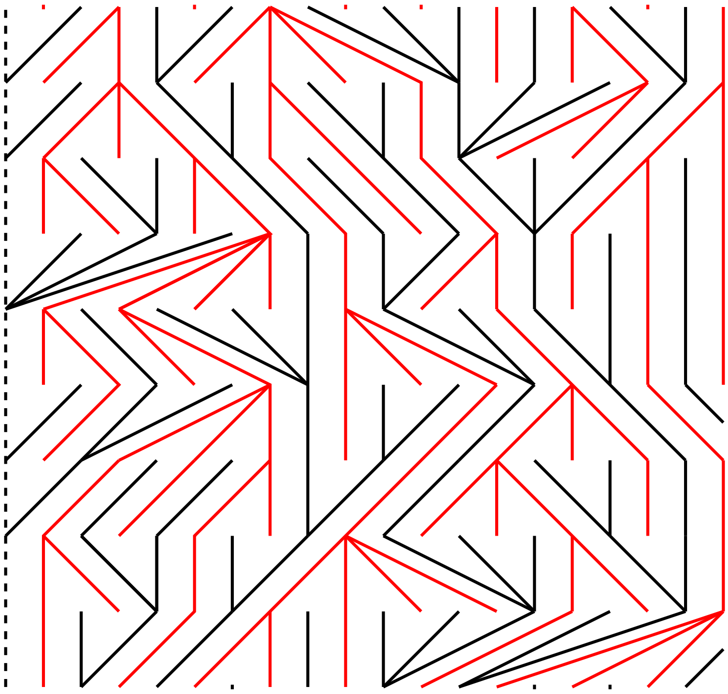
Beyond Galton-Watson Processes:

Forests, Duals, and Ranks

Master's Thesis in Engineering Mathematics and Computational Science



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CHALMERS UNIVERSITY OF TECHNOLOGY
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Cover: A realisation of a random forest and its dual.

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Abstract

A random forest is a random graph (V, E) with a set of vertices $V = \mathbb{N}_0^2$ and a set of edges $E = \{e_v, v \in V\}$ satisfying the following property: if $v = (x, t+1)$, then $e_v = (v, v')$, where $v' = (x', t)$ and $x' = \varphi_t(x)$ is an increasing stochastic process in x. For a given forest, there is a unique way to draw a dual forest. These forests can be used as a graphical representation of discrete time reproduction processes forward and backward in time. They also serve to introduce a new concept, ranked Galton-Watson processes, where individual reproduction depends on the position in the population. A main result is that the dual process to a Galton-Watson process in varying environments with immigration is a Galton-Watson process in varying environments if and only if the reproduction and immigration laws of the first process are linear fractional.

Keywords: Random reproduction processes, Galton-Watson processes, ranked reproduction models, dual Galton-Watson processes, Markov chains, immigration

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Contents

List of Figures **xi**

Notation **xiii**

1 Introduction **1**

2 The Galton-Watson Process **3**
2.1 Definition . . . 3
2.2 Generating functions . . . 4
2.3 Basic properties . . . 5
2.4 Extensions . . . 5

3 Random Forests **7**
3.1 Duality for Markov chains . . . 7
3.2 The primary forest . . . 8
3.3 The dual forest . . . 9
3.4 Random forests of reproduction trees . . . 12

4 Random Forests and the Galton-Watson Process **15**
4.1 The primary process . . . 15
4.2 The dual process . . . 16

5 The Ranked Reproduction Model **19**
5.1 Definition . . . 19
5.2 Expectation of a ranked Galton-Watson process . . . 20

6 Concluding Remarks **23**

Bibliography **25**

List of Figures

2.1 A realisation of a Galton-Watson process. 4

3.1 A realisation of an element in Ψ_0, or simply a function in Φ_0. 8
3.2 A realisation of a random forest, for $0 \leq t \leq 9$ and $0 \leq x \leq 9.5$. . . . 9
3.3 The generalized inverse of the realisation in Figure 3.1 10
3.4 A realisation of the dual forest corresponding to the primary forest in figure 3.2, for $0 \leq t \leq 9$ and $0 \leq x \leq 9$. 11
3.5 A realisation of a random forest in black and its dual forest in red, for $0 \leq t \leq 9$ and $0 \leq x \leq 9.5$. 12

4.1 Two realisations of primary forests and their duals. The t-axis is vertical and the x-axis horizontal. In the right panel $\varphi_t(1) \geq 1$, which must hold if $X_{t,0} \equiv 0$ as in Corollary 1. 16

5.1 A realisation of a random forest in black and its dual forest in red, for $0 \leq t \leq 9$ and $0 \leq x \leq 9.5$. Ranks are highlighted in red and black respectively. 20

Notation

For future reference, we give a list of notation which does not contain all information.

f^t	The function f iterated t times.
$\mathbb{N}_0$	The set of non-negative integers.
Φ	Set of increasing functions $\phi : \mathbb{N}_0 \to \mathbb{N}_0$.
Φ_0	Set of increasing functions $\phi : \mathbb{N}_0 \to \mathbb{N}_0$ such that $\phi(0) = 0$
φ_t	Random process taking values in Φ_0.
$\hat{\varphi}_t$	Generalized inverse of φ_t, taking values in Φ.
$(Z_t^{(x)}, t)$	Dual lineage.
$Z_t^{(x)} = \hat{\varphi}_{t-1}(Z_{t-1}^{(x)})$	Primary reproduction model.
$(\hat{Z}_t^{(y,b)}, t)$	Primary lineage.
$\hat{Z}_t^{(y,b)} = \varphi_t(Z_{t+1}^{(y,b)})$	Dual reproduction model.
$X_{t,i}$	Offspring variable.

1

Introduction

A branching process is a stochastic process in discrete or continuous time. It starts with N individuals and as time elapses each individual may have children or perish independently of each other, according to some probability law. In this thesis we present graphically what we call random forests[1]. These can be used as a graphical representation of discrete time reproduction processes, which are the ones considered here.

In Chapter 2 we present the well known Galton-Watson process, which is a pillar for this work. Main results can be found in Chapters 3, 4, and 5. In Chapter 3 we build up what we call primary and dual forests wich can be interpreted as forests of reproduction trees forward and backward in time. The primary and dual reproduction models are introduced.

Chapter 4 states conditions for the primary forest to have a Galton-Watson structure or in other words, when the primary reproduction model is a Galton-Watson process. We also investigate when the dual process has a Galton-Watson structure if the primary has it. In Chapter 5 we generalize the Galton-Watson processes to what we call ranked Galton-Watson processes. These differ from the classical case in that individuals take more or less favoured positions in the population, with regard to reproduction. We give an upper estimate of the expectation of such processes and some examples. Concluding remarks form a final chapter.

[1]Should not be mistaken for the machine learning concept.

2
The Galton-Watson Process

The simplest discrete time branching processes are the Galton-Watson processes. In these individuals live exactly one time unit. This chapter presents the necessities for coming chapters. There is very much written about Galton-Watson processes. If any derivations or further explanations are desirable see for example [2], [4], or [6].

2.1 Definition

Definition 1 *Let* $((X_{t,i})_{t=0}^{\infty})_{i=1}^{\infty}$ *be a double sequence of iid non-negative integer valued random variables. Define recursively*

$$Z_0 = z_0 \in \mathbb{N}_0 = \{0, 1, 2, ...\}, \quad Z_{t+1} = \sum_{i=1}^{Z_t} X_{t,i}.$$

Then $(Z_t)_{t \geq 0}$ *is called a Galton-Watson process, or a GW process for short. If the distribution of* $X_{t,i}$ *is allowed to depend on* t*, then* $(Z_t)_{t \geq 0}$ *is a Galton-Watson process in a varying environment.*

It is convenient to introduce a random variable $X \stackrel{d}{=} X_{t,i}$ in the time homogeneous setting and $X_t \stackrel{d}{=} X_{t,i}$ otherwise. A GW process is often presented graphically as in Figure 2.1.

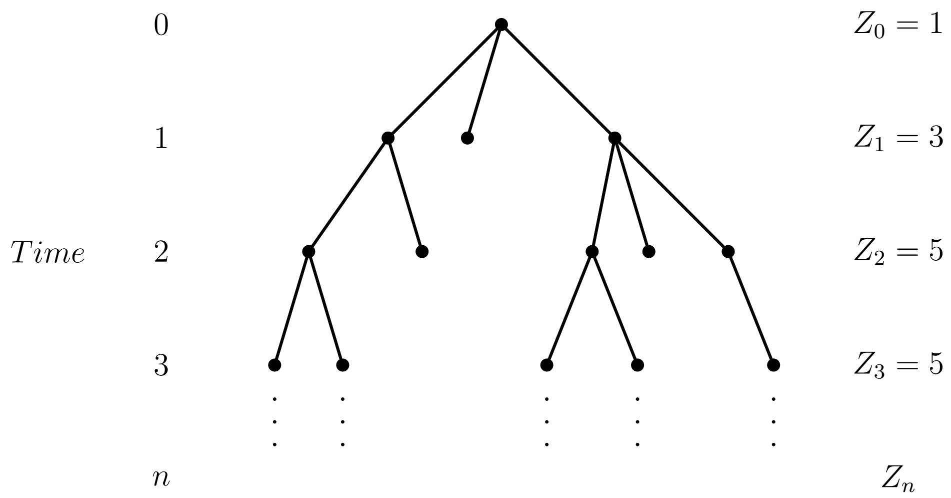

Figure 2.1: A realisation of a Galton-Watson process.

2.2 Generating functions

A fundamental tool in the study of positive integer valued stochastic variables, X, is the probability generating function,

$$f(s) = E[s^X] = \sum_{i=0}^{\infty} s^i P(X = i).$$

It has the following important properties

$$\frac{f^{(k)}(0)}{k!} = P(X = k), \quad f(1) = 1 \quad \text{and } f^{(k)}(1) = E[X(X-1)\cdots(X-k+1)],$$

where $f^{(k)}$ is the k:th derivative. Further, the generating function of a sum of independent random variables is equal to the product of the generating functions of the random variables in question.

If $z_0 = 1$ it holds for a GW process in varying environments that

$$E[s^{Z_t}] = f_1 \circ f_2 \cdots \circ f_t(s), \tag{2.1}$$

where $f_j(s) = E[s^{X_j}]$. In the time homogeneous case equation (2.1) simplifies to

$$E[s^{Z_t}] = \underbrace{f \circ f \cdots \circ f}_{t}(s) = f^t(s). \tag{2.2}$$

Thus, f^t denotes iterations rather than powers. Iterations like those in equation (2.1) and (2.2) are often suitable for computers. There is one well known non trivial case when equation (2.2) is computable analytically. This is when the reproduction law, the distibution of the offspring numbers, is linear fractional[1], which means that

$$P(X = 0) = p_0,\ P(X = k) = (1 - p_0)(1 - p)p^{k-1},\ k = 1, 2, ...,\ p \in (0, 1).$$

[1]Another name is geometric distribution modified at zero.

Generally, the equation $f(s) = s$ has at most two roots, q and 1 (provided $f'(0) < 1$). If $f'(1) = m < 1$, then $q > 1$, if $m = 1$, then $q = 1$ and if $m > 1$, then $q < 1$. In the case of a linear fractional reproduction law the right hand side in (2.2) becomes

$$f^n(s) = 1 - m^n \frac{1-q}{m^n - q} + \frac{m^n (\frac{1-q}{m^n - q})^2 s}{1 - (\frac{m^n - 1}{m^n - q}) s}, \quad m \neq 1,$$
$$f^n(s) = \frac{np - (np + p - 1)s}{1 - p + np - nps}, \quad m = 1.$$

2.3 Basic properties

The GW process is a Markov chain. Writing $E[X] = m$ and $V[X] = \sigma^2$ in the time homogeneous case we have that

$$E[Z_t] = z_0 m^t$$

and

$$V[Z_t] = \begin{cases} \frac{\sigma^2 m^{t-1(m^t - 1)}}{m-1} z_0 & \text{if } m \neq 1 \\ t\sigma^2 z_0 & \text{if } m = 1. \end{cases}$$

This process is called subcritical if $m < 1$, critical if $m = 1$ and supercritical if $m > 1$.

The probability of extinction is the smallest non-negative root Q of the equation $s = f(s)$. In the subcritical and critical cases $Q = 1$, which means that such processes always go extinct. A supercritical process does not necessarily die out.

In varying environment, $E[X_{t,i}] = m_t$ and

$$E[Z_t] = \prod_{j=0}^{t-1} m_j. \tag{2.3}$$

We refrain from a variance formula in the time inhomogeneous case.

2.4 Extensions

A natural extension is to allow immigration into the population. A GW process with immigration, a GWI process, is defined in the same manner as the GW process but with an independent non-negative integer valued immigration component,

$$Z_0 = I_0, \quad Z_{t+1} = \sum_{i=1}^{Z_t} X_{t,i} + I_t.$$

For convenience we write

$$Z_0 = X_{0,0}, \quad Z_{t+1} = \sum_{i=0}^{Z_t} X_{t,i}, \tag{2.4}$$

and interprete $X_{t,0}$ as the number of immigrants.

We can also allow migration, i.e. immigration and emigration. There is not much written about this in textbooks but some in research articles. [9] defines it in the following way

$$Z_0 = 0, \quad Z_{t+1} = \begin{cases} X_{t,1} + \cdots + X_{t,Z_t+\zeta_t} & \text{if } Z_t + \zeta_t > 0 \\ 0 & \text{if } Z_t + \zeta_t \leq 0, \end{cases}$$

where ζ_t is integer valued and independent from $X_{t,i}$ for all $t, i \geq 0$. We will call a GW process with migration a GWM process.

3

Random Forests

Inspired by how [3] interpreted the time-reversed GW process as a random reallocation of balls in boxes together with the stochastic recursion presentend in [1], we construct a random forest on a graph. In a similar manner we build up what we call a dual forest. These forests can be used as a graphical representation of discrete time reproduction processes. We begin with the duality concept.

3.1 Duality for Markov chains

The concept of duality originates in ideas of mirroring a given process, in order better to understand it or to get a new interesting process. It is used in different areas such as interacting particle systems and queueing theory. It is also common to use duality in the study of random systems forward and backward in time. This is what concerns us in this paper. More about the concept of duality can be found in [8] or [5].

In the following definition lower case x and y denote the initial states of the processes X and Y, and T some set of indices.

Definition 2 *Let $X = (X_t^x)_{t\in T}$ and $Y = (Y_t^y)_{t\in T}$ be Markov procesess with state spaces E_1 and E_2 respectively and let $H : E_1 \times E_2 \to \mathbb{R}$ be a bounded function. The process X is H-dual to Y if*

$$E[H(X_t^x, y)] = E[H(x, Y_t^y)] \tag{3.1}$$

for all $x \in E_1$, $y \in E_2$ and $t \in T$.

Clearly, if X is H-dual to Y then Y is dual to X with respect to $F(y, x) := H(x, y)$. If we use $I_{\{x\le y\}}$ (or $I_{\{x\ge y\}}$), the indicator, as the duality function then (3.1) results in

$$P(X_t^x \le y) = P(x \le Y_t^y). \tag{3.2}$$

This is known as Siegmund duality, and closely related to the duality we establish in Section 3.4.

3.2 The primary forest

Let Φ_0 be the set of all functions $\phi : \mathbb{N}_0 \to \mathbb{N}_0$ such that

$$\phi(x+1) \geq \phi(x) \text{ for all } x \in \mathbb{N}_0, \;\; \phi(0) = 0, \;\; \lim_{x\to\infty} \phi(x) = \infty,$$

and let Ψ_0 be the set of all stochastic processes taking values in Φ_0, i.e. a realisation of an element in Ψ_0 is a function in Φ_0. A typical realisation of such a process is presented in Figure 3.1.

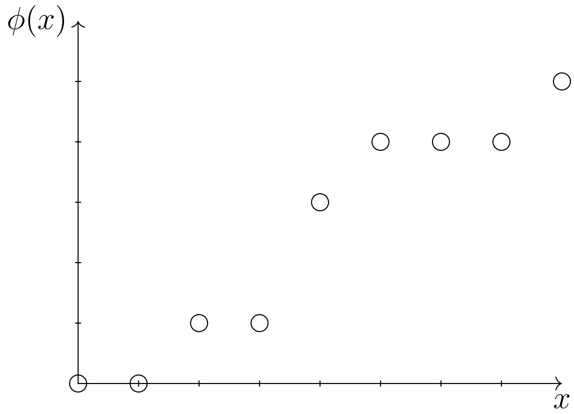

Figure 3.1: A realisation of an element in Ψ_0, or simply a function in Φ_0.

Definition 3 *Let $(\varphi_i)_{i=0}^{\infty}$ be a sequence of elements in Ψ_0. For every $v = (x, t+1) \in \mathbb{N}_0 \times \mathbb{N}$, denote $v_\varphi = (\varphi_t(x), t)$, and define a set of edges in $\mathbb{N}_0^2$ by*

$$E_\varphi = \{(v, v_\varphi), v \in \mathbb{N}_0 \times \mathbb{N}\}.$$

The resulting random graph $(\mathbb{N}_0^2, E_\varphi)$ will be called a primary forest.

A (part of a) realisation of a primary forest is presented in Figure 3.2.

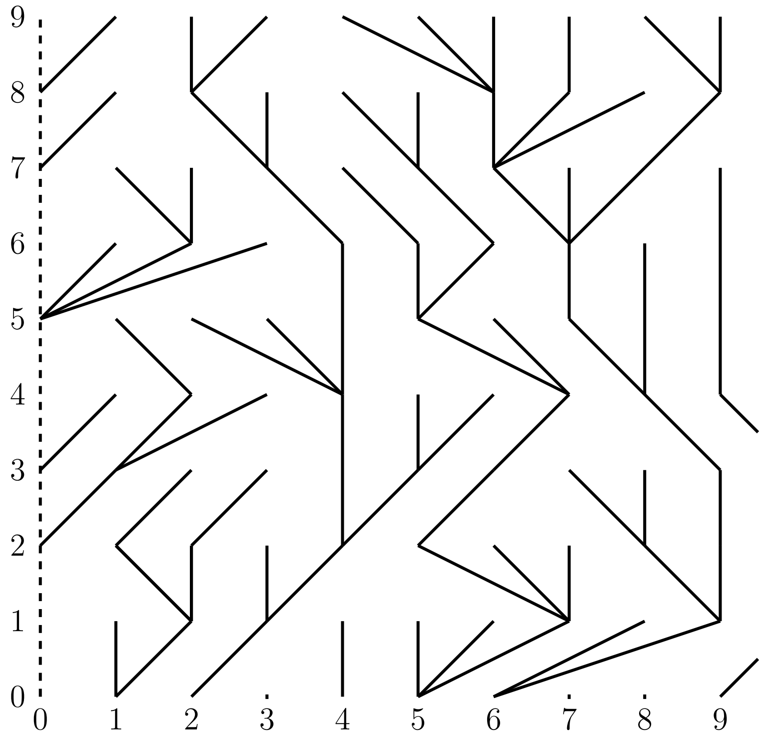

Figure 3.2: A realisation of a random forest, for $0 \leq t \leq 9$ and $0 \leq x \leq 9.5$.

For each $(y, b) \in \mathbb{N}_0 \times \mathbb{N}$, the set of vertices $\{(\hat{Z}_t^{(y,b)}, t), 0 \leq t \leq b\}$, where

$$\hat{Z}_t^{(y,b)} = \varphi_t \circ \varphi_{t+1} \circ \ldots \circ \varphi_{b-1}(y), \quad 0 \leq t \leq b-1, \quad \hat{Z}_b^{(y,b)} = y, \tag{3.3}$$

is connected by the edges E_φ forming a random primary lineage of the corresponding forest. Due to the definition of Φ_0 these lineages branch upwards and merge downwards, and every random forest has a straight vertical lineage going through the vertices $\{(0, t), t \in \mathbb{N}_0\}$. This lineage is dashed in Figure 3.2 and will be called the stem lineage. Following the random forest upwards we see that lineages either split or vanish.

3.3 The dual forest

According to the definition of Φ_0 in the beginning of section 3.2, every function $\phi \in \Phi_0$ has a unique generalized inverse, cf. Lemma 2.1 in [1],

$$\hat{\phi}(y) = \max\{x : \phi(x) \leq y\}. \tag{3.4}$$

Such an inverse is not nessesarily an element of Φ_0 since $\hat{\phi}(0)$ can take any values in $\mathbb{N}_0$. It belongs to the set of functions $\phi : \mathbb{N}_0 \to \mathbb{N}_0$ such that

$$\phi(x+1) \geq \phi(x) \text{ for all } x \in \mathbb{N}_0, \quad \lim_{x \to \infty} \phi(x) = \infty,$$

which we will denote Φ. The inverse of the realisation in Figure 3.1 is presented below.

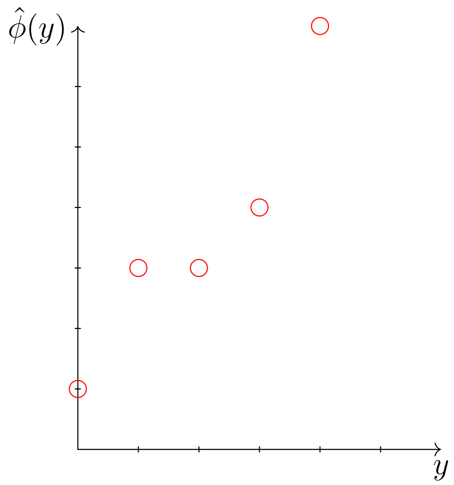

Figure 3.3: The generalized inverse of the realisation in Figure 3.1

Now let Ψ be the set of all stochastic processes taking values in Φ.

Definition 4 *Let $(\varphi_i)_{i=0}^{\infty}$ be a sequence in Ψ_0 and $(\hat{\varphi}_i)_{i=0}^{\infty}$ be a sequence in Ψ such that Equation (3.3) is satisfied for all realisations, and each element in the sequences.*

For every $v = (y,t) \in \mathbb{N}_0^2$, denote $\hat{v}_\varphi = (\hat{\varphi}_t(y), t+1)$, and define a set of edges by

$$\hat{E}_\varphi = \{(v, \hat{v}_\varphi), v \in \mathbb{N}_0^2\}$$

The random graph $(\mathbb{N}_0^2, \hat{E}_\varphi)$ will be called a dual forest.

Note that the dual forest is uniqely determined by the primary forest. In Figure 3.3 below a realisation of the dual forest corresponding to the primary in Figure 3.2 is presented.

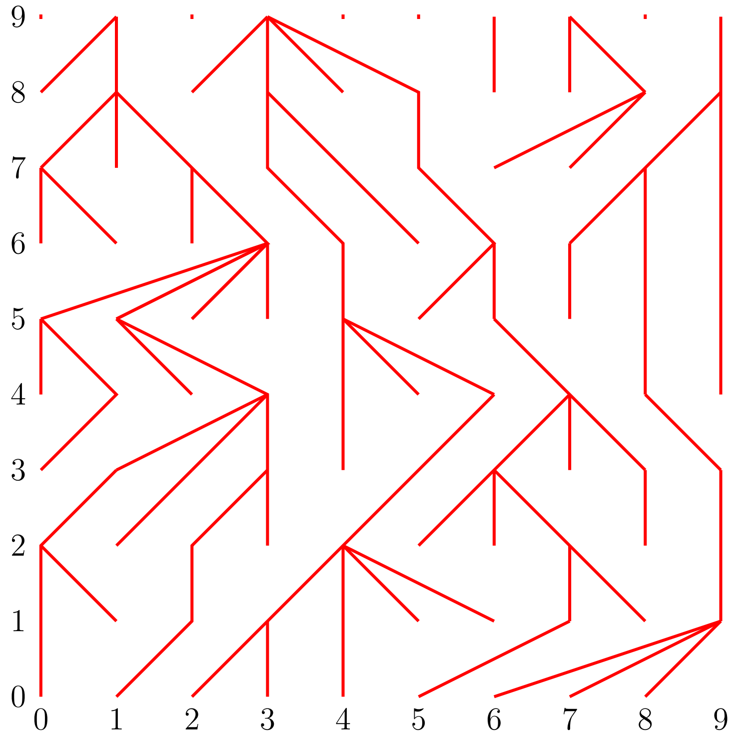

Figure 3.4: A realisation of the dual forest corresponding to the primary forest in figure 3.2, for $0 \leq t \leq 9$ and $0 \leq x \leq 9$.

In the same manner as the primary forest, the dual forest is built by the dual lineages $\{(Z_t^{(x)}, t), t \geq 0\}$, where

$$Z_t^{(x)} = \hat{\varphi}_{t-1} \circ \hat{\varphi}_{t-2} \circ \ldots \circ \hat{\varphi}_0(x), \quad t \geq 1, \quad Z_0^{(x)} = x, \quad x \geq 0. \tag{3.5}$$

Comparing Figure 3.2 and 3.3 we see that the dual forest has a similar structure downwards as the primary has upwards. To clarify the relationship between the primary and dual forest we will depict the dual forest on

$$\hat{\mathbb{N}}_0 \times \mathbb{N}_0, \quad \hat{\mathbb{N}}_0 = \{x + \frac{1}{2}, x \in \mathbb{N}_0\}. \tag{3.6}$$

With (3.6) in our mind we put the two forests together and present them in Figure 3.5.

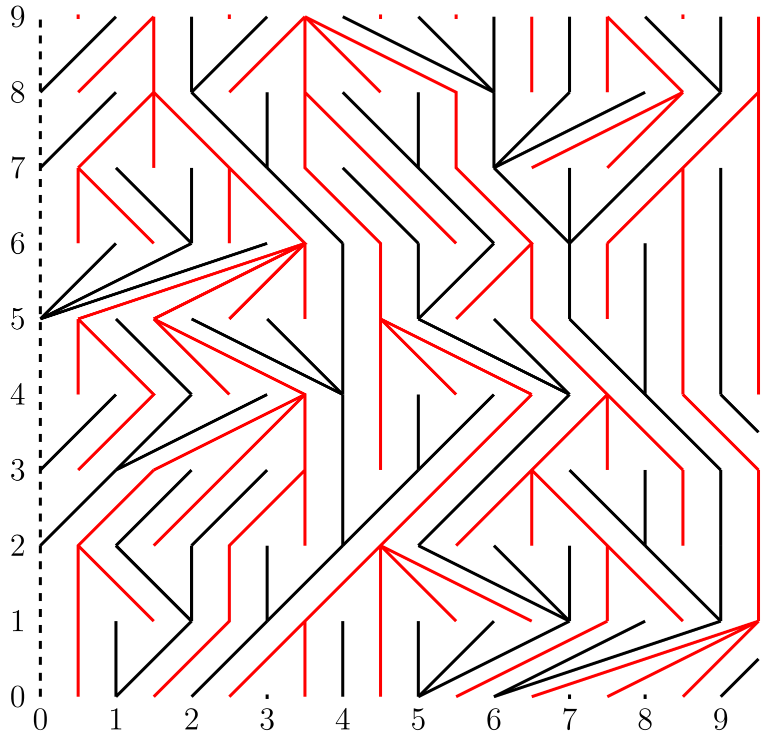

Figure 3.5: A realisation of a random forest in black and its dual forest in red, for $0 \leq t \leq 9$ and $0 \leq x \leq 9.5$.

The structure is clear: black (primary) lineages branch upwards, while red (dual) lineages branch downwards. The black and red lineages do not cross.

3.4 Random forests of reproduction trees

If we look at Figure 3.5 we see that black trees have their roots at each positive integer 1,2,...,9, and that each tree can be interpreted as a reproduction tree starting with one individual. Of course we can consider x black trees at time $t = 0$ and interpret those as one reproduction tree starting with x individuals. The stem lineage at $x = 0$ is an imigration source. In a similar manner, but backward in time, we can interpret the red trees. By convention there is a difference between the red and black forest: the black forest has an immigration source at $x = 0$ (which of course can be identically 0) while the red forest has no immigration. As compensation each individual at $x = 0$ is included in the dual (red) forest.

The key idea when turning to reproduction models is that a dual (red) lineage starting at $(0, x)$ determines the size of the black tree starting at $t = 0$ with x individuals at any time $t \geq 0$. Similarly a primary (black) lineage starting at (b, x) determines the size of the red tree, which branches backward in time, starting at time b with x individuals at any time $t \in \{b, b-1, ..., 0\}$. As example, follow the dual (red) lineage in Figure 3.5 from $(0, 2)$ to $(9, 3)$, the corresponding black population tree starts with 2 individuals and consisting, due to immigration at time $t = 8$, of 3 individuals at time $t = 9$. Let us define the primary and dual reproduction model.

Definition 5 *The primary reproduction model, or for short the primary process, is defined as*

$$Z_t^{(x)} = \hat{\varphi}_{t-1}(Z_{t-1}^{(x)}), \quad t \geq 1, \quad Z_0^{(x)} = x, \quad x \geq 0. \tag{3.7}$$

Definition 6 *The dual reproduction model, or for short the dual process, is defined as*

$$\hat{Z}_t^{(y,b)} = \varphi_t(Z_{t+1}^{(y,b)}), \quad 0 \leq t \leq b-1, \quad \hat{Z}_b^{(y,b)} = y, \tag{3.8}$$

That is, the primary reproduction model (3.7) is defined through dual lineages. These where presented on the line preceding Formula (3.5). The dual reprodcution model (3.8) in its turn was defined through primary lineages, cf. the line before Formula (3.3).

Since primary and dual lineages do not cross it is clear that

$$\hat{Z}_b^{(y,b)} \leq Z_b^{(x)} \iff \hat{Z}_t^{(y,b)} \leq Z_t^{(x)} \text{ for all } t \in \mathbb{N}_0.$$

Hence, trivially

$$P(\hat{Z}_t^{(y,b)} \leq Z_0^{(x)}) = P(\hat{Z}_b^{(y,b)} \leq Z_t^{(x)}),$$

which is a time reversed Siegmund duality, see Equation (3.2).

4

Random Forests and the Galton-Watson Process

In the first section of this chapter we show that the primary reproduction model (3.7) is a GWI or GW process under some assumptions. In the second section we investigate when the dual process (3.8) has a GW structure if the primary has it.

4.1 The primary process

Proposition 1 *If* $(\varphi_i)_{i=0}^{\infty}$ *is a sequence in* Ψ_0 *defined by*

$$\varphi_t(x) = \min\{n : \sum_{j=0}^{n} X_{t,j} \geq x\},$$

where $X_{t,j}$ *are mutually independent, non-negative, non-null, and identically distributed for fixed* t *and* $j \geq 1$ *with* $X_{t,0}$ *possibly having a different distribution. Then, the primary process,* $Z_t^{(x)}$*, is a GWI process.*

Proof. By equation (3.4) we have that

$$\hat{\varphi}_t(y) = \max\{x : \min\{n : \sum_{j=0}^{n} X_{t,j} \geq x\} \leq y\}$$

It is clear that $\min\{n : \sum_{j=0}^{n} X_{t,j} \geq \sum_{j=0}^{y} X_{t,j}\} \leq y$. On the other hand $\min\{n : \sum_{j=0}^{n} X_{t,j} \geq \sum_{j=0}^{y+1} X_{t,j}\} > y$ if $X_{t,y+1} > 0$. If $X_{t,y+1} = 0$ then $\sum_{j=0}^{y} X_{t,j} = \sum_{j=0}^{y+1} X_{t,j}$.

Hence, $\hat{\varphi}_t(y) = \sum_{j=0}^{y} X_{t,j}$, and

$$Z_{t+1}^{(x)} = \hat{\varphi}_t(Z_t^{(x)}) = \sum_{j=0}^{Z_t^{(x)}} X_{t,j}, \quad Z_0^{(x)} = x. \tag{4.1}$$

This is the recursion for the GWI process (2.4). Due to the assumptions on the $X_{t,j}$ the proof is complete. ■

Corollary 1 *If $X_{t,0} \equiv 0$, then the primary process, $Z_t^{(x)}$, reduces to a GW process*

Proof. It follows from (4.1). ∎

The graphical interpretations follows in Figure 3.4 below, for Proposition 1 to the left and its corollary to the right.

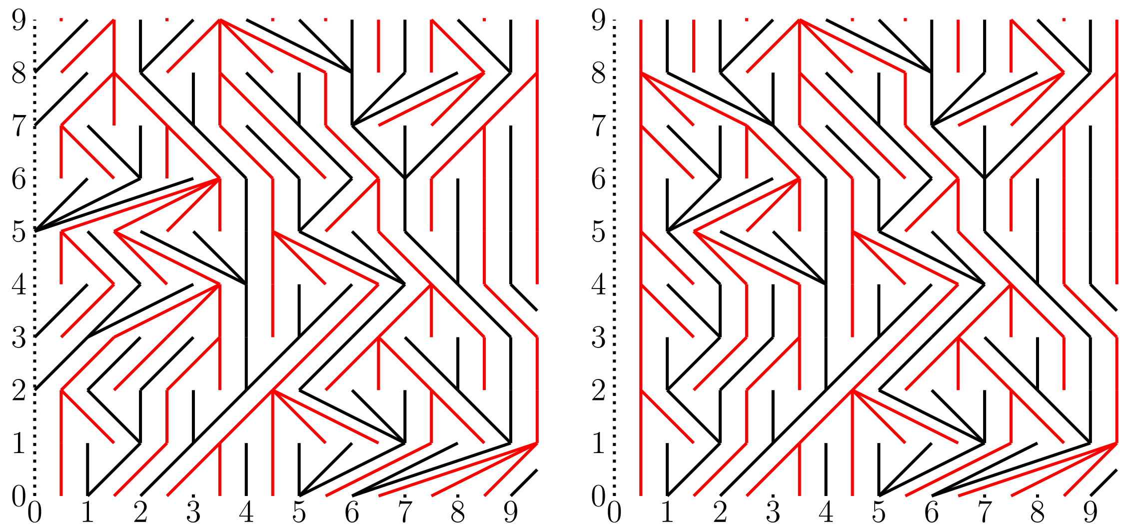

Figure 4.1: Two realisations of primary forests and their duals. The t-axis is vertical and the x-axis horizontal. In the right panel $\varphi_t(1) \geq 1$, which must hold if $X_{t,0} \equiv 0$ as in Corollary 1.

4.2 The dual process

Since the red forest has a similar branching structure downwards as the black forest has upwards, we ask the question: is the dual process a GW process if the primary has a GW structure? The answer is generally no.

Theorem 1 *Let the primary process $(Z_t^{(x)})_{t\geq 0}$ be a GWI process in a varying environment. Its dual $(\hat{Z}_t^{(y,b)})_{t\geq 0}$ is a GW process in a varying environment backward in time if and only if immigration and reproduction laws are linear fractional, i.e.*

$$P(X_{t,j} = 0) = p_{0,t}, \quad P(X_{t,j} = k) = (1 - p_{0,t})p_t(1 - p_t)^{k-1} \quad k \geq 1 \tag{4.2}$$

and

$$P(X_{t,0} = k) = p_t(1 - p_t)^k, \quad p_t \in (0, 1], \quad p_{0,t} \in [0, 1). \tag{4.3}$$

Given (4.2) and (4.3) the dual reproduction laws are also linear fractional, but with parameters $\hat{p}_{0,t} = 1 - p_t$ and $\hat{p}_t = 1 - p_{0,t}$.

Proof. Since $\varphi_t(0) = 0$, we have that

$$\varphi_t(x) = \sum_{j=1}^{x} \varphi_t(j) - \varphi_t(j - 1).$$

Let $Y_{t,j} = \varphi_t(j) - \varphi_t(j-1)$, then

$$\hat{Z}_t^{(y,b)} = \varphi_t(Z_{t+1}^{(y,b)}) = \sum_{j=1}^{Z_{t+1}^{(y,b)}} Y_{t,j}. \tag{4.4}$$

Omitting the index t we note that

$$(X_1, X_2, \ldots)$$

can be uniquely described by

$$(\underbrace{0,\ldots,0}_{\xi_1}, \mu_1, \underbrace{0,\ldots,0}_{\xi_2}, \mu_2, \ldots), \tag{4.5}$$

where $\xi_1 = k$ iff $X_1 = X_2 = ... = X_k = 0$ and $X_{k+1} > 0$. Then the first nonzero component μ_1 has the same distribution as X_{k+1} given that $X_{k+1} > 0$. In the same manner ξ_i, $i = 2, 3, ...$ is the number of zeros between the strictly positive components μ_{i-1} and μ_i.

The dual reproduction variables

$$(Y_1, Y_2, \ldots)$$

can also be uniquely described by zeros and nonzeros. Since they are totally determined by the primary reproduction it must hold that

$$(\underbrace{0,\ldots,0}_{X_0}, \xi_1 + 1, \underbrace{0,\ldots,0}_{\mu_1 - 1}, \xi_2 + 1, \underbrace{0,\ldots,0}_{\mu_2 - 1}) \tag{4.6}$$

describes $(Y_1, Y_2, \ldots)$ uniquely.

If (4.2) and (4.3) hold, then by (4.5) we have that

$$P(\xi_i = k) = P(X > 0)P(X = 0)^k = (1 - p_0)p_0^k$$

and

$$P(\mu_i = k) = P(X = k \mid X > 0) = p(1-p)^{k-1}.$$

That is, for the variables counting zeroes in (4.6) we have,

$$P(\mu_i - 1 = k) = P(X = k \mid X > 0) = p(1-p)^k \text{ and } P(X_0 = k) = p(1-p)^k$$

according to the assumptions. For the non zeros in (4.6) we have that

$$P(\xi_i + 1 = k) = (1 - p_0)p_0^{k-1}.$$

Hence, (4.5) and (4.6) have exactly the same structure, which implies that $Y_1, Y_2, ...$ are iid. Since $P(Y = k \mid Y > 0) = P(\xi_i + 1 = k)$ we have, reinserting t, that

$$P(Y_{t,j} = k) = \begin{cases} 1 - p_t & \text{if } k = 0 \\ p_t(1 - p_{0,t})p_{0,t}^{k-1} & \text{if } k \geq 1. \end{cases}$$

That is, (4.4) is a GW-process in a varying environment with linear fractional reproduction with the parameters $\hat{p}_{0,t} = 1 - p_t$ and $\hat{p}_t = 1 - p_{0,t}$.

If $(\hat{Z}_t^{(y,b)})_{t\geq 0}$ is a GW process in a varying environment, now dropping t again, then Y_1, Y_2,... are iid. Let $p = P(Y > 0)$. We have by (4.6) that $P(X_0 = k) = p(1-p)^k$ that is (4.3) except the index t. On the other hand for $k > 0$, $P(X = k \mid k > 0) = P(\mu_i = k) = P(Y = 0)^{k-1} P(Y > 0) = (1-p)^{k-1} p$. That is (4.2) and (4.3) hold for the primary process. ■

In a similar manner as in Theorem 1 it can be shown that the dual to a GW process with linear fractional reproduction is again a GW process with linear fractional reproduction but with an "eternal" individual. See the right picture in Figure 4.1. These results are related to, but stronger than, the duality relation presented in Proposition 3.6 in [7].

5
The Ranked Reproduction Model

So far we have considered the primary process when the reproduction variables are independent and identically distributed for fixed t. In this chapter the primary reproduction variables may have different distributions even for fixed t. This idea gives a possibility to differentiate the reproductive power of individuals by their ranks, which can be seen as reflecting their position in the population. Due to the rank dependence this process is harder to study. In Section 5.2 we present an upper bound for the expectation of a ranked GW process.

5.1 Definition

Definition 7 *We call the primary process,*

$$Z_{t+1} = \sum_{i=0}^{Z_t} X_{t,i},$$

a ranked GWI (or ranked GW if $X_{t,0} \equiv 0$) if the $X_{t,i}$ are mutually independent and their distributions may depend on i (and may depend on t).

Obviously, for the ranked GW process formulas (2.1) and (2.2) are not valid since the $X_{t,i}$ may have different distributions for fixed t. In Figure 5.1 we present a primary forest with its dual where the ranks are highlighted in black and red.

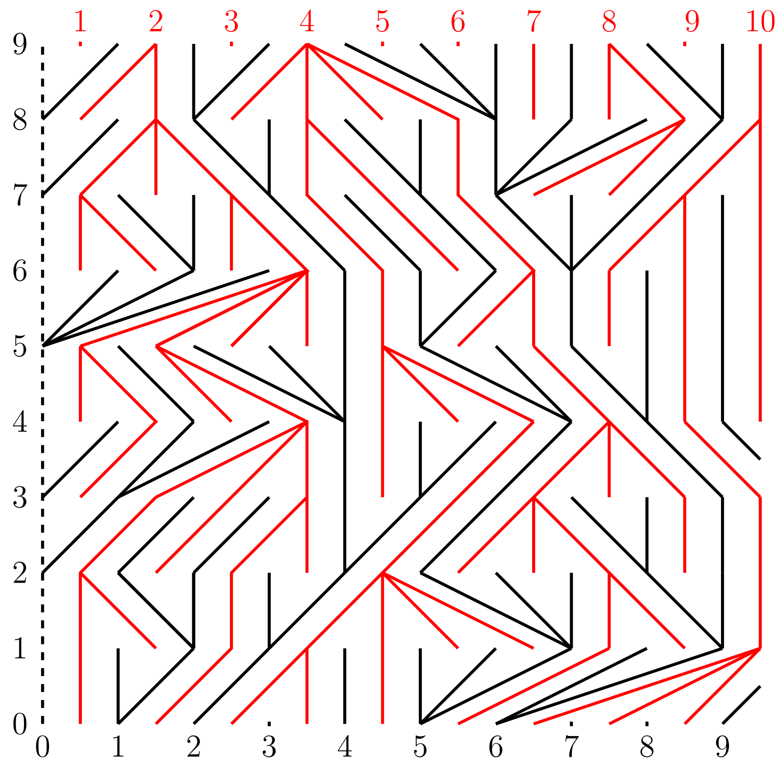

Figure 5.1: A realisation of a random forest in black and its dual forest in red, for $0 \leq t \leq 9$ and $0 \leq x \leq 9.5$. Ranks are highlighted in red and black respectively.

According to the discussion in Section 3.4 primary individuals located at $x = 1$ have rank 1, while the dual individuals located at $x = 0$ have rank 1 and so on.

Example 1 *Let $X_{t,i} \stackrel{d}{=} X_i \sim Poisson(M/i)$ for some $M > 0$ and all $i > 0$, then it is clear that we expect the first ranked individual to have most children. Hence, the individual at rank 1 can be seen as an alpha (fe)male in the population.*

You can think of many kinds of rankings yourself.

5.2 Expectation of a ranked Galton-Watson process

Due to independence between Z_t and $X_{t,i}$, $i = 1, 2, ...,$ the conditional expectation of Z_{t+1} given Z_t is

$$E[Z_{t+1}|Z_t] = E[\sum_{i=1}^{Z_t} X_{t,i}|Z_t] = \sum_{i=1}^{Z_t} m_{t,i}$$

where $E[X_{t,i}] = m_{t,i}$.

For $t \in \mathbb{N}$ define $g_t : \mathbb{N}_0 \to \mathbb{N}_0$ by $g_t(0) = 0$ and

$$g_t(k) = \sum_{i=1}^{k} m_{t,i}.$$

It is clear that

$$E[Z_{t+1}|Z_t] = g_t(Z_t).$$

Now recall Jensen's Inequality for a random variable X and a concave function h

$$E[h(X)] \leq h(E[X]).$$

Proposition 2 *Suppose there exists a concave and increasing function h such that $h(k) \geq g_t(k)$ for all $k \in \mathbb{N}_0$ and all $t \in \mathbb{N}$. Then $E[Z_t] \leq \underbrace{h \circ h \cdots \circ h}_{t}(z_0) = h^t(z_0)$.*

Proof. By the law of total expectation

$$E[Z_t] = E[E[Z_t|Z_{t-1}]] = E[g_{t-1}(Z_{t-1})].$$

Since $h(k) \geq g_t(k)$ for all $k \in \mathbb{N}_0$ and all $t \in \mathbb{N}$

$$E[Z_t] = E[g_{t-1}(Z_{t-1})] \leq E[h(Z_{t-1})].$$

Applying Jensen's Inequality to the concave function h yields

$$E[Z_t] \leq E[h(Z_{t-1})] \leq h(E[Z_{t-1}]).$$

Since h is increasing

$$E[Z_t] \leq E[h(Z_{t-1})] \leq h \circ h(E[Z_{t-2}])$$

and finally

$$E[Z_t] \leq \underbrace{h \circ h \cdots \circ h}_{t}(z_0) = h^t(z_0).$$

∎

Example 2 *If $m_{t,i} \leq 1$ for all t and i let h be the identity function and conclude that $E[Z_t] \leq z_0$.*

This example is obviously trivial since such a process is dominated by a time homogeneous Galton-Watson process with reproduction mean one.

Proposition 3 *For a continuous, increasing and concave function h such that $h(0) \geq 0$ and $h(x^*) = x^*$ for some $x^* \geq 0$ with the property that $h(x)<x$ for all $x > x^*$ we have that $\lim_{t\to\infty} h^t(z_0) \leq x^*$ for any $z_0 \geq 0$.*

Proof. If $z_0 \leq x^*$ it follows that $h(z_0) \leq f(x^*) = x^*$ and $h^t(z_0) \leq x^*$ since h is increasing. Due to the concavity and that $h(0) \geq 0$ we have that $h(z_0) \geq z_0$, and $h^t(z_0) \geq h^{t-1}(z_0)$. Hence, $h^t(z_0)$ is increasing and bounded by x^* which implies its limit exists, and is less than or to equal x^*.

If $z_0 > x^*$ then $h^t(z_0) \geq x^*$ and $h^t(z_0)$ is decreasing. Hence $\lim_{t\to\infty} h^t(z_0) = \bar{x}$, for some $\bar{x} \geq x^*$. Since $h(\bar{x}) = \bar{x}$ it follows by the assumptions that $\bar{x} = x^*$.

∎

Example 3 *Assume that* $m_{t,i} = m_i$ *and there exists a constant* N *such that* $m_i \leq K < 1$ *for all* $i > N$*. Let*

$$h(x) = \begin{cases} x \sum_{i=1}^{N} m_i & \text{if } x \leq N \\ N \sum_{i=1}^{N} m_i + K(x - N) & \text{if } x \geq N. \end{cases}$$

It is easy to see that this function satisfies the assumptions in Proposition 2 and 3. Hence, even though N *is arbitrarily large and* $(m_i)_{i=1}^{N}$ *grows arbitrarily fast,* $\limsup_{n\to\infty} E[Z_n]$ *is finite.*

There is a theorem, see p.110 in [4], which implies that a Markov chain for which zero is the only absorbing state, and the probability to hit zero from each state is strictly positive, then either the process absorbs at zero or goes to infinity. If the ranked GW process has a bounded expectation it will not go to infinity. That is, Propositions 2 and 3 can be used to determine if a ranked GW process goes extinct.

6
Concluding Remarks

The next step in this work should be to reverse time in the dual process, $\hat{Z}_t^{(y,b)}$, in order to fullfill Definition 2. In the ranked setting it would be interesting to see if there are non trivial cases where calculations can be made. At the time of writing, we investigate if the dual to a ranked GWI process can be a GW process. The ranked GW process can be generalized further if we allow dependence between individuals. Here too one can study the dual characteristics. Another interesting question is whether there are cases where the dual process is easier to study and therefore can be used to draw conclusions about the primary process.

Bibliography

[1] Asmussen S and Sigman K, “Monotone Stochastic Recursions and Their Duals” (1996) 10 Probability in the Engineering and Informational Sciences 1

[2] Athreya, Krishna B., Peter E. Ney, SpringerLink (Online service), et al. , ’Branching Processes’, Anonymous Translator, Volume 196(1st edn, Berlin, Heidelberg, Springer Berlin Heidelberg, 1972).

[3] Grosjean, Nicolas, and Thierry Huillet. ’On a Coalescence Process and its Branching Genealogy’, Journal of Applied Probability, vol. 53/no. 4, (2016), pp. 1156-1165.

[4] Haccou, Patsy, Peter Jagers, Vladimir A. Vatutin, et al. , ’Branching Processes: Variation, Growth, and Extinction of Populations’, Anonymous Translator, Volume no. 5.;no. 5;(, Cambridge, Cambridge University Press, 2005).

[5] Jansen, Sabine, and Noemi Kurt. ’On the Notion(s) of Duality for Markov Processes’, Probability Surveys, vol. 11/(2014), pp. 59-120.

[6] Kimmel, Marek, David E. Axelrod, and SpringerLink (e-book collection). , ’Branching Processes in Biology’, Anonymous Translator, Volume 19;19.;(Second;2nd 2015; edn, New York, Springer, 2015).

[7] Klebaner, F. C., U. Rösler, and S. Sagitov. ’Transformations of Galton-Watson Processes and Linear Fractional Reproduction’, Advances in Applied Probability, vol. 39/no. 4, (2007), pp. 1036-1053.

[8] Möhle, Martin. ’The Concept of Duality and Applications to Markov Processes Arising in Neutral Population Genetics Models’, Bernoulli, vol. 5/no. 5, (1999), pp. 761-777.

[9] Nagaev, S. V., and L. V. Khan. ’Limit Theorems for a Critical Galton-Watson Branching Process with Migration’, Theory of Probability and its Applications, vol. 25/no. 3, (1981), pp. 514-525.