Thesis for the Degree of Doctor of Philosophy

On finite element schemes for Vlasov-Maxwell system and Schrödinger equation

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Abstract

This thesis treats finite element schemes for two kind of problems, the Valsov-Maxwell system and the nonlinear Schrödinger equation. We study streamline diffusion schemes applied for numerical solution of the one and one-half dimensional relativistic Vlasov-Maxwell system. The study is made both in a priori and a posteriori settings. In the a priori setting we derive stability estimates and prove optimal convergence rates, due to the maximal available regularity of the exact solution. In addition to this we also prove existence and uniqueness of the numerical solution. In the a posteriori setting we use dual problems to prove error estimates in $L_\infty(H^{-1})$ norm. For the Maxwell equation we also prove error estimates in $H^{-1}(H^{-1})$ norms. Further more we study a *hp*-version of the streamline diffusion scheme for the three dimensional Vlasov-Maxwell system in an a priori setting. A Nitsche type scheme is also introduced and analyzed for Maxwell's equations. For the nonlinear Schrödinger equation a two level time discretization is used. Here we derive a priori error estimates both in L_2 and H^1 norms.

Keywords: Vlasov-Maxwell, streamline diffusion, a priori error analysis, a posteriori error analysis, *hp*-scheme, Nitsche scheme, nonlinear Schrödinger equation

List of included papers

The following papers are included in this thesis:

- **Paper I. Christoffer Standar**. *On Streamline Diffusion schemes for the one and one-half Dimensional Relativistic Vlasov-Maxwell system*, Calcolo, 53 (2016), no. 2, p. 147-169.
- **Paper II.** Mohammad Asadzadeh and **Christoffer Standar**. *A posteriori error estimates for the one and one-half Dimensional Relativistic Vlasov-Maxwell system.* To appear in BIT Numerical Mathematics 2017.
- **Paper III.** Mohammad Asadzadeh, Piotr Kowalczyk and **Christoffer Standar**. *On hp-Streamline Diffusion and Nitsche schemes for the Relativistic Vlasov-Maxwell System.* Preprint.
- **Paper IV.** Mohammad Asadzadeh and **Christoffer Standar**. *Approximating the nonlinear Schrödinger equation by a two level linearly implicit finite element method.* Preprint.

The following papers are not included in this thesis:

- Mohammad Asadzadeh, Larisa Beilina, Muhammad Naseer and Christoffer Standar. *A priori error estimates and computational studies for a Fermi pencil-beam equation*, arXiv:1606.05085, 2016.
- Mohammad Asadzadeh, Larisa Beilina, Muhammad Naseer and Christoffer Standar. *Finite element schemes for Fermi equation*, AIP Conference Proceedings 1863, 370007 (2017).
- Mohammad Asadzadeh, Piotr Kowalczyk and Christoffer Standar. *Convergence of Streamline Diffusion and Nitsche methods for the Vlasov-Maxwell System.* To appear in JCTT, proceeding of ICTT 25, (2017), Monterey, CA.

Contributions. The author of this thesis has contributed in the following manner to the appended papers:

Paper I and **II** is mainly the work of the author.

In **Paper III** the author developed and analyzed the Nitsche scheme.

In **Paper IV** the author constructed and analyzed the time discretization scheme.

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Christoffer Standar
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Contents

Introduction

This thesis work is on construction and analysis of numerical schemes, mostly finite elements, for approximate solution of i) the Vlasov-Maxwell system in different dimensions and ii) the nonlinear Schrödinger equation. The study of the Vlasov-Maxwell system concerns different Galerkin schemes, e.g., streamline diffusion and Nitsche, while for the nonlinear Schrödinger equation a time-splitting approach is comined with a background spatial discretization. Below we first give a description of the equations in the thesis. Then, we introduce the different Galerkin schemes that has been considered. Finally, we present the results of the thesis papers.

1. Vlasov-type systems

The Vlasov equation is a hyperbolic type partial differential equation, which describes the time evolution of the phase-space-time distribution function for plasma of charged particles. The equation was first suggested by Anatoly Vlasov in 1938. The Vlasov equation is often coupled with other equations, to obtain system of equations as mathematical models of charge particle transport. The coupling terms are of crucial importance in describing certain physical phenomenon, e.g., the Vlasov-Maxwell system (VM), which is the focus of our studies in this thesis, describes a self-consistent collisionless plasma under the influence of an electromagnetic field (Vlasov-Poisson is an approximation of the Vlasov-Maxwell system with a Poisson equation describing a potential field). The VM system has a wide range of applications in plasma physics such as lasers, fusion devices and particle accelerators.

Among all Vlasov-type systems, the Vlasov-Poisson system is the one that has been widely studied both in analytical and numerical works. Classical analytical approaches to the Vlasov-Poisson system have been based on Schauder's fixed point theorem which was introduced in a work

on the existence of a classical solution to the two dimensional Vlasov-Poisson system by Ukai and Okabe [**37**]. Another analytic study is due to Bardos and Degond [**8**] where the global existence of solutions of the three dimensional Vlasov-Poisson system with small data has been proved. In early 90s Lions and Perthame [**28**] and Pfaffelmoser [**33**] both proved global existence of the three dimensional Vlasov-Poisson system in independent works. Then, as part of a survey study of classical kinetic models, DiPerna and Lions [**15**] proved the global existence of a weak solution to the Vlasov-Maxwell system in three dimensions. More elaborate analytic study of the VM system is due to Glassey and Schaeffer in several settings and various geometries; see, e.g., [**18**], [**19**], [**21**] or [**22**]. In [**18**] Glassey and Schaeffer proved global existence of classical solution with the assumption of almost neutral initial data. In [**19**] and [**21**] they have studied the global existence with small initial data for VM in one and one-half dimensions (one space variable and two velocity variables) and two and one-half dimensions (two space variables and three velocity variables), respectively. We also mention some other relevant studies e.g., [**14**] and [**30**]. In [**14**] the one and one-half dimensional Vlasov-Maxwell system is studied in an interval, while [**30**] considers the three dimensional Vlasov-Poisson equation with point charges.

As for numerical approximation, the particle methods have long been very popular for discretizing Vlasov-type systems. For studies on particle methods related to the Vlasov equation; see, e.g., [**13**], [**16**], [**38**] or [**20**]. The first three articles concern particle methods for the Vlasov-Poisson system in different settings, while in the fourth one the authors study the one and one-half dimensional Vlasov-Maxwell system. Recently, the discontinuous Galerkin finite element method has become increasingly popular for approximating Vlasov type systems. A discontinuous Galerkin method for the VM system with different choices of the numerical flux has been studied by Cheng et al. in [**12**]. In a series of articles [**26**], [**29**], [**31**] the authors have considered the study of several finite element schemes (including discontinuous Galerkin) for approximating the Maxwell equations. As for a combined system, e.g., in [**5**] the authors construct and analyze a streamline diffusion based DG scheme for the Vlasov-Poisson-Fokker-Planck system.

Our objective is to construct a numerical scheme for the Vlasov-Maxwell system that is both reliable (being highly stable) and efficient

(converges optimally in an L_2-based energy norm due to maximal available regularity of the exact solution). A desirable scheme should also contain a self-adaptivity mechanism that is capable to circumvent over-refinements.

2. The Vlasov-Maxwell system

The main study in this thesis is the relativistic Vlasov-Maxwell system defined by

$$\begin{aligned} &\partial_t f + \hat{v}\cdot\nabla_x f + q(E + c^{-1}\hat{v}\times B)\cdot\nabla_v f = 0,\\ &\partial_t E = c\nabla\times B - j, \qquad \nabla\cdot E = \rho,\\ &\partial_t B = -c\nabla\times E, \qquad \nabla\cdot B = 0. \end{aligned} \tag{2.1}$$

Here f is density, in phase-space-time, of particles with mass m, charge q and velocity

$$\hat{v} = (m^2 + c^{-2}|v|^2)^{-1/2}v \qquad (v \text{ is momentum}).$$

Further, the charge and current densities are given by

$$\rho(t,x) = 4\pi\int qf\,dv \quad \text{and} \quad j(t,x) = 4\pi\int qf\hat{v}\,dv,$$

respectively. The functions E and B describe the electric and magnetic fields, respectively.

For simplicity, we choose to describe the relativistic Vlasov-Maxwell system in the one and one-half dimensional setting ($x\in\mathbb{R}, v\in\mathbb{R}^2$), given by the following system of equations

$$\begin{cases} \partial_t f + \hat{v}_1\partial_x f + (E_1 + \hat{v}_2 B)\partial_{v_1} f + (E_2 - \hat{v}_1 B)\partial_{v_2} f = 0,\\ \partial_x E_1 = \displaystyle\int f dv - \rho_b(x) = \rho(t,x),\\ \partial_t E_1 = -\displaystyle\int \hat{v}_1 f dv = -j_1(t,x),\\ \partial_t E_2 + \partial_x B = -\displaystyle\int \hat{v}_2 f dv = -j_2(t,x),\\ \partial_t B + \partial_x E_2 = 0. \end{cases} \tag{2.2}$$

Considering such geometry, we can derive formulas of d'Alembert type for the field functions, which facilitate the estimation of the field functions. Studies of higher dimensional models require different treatments. To

proceed we set

$$M_1 = \begin{pmatrix} 0 & 0 & 0 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}, \quad M_2 = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix}$$

and let $W = (E_1, E_2, B)$ and $b = (\rho, -j_1, -j_2, 0)$. Then, introducing the notations

$$G(f) = (\hat{v}_1, E_1 + \hat{v}_2 B, E_2 - \hat{v}_1 B)$$

and

$$\nabla f = (\partial_x f, \partial_{v_1} f, \partial_{v_2} f),$$

we can write the Vlasov-Maxwell system in a more compact form as

$$\begin{cases} \partial_t f + G(f) \cdot \nabla f = 0, \\ M_1 \partial_t W + M_2 \partial_x W = b. \end{cases} \tag{2.3}$$

3. The Schrödinger equation

The Schrödinger equation is widely used in quantum mechanics and its solution describes molecular, atomic, subatomic as well as macroscopic systems. In its most general form Schrödinger equation can be written as

$$i\hbar \frac{\partial}{\partial t} \Psi(t, x) = H\Psi(t, x),$$

where H is the Hamiltonian operator for the system. For the case of a single particle moving in an electric field the Hamiltonian is given by

$$H = -\frac{\hbar^2}{2\mu} \Delta + V(t, x)$$

and the Schrödinger equation takes the form

$$i\hbar \frac{\partial}{\partial t} \Psi(t, x) = -\frac{\hbar^2}{2\mu} \Delta \Psi(t, x) + V(t, x)\Psi(t, x), \tag{3.1}$$

where μ is the reduced mass of the particle and $V(t, x)$ is its potential energy. In Paper IV we study a nonlinear version of (3.1). Let $f : [0, \infty) \to \mathbb{R}$ and set, for simplicity, all constants to be equal to 1. Then the nonlinear Schrödinger equation (NLS) can be written as

$$u_t = i\Delta u + if(|u|^2)u + g(t, x). \tag{3.2}$$

For a real λ and with $f(x) = \lambda x$ the above equation is known as the cubic Schrödinger equation, with eigenvalue λ.

Some analytical results on the NLS equation are discussed in a work by Strauss [**35**]. The nonlinearity considered in [**35**] is of the form $f(x) = \lambda x^{(p-1)/2}$ for $p > 1$. Under specific conditions on λ and p there exists a unique global solution to the NLS equation. Results, regarding the blowing up of solution, for the NLS equation can be found in [**17**]. As a classical approach to the continuous problem Brezis and Gallouet [**9**] consider the existence of a solution to the cubic Schrödinger equation in a bounded domain. The nonlinear Schrödinger equation has also been studied in higher dimensions; see, e.g., Tsutsumi and Hayashi [**36**].

In numerical studies of the NLS equation a complex-valued version of Brouwer fixed point theorem is used for proving existence of the discretized form of the equation, while Newton's method has been widely used to handle the nonlinear term numerically. In this setting Akrivis [**1**] propose a finite difference scheme combined with Newton's method for approximation of the cubic Schrödinger equation in one dimension. A modified Newton method is introduced by Akrivis et al. in [**2**]. The main idea of this modified method is to avoid updating the matrix of the linear system at each time step. An alternative approach can be found in [**39**], where a linear two-step finite element method is introduced and the nonlinear term is approximated by the disretized solution in previous time step.

4. The streamline diffusion method

Both Vlasov and Maxwell equations are hyperbolic type PDEs and assuming that the exact solution is in the Sobolev space H^{k+1}, the standard Galerkin method for hyperbolic partial differential equations is suboptimal in the sense that they have a convergence rate of order $\mathcal{O}(h^k)$, where h is the mesh size. On the other hand, with the same regularity assumption (H^{k+1}) the optimal convergence rate for the elliptic and parabolic problems is of order $\mathcal{O}(h^{k+1})$. The streamline diffusion (SD) scheme, in comparison to the classical finite element method for hyperbolic type equations, is more stable and has an improved convergence rate of order $\mathcal{O}(h^{k+1/2})$. The SD method is designed based on a Petrov-Galerkin scheme where an extra convection term in the streamline direction has been added to the test functions. This corresponds to the addition of artificial diffusion to the original equation, which enhance the stability and improves the convergence behavior of the standard Galerkin method

by a factor of order $\mathcal{O}(h^{1/2})$. This is the main reason for studying the SD method for the Vlasov-Maxwell system.

The streamline diffusion method for hyperbolic partial differential equations has been suggested by Hughes and Brooks in [**25**]. For a mathematical analysis of the SD method; see, e.g., [**4**] for Vlasov-Poisson system and Johnson and Saranen [**27**] for Euler and Navier-Stokes equations. To justify for the SD method let us look at an example in one dimensional case and illustrate the method. Suppose we want to approximate the solution of the Dirichlet boundary value problem

$$\begin{cases} u_t + u_x = f(t,x) & \text{for } 0 < x < 1 \text{ and } t > 0, \\ u(t,0) = u(t,1) = 0 & \text{for } t > 0, \\ u(0,x) = g(x) & \text{for } 0 < x < 1. \end{cases}$$

Using the classical finite element method we want to find an approximation U^h in some finite element space V^h, such that the following discrete version of the variational formulation holds true

$$(U_t^h + U_x^h, v) = (f, v) \quad \forall v \in V^h.$$

For the streamline diffusion method we will instead search for $U^h \in V^h$, such that

$$\Big(U_t^h + U_x^h, v + \delta(v_t + v_x)\Big) = \Big(f, v + \delta(v_t + v_x)\Big) \quad \forall v \in V^h.$$

Here δ is a multiple of h (or more general a multiple of h^α for some suitable $\alpha > 0$). In the SD scheme, which is used in this thesis, we will also allow jump discontinuties at grid points in the time direction: Let $0 = t_0 < t_1 < \ldots < t_{M-1} < t_M = T$ be a partition of $[0,T]$ into the subintervals $I_m = (t_m, t_{m+1}]$, $m = 0, \ldots, M-1$. Then, the SD scheme for the example above can be formulated as: find $U^h \in V^h$, such that for $m = 0, 1, \ldots, M-1$,

$$\begin{aligned} &\Big(U_t^h + U_x^h, v + \delta(v_t + v_x)\Big)_{I_m \times \Omega} + \int_\Omega U_+^h(t_m, x)\, v_+(t_m, x)\, dx \\ &\quad = \Big(f, v + \delta(v_t + v_x)\Big)_{I_m \times \Omega} + \int_\Omega U_-^h(t_m, x)\, v_+(t_m, x)\, dx \quad \forall v \in V^h, \end{aligned}$$

where $v_\pm(t,x) = \lim_{s \to 0^\pm} v(t+s, x)$. Roughly speaking, the two extra integrals measure how large the jumps are.

With $v = U^h$ the extra δ term can be interpreted (after partial integration) as $-\delta(U_{tt}^h + U_{xx}^h)$ corresponding to an extra diffusion of order δ

in the streamline direction which is not present in the equation. Hence the name of the method.

5. The *hp*-version

For the classical finite element method (FEM) the accuracy is usually controlled by altering the mesh size h, while the polynomial degree p of the elements are kept fixed, which leads to the so-called h-version of the FEM. Alternatively, one could keep the mesh size fixed and instead increase the spectral order p. Then, this is called the p-version. Combining these two versions leads to the hp-version. The main idea is to split the physical domain of the equation according to the degree of regularity of the exact solution, data and geometry of the domain. Then, e.g., where the exact solution is smooth one uses higher polynomial order and a coarse mesh, while near singularities a more refined mesh is applied together with a low polynomial order. In that sense the hp method may be interpreted as an automatic adaptivity procedure.

An early study of the hp-version can be found in [**7**], where the authors analyze the hp-method for an elliptic equation. In [**23**] and [**24**] Houston et al. combine the hp-version with a streamline diffusion method for hyperbolic equations. A similar study can be found in [**6**] for the Vlasov-Poisson-Fokker-Planck system.

6. Nitsche's method

In [**32**] Nitsche proposed a stabilized penalty method for an elliptic equation. This method has thereafter been widely studied by several authors for different equations. Arnold [**3**] extended Nitsche's method to the heat equation. In a series of articles [**10**], [**11**] and [**34**] the authors study Nitsche's method for different equations. In [**10**] and [**11**] the Nitsche method is applied to a domain decomposition problem for Poisson and Stokes equations, respectively. Sticko and Kreiss [**34**] extended this method to the wave equation.

To illustrate Nitsche's method we look at a simple example as follows. Suppose we are interested in approximating the solution of the Poisson equation

$$\begin{cases} -\Delta u(x) = f(x) & \text{in } \Omega, \\ u(x) = \alpha & \text{on } \partial\Omega, \end{cases} \tag{6.1}$$

for given function f and constant α. Here Ω is a domain in $\mathbb{R}^n$ for $n \geq 2$. For a suitable finite dimensional space V^h the classical finite element method can be formulated as: find $U^h \in V^h$ with $U^h|_{\partial\Omega} = \alpha$, such that

$$(\nabla U^h, \nabla v) = (f, v) \quad \forall v \in V_0^h,$$

where $V_0^h = \{v \in V^h; v = 0 \text{ on } \partial\Omega\}$. If we remove the restriction of U^h being equal to α on the boundary and instead enforce the boundary condition weakly (equivalently assume also $v \in V^h$), then we have the following formulation: find $U^h \in V^h$, such that

$$(\nabla U^h, \nabla v) - \int_{\partial\Omega} \frac{\partial U^h}{\partial n} v \, ds - \int_{\partial\Omega} (U^h - \alpha) \frac{\partial v}{\partial n} \, ds = (f, v) \quad \forall v \in V^h.$$

This, however, is not satisfactory enough, since the corresponding bilinear form is not gauranteed to be coercive. To compensate for the loss of coercivity a particular additional boundary term is added. This yields the Nitsche scheme, which can be formulated as follows: find $U^h \in V^h$, such that

$$\begin{aligned}(\nabla U^h, \nabla v) - \int_{\partial\Omega} \frac{\partial U^h}{\partial n} v \, ds - \int_{\partial\Omega} (U^h - \alpha) \frac{\partial v}{\partial n} \, ds \\ + \frac{\gamma}{h} \int_{\partial\Omega} (U^h - \alpha) v \, ds = (f, v) \quad \forall v \in V^h.\end{aligned} \tag{6.2}$$

Here h is the mesh size and γ is a penalty parameter to be chosen large enough for the bilinear form to be coercive. A simple calculation shows that the exact solution of (6.1) satisfies (6.2) meaning that Nitsche's method is consistent. Since the boundary condition is enforced through the forth term of (6.2) the third term could be removed, but then the bilinear form will not be symmetric.

7. Summary of papers I-III

Papers I and II are devoted to the analysis of the streamline diffusion scheme for the one and one-half dimensional relativistic Vlasov-Maxwell system in the *a priori* and *a posteriori* settings, respectively. Since the system is both nonlinear and coupled, we perform an iteration procedure to approximate the solution of the Vlasov-Maxwell system (2.1). Starting with an initial guess $f^{h,0}$ we compute the fields $E^{h,1}$ and $B^{h,1}$ and insert them into the Vlasov equation to get the numerical approximation $f^{h,1}$ at the next step. This will then be inserted in the Maxwell equations to

get the corresponding fields $E^{h,2}$ and $B^{h,2}$ and so on. The iteration step i yields a Vlasov equation for $f^{h,i}$ with the fields $E^{h,i}$ and $B^{h,i}$.

In Paper I we focus on the a priori error analysis of the scheme described above, while Paper II is devoted to the a posteriori error analysis. In Paper I we also consider a SD-based discontinuous Galerkin scheme, where in addition to grid points in time, jumps are also allowed across inter-element boundaries in phase-space variables. The a priori error estimates are derived in two different triple norms (energy norms), one for the Maxwell part and another for the Vlasov part. Paper III generalize the SD scheme introduced in Paper I to a hp-scheme for the three dimensional VM system. This paper contains also the study of a Nitsche type scheme for the Maxwell equations. As in Paper I, the focus is on a priori error estimates. A concise numerical implementation is justifying the results.

Before stating our results from the three VM papers, we need to introduce some notations. Let $\Omega_x \subset \mathbb{R}$ and $\Omega_v \subset \mathbb{R}^2$ denote the space and velocity domains, respectively. Now we will introduce a finite element structure on $\Omega_x \times \Omega_v$. Let $T_h^x = \{\tau_x\}$ and $T_h^v = \{\tau_v\}$ be finite elements subdivisions of Ω_x with elements τ_x and Ω_v with elements τ_v, respectively. Then $T_h = T_h^x \times T_h^v = \{\tau_x \times \tau_v\} = \{\tau\}$ is a subdivision of $\Omega_x \times \Omega_v$. Let $0 = t_0 < t_1 < \ldots < t_{M-1} < t_M = T$ be a partition of $[0,T]$ into sub-intervals $I_m = (t_{m-1}, t_m]$, $m = 1, 2, \ldots, M$. Further let $\mathcal{C}_h$ be the corresponding subdivision of $Q_T = [0,T] \times \Omega_x \times \Omega_v$ into elements $K = I_m \times \tau$, with $h = \operatorname{diam} K$ as the mesh parameter. We also introduce $\tilde{\mathcal{C}}_h$ as the finite element subdivision of $\tilde{Q}_T = [0,T] \times \Omega_x$, which is considered for the Maxwell part solely. Before we define our finite dimensional spaces we need to introduce some function spaces, viz

$$\mathcal{H}_0 = \prod_{m=1}^{M} H_0^1(I_m \times \Omega_x \times \Omega_v) \quad \text{and} \quad \tilde{\mathcal{H}}_0 = \prod_{m=1}^{M} H_0^1(I_m \times \Omega_x),$$

where

$$H_0^1(I_m \times \Omega) = \{w \in H^1; w = 0 \text{ on } \partial\Omega\}$$

with $\Omega = \Omega_x \times \Omega_v$ for VM or $\Omega = \Omega_x$ for the Maxwell part. For $k = 0, 1, 2, \ldots$, we define the finite element spaces

$$V_h = \{w \in \mathcal{H}_0; w|_K \in P_k(I_m) \times P_k(\tau_x) \times P_k(\tau_v), \forall K = I_m \times \tau \in \mathcal{C}_h\}$$

and

$$\tilde{V}_h = \{g \in [\tilde{\mathcal{H}}_0]^3; g_i|_{\tilde{K}} \in P_k(I_m) \times P_k(\tau_x), \forall \tilde{K} = I_m \times \tau_x \in \tilde{\mathcal{C}}_h,\ i = 1, 2, 3\},$$

where $P_k(\cdot)$ is the set of polynomial of degree at most k on the given set. We shall also use some notation as

$$(f,g)_m = (f,g)_{S_m} = \int_{S_m} f(t,x,v)g(t,x,v)\,dt\,dx\,dv, \qquad \|g\|_m = (g,g)_m^{1/2}$$

and

$$\langle f,g\rangle_m = (f(t_m,\ldots),g(t_m,\ldots))_\Omega, \qquad |g|_m = \langle g,g\rangle_m^{1/2},$$

where $S_m = I_m \times \Omega$, is the slab at m-th time interval I_m, $m = 1,2,\ldots,M$. Here, the domain Ω equals Ω_x for the Maxwell equations, while $\Omega = \Omega_x \times \Omega_v$ for the Vlasov equation.

To proceed we define $f^{h,i}$, $b^{h,i}$ and $W^{h,i}$ as the approximation on the ith step for f, b and W, respectively. We are now ready to formulate the SD scheme for (2.2). This is stated separately for the Maxwell equations and the Vlasov equation. The streamline diffusion method on the ith step for the Maxwell part can now be formulated as follows: Find $W^{h,i} \in \tilde{V}_h$ such that for $m = 1,2,\ldots,M$,

$$\begin{aligned}(7.1)\quad &(M_1 W_t^{h,i} + M_2 W_x^{h,i}, \hat{g} + \delta(M_1 g_t + M_2 g_x))_m + \langle W_+^{h,i}, g_+\rangle_{m-1} \\ &\quad = (b^{h,i-1}, \hat{g} + \delta(M_1 g_t + M_2 g_x))_m + \langle W_-^{h,i}, g_+\rangle_{m-1}, \quad \forall\, g \in \tilde{V}_h,\end{aligned}$$

where $\hat{g} = (g_1, g_1, g_2, g_3)^T$. The corresponding streamline diffusion method on the ith step for the Vlasov equation can be formulated as follows: Find $f^{h,i} \in V_h$ such that for $m = 1,2,\ldots,M$,

$$\begin{aligned}(7.2)\quad &(f_t^{h,i} + G(f^{h,i-1})\cdot\nabla f^{h,i}, g + \delta(g_t + G(f^{h,i-1})\cdot\nabla g))_m \\ &\qquad + \langle f_+^{h,i}, g_+\rangle_{m-1} = \langle f_-^{h,i}, g_+\rangle_{m-1}, \quad \forall g \in V_h,\end{aligned}$$

where

$$G(f^{h,i-1}) = (\hat{v}_1, E_1^{h,i} + \hat{v}_2 B^{h,i}, E_2^{h,i} - \hat{v}_1 B^{h,i}).$$

As stated above the a priori error estimates will be derived in two different triple norms. These triple norms are defined by

$$|||g|||_{\mathrm{M}}^2 = \frac{1}{2}\left(|g_+|_0^2 + |g_-|_M^2 + \sum_{m=1}^{M-1}|[g]|_m^2 + 2\delta\sum_{m=1}^{M}\|M_1 g_t + M_2 g_x\|_m^2\right)$$

for the Maxwell equations and

$$|||g|||_{\mathrm{V}}^2 = \frac{1}{2}\left(|g_+|_0^2 + |g_-|_M^2 + \sum_{m=1}^{M-1}|[g]|_m^2 + 2\delta\sum_{m=1}^{M}\|g_t + G(f^{h,i-1})\cdot\nabla g\|_m^2\right)$$

for the Vlasov part.

The main result of Paper I is the following convergence theorem for the density function f.

THEOREM 7.1. *Let $f^{h,i}$ be the solution to (7.2) and assume that the exact solution f for (2.3) is in the Sobolev space H^{k+1} and that some additional conditions holds (see Paper I for details), then there exists a constant C such that*

$$|||f - f^{h,i}|||_{\mathrm{V}} \leq Ch^{k+\frac{1}{2}} + Ch^{i-\frac{1}{2}}.$$

We have a corresponding error estimate for the solution $W^{h,i}$ of (7.1) and the exact solution W for (2.3). If we assume that W is in the Sobolev space H^{k+1}, then there exists a constant C such that

$$|||W - W^{h,i}|||_{\mathrm{M}} \leq Ch^{k+\frac{1}{2}} + Ch^{i-1}.$$

In the a posteriori setting in Paper II, the errors are measured in terms of the residuals. Due to the nature of the approximating scheme, two kinds of residuals will appear in our estimates. The first kind measures how well the approximate solution satisfies the equation. This is done replacing the exact solution by a symbol for an approximate solution into the equation and taking the difference between the right and the left hand side. The second kind of residual measures how large the jumps of approximated solution is. Consequently, the residuals for the Maxwell equations are defined as

$$\tilde{R}_1^i = b^{h,i-1} - M_1 W_t^{h,i} - M_2 W_x^{h,i}$$

and

$$\tilde{R}_2^i|_{S_m} = \Big(W_+^{h,i}(t_m, x) - W_-^{h,i}(t_m, x)\Big)/h.$$

The corresponding residuals for the Vlasov equation are

$$R_1^i = f_t^{h,i} + G(f^{h,i-1}) \cdot \nabla f^{h,i}$$

and

$$R_2^i|_{S_m} = \big(f_+^{h,i}(t_m, x, v) - f_-^{h,i}(t_m, x, v)\big)/h,$$

where the residuals with index 2 are considered to be constant in time on each slab.

Let us decompose the error into two parts

$$W - W^{h,i} = \underbrace{W - W^i}_{\text{analytical iteration error}} + \underbrace{W^i - W^{h,i}}_{\text{numerical error}} = \tilde{\mathcal{E}}^i + \tilde{e}^i,$$

where W^i is the exact solution to the approximated Maxwell equations at the ith iteration step:

$$M_1 W_t^i + M_2 W_x^i = b^{h,i-1}.$$

We do a similar decomposition of the error

$$f - f^{h,i} = \underbrace{f - f^i}_{\text{analytical iteration error}} + \underbrace{f^i - f^{h,i}}_{\text{numerical error}} = \mathcal{E}^i + e^i,$$

where f^i is the exact solution of the approximated Vlasov equation at the ith iteration step:

$$f_t^i + G(f^{h,i-1}) \cdot \nabla f^i = 0.$$

We will now assume that W^i and f^i converge to the analytical solutions W and f, respectively. For a sufficiently large i, $\tilde{e}^i$ and e^i are the dominating part of the errors $W - W^{h,i}$ and $f - f^{h,i}$. Therefore, for large enough i, we have that

$$\|\tilde{\mathcal{E}}^i\|_{H^{-1}(\tilde{Q}_T)} << \|\tilde{e}^i\|_{H^{-1}(\tilde{Q}_T)}.$$

As the main results in Paper II we have the following a posteriori error estimates in $L_\infty(H^{-1})$-norms

THEOREM 7.2. *There exists a constant C such that*

$$\|W(T,\cdot) - W^{h,i}(T,\cdot)\|_{H^{-1}(\Omega_x)} \leq C\Big(\|h\tilde{R}_1^i\|_{L_2(\tilde{Q}_T)} + \|h\tilde{R}_2^i\|_{L_2(\tilde{Q}_T)}\Big).$$

and

THEOREM 7.3. *There exists a constant C such that*

$$\|f(T,\cdot) - f^{h,i}(T,\cdot)\|_{H^{-1}(\Omega)} \leq C\Big(\|hR_1^i\|_{L_2(Q_T)}(2 + \|G(f^{h,i-1})\|_{L_\infty(Q_T)}) + \|hR_2^i\|_{L_2(Q_T)}\Big).$$

For the Maxwell part we also have the following a posteriori estimate in the $H^{-1}(H^{-1})$-norm

THEOREM 7.4. *There exists a constant C such that*

$$\|W - W^{h,i}\|_{H^{-1}(\tilde{Q}_T)} \leq C\Big(\|h\tilde{R}_1^i\|_{L_2(\tilde{Q}_T)} + \|h\tilde{R}_2^i\|_{L_2(\tilde{Q}_T)}\Big).$$

In Paper III the SD scheme, introduced in Paper I, is generalized to a *hp*-scheme for the three dimensional VM system. The a priori analysis of the *hp*-scheme in Paper III follows the analysis in Paper I, with some modification because of the use of a *hp* method. An important ingredient is the interpolation error estimates in Theorem 3.2 of [**6**]. The following

bounds hold for the interpolation error $\eta = g - \tilde{g}$ of a function $g \in H^{k+1}([0,T] \times \Omega)$ (where Ω stands for Ω_x in the Maxwell's equations and $\Omega_x \times \Omega_v$ for the Vlasov case) and its gradient:

$$\|\eta\|^2 \leq C \sum_K \left(\frac{h_K}{2}\right)^{2s_K+2} \Phi_1(p_K, s_K) \|g\|^2_{s_K+1,K},$$

$$\|\mathcal{D}\eta\|^2 \leq C \sum_K \left(\frac{h_K}{2}\right)^{2s_K} \Phi_2(p_K, s_K) \|g\|^2_{s_K+1,K},$$

where the sums are taken over all space-time elements of the triangulation of the domain, $[0,T] \times \Omega$, $0 \leq s_K \leq \min(p_K, k)$, with p_K being the spectral order of the element K. The norm $\|\cdot\|_{s,K}$ is the standard norm on $H^s(K)$. Closed form formulas for Φ_1 and Φ_2 are given in Theorem 3.2 of [**6**]. A less involved formula for Φ_1 can be found in [**24**]. In the error estimates we use $\Phi_{\mathrm{M}} = \max(\Phi_1, \Phi_2)$ with dimension $\mathcal{N} = \dim \Omega_x + 1$ and $\Phi_{\mathrm{V}} = \max(\Phi_1, \Phi_2)$ with dimension $\mathcal{N} = \dim \Omega_x + \dim \Omega_v + 1$. The main theorems of the *hp*-scheme for the Vlasov-Maxwell system consist of the error estimates

$$\begin{aligned} |||W - W^{h,i}|||^2_{\mathrm{M}} \leq & C \sum_{K \in \tilde{\mathcal{C}}_h} h_K^{2s_K+1} p_K^{-1} \Phi_{\mathrm{M}}(p_K, s_K) \|W\|^2_{s_K+1,K} \\ & + C \|f - f^{h,i-1}\|^2_{Q_T}, \end{aligned}$$

for the fields and

$$\begin{aligned} |||f - f^{h,i}|||^2_{\mathrm{V}} \leq C \Bigg(& \sum_{K \in \tilde{\mathcal{C}}_h} h_K^{2s_K+1} p_K^{-1} \Phi_{\mathrm{M}}(p_K, s_K) \|W\|^2_{s_K+1,K} \\ & + \sum_{K \in \mathcal{C}_h} h_K^{2s_K+1} p_K^{-1} \Phi_{\mathrm{V}}(p_K, s_K) \|f\|^2_{s_K+1,K} \Bigg) \\ & + Cph \|f - f^{h,i-1}\|^2_{Q_T} \end{aligned}$$

for the density function. Note that the triple norms appearing here are not exactly the same as in Paper I. The explicit expressions of the triple norms and the relevant conditions for the above error estimates can be found in Paper III. The results are justified through numerical simulations in a lower dimensional case.

In Paper III a Nitsche scheme is studied for the Maxwell equations as well. This can be viewed as an alternative method to the SD scheme for

the Maxwell part. By assuming extra regularity and differentiating the Maxwell's equations

$$E_t - \nabla_x \times B = -j,$$
$$B_t + \nabla_x \times E = 0,$$

with respect to t, we obtain second order differential equations

$$E_{tt} + \nabla_x \times (\nabla_x \times E) = -j_t,$$
$$B_{tt} + \nabla_x \times (\nabla_x \times B) = \nabla_x \times j.$$

Since the equations for E and B are similar, it suffices to analyze the equation for one of them: E. To this end we define the bilinear form

$$a(E,g) := \int_{\Omega_x} \nabla_x \times E \cdot \nabla_x \times g \, dx - \int_{\Gamma_x} \nabla_x \times E \cdot (g \times n) \, ds$$
$$- \int_{\Gamma_x} (E \times n) \cdot \nabla_x \times g \, ds + \frac{\gamma}{h} \int_{\Gamma_x} E \cdot g \, ds$$

and the finite element space

$$V_h^x = \{g \in H(\mathrm{curl}, \Omega_x) : g|_{\tau_x} \in P_1(\tau_x), \, \forall \tau_x \in T_h^x\}.$$

The semi-discrete problem can now be formulated as: for each fixed t, find $E^h(t,\cdot) \in V_h^x$, such that

$$(E_{tt}^h, g)_{\Omega_x} + a(E^h, g) = -(j_t, g)_{\Omega_x} \quad \forall g \in V_h^x.$$

The continuity and coercivity for the bilinear form $a(\cdot,\cdot)$ hold in the mesh dependent norm

$$|||g|||_h^2 := \|\nabla_x \times g\|_{\Omega_x}^2 + \|h^{-1/2} g\|_{\Gamma_x}^2 + \|h^{1/2} \nabla_x \times g\|_{\Gamma_x}^2.$$

However, the coercivity is only valid when the penalty parameter γ is chosen large enough (see Paper III for more details on how to choose γ). An important feature in the analysis of the Nitsche method is the use of the projection $Q_h : H(\mathrm{curl}, \Omega_x) \to V_h^x$ defined by

$$a(Q_h u, v) = a(u, v) \quad \forall v \in V_h^x,$$

with the following error estimates

LEMMA 7.5. *There exists a constant C, such that*

$$\|u - Q_h u\|_{\Omega_x} + h|||u - Q_h u|||_h \leq C h^2 \|u\|_{H^2(\Omega_x)}.$$

To formulate the fully discrete problem we define the notations

$$\bar{\partial}_t^2 u^m = \frac{u^m - 2u^{m-1} + u^{m-2}}{k^2},$$
$$\hat{u}^m = \frac{u^m + 2u^{m-1} + u^{m-2}}{4},$$

where $u^m = u(t_m)$. Then, the fully discrete problem for the Maxwell part, reads as follows: for $m = 2, 3, \ldots, M$, find $\mathcal{E}^m$ such that

$$(\bar{\partial}_t^2 \mathcal{E}^m, \chi) + a(\hat{\mathcal{E}}^m, \chi) = -(j_t^{m-1}, \chi) \quad \forall \chi \in V_h^x.$$

Here $\mathcal{E}^0 = Q_h E^0$ and $\mathcal{E}^1$ is assumed to be chosen as an approximation of E^1 satisfying $\|E^1 - \mathcal{E}^1\|_{\Omega_x} \leq C(h^2 + k^2)$. Then we have the following error estimate for the fully discrete problem:

$$\|E(t_m) - \mathcal{E}^m\|_{\Omega_x} \leq C(h + k).$$

8. Summary of paper IV

Paper IV is devoted to a two level time stepping scheme for the nonlinear Schrödinger equation (3.2). To formulate the discretization procedure, we introduce a not necessarily uniform partition $\{t_n\}_{n=0}^N$ of the time interval $[0, T]$, i.e. $t_n < t_{n+1}$ for $n = 0, 1, \ldots, N-1$, $t_0 = 0$ and $t_N = T$. Then, we set $k_n := t_n - t_{n-1}$ for $n = 0, 1, \ldots, N$. For a suitable finite element space S_h, the following two time-step numerical scheme is analyzed in Paper IV:

Step 1. Set

$$U_h^0 = u_{0,h},$$

where $u_{0,h} = R_h u_0$ is the Ritz projection of the initial data.

Step 2. For $n = 1, 2, \ldots, N$, first find $U_h^{n-\frac{1}{2}} \in S_h$ such that

$$\left(\frac{U_h^{n-\frac{1}{2}} - U_h^{n-1}}{k_n/2}, \chi \right) = i \left(\frac{\nabla U_h^{n-\frac{1}{2}} + \nabla U_h^{n-1}}{2}, \nabla \chi \right) + i \left(f(|U_n^{n-1}|^2) \frac{U_h^{n-\frac{1}{2}} + U_h^{n-1}}{2}, \chi \right) + (g(t_{n-1}, \cdot), \chi),$$

for all $\chi \in S_h$ and then find $U_h^n \in S_h$ such that

$$\left(\frac{U_h^n - U_h^{n-1}}{k_n}, \chi\right) = i\left(\frac{\nabla U_h^n + \nabla U_h^{n-1}}{2}, \nabla\chi\right) + i\left(f(|U_h^{n-\frac{1}{2}}|^2)\frac{U_h^n + U_h^{n-1}}{2}, \chi\right) + \left(g(t_{n-\frac{1}{2}}, \cdot), \chi\right),$$

for all $\chi \in S_h$.

An important step in the analysis of the above scheme is the consistency analysis. For $n = 1, 2, \ldots, N$, we define $r^{n-\frac{1}{2}}$ and r^n by

$$\frac{u^{n-\frac{1}{2}} - u^{n-1}}{k_n/2} = i\Delta\Big(\frac{u^{n-\frac{1}{2}} + u^{n-1}}{2}\Big) + if(|u^{n-1}|^2)\frac{u^{n-\frac{1}{2}} + u^{n-1}}{2} + g(t_{n-1}, \cdot) + r^{n-\frac{1}{2}}$$

and

$$\frac{u^n - u^{n-1}}{k_n} = i\Delta\Big(\frac{u^n + u^{n-1}}{2}\Big) + if(|u^{n-\frac{1}{2}}|^2)\frac{u^n + u^{n-1}}{2} + g(t_{n-\frac{1}{2}}, \cdot) + r^n,$$

respectively, where $u^n = u(t_n, \cdot)$ for $n = 0, 1, \ldots, N$. The terms $r^{n-\frac{1}{2}}$ and r^n measure how consistent our time discretization is with the original equation. We have the following estimates for $r^{n-\frac{1}{2}}$ and r^n:

PROPOSITION 8.1. *Assume that there is a constant C_1 such that*

$$\max\Big(\|\partial_t u\|_\infty,\, \|\partial_t^2 u\|_\infty,\, \|\Delta\partial_t u\|_\infty\Big) < C_1, \tag{8.1}$$

then

$$\|r^{n-\frac{1}{2}}\| \le Ck_n.$$

Further if, in addition to (8.1)*, we have that there is a constant C_2 such that*

$$\max\Big(\|\partial_t^3 u\|_\infty,\ \|\Delta\partial_t^2 u\|_\infty\Big) < C_2, \tag{8.2}$$

then

$$\|r^n\| \le Ck_n^2.$$

The main result of Paper IV is given by the following theorem.

THEOREM 8.2. *Let $e^n := U_h^n - u^n$ be the error at the time level $t = t_n$. Assume that u satisfies the conditions* (8.1) *and* (8.2)*. Then there is a constant C such that*

$$\|e^n\| \leq C(k^2 + h^r),$$

and

$$\|\nabla e^n\| \leq C(k + h^{r-1}),$$

with $k := \max_{1 \leq n \leq N} k_n$.

One would expect an order of $\mathcal{O}(k^2 + h^{r-1})$ for the error estimate of the gradient, since we only have spatial derivatives (cf. [**39**]).

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