. The characteristic quantities of the space-time are $\mu_c = -1.231$, $E_0 = 1.041$, $\Gamma = 0.801$, $R_1 = 0.028$, $R_2 = 0.052$. The initial data for the geodesic is $\mathring{r} = 0.04$, $\mathring{L} = 0.011$, $\mathring{w} = 1.6$. The energy of the particle is $E = 1.035$.

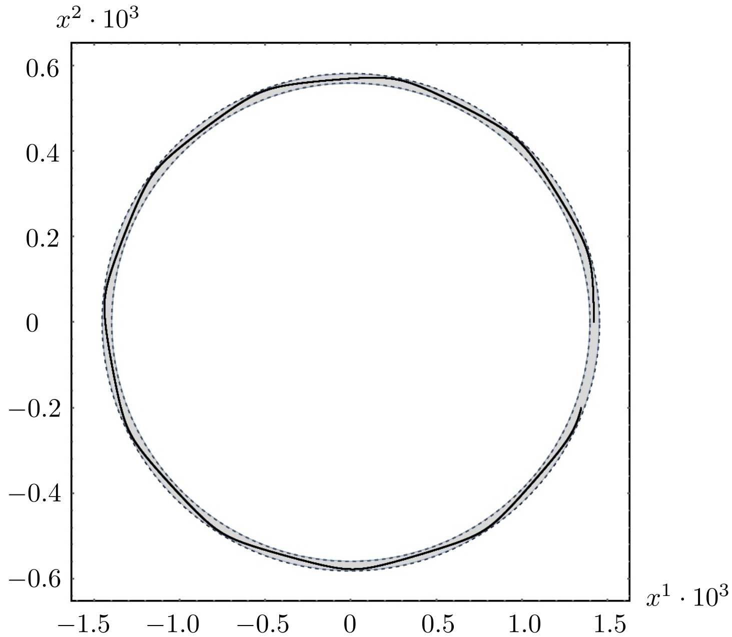

Fig. 5: The spatial support of the matter distribution function f is shaded. The characteristic quantities of the space-time are $\mu_c = -1.11$, $E_0 = 23.58$, $\Gamma = 0.883$, $R_1 = 1.34 \cdot 10^{-3}$, $R_2 = 1.45 \cdot 10^{-3}$. The initial data for the geodesic is $\mathring{r} = 1.42 \cdot 10^{-3}$, $\mathring{L} = 0.0102$, $\mathring{w} = 0$. The energy of the considered particle is $E = 23.459$.

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