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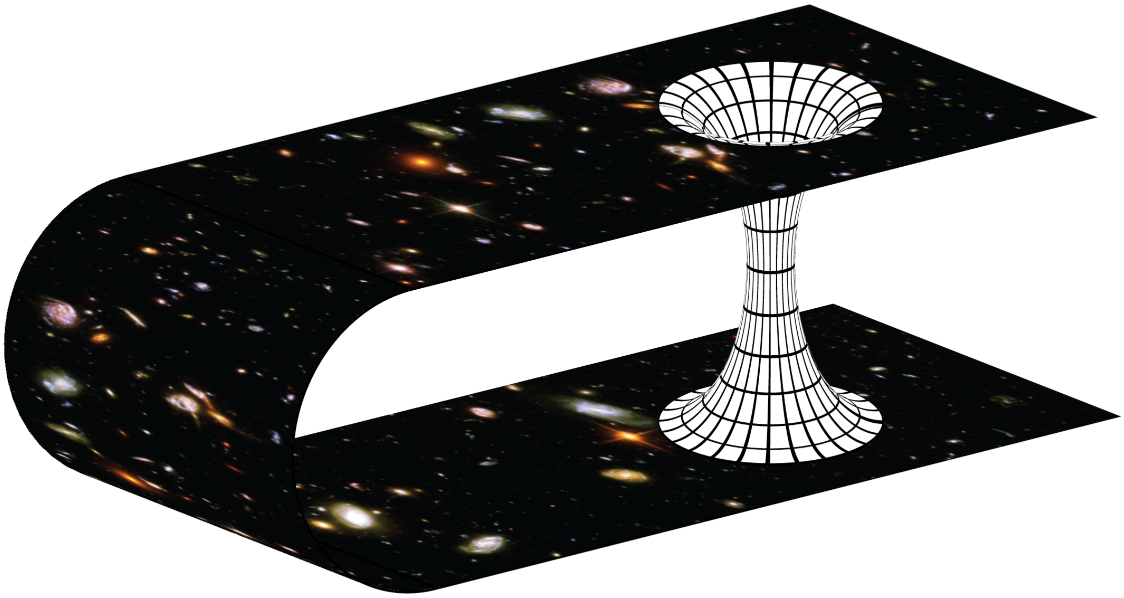

From Black Holes to Wormholes in Higher Spin Gravity

2+1-dimensional gravity in a Chern-Simons formulation

Bachelor's thesis for Engineering Physics

Simon Ekhammar, Daniel Erkensten, Marcus Lassila, Torbjörn Nilsson

Department of Physics
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Cover: Embedding diagram of wormhole connecting two distant regions of space. The wormhole provides a shorter path between two points than the route going through the regular spacetime, here represented as a folded sheet. The image mapped on the sheet is the Hubble Ultra Deep Field, courtesy of NASA. Image created by Torbjörn Nilsson.

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Abstract

The recent ER=EPR conjecture as well as advances in string theory have spurred the interest in wormholes and their relation to black holes. The complexity of these phenomena motivates the restriction to a tensionless limit of string theory, which is believed to be described by a higher spin theory. This thesis investigates the specific case of three-dimensional black holes and wormholes in Einstein- and higher spin gravity. We first study black holes and wormholes in general relativity, and it is found that to have traversable wormholes we must introduce exotic matter, i.e. matter with negative energy density. It is then shown that the Einstein-Hilbert action in 2+1 dimensions can be expressed as a Chern-Simons action with the gauge group $SL(2) \times SL(2)$. By extending the gauge group to $SL(3) \times SL(3)$ a generalization of Einstein gravity is obtained, yielding a higher spin gravity theory. In the extended theory the metric, which was previously fundamental, has become gauge dependent. Because of this we must instead classify solutions through the holonomy. We show that in this theory we can resolve a conical singularity without changing its holonomy. What is more interesting, is that we show that this precise transformation transforms both the cone and the black hole to a traversable wormhole. Of course, this possibility brings into question the interpretation of matter, and more fundamentally, the stress-energy tensor in a higher spin gravity theory. Even without fully resolving these issues the theory can be useful, as it provides a possible way of simplifying calculations by changing the geometry of a problem without changing its solution. As this text is geared towards undergraduates in physics, it begins with an introductory chapter on Maxwell's theory of electromagnetism as it is an example of what might possibly be the simplest gauge theory. Generalizing this theory to non-abelian groups leads to Yang-Mills and Chern-Simons theories. An introduction to general relativity with a special emphasis on the Cartan formulation, is provided as well. In addition to this there are multiple appendices detailing the necessary mathematical background needed to understand the later parts of the work.

Keywords: Gauge Theory, 2+1-Dimensional Gravity, Higher Spin Gravity, Wormholes, Black Holes, Chern-Simons Theories

Sammandrag

På senare tid har intresset för maskhål och deras relation till svarta hål ökat, framförallt på grund av ER=EPR-hypotesen och framsteg inom strängteori. Komplexiteten hos dessa fenomen motiverar restriktionen till den spänningslösa gränsen av strängteori, vilken tros beskrivas av en högre spinn-teori. I detta arbete undersöks tredimensionella svarta hål och maskhål i Einsteinsk gravitation och i en högre spinn-gravitationsteori. Vi studerar först svarta hål och maskhål i allmän relativitetsteori, den konventionella gravitationsteorin, och vi finner att vi behöver introducera exotisk materia för att åstadkomma maskhål som tillåter genomfärd. Det visas sedan att Einstein-Hilbert-verkan i 2+1 dimensioner kan uttryckas som en Chern-Simons-gaugeteori med gaugegruppen $SL(2) \times SL(2)$. Genom att utöka gaugegruppen till $SL(3) \times SL(3)$ generaliserar vi Einsteinsk gravitation och erhåller en högre spinn-gravitationsteori. I denna teori är metriken, vilken tidigare givit oss den fundamentala geometriska tolkningen av lösningar, gaugeberoende. På grund av detta klassificerar vi istället våra lösningar genom holonomier. Explicit demonstreras hur vi i denna teori kan upplösa singulariteten hos en kon utan att ändra dess holonomi. Ännu mer intressant är att denna transformation visar sig transformera både konen samt ett svart hål till ett maskhål. Detta leder oss till att ifrågasätta huruvida en väldefinierad stress-energitensor kan existera i en sådan teori. Trots dessa svårigheter kan teorin emellertid visa sig vara användbar, eftersom den kan nyttjas för att förenkla beräkningar genom att förändra geometrin utan att deformera lösningen. Då målgruppen till denna text är studenter med motsvarande tre års kandidatstudier i fysik inleds arbetet med ett grundläggande kapitel om Maxwells elektromagnetism vilket kanske är det enklaste exemplet på en teori med gaugesymmetri, ett begrepp som genomsyrar allt som behandlas i detta arbete. En generalisering av denna teori till icke-abelska grupper fås ur Yang-Mills och Chern-Simons teorier. Vidare ges en introduktion till allmän relativitetsteori med särskilt fokus på Cartans formulering. Dessutom återfinns ett stort antal appendix i rapporten, i vilka den nödvändiga matematiska bakgrunden ges för att kunna tillgodogöra sig och förstå de senare delarna i arbetet.

Nyckelord: Gaugeteori, 2+1-dimensionell gravitation, Högre spinn-gravitation, Maskhål, Svarta hål, Chern-Simons-teorier

Acknowledgements

This thesis is written in a similar fashion as another bachelor's thesis on the same topic, "Resolution of the Big Bang Singularity in a Higher Spin Toy Model" by Dahlén, Hallenbert, Nilsson and Rasmusson [1]. Their thesis has served as a great source of inspiration for the introductory chapters with background theory.

We must not forget to show our most sincere gratitude to our supervisor, Bengt E W Nilsson. Thank you for your support and commitment and for introducing us to the fascinating field of gravitational physics. Our journey from classical electromagnetism to higher spin gravity would not have been possible without the many meetings and private conversations.

Göteborg, May 2017
The Authors

Contents

List of Symbols

$\mathcal{L}$	Lagrangian density
ds^2	Line element, $ds^2 = g_{\mu\nu}dx^\mu dx^\nu$
$\sqrt{-g}$	Integration measure, $\sqrt{-\det(g_{\mu\nu})}$
AdS_3	Anti-de Sitter space in three (3) dimensions.
tr[...]	Trace or invariant bilinear form
$T_{(\mu\ldots\rho)}$	Index symmetrisation, $T_{(\mu\nu)} = \frac{1}{2!}(T_{\mu\nu} + T_{\nu\mu})$
$T_{[\mu\ldots\rho]}$	Index antisymmetrisation, $T_{[\mu\nu]} = \frac{1}{2!}(T_{\mu\nu} - T_{\nu\mu})$
A	Gauge connection
$F_{\mu\nu}$	Field strength tensor
$\mu,\nu,\rho...$	Curved space indices (denoted with Greek letters)
$a,b,c...$	Flat space indices (denoted with Latin letters)
ε^{abc}	Levi-Civita symbol: $\varepsilon^{012} = 1$
ϵ^{abc}	Levi-Civita tensor: $\epsilon^{abc} = \frac{1}{\sqrt{-g}}\varepsilon^{abc}$
$g_{\mu\nu}$	Metric tensor
$\eta_{\mu\nu}$	Minkowski metric, $\eta_{\mu\nu} = \mathrm{diag}(-1,1,1...)$
$*$	Hodge dual operator
$\Box$	d'Alembert operator
$\delta^{\mu_1\ldots\mu_n}_{\nu_1\ldots\nu_n}$	Generalized Kronecker delta, $\delta^{\mu_1}_{[\nu_1}\delta^{\mu_2}_{\nu_2}...\delta^{\mu_n}_{\nu_n]}$
d	Exterior derivative
$\Gamma^\mu_{\nu\rho}$	Christoffel symbol
Λ	Cosmological constant
$\mathcal{H}$	Hamiltonian density
$\mathcal{S}_{CS}$	Chern-Simons action
$\mathcal{S}_{EH}$	Einstein-Hilbert action
D	Covariant derivative, $\mathrm{D} = \mathrm{d} + A$
Hol(...)	Holonomy
${\omega_\mu}^a{}_b$	Spin connection
$\psi_{\mu\nu\rho}$	Spin-3 field
$\wedge$	Wedge operator
D_μ	Covariant derivative
${e_\mu}^a$	Vielbien, frame field, dreibein
${f^a}_{bc}$	Structure constant
G	Newton's constant of gravity
l	Radius of curvature
R	Ricci scalar, $R = {R^\mu}_\mu$
${R^\mu}_{\nu\rho\sigma}$	The Riemann curvature tensor
R^a	The curvature two-form
$R_{\mu\nu}$	Ricci tensor, $R_{\mu\nu} = {R^\sigma}_{\mu\sigma\nu}$
$SL(n,R)$	Special Linear Group
$SO(n)$	Special Orthogonal Group
$SU(n)$	Special Unitary Group
T^a	Torsion tensor

Sammanfattning

I snart ett århundrade har forskare och fysiker försökt förena de två hörnstenar som den fundamentala fysiken vilar på: kvantmekanik och allmän relativitetsteori. Denna förening refererar dagens fysiker till som *kvantgravitation* (engelska *quantum gravity*). I 2+1 dimensioner (två rumsliga dimensioner och en tidsdimension) verkar en sådan förening vara möjlig. Detta undersöktes av Witten redan 1988, då han utnyttjade att gravitation kan uttryckas som en Chern-Simons gaugeteori, vilken han påvisade kunde kvantiseras.

Med grund i detta studerar vi kopplingen mellan gravitation och gaugeteori i 2+1 dimensioner och utvecklar en tredimensionell modell som vi kan använda för att studera intressanta fenomen som svarta hål och maskhål. Vidare utökar vi teorin till en *högre spinn-teori*, vilken kan användas till att bland annat lösa upp singulariteter. I högre spinn-teorin är den geometriska tolkningen av svarta hål och maskhål, till skillnad från i konventionell gravitationsteori, inte fullständigt klarlagd. Detta gör högre spinn till ett spekulativt men intressant område att studera. Vår förhoppning är att ge en inblick i hur svarthåls- och maskhålslösningar kan tolkas och påvisa de problem som uppstår vid övergången från den konventionella gravitationsteorin till en högre spinn-teori. I synnerhet intresserar vi oss för så kallade *traversable* maskhål, det vill säga maskhål som kan korsas. Då det i Einsteinsk gravitation visar sig att sådana inte kan existera utan tillgång till materia med negativ energidensitet, ofta refererat till som *exotisk materia*, utforskar vi istället om dessa maskhål kan förekomma i en högre spinn-teori.

En målsättning i detta arbete är att introducera läsaren till viktiga fysikaliska koncept som gaugeteorier och allmän relativitetsteori, som båda kräver ett stort antal matematiska verktyg för att kunna ta till sig. För att kunna ge en grundläggande förståelse för dessa teorier återfinns den centrala matematiken, som i huvudsak kan sammanfattas i gruppteori och differentialgeometri, i form av appendixavsnitt som komplement till huvudtexten. I rapportens första del ges en introduktion till gaugeteorier vilken påbörjas med ett inledande kapitel om Maxwells elektromagnetism som den första gaugeteorin. Detta följs naturligt av förlängningen till gaugeteorier med icke-abelska grupper, där Yang-Mills gaugeteori är ett fundamentalt exempel. Det hela kulminerar i Chern-Simons gaugeteori, vilken är den teori som vi senare kopplar till gravitation.[1] Därefter introduceras det andra viktiga huvudspåret i rapporten, allmän relativitetsteori och gravitation i 2+1 dimensioner. I synnerhet använder vi Cartans formulering av teorin som utnyttjar ekvivalensprincipen, för att beskriva rumtiden med ett lokalt platt tangentplan i varje punkt.

De nyvunna kunskaperna om Einsteinsk gravitation och gaugeteorier använder vi sedan för att, likt Witten, uttrycka gravitation i 2+1 dimensioner som en Chern-Simons-teori. I rapportens senare del betraktas maskhål- och svarthålslösningar till Einsteins ekvationer, dels i vanlig Einsteinsk gravitation men även i den generaliserade högre spinn-teorin. Här läggs särskilt fokus på den geometriska tolkningen av maskhål och svarta hål i högre spinn. Vi finner bland annat att lösningar som beskriver en maskhålsgeometri samtidigt kan vara svarta hål i en sådan teori. Nedan återfinns en mer detaljerad summering av de viktigaste delarna i den ordning de uppträder i arbetet.

Gaugeteorier

Ett av de största framstegen inom teoretisk fysik var sammanslagningen av de elektriska och magnetiska krafterna som kan sammanfattas i Maxwells ekvationer. En egenhet med teorin är att den är oförenlig med Newtonsk mekanik, eftersom ekvationerna inte är invarianta under en Galileisk transformation. Istället hade Maxwell funnit en Lorentzinvariant teori, långt före Einstein formulerade sin berömda relativitetsteori. Förutom detta hade Maxwell upptäckt den första *gaugeteorin*. I kvantmekanik saknar fasen hos ett kvanttillstånd mening, och denna symmetri syns redan i Maxwells teori. Det visar sig även möjligt att utifrån denna symmetri konstruera Maxwells ekvationer genom att finna en Lagrangian med samma symmetriegenskaper.

Yang-Mills-teori är en generalisering av elektromagnetismen till en godtycklig symmetri som även tillåts vara *icke-abelsk*, det vill säga att två skilda transformer inte kommuterar. Den elektromagnetiska fältstyrkan F generaliseras till en två-form med samma namn medan fyrvektorpotentialen A och ström-

[1] Chern-Simons gaugeteori utgör även en topologisk knutteori, vilket påvisades av Witten i slutet av 1980-talet.

men J blir ett-former.

Genom att tillämpa minsta verkans princip på Yang-Mills-Lagrangianen finner vi Yang-Mills ekvationer:

$$\mathrm{D}F = 0 \ ,$$
$$\mathrm{D} * F = *J \ ,$$

där D är den en kovarianta yttre derivatan, $\wedge$ kilprodukten och $*$ är Hodge-dualen. Maxwells ekvationer är ett specialfall av dessa då symmetrigruppen består av fastransformationer.

För att få en gaugeteori som är rumtidsoberoende, så som allmän relativitetsteori, konstrueras en Lagrangian som är ett specialfall av $\mathcal{L}_{YM}$ då $F = *F$ och strömmen J är avslagen. Teorin, i vilken denna Lagrangian återfinns, kallas för en *Chern-Simons-teori*, och dess rörelseekvationer ges av platthetsvillkoret

$$F = \mathrm{d}A + A \wedge A = 0 \ .$$

Vi behandlar i senare delar av texten Chern-Simons-teori för att uttrycka gravitation i 2+1 dimensioner.

Gravitation i 2+1 dimensioner

Den fundamentala teorin för att beskriva gravitation på en makroskopisk skala är Einsteins allmänna relativitetsteori. I denna teori kombineras tid och rum till *rumtiden*, vilken modelleras som en mångfald. Energi och materia kan kröka rumtiden lokalt för att ge upphov till gravitation. Krökningen av rumtiden beskrivs av *Riemanns krökningstensor* $R^{\mu}{}_{\nu\rho\sigma}$. Ofta används istället *Riccitensorn* eller *Ricciskalären* för att beskriva krökningen av rumtiden. Riccitensorn, $R_{\mu\nu}$, fås genom att ta ett visst spår av Riemanntensorn, och Ricciskalären, R, fås i sin tur genom att ta spåret av Riccitensorn.

I likhet med avsnittet om gaugeteorier kan vi härleda rörelseekvationerna från en verkan:

$$\mathcal{S} = \underbrace{\frac{1}{2\kappa}\int d^3x\sqrt{-g}(R-2\Lambda)}_{\mathcal{S}_{EH}} + \mathcal{S}_{\text{materia}} \ ,$$

där κ är Einsteins konstant, Λ är den kosmologiska konstanten och $\mathcal{S}_{\text{materia}}$ är en verkan som beskriver distributionen av materia. Vi har även indikerat att den första termen ofta refereras till som $\mathcal{S}_{EH}$, *Einstein-Hilbert-verkan*. Variationen av den totala verkan ger Einsteins fältekvationer:

$$R_{\mu\nu} - \frac{1}{2}g_{\mu\nu}R + \Lambda g_{\mu\nu} = \kappa T_{\mu\nu} \ , \tag{0.1}$$

där $T_{\mu\nu}$ är stressenergitensorn som beskriver flödet av energi i rumtiden och utgör källan till gravitationsfältet.

Det är även möjligt att uttrycka allmän relativitetsteori med den alternativa *Einstein-Cartan*-formalismen. I denna formalism är en lokal ortogonal bas e^a, vielbeins[2] $e_{\mu}{}^{a}$ och *spinn-kopplingen* $\omega_{\mu}{}^{a}{}_{b}$ de fundamentala objekten. Vielbeinen lever i det lokala tangentrummet och relateras till den krökta mångfaldens metrik genom

$$g_{\mu\nu} = e_{\mu}{}^{a} e_{\nu}{}^{b} \eta_{ab} \ ,$$

där η_{ab} är Minkowskimetriken, som därmed ger en relation mellan den lokala ortogonala basen och de mer generella kroklinjiga koordinaterna.

Cartans strukturekvationer beskriver krökningen av rumtiden i termer av den lokala ortogonala basen och spinn-kopplingen:

$$T^a = \mathrm{d}e^a + \epsilon^{a}{}_{bc}\omega^b \wedge e^c, \tag{0.2}$$
$$R^a = \mathrm{d}\omega^a + \frac{1}{2}\epsilon^{a}{}_{bc}\omega^b \wedge \omega^c \ , \tag{0.3}$$

där T^a är torsionsformen och R^a är krökningsformen. Torsionsformen beskriver *torsionen* av rumtiden, vilket kan ses beskriva till vilken grad en skalärprodukt bevaras mellan två vektorer då de transporteras i rumtiden. I allmän relativitetsteori gäller att torsionen är noll.

Einstein-Hilbert-verkan kan uttryckas i Cartan-formalism på följande sätt:

$$\mathcal{S}_{EH}[e,\omega] = -\frac{1}{2\kappa}\int\left[2e^a \wedge (d\omega_a + \frac{1}{2}\epsilon_{abc}\omega^b \wedge \omega^c) - \frac{1}{3}\Lambda\epsilon_{abc}e^a \wedge e^b \wedge e^c\right] \ . \tag{0.4}$$

[2]Ordet vielbein översätts från tyska till svenska som "flera ben".

Varierar vi denna verkan med avseende på ω erhålls den första strukturekvationen med $T^a = 0$. Varierar vi istället med avseende på e erhålls analogen till Einsteins fältekvationer i vakuum:

$$\mathrm{d}\omega^a + \frac{1}{2}\epsilon^a{}_{bc}\omega^b \wedge \omega^c = \frac{\Lambda}{2}\epsilon^a{}_{bc}e^b \wedge e^c \ . \tag{0.5}$$

Lösningar till Einsteins ekvationer

Den allra första kända lösningen till Einsteins ekvationer hittades 1915 av Karl Schwarzschild. *Schwarzschildlösningen* i 3+1 dimensioner är en sfäriskt symmetrisk metrik med två mycket intressanta egenskaper: en händelsehorisont samt en singularitet i origo. Händelsehorisonten är i sig ingen kurvatursingularitet utan endast en koordinatsingularitet som utgör en yta från vilken inget ljus kan ta sig ut, varför Schwarzschildlösningen senare kom att kallas ett svart hål. Med ett koordinatbyte finner vi att det svarta hålet kan ses som en brygga mellan två olika rymdtider, en *Einstein-Rosen-brygga* eller mer konventionellt ett maskhål. Detta maskhål tillåter dock inte information att färdas från den ena sidan till den andra och är därmed ett exempel på ett *non-traversable* maskhål.

Inspirerade av Schwarzschildlösningen undersöks en radiellt symmetrisk metrik i 2+1 dimensioner:

$$ds^2 = -B^2(r)dt^2 + A^2(r)dr^2 + r^2d\phi^2 \ ,$$

och genom att lösa Cartans strukturekvationer (0.5) med en kosmologisk konstant $\Lambda = -\frac{1}{l^2}$ där l är en karakteristisk längd, finner vi

$$ds^2 = -\Big(\frac{r^2}{l^2} - M\Big)dt^2 + \Big(\frac{r^2}{l^2} - M\Big)^{-1}dr^2 + r^2d\phi^2 \ .$$

Här är M en integrationskonstant. Om $M \in (-1,0)$ har vi en konisk singularitet i origo, vilket kan tolkas som en punktpartikel med en massa som fixeras av värdet på M. Specialfallet $M = -1$ är ekvivalent med AdS_3, det vill säga ett negativt krökt 2+1-dimensionellt universum.[3] För $M > 0$ får vi ett svart hål, känt som ett *BTZ-svart hål* efter fysikerna Bañados, Teitelbom and Zanelli som fann lösningen på 1990-talet. BTZ-hålet har precis som Schwarzschild-hålet en händelsehorisont, men singulariteten är ingen kurvatursingularitet, utan en kausalitetssingularitet. Konstanten M kan för BTZ-hålet relateras både till omkretsen på det svarta hålet samt dess massa.

En annan intressant radiellt symmetrisk geometri är *Morris och Thorne-maskhålet*, vilket kan beskrivas med metriken:

$$ds^2 = e^{2\Phi(r)}dt^2 + \frac{dr^2}{1 - \frac{b(r)}{r}} + r^2d\phi^2 \ ,$$

där $\Phi(r)$ betecknar *rödskiftsfunktionen* och $b(r)$ är *formfunktionen*. Genom att konstruera Einsteintensorn finner vi att för att lösa Einsteins fältekvationer (0.1) krävs en stresstensor med negativ energidensitet. Ett material med en sådan egenskap refereras ofta till som *exotiskt*. Vi undersöker huruvida vi kan konstruera *exotisk materia* från det elektromagnetiska fältet och ett konformt kopplat skalärfält. Det elektromagnetiska fältet visar sig endast tillåta positiv energidensitet. Det konformt kopplade skalärfältet visar sig vara mycket svårt att analysera. Vi visar emellertid att det inte går att erhålla icke-triviala lösningar om skalärfältet är masslöst i 2+1 dimensioner.

Högre spinn

För att vidare undersöka maskhål samt generaliseringar av svarta hål relateras gaugeteori till allmän relativitetsteori i 2+1 dimensioner med en negativ kosmologisk konstant. Vi finner att AdS_3:s isometrigrupp $SO(2,2)$ är isomorf med $SL(2) \times SL(2)$. Detta leder oss till att undersöka verkan

$$\mathcal{S}_{CS}[A] - \mathcal{S}_{CS}[\overline{A}] \ , \tag{0.6}$$

där $\mathcal{S}_{CS}$ är en *Chern-Simons-verkan* och A samt $\overline{A}$ ges som

$$A = (\omega^a_\mu + \frac{e^a_\mu}{l})T_a dx^\mu \ , \quad \overline{A} = (\omega^a_\mu - \frac{e^a_\mu}{l})T_a dx^\mu \ , \tag{0.7}$$

[3] Negativ krökning kan liknas vid ytan av en sadel.

där T_a är generatorer till $sl(2)$, det vill säga Liealgebran till $SL(2)$. Detta val innebär att (0.6) blir ekvivalent med en variant av ekvation (0.4), mer precist en Einstein-Hilbert-verkan med negativ kosmologisk konstant.

Det är rättframt att generalisera genom att istället välja gruppen $SL(N) \times SL(N)$ som med $N > 2$ innehåller $SL(2) \times SL(2)$. Detta är en högre spinn-gravitationsteori som besitter en utökad mängd gaugesymmetrier. I detta arbete studerar vi dock endast generaliseringen $SL(3) \times SL(3)$ som visar sig vara högst intressant. Detta innebär således att generatorerna i (0.7) nu tillhör $sl(3)$. Denna generalisering tillåter inte bara den spinn-2-metrik som diskuterats ovan, utan även en högre-dimensionell spinn-3-metrik. Med de utökade gaugetransformationerna kan man införa singulariteter i tidigare väldefinerade metrikerna.

Då metriken inte längre är gaugeinvariant används istället *holonomier* för att klassificera lösningar. Dessa beskrivs matematiskt enligt:

$$\mathrm{Hol}(A) = \mathcal{P} \exp\left(\oint_\gamma A\right) ,$$

där γ är en sluten kurva, $\mathcal{P}$ betecknar en vägordnad integral och A ges av (0.7).

Med dessa verktyg går det att gaugetransformera både en konisk singularitet och ett svart hål till ett maskhål utan att ändra holonomin. På detta vis möjliggör högre spinn att singulariteter i geometrin kan upplösas. Generaliseringen av ett svart hål i en högre spinn-gravitationsteori är ej trivial då vi ej längre kan göra en geometrisk tolkning av fenomenet. Istället kan dessa objekt defineras genom krav på holonomier som implicerar välkända geometriska egenskaper, såsom händelsehorisonter, om högre spinn-fält slås av.

Slutsats

Genom att använda gaugeteori har vi lyckats konstruera en generalisering av allmän relativitetsteori. Vår modell är en så kallad högre spinn-teori och är därmed en del av en större grupp av teorier, framförallt utvecklade av Vasiliev i slutet av 1990-talet. Högre spinn-teorier är intressanta av flera anledningar men kanske framförallt då de anses beskriva strängteori i gränsen då strängspänningen försvinner. Förhoppningen är att genom högre spinn förenkla svåra problem i strängteori.

Ett problem som fysiker brottats länge med är singulariteter. Singulariteter anses ofta vara ofysikaliska och signalerar områden i vilka ny fysik krävs. Vi har i detta arbete visat att i en högre spinn-teori kan singulariteter ofta tolkas som *gaugeartefakter*, det vill säga effekter av gaugeval. Explicit visades hur singulariteten för en punktpartikel samt ett svart hål kan transformeras bort. Dessa transformationer var även valda så att holonomin var bevarad och därmed kan den transformerade metriken anses vara ekvivalent med den första. Anmärkningsvärt var att den nya metriken vi erhöll efter transformationen kunde tolkas som ett maskhål. Detta innebär att vi har lyckats kringgå kravet på exotisk materia för ett maskhål. Givetvis ställer detta även frågan hur massa, eller kanske mer exakt, stress-energitensorn bör tolkas i högre spinn. Detta eftersom den är intimt förknippad med den nu gaugeberoende geometrin.

Möjligheten att förändra geometrin genom gaugetransformationer är mycket intressant då den kräver nya definitioner av objekt som annars klassifieras med geometrin. Vi har framförallt undersökt hur ett svart hål kan definieras utan direkt referens till metriken och hur dessa krav kan formuleras i termer av holonomier. Problemet med den geometriska tolkningen av metriken är komplext och framförallt finns det ännu inte en klar tolkning av den extra spinn-3-metriken och hur denna interagerar med spinn-2-metriken. Här finns en möjlighet till nya upptäckter och insikter. Även utan en klar förståelse för dessa problem finns möjligheter att utnyttja de teorier vi har studerat. Ett möjligt område är partikeldynamik i AdS_3, där genombrott gjorts med geometriska modeller. Möjligtvis kan högre spinn-transformationer förenkla dessa geometrier och underlätta beräkningar. Högre spinn skulle även kunna fungera som en typmodell av hur vi kan generalisera definitionen av geometriska begrepp till mer allmänna sammanhang.

Möjligtvis kan den hyllade AdS/CFT-korrespondensen kasta ljus på om en geometrisk tolkning av högre spinn är möjlig. AdS/CFT relaterar strängteori i AdS med en konform fältteori på randen. Detta innebär alltså en ekvivalens mellan objekt definerade i olika antal dimensioner, något som mycket väl kan visa sig vara högst användbart i högre spinn.

Chapter 1

Introduction

Since the last century several attempts have been made to unite the two cornerstones of fundamental physics: quantum mechanics and general relativity. Physicists refer to the unification of quantum mechanics and spacetime as the theory of *quantum gravity*. This kind of theory is of great importance when it comes to understanding the more enigmatic aspects of the universe, e.g. such as the big bang and black holes. The reconciliation of quantum theory and gravity in ordinary spacetime has remained elusive but the search has spawned different toy models. One such toy model, proposed by Witten[1] in 1988, is quantum gravity in 2+1 dimensions as a Chern-Simons gauge theory [2]. This theory has the attractive attribute of being exactly solvable.[2]

In 2013, the famous theoretical physicists Maldacena and Susskind[3] proposed a conjecture which may lead to pioneering contributions in the fields of black holes and quantum entanglement [4]. The conjecture can be simply stated as ER=EPR and is founded upon Einstein's and Rosen's theory of wormholes (ER) and the Einstein-Podolsky-Rosen theory of quantum entanglement of particles (EPR)[4].

Maldacena and Susskind proposed that a non-traversable wormhole in general relativity is equivalent to entanglement in quantum mechanics [7]. In order to understand their conjecture one has to be familiar with properties of black holes. In short, a black hole is a curved spacetime with a geometry consisting of two regions: the exterior and the interior, separated by a surface, the so called *event horizon*. Black holes are predicted by Einstein's equations, from which the fully extended geometry of the solutions describe a pair of black holes joined by a wormhole. The wormhole can be seen as the interior of the joined black holes and as a "bridge" between them, and is therefore often referred to as an Einstein-Rosen Bridge.

It may be an appealing idea that one can use wormholes, if they constitute a bridge between black holes, to travel from one black hole to another by travelling through them. However, nothing can escape the interior of black holes and, furthermore, the bridge may collapse after a very short period of time. In fact, the wormhole fails to serve the purpose of a bridge as it can not be crossed. This is a consequence of the fact that the wormhole gets thinner as time progresses and the horizons of the black holes separate extremely rapidly [7]. In other words conventional Einstein-Rosen bridges are said to be *non-traversable* and do not constitute a particular good example of time machines. However, they are yet of today seen as prototypes of wormholes and spurred Morris and Thornes' (abbreviated MT) idea of *traversable* wormholes, theoretically realized in 1988 [8]. In their paper they proposed that traversable wormholes could exist according to the theory of general relativity with the use of *exotic matter*, i.e. matter with a negative mass. In this paper we show that Morris and Thorne were indeed correct, by noting that the MT-solution, upon solving Einstein's equations, yields a negative component in the stress-energy tensor.[5] It may now seem, due to the need for exotic matter, that traversable wormholes are purely mathematical constructions and have no correspondence in the physical world and our universe. There is still hope, however, since in quantum field theory negative energy density is possible as seen in the Casimir effect, so the construction of a traversable wormhole might be possible after all, but then as a quantum object [9].

Another way to allow for traversable wormholes is to introduce new degrees of freedom and thereby generalize our regular theory, in our case yielding what is called a *higher spin gravity theory* [10]. Eventually new problems arise when working in this theory - for instance we may not have a stress-energy tensor with the same physical interpretation as in conventional Einstein gravity. In fact, even defining

[1] Edward Witten (1951-), greatly acclaimed professor of mathematical physics at Princeton University. Awarded the Fields medal in 1990, partly for his work on topological quantum field theory. In particular he connected Chern-Simons theory to knot theory.

[2] Witten claimed in 2007 that this model is in fact incorrect and instead proposed a different model related to the monster group [3].

[3] Juan Maldacena (1968-), Argentinian professor of theoretical physics at Princeton University, Leonard Susskind (1940-), American professor of theoretical physics at Stanford University.

[4] The acronyms ER and EPR can be derived from the last names of the physicists who wrote the first papers on wormholes and entanglement, Einstein-Rosen [5] and Einstein-Podolsky-Rosen [6] respectively.

[5] The stress energy tensor, $T^{\mu\nu}$, is a tensor which describes the density and flux of energy in spacetime. More specifically $T^{00} = \rho$, where ρ is the energy density.

a proper stress tensor is difficult. Moreover, the extended symmetries, as opposed to the symmetries of the regular theory of gravity, may dramatically alter the geometry of spacetime. As such, higher spin theory requires refined mathematical tools in order to be properly interpreted.

1.1 Motivation of thesis

In this thesis we aim to provide an understanding of how Einstein's theory of general relativity is connected to internal symmetries, and how this facilitates its generalization to higher spin gravity. A theory that possesses internal symmetries is invariant under some set of transformations, in this case gauge transformations[6], and therefore the theories we study are referred to as gauge theories. In technical terms the gauge invariant object of interest is the Lagrangian, which by the principle of stationary action gives us proper equations of motion. In 2+1 dimensions it is particularly simple[7] to connect gauge theories to the theory of gravity, which is why we restrict to this case in our thesis.

Since we study gravity in 2+1 dimensions, rather than four dimensions (space and time included), the model we develop constitutes nothing but a toy model for our universe. Remarkably, interesting solutions (e.g. black holes and wormholes) from four-dimensional gravity appear in 2+1 dimensions as well, allowing us to study such solutions in our model. In particular we aim to investigate black hole and wormhole solutions in a higher spin gauge theory, which is a generalization of the conventional gravitational theory.[8] Higher spin gravity is yet of today a not fully understood area in gravitational physics, especially when it comes to the geometrical interpretation of the solutions.

In general, literature on this subject is very advanced, and thus we aim to simplify and make more explicit other work on the subject, as well as provide some potentially new analysis. To make sure information is not obscured by tedious equations, some calculations are moved to the Appendix, but are seldom omitted completely. Our objective is that the end product shall provide a solid introduction to the subject of higher spin gravity in 2+1 dimensions suitable for an undergraduate in physics, maintaining clarity throughout.

1.2 Reading guide

We start off this text where basic courses in special relativity and electromagnetism left off and then introduce and develop the theories needed to provide a basic understanding of black holes and wormholes. There is a lot of mathematical background structure involved in these theories, and an introduction to all of the necessary mathematical tools is given in appendices A-E, in case the reader is not familiar with these.

In the beginning of each chapter a summary of the chapter is given, along with its most important results. The more casual reader can get a basic understanding of the material and our results just by reading the introductory paragraphs.

In chapter 2 and 3 we introduce gauge theories, the basis of all of modern physics. Gauge theories describe fields with a set of internal symmetries, which mathematically are expressed through group theory. These sections culminate in the introduction of Chern-Simons gauge theory which we can use to construct a theory of gravity in 2+1 dimensions. In chapter 4 we move on to general relativity and in particular Elié Cartan's formulation. We present black hole and wormhole solutions to Einstein's equations in chapter 5. In particular, we find solutions that contain singularities, that is, points where the theory is not well defined. To resolve the problem of these singularities we then use Chern-Simons gauge theory to reformulate the theory of general relativity in chapter 6. In chapter 7 we perform a generalization of Einstein gravity by extending the symmetries possessed by general relativity resulting in a higher spin theory. Using this enhanced symmetry we dedicate chapter 8 to studying singularity resolution and higher spin black holes. In particular we construct a solution that admits a traversable wormhole geometry.

[6]A gauge can be seen, in simple terms, as a coordinate system which varies depending on your location with respect to some base space. Thus a gauge transform corresponds to a change of coordinates on the location.

[7]As pointed out by Witten in his article "2 + 1 Dimensional Gravity as an Exactly Soluble System", for instance the Einstein-Hilbert action, from which Einstein's field equations are derived, is perturbatively non-renormalisable in four dimensions, which is not the case in 2+1 dimensions[2].

[8]Regular Einsteinian gravity is mediated by the spin-2 boson, the graviton and therefore constitutes a spin-2 theory. In higher spin theories the fundamental constituents have spins larger than two. We will exclusively study the case spin-3, in which the spin-2 boson is coupled to a spin-3 field.

Chapter 2

Electromagnetism

In this chapter we treat electromagnetism formulated in the framework of special relativity and discuss it from two different points of view. First we revisit the standard Maxwell equations and rewrite these in the language of tensors. Doing so we introduce the four-vector potential A^μ and the field strength tensor $F^{\mu\nu}$. A reminder of tensors and assorted operations such as lowering and raising indices can be found in Appendix A. We then start from a general Lagrangian and by insisting that our Lagrangian shall be invariant under local phase transformations, we introduce the covariant derivative and the gauge connection. Using these we can recover the four-vector potential and the field strength without ever discussing the electric and magnetic field. This procedure sets the stage for more general gauge theories to be discussed in the next chapter. We end this chapter with a discussion of the Lagrangian for the electromagnetic field and show that its equations of motion together with the *Bianchi identity* gives us Maxwell's equations.

2.1 Maxwell's equations in tensor formalism

One of the milestones in theoretical physics was the unification of the electric and magnetic forces at the end of the 19th century. The work of several great physicists reached its culmination when James Clerk Maxwell published his "A treatise on Electricity and Magnetism" in 1873 [11]. Here he introduced for the first time the four governing equations of electricity and magnetism. Because of his achievement we therefore speak of them as Maxwell's equations. In natural units these can be written, in a form due to Oliver Heaviside[1], as

$$\nabla \cdot \mathbf{E} = \rho \, , \qquad \nabla \times \mathbf{B} = \mathbf{J} + \frac{\partial \mathbf{E}}{\partial t} \, , \tag{2.1}$$

$$\nabla \times \mathbf{E} = -\frac{\partial \mathbf{B}}{\partial t} \, , \qquad \nabla \cdot \mathbf{B} = 0 \, . \tag{2.2}$$

Here $\mathbf{E}$ is the electric field, $\mathbf{B}$ the magnetic field, $\mathbf{J}$ the electric current and ρ the charge density. A remarkable fact is that these equations cannot be fit into a Newtonian theory. In other words, the equations are not invariant under *Galilean transformations*, instead, what Maxwell had found more than 30 years before Einstein was a theory that fits perfectly in the framework of special relativity. The Lorentz invariance of Maxwell's equations is by no means obvious from the way we expressed them above. To see that this is the case we must recast them into the language of relativity, that is, the language of four-vectors and tensors. If we can manage to do this, Lorentz invariance will be a consequence [12].

We will start our endeavour of rewriting Maxwell's equations in terms of tensors by defining a scalar potential ϕ and a vector potential $\mathbf{A}$ such that

$$\mathbf{E} = -\nabla\phi - \frac{\partial \mathbf{A}}{\partial t} \, , \qquad \mathbf{B} = \nabla \times \mathbf{A} \, . \tag{2.3}$$

Notice that given $\mathbf{E}$ and $\mathbf{B}$, the scalar potential and the vector potential, are not uniquely defined. To see this explicitly we can transform our scalar and vector potential according to

$$\phi \mapsto \phi' = \phi - \frac{\partial \alpha}{\partial t} \, , \qquad A \mapsto A' = A + \nabla\alpha \, , \tag{2.4}$$

where α is an arbitrary scalar function. While these transformations obviously alter the scalar potential and the vector potential, the electric and magnetic field remain unchanged. This can be seen by simply

[1] Oliver Heaviside (1850-1925), English mathematician, engineer and physicist. He did not only rewrite Maxwell's equations in the form as we know them today, but he actually employed the divergence and curl operators to vector calculus.

computing the fields for both potentials with the definition (2.3). What we have encountered is a redundancy in the formulation of the theory. We call transformations with this property *gauge transformations* and the procedure of eliminating the freedom to do these transformations is referred to as *fixing a gauge.*

A simple example of a tensor is a vector. Since the two potentials ϕ and $\mathbf{A}$ together have 4 components they are the prime candidates for constructing a vector according to

$$A^\mu = (\phi, \mathbf{A}) \ .$$

Here μ is an index taking the values 0,1,2,3. This notation should be familiar from a course in special relativity and will be used heavily in this thesis. The vector A^μ is often referred to as the (four)-vector potential.

We certainly would like to somehow relate $\mathbf{E}$ and $\mathbf{B}$ in a similar way but this is a bit trickier. The reason is that $\mathbf{E}$ and $\mathbf{B}$ both have three components while a four-vector has four. However, we do have a four-potential and from the defining relation (2.3) we can guess that we somehow want to form a combination of derivatives and scalar potentials. There are three main candidates; the full tensor, $\partial_\nu A_\mu$, the symmetric tensor $2\partial_{(\nu}A_{\mu)}$ or the antisymmetric tensor $2\partial_{[\nu}A_{\mu]}$.[2] However, the full tensor has 16 independent components, the symmetric 10 but the antisymmetric 6 components, precisely as many as the electric and magnetic field. Thus the only plausible tensor is the antisymmetric and we refer to this tensor as the *field strength*, $F^{\mu\nu}$ [13]. In matrix form we have

$$F^{\mu\nu} = \partial^\mu A^\nu - \partial^\nu A^\mu = \begin{pmatrix} 0 & E_x & E_y & E_z \\ -E_x & 0 & B_z & -B_y \\ -E_y & -B_z & 0 & B_x \\ -E_z & B_y & -B_x & 0 \end{pmatrix} \ , \tag{2.5}$$

meaning that we can extract the electric and the magnetic field by the identities $F^{0i} = E^i$ and $\frac{1}{2}\epsilon_{jk}{}^{i}F^{jk} = B^i$, where $i, j, k \in \{1,2,3\}$. Having identified the field strength we can now express Maxwell's equations as tensor equations:

$$\partial_\nu F^{\mu\nu} = J^\mu \ , \tag{2.6}$$

$$\epsilon_{\mu\nu\rho\sigma}\partial^\nu F^{\rho\sigma} = 0 \ . \tag{2.7}$$

Not only are the equations manifestly Lorentz invariant, we now only have two equations. If written out in terms of the electric and magnetic field we find that the first equation is equivalent to (2.1) and the second to (2.2). The second equation is also known as the *Bianchi Identity.* The proof is rather short:

$$\epsilon_{\mu\nu\rho\sigma}\partial^\nu F^{\rho\sigma} = 2\epsilon_{\mu\nu\rho\sigma}\partial^\nu\partial^{[\rho}A^{\sigma]} = 2\epsilon_{\mu\nu\rho\sigma}\partial^{(\nu}\partial^{\rho)}A^\sigma = 0 \ ,$$

where we in the last step used that the contraction of a symmetric and anti-symmetric tensor vanishes.

Let us take a closer look at the field strength, $F^{\mu\nu}$, or more precisely: its definition. To define the field tensor and then retrieve all of Maxwell's equations we are only interested in the derivatives of the four-potential. Consider the new four-potential $(A')^\mu = A^\mu + \partial^\mu\alpha$, where $\alpha = \alpha(x)$ is an arbitrary scalar field. The new field strength then becomes

$$(F')^{\mu\nu} = \partial^\mu(A^\nu + \partial^\nu\alpha) - \partial^\nu(A^\mu + \partial^\mu\alpha) = 2\partial^{[\mu}A^{\nu]} + \underbrace{[\partial^\mu,\partial^\nu]\alpha}_{0} = F^{\mu\nu},$$

and hence the field strength is gauge invariant.

2.2 From symmetry to electromagnetism

In this section we will with the use of symmetry retrieve Maxwell's equations. To do this we will use the concept of a Lagrangian. Readers unfamiliar with the Lagrangian and its properties will find a more thorough description in Appendix D. Since all dynamics and interactions can be derived from the Lagrangian we can be certain that if our Lagrangian is invariant under a symmetry, so is our theory. Now let us attack this problem from a different viewpoint. Given a Lagrangian let us modify it to become invariant under a symmetry transformation. To proceed we need to know the general form of a

[2] The parentheses and brackets denote symmetrization and anti-symmetrization, $2\partial_{(\mu}A_{\nu)} = \partial_\mu A_\nu + \partial_\nu A_\mu$ and $2\partial_{[\mu}A_{\nu]} = \partial_\mu A_\nu - \partial_\nu A_\mu$.

Lagrangian. In order not to stray too far from our path we will consider a complex scalar field $\phi(x)$ and the *Klein-Gordon Lagrangian*

$$\mathcal{L}_{KG} = -\frac{1}{2}\partial^\mu\overline{\phi}\partial_\mu\phi - \frac{1}{2}m^2\overline{\phi}\phi \ .$$

This is by no means the only fundamental Lagrangian but it possesses the properties we want to examine. A short "derivation" of the Klein-Gordon Lagrangian can be found in Appendix D. Notice that the Lagrangian has two terms, one with derivatives and one without. If we want our Lagrangian to be invariant under a symmetry transformation we simply need to make sure that each of the terms are. If some term should not be invariant under the transformation we will have to modify it.

The first symmetries we will consider in this thesis are those of the group $U(1)$. A full discussion on groups and their properties can be found in Appendix E. Here we simply note that $U(1)$ is the group that contains all phase transformations. These transformations form an (abelian) group and we can write a general element, $g \in U(1)$, as $g = e^{i\alpha(x)}$. The reason for studying this symmetry is that quantum mechanics tells us that the phase of a state lacks physical meaning. If this should also be the case for our theory we better have a Lagrangian that is invariant under phase transformations.

Let us first consider a global phase transformation that is

$$\phi \mapsto \phi' = e^{i\alpha}\phi \ ,$$

where α is a constant. It is straightforward to check that both $|\phi|^2$ and $|\partial_\mu\phi|^2$ are invariant under this transformation. To deal with the local case we must make our transformation become a function of spacetime, thus we let our constant α become a scalar field $\alpha(x)$. The mass term, $m^2\overline{\phi}\phi$, in our Lagrangian is trivially invariant under a local phase transformation. However, terms with a derivative are not. We can view this problem in two different ways, the first is strictly algebraic. By explicitly writing out the transformation we find

$$\partial_\mu\phi \mapsto \partial_\mu\phi' = \partial_\mu e^{i\alpha(x)}\phi = (i\partial_\mu(\alpha(x))\phi + \partial_\mu\phi)e^{i\alpha(x)} \ . \tag{2.8}$$

The problematic part is of course the first term since this will produce a non-vanishing contribution to all derivatives and $|\partial\phi|^2$ is no longer invariant. Another way to understand this problem is to study the definition of the derivative, as done in Peskin and Schroeder [14] on which we will base our discussion.

Remember that the derivative of $\phi(x)$ in the direction n^ν is defined as

$$n^\nu\partial_\nu\phi = \lim_{h\to 0}\frac{\phi(x+hn) - \phi(x)}{h} \ ,$$

but here we are subtracting fields from points that differ in their transformation property. This is not very physical at all. If we want to subtract these fields we need some way of relating them to each other. A way of doing this is to introduce a scalar function $U(x,y)$ with the transformation property

$$U(x,y) \mapsto e^{i\alpha(x)}U(x,y)e^{-i\alpha(y)} \ .$$

Then both terms in the difference $\phi(x) - U(x,y)\phi(y)$ transform the same way. Armed with this function we choose to abandon our old derivative and define a new, meaningful one. This derivative takes the form

$$n^\mu D_\mu\phi(x) = \lim_{h\to 0}\frac{\phi(x+hn) - U(x+hn,x)\phi(x)}{h} \ ,$$

and is called the *covariant derivative*. To proceed we would like to find a new simpler expression for this derivative, preferably without the limit. It is clear that we must have $U(x,x) = 1$ since the field transforms identically at the same point. Thus we may expand our scalar function $U(x,y)$ around the identity according to

$$U(x+hn,x) = 1 - ihn^\mu A_\mu + \mathcal{O}(h^2) \ .$$

By using this expansion in the definition of the covariant derivative we find

$$n^\mu D_\mu\phi(x) = \lim_{h\to 0}\frac{\phi(x+hn) - (1 - in^\mu A_\mu h + \mathcal{O}(h^2))\phi(x)}{h} = (n^\mu\partial_\mu + in^\mu A_\mu)\phi(x) \ .$$

Cancelling the $n^\mu\phi(x)$ on both sides we reach a much more practical expression

$$D_\mu = \partial_\mu + iA_\mu \ .$$

We now have a meaningful derivative, but to get a Lagrangian that is invariant under the phase transformation we must specify the transformation of the field A_μ. By studying the appearance of the terms containing the derivatives we can see that D_μ must satisfy

$$D_\mu\phi \mapsto (D_\mu\phi)' = e^{i\alpha(x)}(D_\mu\phi) \ .$$

Since we have that $(D_\mu\phi)' = (\partial_\mu + iA'_\mu)\phi'$ we can use the transformation property of the covariant derivative and the algebraic expression from equation (2.8) to write down an equation for the transformed field A'_μ. We find that

$$(D_\mu\phi)' = e^{i\alpha(x)}(i\partial_\mu(\alpha(x)) + \partial_\mu + iA'_\mu)\phi = e^{i\alpha(x)}(\partial_\mu + iA)\phi \ ,$$

and solving for A'_μ we reach the conclusion that A_μ must transform as

$$A_\mu \mapsto A'_\mu = A_\mu - \partial_\mu\alpha(x) \ .$$

Let us first study our new Lagrangian, the one obtained by simply replacing all the ordinary derivatives with the covariant derivative:

$$\begin{aligned}\mathcal{L} = -\frac{1}{2}\overline{D_\mu\phi}D^\mu\phi - \frac{1}{2}m^2\overline{\phi}\phi = \mathcal{L}_{KG} + \frac{1}{2}(-i\overline{\partial_\mu\phi}\phi + i\overline{\phi}\partial_\mu\phi)A^\mu - \frac{1}{2}A^\mu A_\mu\phi^2 \\ = \mathcal{L}_{KG} + J_\mu A^\mu - \frac{1}{2}A^\mu A_\mu\phi^2 \ .\end{aligned} \tag{2.9}$$

Thus we obtain a coupling term in the Lagrangian between the original field and our connection, $J_\mu A^\mu$ and a second term $A^\mu A_\mu\phi^2$. The J_μ can be interpreted as a current as we will se in the next section when we study the Maxwell Lagrangian. The term $A^\mu A_\mu\phi^2$ is certainly interesting as well, it has the appearance of a mass term due to the squares. It is terms like this that, very simplified, through the Higgs mechanism can make scalar fields give mass to otherwise massless fields such as A^μ [14]. While this certainly is intriguing the discussion of the Higgs field lies outside of the scope of this thesis.

We have thus made our theory invariant under local phase transformations, but that also led us to introduce a new field A_μ. Because of how the field arose we sometimes refer to it as a *connection*. Of course, we didn't pick the name A^μ by accident, this is exactly the four-potential we studied earlier. However, we are not truly ready to make that claim. It is natural to look for more terms to add to the Lagrangian. Especially interesting would be a term containing only A^μ. This term would then describe the dynamics of the field A^μ in the absence of ϕ. We should also demand that this term is invariant under the local phase transformation, otherwise it would be pointless to introduce A_μ from the very beginning. Since we know that the covariant derivative contains the field A^μ and transforms in the correct way, we can study the commutator between two covariant derivatives:

$$[D_\mu,D_\nu]\phi = \Big(\underbrace{[\partial_\mu,\partial_\nu]}_{0} + i[\partial_\mu,A_\nu] + i[A_\mu,\partial_\nu] - \underbrace{[A_\mu,A_\nu]}_{0}\Big)\phi = 2i\partial_{[\mu}A_{\nu]}\phi = iF_{\mu\nu}\phi \ ,$$

where we have once again found the field strength $F^{\mu\nu}$. This is of course the tensor introduced in the previous section, but it and A^μ has now been recovered using only symmetry arguments. We have yet to even mention the electric and magnetic field. Notice that because we know how the covariant derivative transforms we must have

$$[D_\mu,D_\nu]\phi \mapsto ([D_\mu,D_\nu]\phi)' = e^{-\alpha(x)}[D_\mu,D_\nu]\phi \ .$$

Using this we immediately see that $F_{\mu\nu}$ must transform as

$$F_{\mu\nu} \mapsto (F_{\mu\nu})' = e^{-\alpha(x)}F_{\mu\nu}e^{\alpha(x)} = F_{\mu\nu} \ ,$$

thus our tensor $F_{\mu\nu}$ is invariant under the transformation and is a perfect candidate to use for constructing an additional term in the Lagrangian. Since we want to add a scalar term to our Lagrangian we contract $F_{\mu\nu}$ with itself before we add it to Lagrangian. We thus have

$$\mathcal{L} = \mathcal{L}_{KG} + J_\mu A^\mu + A^\mu A_\mu\phi^2 - \frac{1}{4}F_{\mu\nu}F^{\mu\nu} \ ,$$

the specific choice of sign and factor will be explained shortly. While we have so far only studied the Klein-Gordon Lagrangian the form above is rather general and in it we have found what is often referred to as the electromagnetic Lagrangian, defined as

$$\mathcal{L}_{EM} = -\frac{1}{4}F_{\mu\nu}F^{\mu\nu} + J_\mu A^\mu \ .$$

We have dropped the terms $\mathcal{L}_{KG}$ and $A^\mu A_\mu \phi^2$ because they are results of the special case of a scalar field and does not in general affect the dynamics of an electromagnetic field.

2.3 The electromagnetic Lagrangian

In the previous section we derived a Lagrangian containing a field A^μ and a tensor $F^{\mu\nu}$. As mentioned these quantities are the four-potential and field strength of Maxwell's theory of electromagnetism. But to see this we need to find the equations of motion for these fields. In order to do so we will study the electromagnetic Lagrangian as

$$\mathcal{L}_{EM} = -\frac{1}{4}F_{\mu\nu}F^{\mu\nu} + J_\mu A^\mu \ . \tag{2.10}$$

Let us first make a quick remark about sign convention. In analytical mechanics, i.e in a non-field theory, we define the Lagrangian as the difference between the kinetic energy and the potential energy, $L = T - V$. Using the anti-symmetry of the field strength we may write

$$-\frac{1}{4}F_{\mu\nu}F^{\mu\nu} = -\frac{1}{2}F_{\mu\nu}\partial^{[\mu}A^{\nu]} = -\frac{1}{2}F_{\mu\nu}\partial^\mu A^\nu = -\frac{1}{2}(\partial_\mu A_\nu - \partial_\nu A_\mu)\partial^\mu A^\nu \ .$$

Let us look at the particular terms with $\mu = 0$ and $\nu = i \in \{1,2,3\}$:

$$-\frac{1}{2}(\partial_0 A_i - \partial_i A_0)\partial^0 A^i = \frac{1}{2}(\partial_0 A_i - \partial_i A_0)\partial_0 A_i = \frac{1}{2}(\partial_0 A_i)^2 - \frac{1}{2}\partial_i A_0 \partial_0 A_i \ .$$

As we associate ∂_0 with $\frac{\partial}{\partial t}$ we see that the first term is a kinetic term with the correct sign. This motivates the sign convention used in (2.10).

We now proceed to find the equations of motion. Lagrangian mechanics tells us that to find the equations of motion we need to find the stationary point of the action

$$\mathcal{S}[A] = \int \mathrm{d}^4x(-\frac{1}{4}F_{\mu\nu}F^{\mu\nu} + J_\mu A^\mu) \ ,$$

The action is as usual stationary if a small perturbation leaves the action invariant to first order, $\delta S[A] = 0$. Notice that we now treat A^μ as our variable instead of x^μ. To find the stationary point we want to study a small variation δA^μ. Varying the two contracted field strengths gives

$$\delta\Big(\frac{1}{4}F_{\mu\nu}F^{\mu\nu}\Big) = \frac{1}{2}F_{\mu\nu}\delta F^{\mu\nu} = F_{\mu\nu}\partial^{[\mu}\delta A^{\nu]} = F_{\mu\nu}\partial^\mu \delta A^\nu = \partial^\mu(F_{\mu\nu}\delta A^\nu) - (\partial^\mu F_{\mu\nu})\delta A^\nu \ .$$

The variation of the coupling term is much easier:

$$\delta(J_\mu A^\mu) = J_\mu \delta A^\mu \ .$$

We can now compute the variation of the action:

$$\delta\mathcal{S}[A] = \int \mathrm{d}^4x\Big(- \partial^\mu(F_{\mu\nu}\delta A^\nu) + (\partial^\mu F_{\mu\nu} + J_\nu)\delta A^\nu\Big) = \underbrace{[F_{\mu\nu}\delta A^\nu]}_{0} + \int \mathrm{d}^4x\Big(\partial^\mu F_{\mu\nu} + J_\nu\Big)\delta A^\nu = 0 \ .$$

Here we have used the fact that we have a total derivative and that we demand the variation to vanish identically on the boundary. Now since δA is arbitrary the other term must be zero over all space. We also note that the Bianchi Identity holds due to the definition of $F^{\mu\nu}$. Thus we recover Maxwell's equations

$$\partial_\mu F^{\nu\mu} = J^\nu \ ,$$
$$\epsilon_{\mu\nu\sigma\rho}\partial^\nu F^{\sigma\rho} = 0 \ .$$

In the next chapter we will continue to discuss gauge theories and we will make use of differential forms. It is also possible to express Maxwell's equations using differential forms. This is done in Appendix C.

Chapter 3

Non-Abelian Gauge Theories

We begin this chapter by extending the symmetry approach in the previous chapter to non-abelian symmetry groups, i.e. groups with elements that do not commute. This approach is intimately connected with *group theory*, which we give an introduction to in Appendix E. We then introduce two different approaches for constructing the Lagrangian of the theory, called *Yang-Mills gauge theory* and *Chern-Simons gauge theory*, respectively. Yang-Mills theory is formulated in terms of the *Hodge dual operator*, that depends on the metric of the space in which the theory is formulated. Chern-Simons theory is a limiting case of Yang-Mills theory that is formulated in a coordinate independent fashion, making it compatible with general relativity.

To formulate Chern-Simons gauge theory we need to use the formalism of *differential forms*, which is covered in Appendix C. For instructive purposes the transition to differential forms is performed in section 3.1, as we can begin the section on familiar ground, and accompany the transition with helpful examples.

3.1 Yang-Mills gauge theory

The generalization of Maxwell's equations to a non-abelian gauge symmetry was first performed in 1954 by Chen-Ning Yang (1922-) and Robert Mills (1927-1999). This section details this generalization, and culminates in the Yang-Mills equations of motion. At first Yang-Mills theories were criticized, because the quanta of a Yang-Mills field are required to be massless to preserve gauge invariance. With the proposition of the Higgs-mechanism in the 1960s Yang-Mills theory gained traction, and *the Standard Model* is now understood as a Yang-Mills theory of the gauge group $SU(2) \times U(1) \times SU(3)$.

This theory will be formulated in terms of a matrix-valued transformation g acting on an abstract vector field ϕ. The choice of the $U(1)$ symmetry of electromagnetism is motivated by the fact that the probability density of the electron is independent of the phase of its wavefunction. In a similar fashion, observing that the neutron and proton are very similar in the absence of electromagnetic interaction, it was concluded that the observables of proton-neutron system are invariant under a $SU(2)$ transformation acting on a two-vector containing their respective wavefunctions. The quantized Yang-Mills theory of $SU(2)$ describes the weak interaction.

As we saw in the previous chapter, the equations of motion for a system can be completely derived from its Lagrangian. Because of this, we construct a gauge theory by demanding symmetries of the Lagrangian, and the equations of motion that follow will inherit them. To begin with, we do not write down an explicit Lagrangian, instead we assume it to be invariant under some global group of transformations g. We can write a general local transformation as

$$\phi(x) \to g(x)\phi(x) = e^{\alpha(x)^a T_a}\phi(x) \ , \tag{3.1}$$

following the notation of Appendix E, where the T_a are the Lie algebra-valued generators of the transformation group g. Because the transformation is local, partial derivatives acting on the transformed scalar field will produce an additional term per group generator in g by the product rule:

$$\partial_\mu \phi(x) \to \partial_\mu(e^{\alpha^a T_a}\phi(x)) = e^{\alpha^a T_a}(\phi(x)\partial_\mu \alpha(x)^a T_a + \partial_\mu \phi(x)) \ . \tag{3.2}$$

Just like in the case of Maxwell's equations we now need to find a covariant derivative, D_μ, that acts directly on the field, even after a transformation. That is, we want D_μ to fulfill the relation:

$$D_\mu \phi(x) \to D'_\mu(e^{\alpha^a T_a}\phi(x)) = e^{\alpha^a T_a}(D_\mu \phi(x)) \ . \tag{3.3}$$

Looking at the invariance-breaking term $\partial_\mu \alpha(x)^a T_a$ in equation (3.2) we note that it is Lie algebra-valued because $\partial_\mu \alpha(x)^a$ is just a scalar. This implies that our correction term must also be Lie algebra-valued. We define

$$D_\mu = (\partial_\mu + A^a_\mu T_a) \ ,$$

where A^a_μ is a collection of new gauge fields analogous to the *connection* field found in the previous section. Inserting this D_μ into (3.3) we obtain:

$$e^{\alpha^a T_a}(\partial_\mu + (\partial_\mu \alpha(x)^a)T_a + A'^a_\mu T_a)\phi(x) = e^{\alpha^a T_a}(\partial_\mu + A^a_\mu T_a)\phi(x) \ .$$

For the equality to hold we see that A^a_μ must transform as

$$A^a_\mu \to A'^a_\mu = A^a_\mu - \partial_\mu \alpha(x)^a \ .$$

Making a callback once again to Maxwell's equations we form the commutator of the covariant derivatives:

$$[D_\mu, D_\nu]\phi = 2(\frac{1}{2}\underbrace{[\partial_\mu,\partial_\nu]}_{0} + \partial_{[\mu}A^a_{\nu]}T_a + \underbrace{A^a_{[\nu}\partial_{\mu]}T_a + A^a_{[\mu}\partial_{\nu]}T_a}_{0} + \frac{1}{2}[A^a_\mu T_a, A^b_\nu T_b])\phi \ ,$$

where the second and third terms come from the product rule of derivatives. We are left with terms two and five, and the only difference to Maxwell's equations is that the fifth term doesn't cancel out because the group generators do not commute. We are left with

$$[D_\mu, D_\nu] = \partial_\mu A^a_\nu T_a - \partial_\nu A^a_\mu T_a + A^b_\mu A^c_\nu f^a{}_{bc} T_a = F^a_{\mu\nu} T_a \ , \tag{3.4}$$

where $f^a{}_{bc}$ is the structure factor belonging to the symmetry group of our gauge theory. We define this commutator as the field strength tensor, $F^a_{\mu\nu}T_a$. Contracting the a's allows for the cleaner notation $F_{\mu\nu}$, but the reader must stay aware that F is Lie algebra-valued.

Because the field strength is the commutator of covariant derivatives it has the same transformation properties when acting on ϕ. This lets us determine the transformation of the field strength tensor:

$$F'_{\mu\nu} g = g F_{\mu\nu} \implies F'_{\mu\nu} = g F_{\mu\nu} g^{-1} \ .$$

Unlike the case with the $U(1)$ symmetry g does not necessarily commute with $F^a_{\mu\nu}$, so our field strength tensor is not gauge invariant. The field strength tensor not being an invariant might raise some worries that we cannot use it to construct a gauge invariant Lagrangian. We can, however, pick some particular matrix representation of the group g and form the trace of both sides:

$$\mathrm{tr}\,[F_{\mu\nu}] \to \mathrm{tr}\left[g F_{\mu\nu} g^{-1}\right] = \mathrm{tr}\,[F_{\mu\nu}] \ ,$$

where the last identity comes from the cyclicity of the trace. By applying the trace to the field strength tensor we have obtained an invariant property which can be used for the Lagrangian.

For the next few chapters we will be considering only compact groups, which means that the basis of their Lie algebra can be rescaled in such a way that $\mathrm{tr}\,[T_a T_b] = \delta_{ab}$. We provide no general proof for this, but instructive examples can be obtained from appendices E.5.1 and E.5.3. What this means, in practice, is that all terms of $\mathrm{tr}\,[F_{\mu\nu}F^{\mu\nu}]$ will have the same sign so that they can all correspond to kinetic energy. This allows us to construct a meaningful Lagrangian:

$$\mathcal{L} = -\frac{1}{4}\,\mathrm{tr}\,[F_{\mu\nu}F^{\mu\nu}] \ .$$

To successfully make this theory a generalization of Maxwell's theory of electromagnetism we need to include the *Noether current*, J. This term was found for the special case of Maxwell's equations in section (2.2). It is expressed as $J_\mu A^\mu$, defined as

$$J_\mu A^\mu = \frac{1}{2}\overline{D_\mu \phi} D_\mu \phi - \frac{1}{2}\overline{\partial_\mu \phi}\partial_\mu \phi - \frac{1}{2}A_\mu A^\mu |\phi|^2 = \frac{1}{2}((\partial_\mu \overline{\phi})T^a \phi + T^a \overline{\phi}\partial_\mu \phi)A^\mu_a \ , \tag{3.5}$$

and can be seen as the coupling term between the connection and the derivative of the scalar field. For this rewriting to work we note that the generators T must be attached to the current, J_μ, instead of the connection field A^μ. That is, the Noether current is Lie algebra-valued, and the generators of the algebra correspond to the preserved charges of the gauge theory. In the case of the quantum weak interaction, the three generators of $SU(2)$ roughly correspond to the $W^\pm$ and Z^0 bosons respectively.

Just like the field strength was no longer a gauge invariant, the gauge current is not invariant. The solution to constructing a gauge invariant current term is exactly the same as for the field strength; we need only form the trace of $J_\mu A^\mu$ over the generators of the gauge group. At this point the Yang-Mills Lagrangian is

$$\mathcal{L}_{YM} = \mathrm{tr}\,[-\frac{1}{4}F_{\mu\nu}F^{\mu\nu} + J_\mu A^\mu] \ .$$

What we have arrived at is a generalization of the Maxwell Lagrangian, called the *Yang-Mills Lagrangian*. To finish our preparations for finding the equations of motion for the Yang-Mills Lagrangian we will make the transition to the language of differential forms. We first express the relevant objects of our theory as differential forms:

$$A = A_\mu dx^\mu \ , \qquad F = \frac{1}{2} F_{\mu\nu} dx^\mu \wedge dx^\nu \ , \qquad J = J_\mu dx^\mu \ .$$

We can also express F as a function of A according to:

$$\begin{aligned} F &= \mathrm{d}A + A \wedge A \\ &= \partial_{[\mu} A_{\nu]} dx^\mu \wedge dx^\nu + A_{[\mu} A_{\nu]} dx^\mu \wedge dx^\nu \\ &= \left(\partial_{[\mu} A_{\nu]} + A_{[\mu} A_{\nu]} \right) dx^\mu \wedge dx^\nu \ , \end{aligned}$$

where a quick comparison of the term in parentheses with (3.4) shows that what we have is really $\frac{1}{2} F_{\mu\nu}$, because of the normalization factor in the symmetrization brackets.

Recalling Maxwell's equations (2.7), we see that for an abelian connection the Bianchi identity can be formulated in differential forms as:

$$\epsilon^{\mu\nu\sigma\rho} \partial_\nu F_{\sigma\rho} d^4x = \partial_{[\nu} F_{\rho\sigma]} dx^\mu \wedge dx^\nu \wedge dx^\sigma \wedge dx^\rho \Rightarrow \mathrm{d}F = \mathrm{dd}A \equiv 0 \ ,$$

by the definition of the exterior derivative. We would like to generalize this identity to the non-abelian case by introducing the exterior covariant derivative. Using the graded commutator defined in (C.8), when acting on Lie algebra-valued forms, ω, of order p, the covariant exterior derivative is

$$\mathrm{D}\omega = \mathrm{d}\omega + [A, \omega] = \mathrm{d}\omega + A \wedge \omega - (-1)^p \omega \wedge A \ . \tag{3.6}$$

Letting D act on F we obtain the Yang-Mills generalization of the Bianchi identity,

$$\begin{aligned} \mathrm{D}F &= \mathrm{d}(\mathrm{d}A + A \wedge A) + [A, (\mathrm{d}A + A \wedge A)] \\ &= dA \wedge A - A \wedge dA + [A, dA] + [A, A \wedge A] \\ &= dA \wedge A - A \wedge dA + A \wedge dA - dA \wedge A + [A, A \wedge A] \\ &= [A, A \wedge A] = 0 \ . \end{aligned} \tag{3.7}$$

Finally, the Yang-Mills Lagrangian, $\mathrm{tr}\,[-\frac{1}{4} F_{\mu\nu} F^{\mu\nu} + J_\mu A^\mu]$, is expressed in terms of differential forms as

$$\mathcal{L}_{YM} = \mathrm{tr}\,[\frac{1}{2} F \wedge *F - A \wedge *J] \ .$$

To make clear that this expression is indeed correct we evaluate it term by term.

$$\begin{aligned} F \wedge *F &= \left(\frac{1}{2} F_{\mu\nu} dx^\mu \wedge dx^\nu \right) \wedge \left(\frac{1}{4} F^{\rho\sigma} \epsilon_{\rho\sigma\alpha\beta} dx^\alpha \wedge dx^\beta \right) \\ &= \frac{1}{8} F_{\mu\nu} F^{\rho\sigma} \epsilon_{\rho\sigma\alpha\beta} dx^\mu \wedge dx^\nu \wedge dx^\alpha \wedge dx^\beta \\ &= \frac{1}{8} F_{\mu\nu} F^{\rho\sigma} \epsilon_{\rho\sigma\alpha\beta} \varepsilon^{\mu\nu\alpha\beta} d^4x \quad = -\frac{1}{2} F_{\mu\nu} F^{\rho\sigma} \delta^{\mu\nu}_{\rho\sigma} \sqrt{-g} d^4x \\ &= -\frac{1}{2} F_{\mu\nu} F^{\mu\nu} \sqrt{-g} d^4x \ , \end{aligned}$$

where $\sqrt{-g} d^4x$ is the integration measure, as defined in Appendix C.

For the term $A \wedge *J$ we have, using the same identities

$$A \wedge *J = A_\mu dx^\mu \wedge \left(\frac{1}{6} J^\alpha \epsilon_{\alpha\beta\rho\sigma} dx^\beta \wedge dx^\rho \wedge dx^\sigma \right) = -A_\mu J^\mu \sqrt{-g} d^4x \ .$$

At this point we are ready to formulate the Yang-Mills action in terms of differential forms:

$$\mathcal{S}_{YM} = \int \mathrm{tr}\,[\frac{1}{2} F \wedge *F - A \wedge *J] \ \sqrt{-g} d^4x. \tag{3.8}$$

The equations of motion are found by demanding the principle of least action, i.e. $\delta\mathcal{S} = 0$. We thus vary the action with respect to the connection A:

$$\delta\mathcal{S}_{YM} = \int \mathrm{tr}\,[\frac{1}{2}\delta F \wedge *F + \frac{1}{2}F \wedge \delta(*F) - \delta A \wedge *J]\sqrt{-g}d^4x = \int \mathrm{tr}\,[\delta F \wedge *F - \delta A \wedge *J]\,\sqrt{-g}d^4x,$$

where we used the property (C.11) in Appendix C. First, we vary F:

$$\delta F = \mathrm{d}\delta A + \delta A \wedge A + A \wedge \delta A = \mathrm{d}\delta A + [A, \delta A] = \mathrm{D}\delta A \tag{3.9}$$

where it was used that δA is a one-form. Inserting this result into the action gives us the equation

$$\mathrm{D}\delta A \wedge *F + F \wedge *\mathrm{D}\delta A - \delta A \wedge *J = 0\ . \tag{3.10}$$

Using the fact that we are dealing with an expression inside an integral we can write

$$\mathrm{D}\delta A \wedge *F = \mathrm{D}(\delta A \wedge *F) + \delta A \wedge \mathrm{D} * F = \delta A \wedge \mathrm{D} * F\ ,$$

where we used the fact that the $\mathrm{D}(\cdot \wedge \cdot)$-terms are vanishing boundary terms. Inserted into equation (3.10) this gives us

$$\delta A \wedge (\mathrm{D} * F - *J) = 0 \implies \mathrm{D} * F = *J\ , \tag{3.11}$$

which together with the Bianchi identity

$$\mathrm{D}F = 0 \tag{3.12}$$

constitutes the Yang-Mills equations of motion in differential form.

Assuming the background metric to be Minkowski in 3+1 dimensions, and picking *U(1)* as the gauge group of the theory, one obtains Maxwell's equations from (3.11) and the Bianchi identity (3.12). As a final rematrk we note that the calculations here were carried out under the assumption of a 4-dimensional background space. However, the definition of the Hodge dual as well as the contraction of the ϵ-symbols will cancel out, yielding the same equations of motion regardless of the dimension of the background space.

3.2 Chern-Simons gauge theory

A big problem in formulating a grand unified theory of quantum physics and gravity lies in a fundamental incompatibility between the Yang-Mills action and Einstein's theory of general relativity. As will be made clear in section 4 and 5, general relativity is formulated in such a way that the metric tensor, $g_{\mu\nu}$, is not postulated, it has to be a solution to Einstein's equations. In contrast, the Yang-Mills action

$$\mathcal{S}_{YM} = \int \mathrm{tr}\,[F \wedge *F - A \wedge *J]\ ,$$

depends on the Hodge dual operator $(*)$ that in turn depends on the metric of the underlying space.[1]

To construct a gauge theory that is not dependent on the background metric we can formulate the Yang-Mills Lagrangian without the current term J^μ, and drop the Hodge dual operator. Equality with the Yang-Mills Lagrangian then holds if F is *self-dual* ($F = *F$) and the current J^μ is 0. This approach, which we will detail here is called *Chern-Simons gauge theory* and is named after physicists Shiing-Shen Chern (1911-2004) and James Harris Simons (1938-).

In $2j$ dimensions we can try a Lagrangian, called the *j:th Chern character*

$$\mathcal{L}_C = \frac{1}{j!}\,\mathrm{tr}\,[F \wedge F \wedge ... \wedge F] = \mathrm{tr}\,[F^j]\ ,$$

where F appears j times in the wedge product. Since $\mathrm{tr}[F]$ is gauge invariant this Lagrangian is also gauge invariant. However, we will find that this Lagrangian will yield no proper equations of motion because all possible A are stationary points of the Lagrangian. From the previous section we have the result $\delta F = \mathrm{D}\delta A$, see (3.9), where "D" is the covariant exterior derivative. Applying this result to the second Chern character we have

$$\delta\,\mathrm{tr}\,[F \wedge F] = \mathrm{tr}\,[\delta F \wedge F + F \wedge \delta F] = 2\,\mathrm{tr}\,[\mathrm{D}\delta A \wedge F]\ ,$$

[1] The discussion here closely follows the one in "*Gauge Fields, Knots and Gravity*" pg. 279 onwards [15].

where the cyclicity and linearity of the trace was used. The term $\mathrm{D}\delta A \wedge F$ is actually when inserted into the action,

$$\mathrm{D}\delta A \wedge F = \mathrm{D}(\delta A \wedge F) - \delta A \wedge \mathrm{D}F = 0 \ .$$

The first term on the right side is a vanishing boundary term while the second term is $\mathrm{D}F = 0$ by the Bianchi identity. This means we cannot obtain proper equations of motion since the equations will just read $0 = 0$. This result turns out to hold for a general Chern-character.

To obtain proper equations of motion some more work is needed. The Chern n-form can be locally expressed as the exterior derivative of a $(2n-1)$-form Ω, because all closed forms fulfilling $\mathrm{d}F^n = 0$ are locally *exact* by the Poincaré lemma. Checking this for $\mathrm{tr}[F^n]$ we obtain

$$\mathrm{d}\,\mathrm{tr}\,[F^n] = n\,\mathrm{tr}\,[F^{n-1} \wedge \mathrm{d}F] = n\,\mathrm{tr}\,[F^{m-1} \wedge (\mathrm{d}\,\mathrm{d}A + \mathrm{d}A \wedge A - A \wedge \mathrm{d}A)] = 0 \qquad \Longrightarrow \ \mathrm{tr}[F^n] = \mathrm{d}\Omega \ ,$$

where we in the last step used the cyclicity of the trace.

An action using $\mathrm{tr}\,[\Omega]$ as a Lagrangian does in fact generate proper equations of motion which we will show explicitly for the second Chern character. Some statements from the previous paragraph will also be made a lot more clear by way of the example.

To find the equations of motion we need to find Ω such that $\mathrm{d}\Omega = F \wedge F$. Assuming the symmetry group G is simply connected, we can let $A_s = sA, F_s = s\mathrm{d}A + s^2 A \wedge A$ and use that parametrisation to find Ω. This method works as follows:

$$\begin{aligned}\mathrm{tr}\,[F \wedge F] &= \int_0^1 \frac{\mathrm{d}}{\mathrm{d}s}\,\mathrm{tr}\,[F_s \wedge F_s]\mathrm{d}s = 2\int_0^1 \mathrm{tr}\,[\frac{\mathrm{d}F_s}{\mathrm{d}s} \wedge F_s]\mathrm{d}s \\ &= 2\mathrm{d}\int_0^1 \mathrm{tr}\,[A \wedge F_s]\mathrm{d}s = 2\mathrm{d}\int_0^1 \mathrm{tr}\,[sA \wedge \mathrm{d}A + s^2 A \wedge A \wedge A]\mathrm{d}s \\ &= \mathrm{d}\,\mathrm{tr}\,[A \wedge \mathrm{d}A + \frac{2}{3}A \wedge A \wedge A] \ . \end{aligned} \tag{3.13}$$

Thus, the three-form $\mathrm{tr}\,[A \wedge \mathrm{d}A + \frac{2}{3}A \wedge A \wedge A]$ can be related to $\mathrm{tr}\,[F \wedge F]$ as a boundary term that appears when when $F \wedge F$ is integrated across the manifold $[0,1] \times M$ where M is some three-dimensional manifold. This means that what we have obtained is the self-dual sourceless limit of a Yang-Mills theory, formulated in terms of the boundary. We have, from Stokes' theorem (C.13), the equality

$$\int_{[0,1]\times M} \mathrm{tr}\,[F \wedge F] = \int_{1\times M} \mathrm{tr}\,[A \wedge \mathrm{d}A + \frac{2}{3}A \wedge A \wedge A] - \int_{0\times M} 0 \ ,$$

where the right-hand side is the sum of two integrals at the boundaries 0 and 1 of $[0,1] \times M$ as parametrized in equation (3.13). We can finally introduce the *Chern-Simons action* as follows:

$$\mathcal{S}_{CS}[A] = \frac{1}{2}\int_M \mathrm{tr}\,[A \wedge \mathrm{d}A + \frac{2}{3}A \wedge A \wedge A] \ .$$

This action should have the reader a little worried, as it depends not on the gauge invariant F, but instead on the gauge-variant connection A. It can be shown that under a general gauge transform the action always changes by an integer multiple of $8\pi^2$. This action is often rescaled by a factor $\frac{k}{4\pi}$, so that this change turns into an integer multiple of 2π, turning the expression $\exp\left(\frac{i}{k}S_{CS}\right)$ gauge invariant. In this thesis we will use this rescaled Chern-Simons action and treat it as gauge invariant:

$$\mathcal{S}_{CS}[A] = \frac{k}{4\pi}\int_M \mathrm{tr}\,[A \wedge \mathrm{d}A + \frac{2}{3}A \wedge A \wedge A] \ . \tag{3.14}$$

The equations of motion for the Chern-Simons action can be obtained in the usual way by varying A:

$$\delta\mathcal{S}_{CS} = \frac{1}{2}\int_M \delta\,\mathrm{tr}\,[A \wedge \mathrm{d}A + \frac{2}{3}A \wedge A \wedge A] = \int_M \mathrm{tr}\,[(\mathrm{d}A + A \wedge A) \wedge \delta A] = \int_M \mathrm{tr}\,[F \wedge \delta A] \ ,$$

which vanishes for all variations if and only if $F = 0$. It might seem trivial that the field stength tensor F is identically zero, however in a classical setting this action can be related to the Einstein-Hilbert action in Cartan formalism. This relation is achieved by coupling the gauge connections A_μ to the vielbeins ${e_\mu}^a$ and spin connections ${\omega_\mu}^a$ of Einstein-Cartan theory that will be introduced in the next chapter.

Chapter 4

General Relativity in Cartan Formalism

The most accurate theory as of today for describing gravity at a macroscopic level is Einstein's theory of general relativity. This is a theory describing how the presence of matter and energy affects the curvature of spacetime, which in turn is the cause of gravity. As such it is a highly geometrical theory and its mathematical foundation is differential geometry. A brief and non-rigorous treatment of the most important concepts of differential geometry is given in Appendix C. Because the metric is a fundamental quantity in general relativity, we provide a reminder in Appendix B. Since our goal is to develop a theory of gravity as a Chern-Simons gauge theory, we need to adapt Cartan formalism to general relativity[1]. The Cartan formalism, developed by the French mathematician Élie Cartan (1869-1951), is not the standard approach to general relativity. Besides being necessary for describing gravity as a Chern-Simons gauge theory, the Cartan formalism has its advantages; it facilitates a lot of calculations and allows spinors to be incorporated into the theory. However, this comes at a cost of the theory being somewhat more conceptually difficult.

This chapter is devoted to developing the theory of general relativity in Cartan formalism. We start off by discussing some general features of the mathematical framework of general relativity and then proceed with developing the necessary mathematical tools for describing the theory. We introduce Cartan's structure equations and Einstein's field equations, the latter partly by considering an action formulation of the theory. Finally we present the Einstein-Hilbert action in Cartan formalism.

4.1 The mathematical framework of general relativity

In the theory of general relativity, spacetime is modeled as a differential manifold with a Lorentzian metric. Physical quantities are described by scalars, vectors, more general tensors or spinors[2]. We will equip the manifold with a set of smooth coordinates x^μ.

The manifold will in general be curved and a subtle issue arises regarding how to compare vectors, or more general tensors, at different points on the manifold. In particular, we might like to be able to add, subtract and perform various other operations involving tensors at different points on the manifold. In ordinary flat Euclidean space there is no problem here since we can simply move a vector from one point to another while keeping the vector constant. Thus, we can for example add two vectors by moving one of the vectors to the other, keeping it constant, and then performing the addition. However, when the manifold is curved there is no longer any unambiguous way of comparing vectors defined on different points on the manifold. One intuitive way of visualizing this is to consider transporting a geometrical vector on a two-dimensional sphere embedded in a three-dimensional space, see Figure 4.1. We take the vector to be a tangential vector to the sphere. Consider transporting the vector along the path on the surface of the sphere indicated by the dotted line, starting and ending at the north pole, while keeping the vector tangential to the surface and without rotating it. This is known as a *parallel transport* of the vector. From the figure we see that the vector, after completing the loop once, does not point in the same direction as it initially did. Parallel transporting a vector from a given point to another on the sphere, will in general depend on the path taken. Furthermore, the parallel transport around a closed loop will effectively rotate the vector by some angle. This is an inherent feature of curved spaces. Clearly, the three-dimensional embedding space is merely a tool for us to visualize the sphere and is not necessary to perform the parallel transport and reach the same conclusions.

Since transporting a vector from one point to another on our curved manifold will depend on the path taken, and since there is no preferred path to choose, we can not find a meaningful way of comparing vectors at different points on the manifold, i.e. at different tangent spaces.

[1] For a more complete treatment of general relativity see for example [16] or [12].

[2] We will not be considering spinors in this thesis.

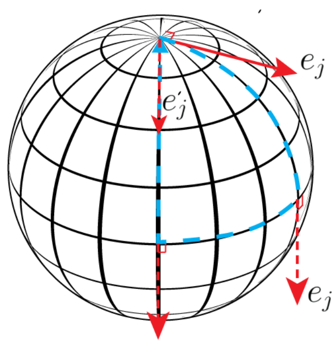


Figure 4.1: Parallel transport of a vector on a sphere. The vector maintains its orientation relative to the surface of the sphere throughout the transportation process. As is apparent, a vector is not preserved when transported around a closed loop on the sphere.

Having shed some light on this phenomenon of curved manifolds, we now proceed with developing some fundamental quantities needed to formulate the theory of general relativity in Cartan formalism.

4.2 Frame fields, connections and the covariant derivative

Our chosen coordinates x^μ on the manifold induces a natural and local covariant basis set ∂_μ and a local set of basis one-forms dx^μ. These are basis sets for the tangent space and cotangent space at a given point on the manifold, respectively. One of the key concepts of the Cartan formalism, as applied to general relativity, is the fact that the manifold describing spacetime has a Lorentzian metric, and as such we can always transform to a local Lorentz frame of flat spacetime. Thus we can form a local orthonormal covariant basis at each point in the manifold by

$$\tilde{e}_a(x) = \tilde{e}_a{}^\mu \partial_\mu \ , \tag{4.1}$$

and a local orthogonal basis one-forms at each point on the manifold by

$$e^a(x) = e_\mu{}^a dx^\mu \ . \tag{4.2}$$

These local orthonormal bases are called frame fields. The components $\tilde{e}_a{}^\mu$ and $e_\mu{}^a$ each form a $n \times n$ matrix, which in the classical theory of general relativity is invertible. The entire object $e_\mu{}^a$ is called vielbein[3]. Their inverses will be denoted by switching the upper index with the lower, i.e. $\tilde{e}_\mu{}^a$ and $e_a{}^\mu$ are the inverses of $\tilde{e}_a{}^\mu$ and $e_\mu{}^a$, respectively. This means they satisfy $\tilde{e}_a{}^\mu \tilde{e}_\nu{}^a = e_\nu{}^a e_a{}^\mu = \delta^\mu_\nu$.

We will use the convention that Greek indices indicate an object in the curved coordinate basis ∂_μ or dx^μ and Latin indices indicate an object in a flat orthonormal coordinate basis. Appropriately, the Greek indices will be referred to as curved indices and the Latin indices as flat indices.

The local set of basis one-forms $e^a(x)$ may be chosen to be compatible with the local covariant basis set $\tilde{e}_a(x)$, in a sense that

$$< e^a(x) \ , \tilde{e}_b(x) >= \delta^a_b \ .$$

where $< . \, , . >$ denotes a properly defined scalar product on the manifold. However, we will not need to go any deeper into this issue nor define such a scalar product. The important observation is that we can choose the local set of basis one-forms to be compatible with the local coordinate basis, and as a direct consequence $\tilde{e}_a{}^\mu$ becomes the inverse of $e_\mu{}^a$, and we can omit the tilde on the matrix components $\tilde{e}_a{}^\mu$ of (4.1). However, we should keep the tilde on the local orthonormal covariant basis $\tilde{e}_a(x)$ to separate it from $e_a(x)$, which is the local orthogonal set of basis one-forms with a lowered flat index. Indeed, these two objects can not be the same as $\tilde{e}_a(x)$ is not even a differential form.

The vielbeins are simply the transformation matrices going from curved indices to flat indices, i.e. the vielbeins can be used to convert between curved indices and flat indices of a vector, or a more general tensor for that matter. Suppose we have a contravariant vector, V^μ, with a curved index. Then we can use the vielbein $e_\mu{}^a$ to express the vector with a flat index,

$$V^a = e_\mu{}^a V^\mu \ .$$

[3]Vielbein is German for many legs.

The generalisation to any tensor having a mixture of curved and flat, upper and lower indices, should be clear.

Since the metric in the local flat orthonormal basis is just the usual Minkowski metric, η_{ab}, we can write the general curved metric in terms of the vielbeins as

$$g_{\mu\nu}(x) = e_\mu{}^a(x) e_\nu{}^b(x) \eta_{ab} \ . \tag{4.3}$$

Here we have explicitly indicated the dependence on the coordinates of the general metric $g_{\mu\nu}$ and the vielbeins. This explicit coordinate dependence will often be omitted for brevity. Analogously the flat metric can be expressed in terms of the general metric as

$$\eta_{ab} = e_a{}^\mu e_b{}^\nu g_{\mu\nu} \ . \tag{4.4}$$

The metrics $g_{\mu\nu}$ and η_{ab} can be used to raise or lower curved indices and flat indices, respectively. In conclusion, tensors with curved indices transform differently than tensors with flat indices. More precisely, a tensor with curved indices transforms tensorially under a general coordinate transformation $x^\mu \to x'^\mu$, while a tensor with flat indices transforms tensorially under local Lorentz transformations $x^a \to x'^a = \Lambda^a{}_b x^b$. Tensors with mixed indices transforms accordingly in its respective type of indices, and a curved index does not transform under a local Lorentz transformation and similarly for a flat index under a general coordinate transformation. For example, a rank-2 tensor with one lower and one upper curved index transforms as

$$T'_\mu{}^\nu = \frac{\partial x^\rho}{\partial x'^\mu} \frac{\partial x'^\nu}{\partial x^\sigma} T_\rho{}^\sigma \ , \tag{4.5}$$

under a general coordinate transformation, while a similar tensor with flat indices transforms as

$$T'_a{}^b = \Lambda_a{}^c \Lambda_d{}^b T_c{}^d \ , \tag{4.6}$$

under local Lorentz transformations.

The general metric tensor $g_{\mu\nu}$ can be written in terms of the spacetime interval ds^2 and the local basis set dx^μ by the defining relation

$$ds^2 = g_{\mu\nu} dx^\mu dx^\nu \ . \tag{4.7}$$

Substituting the relation between the general metric tensor and the Minkowski metric tensor given by (4.3) and using the definition of the frame field (4.2), we arrive at a particularly useful relation between the spacetime interval and the frame field,

$$ds^2 = \eta_{ab} e^a e^b = e^a e_a \ . \tag{4.8}$$

This relation provides a simple way of determining the vielbeins of a given metric when expressed in terms of the spacetime interval.

If we take the determinant of each side of (4.2) we get

$$\det(g_{\mu\nu}) = \det(e_\mu{}^a e_\nu{}^b \eta_{ab}) = \det(e_\mu{}^a)\det(e_\nu{}^b)\det(\eta_{ab}) = -e^2 \ .$$

Here we have introduced the notation $e = \det(e_\mu{}^a)$ as well as the fact that $\det(\eta_{ab}) = -1$. Writing $g = \det(g_{\mu\nu})$ we can restate the relation above as

$$\sqrt{-g} = e \ . \tag{4.9}$$

Note that since the metric is Lorentzian, its determinant is negative, hence we need to insert a minus sign under the square root above.

Since we need a way of relating vectors at different points on a curved manifold we need to introduce a connection between the local tangent spaces as well as the local flat Minkowski spaces. For this purpose we construct the covariant derivative operator as the partial derivative operator plus a linear transformation[4]. We start by forming a covariant derivative operating on vector fields with curved indices as

$$D_\mu V^\nu = \partial_\mu V^\nu + \Gamma^\nu_{\mu\rho} V^\rho \ . \tag{4.10}$$

The second term is a way of compensating for the fact that our space may be curved and it is called a connection term. It is a linear transformation of the vector V and $\Gamma^\nu_{\mu\rho}$ are the connection coefficients.

[4]If we require our covariant derivative operator to obey the Leibniz product rule on tensor products, then we can always write our covariant derivative as a partial derivative plus a linear transformation. We will not prove this statement.

We now require that the object $D_\mu V^\nu$ be invariant under general smooth coordinate transformations, so that it constitutes a proper tensor. Let us therefore apply a general coordinate transformation, $x^\mu \to x'^\mu$, to the above expression;

$$\begin{aligned} D_\mu V^\nu \longrightarrow (D_\mu V^\nu)' &= \left(\frac{\partial x^\rho}{\partial x'^\mu}\frac{\partial}{\partial x^\rho}\right)\frac{\partial x'^\nu}{\partial x^\sigma}V^\sigma + (\Gamma^\nu_{\mu\rho}V^\rho)' \\ &= \frac{\partial x^\rho}{\partial x'^\mu}\frac{\partial x'^\nu}{\partial x^\sigma}\partial_\rho V^\sigma + \frac{\partial x^\rho}{\partial x'^\mu}\frac{\partial^2 x'^\nu}{\partial x^\rho \partial x^\sigma}V^\sigma + (\Gamma^\nu_{\mu\rho}V^\rho)' \ . \end{aligned} \tag{4.11}$$

Since we are dealing with general coordinate transformations, the second order derivative in the second term in the last line of the expression above is not necessarily zero. Thus we need the connection term in the definition of the covariant derivative to cancel this second order derivative in order to make the covariant derivative transform covariantly. Hence, we must impose a certain transformational property on the object $\Gamma^\nu_{\mu\rho}$. We define this object to transform as

$$\Gamma^\nu_{\mu\rho} \longrightarrow \Gamma'^\nu_{\mu\rho} = \frac{\partial x^\sigma}{\partial x'^\mu}\frac{\partial x'^\nu}{\partial x^\lambda}\frac{\partial x^\tau}{\partial x'^\rho}\Gamma^\lambda_{\sigma\tau} - \frac{\partial x^\sigma}{\partial x'^\mu}\frac{\partial x^\tau}{\partial x'^\rho}\frac{\partial^2 x'^\nu}{\partial x^\sigma \partial x^\tau} \ , \tag{4.12}$$

under a general coordinate transformation. This object is the *Christoffel symbol*[5], and the corresponding connection is sometimes referred to as the *affine connection.* It is not a proper tensor as it does not transform as such. However, the defined transformation property of the Christoffel symbol does indeed make the object $D_\mu V^\nu$ transform as a tensor, which can be verified by inserting (4.12) into the last line of (4.11). The result is

$$\begin{aligned} D_\mu V^\nu \longrightarrow (D_\mu V^\nu)' &= \frac{\partial x^\rho}{\partial x'^\mu}\frac{\partial x'^\nu}{\partial x^\sigma}\partial_\rho V^\sigma + \frac{\partial x^\rho}{\partial x'^\mu}\frac{\partial^2 x'^\nu}{\partial x^\rho \partial x^\sigma}V^\sigma + \frac{\partial x^\rho}{\partial x'^\mu}\frac{\partial x'^\nu}{\partial x^\sigma}\frac{\partial x^\lambda}{\partial x'^\rho}\Gamma^\sigma_{\rho\lambda}\frac{\partial x'^\rho}{\partial x^\tau}V^\tau \\ &\quad - \frac{\partial x^\sigma}{\partial x'^\mu}\frac{\partial x^\lambda}{\partial x'^\rho}\frac{\partial^2 x'^\nu}{\partial x^\sigma \partial x^\lambda}\frac{\partial x'^\rho}{\partial x^\tau}V^\tau \\ &= \frac{\partial x^\rho}{\partial x'^\mu}\frac{\partial x'^\nu}{\partial x^\sigma}\partial_\rho V^\sigma + \frac{\partial x^\rho}{\partial x'^\mu}\frac{\partial x'^\nu}{\partial x^\sigma}\Gamma^\sigma_{\rho\lambda}V^\lambda \\ &= \frac{\partial x^\rho}{\partial x'^\mu}\frac{\partial x'^\nu}{\partial x^\sigma}D_\rho V^\sigma \ . \end{aligned}$$

Note that we have written the Christoffel symbol with the upper index right above the first lower index. Since it is not a tensor there is no meaningful way of raising or lowering any of its indices.

The covariant derivative operating on a covariant vector (or a one -form) can analogously be expressed as a partial derivative plus a connection term,

$$D_\mu V_\nu = \partial_\mu V_\nu + \tilde{\Gamma}^\rho_{\mu\nu}V_\rho \ .$$

In general the connection coefficients $\tilde{\Gamma}^\rho_{\mu\nu}$ do not need to be related to the earlier connection coefficients $\Gamma^\nu{}_{\mu\rho}$, although they obviously must satisfy the same transformation property (4.12). By requiring that the covariant derivative commutes with contraction and reduces to the partial derivative when operating on scalars, it is straightforward to show that the connection coefficients must be related through $\tilde{\Gamma}^\rho_{\mu\nu} = -\Gamma^\rho_{\mu\nu}$. The expression for the covariant derivative acting on a covariant vector with a curved index is then given by

$$D_\mu V_\nu = \partial_\mu V_\nu - \Gamma^\rho_{\mu\nu}V_\rho \ . \tag{4.13}$$

We now proceed with developing the covariant derivative acting on tensors with flat indices. The procedure is entirely analogous to the way we formed a covariant derivative acting on tensors with curved indices. Acting on a vector with a contravariant flat index, we take the covariant derivative to be the partial derivative plus a connection term,

$$D_\mu V^a = \partial_\mu V^a + \omega_\mu{}^a{}_b V^b \ . \tag{4.14}$$

The object $\omega_\mu{}^a{}_b$ is called a *spin connection* and can be regarded as a linear transformation matrix for every value of the index μ. These are clearly seen to play the same role as the Christoffel symbol $\Gamma^\nu_{\mu\rho}$ does for the case of curved indices. Requiring that the covariant derivative should transform as a tensor under

[5] The Christoffel symbol is named after the German mathematician and physicist Elwin Bruno Christoffel (1829-1900), one of many who made great contribution to the field of differential geometry.

a local Lorentz transformation, the spin connection must transform in a similar way as the Christoffel symbol, see equation (4.12). Demanding that the covariant derivative commutes with contraction and reduces to the partial derivative when operating on scalars, the expression for the covariant derivative acting on a vector with a lower flat index is

$$D_\mu V_a = \partial_\mu V_a - \omega_\mu{}^b{}_a V_b \ . \tag{4.15}$$

The covariant derivative as defined above can be generalized to act on more general tensors of arbitrarily many indices, upper and lower, curved and flat. As an illustrative example we write the explicit expression for the covariant derivative acting on a tensor with two upper and two lower indices, one curved and one flat each,

$$D_\mu T^{\rho a}{}_{\nu b} = \partial_\mu T^{\rho a}{}_{\nu b} + \Gamma^\rho_{\mu\sigma} T^{\sigma a}{}_{\nu b} - \Gamma^\sigma_{\mu\nu} T^{\rho a}{}_{\sigma b} + \omega_\mu{}^a{}_c T^{\rho c}{}_{\nu b} - \omega_\mu{}^c{}_b T^{\rho a}{}_{\nu c} \ . \tag{4.16}$$

Hopefully, the generalization to more general tensors is clear. For each index of a tensor a corresponding connection term is added to the covariant derivative of the tensor.

We can establish a relation between the Christoffel symbol, the spin connection and the vielbeins. This is done by considering the covariant derivative of a vector expressed in the curved coordinate basis as well as the local orthonormal basis. Exploiting the fact that a vector is an invariant object, independent of the coordinates we choose to represent it in, we can then equate the different expressions. For this purpose we make use of the tensor product explicitly, see Appendix A for a brief treatment of tensors and the tensor product. The covariant derivative of a vector V in a curved coordinate basis may be written as

$$DV = (D_\mu V^\nu) dx^\mu \otimes \partial_\nu = (\partial_\mu V^\nu + \Gamma^\nu_{\mu\rho} V^\rho) dx^\mu \otimes \partial_\nu \ . \tag{4.17}$$

In a local orthonormal basis, the covariant derivative of V may be expressed as

$$\begin{aligned} DV &= (D_\mu V^a) dx^\mu \otimes \tilde{e}_a \\ &= (\partial_\mu V^a + \omega_\mu{}^a{}_b V^b) dx^\mu \otimes \tilde{e}_a \\ &= (\partial_\mu (e_\nu{}^a V^\nu) + \omega_\mu{}^a{}_b e_\nu{}^b V^\nu) dx^\mu \otimes e_a{}^\rho \partial_\rho \\ &= e_a{}^\rho (e_\nu{}^a \partial_\mu V^\nu + V^\nu \partial_\mu e_\nu{}^a + e_\nu{}^b \omega_\mu{}^a{}_b V^\nu) dx^\mu \otimes \partial_\rho \\ &= (\delta^\rho_\nu \partial_\mu V^\nu + (e_a{}^\rho \partial_\mu e_\nu{}^a) V^\nu + e_a{}^\rho e_\nu{}^b \omega_\mu{}^a{}_b V^\nu) dx^\mu \otimes \partial_\rho \\ &= (\partial_\mu V^\nu + (e_a{}^\nu \partial_\mu e_\rho{}^a) V^\rho + e_a{}^\nu e_\rho{}^b \omega_\mu{}^a{}_b V^\rho) dx^\mu \otimes \partial_\nu \ . \end{aligned} \tag{4.18}$$

Comparing (4.17) with (4.18), we find the following expression for the Christoffel symbol in terms of the vielbeins and the spin connection:

$$\Gamma^\nu_{\mu\rho} = e_a{}^\nu \partial_\mu e_\rho{}^a + e_a{}^\nu e_\rho{}^b \omega_\mu{}^a{}_b \ . \tag{4.19}$$

Equivalently, we can write this relation as an expression of the spin connection in terms of the vielbeins and the Christoffel symbol,

$$\omega_\mu{}^a{}_b = e_b{}^\rho e_\nu{}^a \Gamma^\nu_{\mu\rho} - e_b{}^\rho \partial_\mu e_\rho{}^a \ . \tag{4.20}$$

Here we simply multiplied (4.2) by $e^\rho{}_b e_\nu{}^a$ and rearranged terms. If we instead multiply by $e_\nu{}^a$ and rearrange terms we find

$$0 = \partial_\mu e_\rho{}^a + \omega_\mu{}^a{}_b e_\rho{}^b - \Gamma^\nu_{\mu\rho} e_\nu{}^a = D_\mu e_\rho{}^a \ . \tag{4.21}$$

The vanishing of the covariant derivative of the vielbein is known as the *tetrad postulate*. However, by the way we defined the frame fields and the covariant derivative, the vanishing of the covariant derivative of the vielbein came as a consequence and we did not need to postulate this at all.

4.3 Cartan's structure equations

A more classical treatment of general relativity might go on to define the torsion tensor as twice the antisymmetric part of the affine connection, and the Riemann curvature tensor describing the curvature of the space from the commutator of covariant derivative in the affine connection, as suggested by the concept of parallell transport. Such a treatment is presented in Appendix G and some other useful results are developed in the process. However, here we will present an alternative description of the theory of general relativity, a formalism that utilize the frame fields developed at the beginning of this chapter.

This is known as the *Cartan formalism.* As opposed to the rather cumbersome calculations that are necessary in order to determine the curvature of space from its metric in the ordinary formulation of general relativity, the Cartan formalism provides a fairly easy way of accomplishing this. This is due to *the Cartan structure equations*,

$$T^a = \mathrm{d}e^a + \epsilon^a{}_{bc}\omega^b \wedge e^c \ , \tag{4.22}$$

$$R^a = \mathrm{d}\omega^a + \frac{1}{2}\epsilon^a{}_{bc}\omega^b \wedge \omega^c \ . \tag{4.23}$$

where T^a is the torsion two-form, R^a is the curvature two-form and ω^b is the spin connection one-form expressed with one flat index,

$$\omega^a = \frac{1}{2}\epsilon^{abc}\omega_{\mu bc}dx^\mu \ . \tag{4.24}$$

The Cartan structure equations can be seen as defining equations for the torsion and curvature of spacetime. Physically, the torsion is a measure of the anti-symmetry of the connection on our space. To establish a unique connection on our manifold, we must require our connection to be torsion-free, meaning that $T^a = 0$. The first of the Cartan structure equations above then essentially reduces to a differential equation for determining the spin connection from the frame field,

$$\mathrm{d}e^a + \epsilon^a{}_{bc}\omega^b \wedge e^c = 0 \ . \tag{4.25}$$

Now we have a way of determining the curvature from the metric tensor; first the frame field can be determined from the metric by (4.8), then we can solve (4.25) for the spin connection and finally we solve (4.23) for the curvature. At this point a more physical description of curvature is in order.

The curvature tensor measures how the space locally deviates from a flat Minkowski space. It can be defined as the commutator of the covariant derivative. The Riemann curvature tensor $R^a{}_{b\mu\nu}$ is defined in this way by

$$[D_\mu, D_\nu]V^a = R^a{}_{b\mu\nu}V^b \ , \tag{4.26}$$

where V^a is an arbitrary (differentiable) vector. By evaluating how the commutator of the covariant derivative acts on V^a it is possible to derive an explicit formula for the Riemann curvature tensor in terms of the connections, and also a number of symmetry properties. However, we will not be needing such an expression for the Riemann curvature tensor when performing any calculations later on. Therefore we simply refer the interested reader to Appendix G where such derivations are performed for the affine connection.

As a final note, the relation between the curvature two-form and the Riemann curvature tensor is

$$R^a = \frac{1}{2}\epsilon^{abc}R_{bc\mu\nu}dx^\mu \wedge dx^\nu = \frac{1}{2}\epsilon^{abc}R_{bcde}e^d \wedge e^e \ , \tag{4.27}$$

and defining the curvature by (4.26) is indeed equivalent to defining the curvature by (4.23), although we will not prove this, however, compare (4.23) with (G.3).

The Cartan structure equations (4.22) and (4.23) describe the torsion and curvature of spacetime, respectively, in terms of the frame field and spin connection. However, they do not give any information about the physical sources which generate the curvature of spacetime. The theory of general relativity tells us how gravitating sources curve spacetime, and we have yet to give a more precise description of this. This description is given by the Einstein field equations. Before deriving these equations we need to define the Ricci tensor and the Ricci scalar.

The Ricci tensor R_{ab} is defined by contracting two of the indices of the Riemann curvature tensor,

$$R_{ab} = R^c{}_{acb} \ . \tag{4.28}$$

Some authors define the Ricci tensor by contracting the first index with the last index, and this differs from our definition only by a sign. The contraction of the first with the second index of the Riemann curvature tensor is identically zero since it is anti-symmetric in these two indices. Furthermore, the Ricci tensor is symmetric in its two indices, a consequence of the symmetry properties of the Riemann curvature tensor, see Appendix G.

The Ricci scalar is formed by contracting the two indices of the Ricci tensor,

$$R = R^a{}_a = \eta^{ab}R_{ab} \ , \tag{4.29}$$

i.e. it is the trace of the Ricci tensor, which in turn is a kind of trace of the Riemann curvature tensor. As such, the Ricci tensor contains less information about the curvature than the Riemann curvature tensor, and the Ricci scalar contains less information than the Ricci tensor.

4.4 Einstein's field equations and the stress-energy tensor

Having developed the formalism of general relativity we now turn our attention to the Einstein field equations, the Eimstein hilbert action and the stress-energy tensor.

We will be considering the specific case of 2+1 dimensions, but most of the results to be derived are, however, easily generalized to higher dimensions. In order to derive the equations we will once again turn to the principle of stationary action. Let us first consider the most simple case, a space without matter, i.e. vacuum space. It is important to understand that the vacuum field equations are fully compatible with the notion of matter, but they only describe the space where there is none. For example, if we put a lot of mass in one place we will eventually create a black hole, but to describe this black hole we only need the vacuum equations as long as we are not interested in the point where all the mass is concentrated.

To find the vacuum equations we need an action and a Lagrangian. Now, a Lagrangian is by definition a scalar and we have already come across a scalar associated with the curvature of space, the Ricci scalar R. We consider the action:

$$\mathcal{S}_{EH} = \frac{1}{2\kappa}\int d^3x\sqrt{-g}(R-2\Lambda) \ . \tag{4.30}$$

This is the *Einstein-Hilbert action*, originally proposed by Hilbert.[6] Here $\kappa = 8\pi G c^{-4}$ is Einstein's constant, G is Newtons gravitational constant, c the speed of light in vacuum, g the determinant of the metric, i.e. $g = \det(g_{\mu\nu})$ and Λ the *cosmological constant.* The cosmological constant was not present in the original action but was later introduced by Einstein in order to achieve a theoretical model which described a static universe. However, Einstein later abandoned the idea of a static universe for the concept of an expanding universe, which physicists of today agree on. Nevertheless, the cosmological constant is believed to describe the energy density of vacuum space and is yet of great importance, nowadays mostly in the field of dark energy [17]. Furthermore, the *Anti-de Sitter*, or AdS, space will be of great importance when considering black hole and wormhole solutions, as well as for formulating a theory of gravity as a Chern-Simons gauge theory. The AdS space is defined as a spacetime with a constant negative curvature and therefore we need to account for both terms in our Einstein-Hilbert action.

To find the equations of motion we have to vary the action with respect to the metric. In order to do this we make use of the following two identities:

$$\delta\sqrt{-g} = -\frac{1}{2}\sqrt{-g}g_{\mu\nu}\delta g^{\mu\nu} \ , \tag{4.31}$$

$$\delta R = \delta g^{\mu\nu}R_{\mu\nu} + D^2 g_{\mu\nu}\delta g^{\mu\nu} - D_\mu D_\nu \delta g^{\mu\nu} \ . \tag{4.32}$$

Proof of these identities can be found in Appendix H. Using these we may now vary the action

$$\delta\mathcal{S}_{EH} = \frac{1}{2\kappa}\int d^3x\Big(\delta\sqrt{-g}(R-2\Lambda) + \sqrt{-g}\delta R\Big) = \frac{1}{2\kappa}\int d^3x\sqrt{-g}\Big(\frac{-g_{\mu\nu}}{2}(R-2\Lambda) + R_{\mu\nu}\Big)\delta g^{\mu\nu} + \underbrace{\frac{1}{2\kappa}\int d^3x\sqrt{-g}(D^2 g_{\mu\nu} - D_\mu D_\nu)\delta g^{\mu\nu}}_{0} \ , \tag{4.33}$$

where the last term is a vanishing boundary term. Since we demand that $\delta\mathcal{S}_{EH} = 0$ for all variations $\delta g^{\mu\nu}$ the quantity in the parenthesis must vanish identically. We have thus found Einstein's vacuum equations:

$$G_{\mu\nu} + \Lambda g_{\mu\nu} = R_{\mu\nu} - \frac{1}{2}R g_{\mu\nu} + \Lambda g_{\mu\nu} = 0 \ , \tag{4.34}$$

where we have introduced *the Einstein tensor*, $G_{\mu\nu}$. The Einstein tensor is a symmetric tensor since both the Ricci tensor and the metric tensor are symmetric. Furthermore, it is divergence free: $D^\mu G_{\mu\nu} = 0$, which follows from the fact that the Riemann curvature tensor satisfies the Bianchi identity, see Appendix G.

These equations may be simplified even further. To do this we take the trace of the equations. Remembering that $\mathrm{tr}[g_{\mu\nu}] = g_{\mu\nu}g^{\mu\nu} = 3$ since we are working in 2+1 dimensions,

$$\mathrm{tr}[R_{\mu\nu} - \frac{R}{2}g_{\mu\nu}] = -\,\mathrm{tr}[\Lambda g_{\mu\nu}] \implies R = 6\Lambda \ ,$$

[6]There were several disputes between Hilbert and Einstein concerning the development of the field equations. No doubt, Einstein got most of the glory.

where we used the fact that $\mathrm{tr}\,[R_{\mu\nu}] = R^{\mu}{}_{\mu} = R$. Substituting this into the vacuum equations we find

$$R_{\mu\nu} = 2\Lambda g_{\mu\nu} \ . \tag{4.35}$$

We are now ready to add the effect of matter. To do this we add a term to the action so that our full action reads $\mathcal{S} = \mathcal{S}_{EH} + \mathcal{S}_{\text{matter}}$. To find the equations of motion we have to vary $\mathcal{S}_{\text{matter}} = \int d^3x\sqrt{-g}\mathcal{L}_{\text{matter}}$ and we write this variation as

$$\delta\mathcal{S}_{\text{matter}} = \int d^3x\sqrt{-g}\frac{\delta\left(\sqrt{-g}\mathcal{L}_{\text{matter}}\right)}{\delta g^{\mu\nu}}\frac{\delta g^{\mu\nu}}{\sqrt{-g}} \ . \tag{4.36}$$

The expression $\frac{\delta(\sqrt{-g}\mathcal{L}_{\text{matter}})}{\delta g^{\mu\nu}}$ is to be interpreted as a functional derivative. If we now require that $\delta\mathcal{S} = \delta\mathcal{S}_{EH} + \delta\mathcal{S}_{\text{matter}} = 0$ for all variations we find:

$$\frac{1}{2\kappa}(G_{\mu\nu} + \Lambda g_{\mu\nu}) = -\frac{1}{\sqrt{-g}}\frac{\delta(\mathcal{L}_{\text{matter}})}{\delta g^{\mu\nu}} \ ,$$

where we have used (4.33) and (4.36). This equation can be written in a more sophisticated way by defining the *stress-energy tensor* as[7]

$$T_{\mu\nu} \equiv \frac{-2}{\sqrt{-g}}\frac{\delta S_{\text{matter}}}{\delta g^{\mu\nu}} \ . \tag{4.37}$$

We may then write

$$G_{\mu\nu} + \Lambda g_{\mu\nu} = \kappa T_{\mu\nu} \ . \tag{4.38}$$

These are Einstein's famous field equations that tells us how the distribution of matter and energy curves spacetime.

Since the stress-energy tensor might be an unfamiliar concept we will discuss its interpretation and give an example in the familiar setting of electromagnetism. The diagonal elements of the stress-energy tensor will be of particular interest to us, especially T^{00}. Formally we can think of $T^{\mu\nu}$ as the flux of momentum p^{μ} through a surface. Since we know from the theory of special relativity that p^0 can be interpreted as the energy we will think of T^{00} as the *energy density* of a system. The stress-energy tensor also satisfies the conservation law $D_{\mu}T^{\mu\nu} = 0$. This of course reduces to the familiar equation of continuity in flat space time. For more details and a proof of the conservation law, we refer the reader to Carroll [16].

Let us return to the electromagnetic field tensor and derive the stress-energy tensor in a curved space. The electromagnetic Lagrangian in flat space is $\mathcal{L} = -\frac{1}{4}F^{\mu\nu}F_{\mu\nu} = -\frac{1}{4}\eta^{\mu\rho}\eta^{\nu\sigma}F_{\rho\sigma}F_{\mu\nu}$ as discussed in section 2.3. The prescription to promote a flat action to an action in a curved spacetime is straightforward. First we replace all the $\eta^{\mu\nu}$ with the general metric tensor $g^{\mu\nu}$ and then we add a factor of $\sqrt{-g}$ to make an invariant volume measure $d^3x\sqrt{-g}$. Explicitly, we find

$$\mathcal{S}_{EM} = -\frac{1}{4}\int d^4x\sqrt{-g}g^{\mu\rho}g^{\nu\sigma}F_{\rho\sigma}F_{\mu\nu} \ .$$

To vary the action we need the identity (4.31) as well as

$$\delta g^{\rho\sigma} = -g^{\rho\mu}g^{\nu\sigma}\delta g_{\mu\nu} \ . \tag{4.39}$$

The proof of these may be found in Appendix H. Using this we now calculate the variation of the action:

$$\begin{aligned}\delta\mathcal{S}_{\text{Maxwell}} &= -\frac{\sqrt{-g}}{4}\Big(\frac{1}{2}g^{\mu\nu}g^{\alpha\beta}g^{\gamma\lambda}F_{\alpha\gamma}F_{\beta\lambda} - g^{\alpha\mu}g^{\nu\beta}g^{\gamma\lambda}F_{\alpha\gamma}F_{\beta\lambda} - g^{\alpha\beta}g^{\gamma\mu}g^{\nu\lambda}F_{\alpha\gamma}F_{\beta\lambda}\Big)\delta g_{\mu\nu} \\ &= -\frac{\sqrt{-g}}{4}\Big(\frac{1}{2}g^{\mu\nu}F^{\alpha\beta}F_{\alpha\beta} - F^{\mu\lambda}F^{\nu}{}_{\lambda} - F^{\beta\mu}F_{\beta}{}^{\nu}\Big)\delta g_{\mu\nu} = \frac{\sqrt{-g}}{2}F^{\mu\alpha}F^{\nu}{}_{\alpha} - \frac{\sqrt{-g}}{8}g^{\mu}F^{\alpha\beta}F_{\alpha\beta} \ ,\end{aligned}$$

and we may now compute the stress-energy tensor from our definition (4.37):

$$T^{\mu\nu}_{EM} = F^{\mu\alpha}F^{\nu}{}_{\alpha} - \frac{1}{4}g^{\mu\nu}F^{\alpha\beta}F_{\alpha\beta} \ . \tag{4.40}$$

[7] The stress-energy tensor is also commonly referred to as the *energy-momentum tensor*.

Notice the change in signs by the definition depending on if we consider $T^{\mu\nu}$ or $T_{\mu\nu}$. Let us now investigate the 00 component of the stress tensor in 3+1 dimensional flat space, i.e. $g^{00} = \eta^{00} = -1$,

$$T^{00}_{EM} = F^{0\alpha}F^{0}{}_{\alpha} + \frac{1}{4}F^{\alpha\beta}F_{\alpha\beta} \ .$$

We saw in the chapter on electromagnetism that $F^{0\alpha} = \mathbf{E}^{\alpha}$, and it thus follows that $F^{0\alpha}F^{0}{}_{\alpha} = \mathbf{E}\cdot\mathbf{E} = E^2$. To evaluate $F^{\alpha\beta}F_{\alpha\beta}$ we note that this is the trace of a matrix-multiplication between the matrices $F^{\alpha\beta}$ and $-F^{\beta\alpha}$. Using the explicit matrix representation 2.5, we find that

$$F^{\alpha\beta}F_{\alpha\beta} = -E^2 + (B_z^2 + B_y^2 - E_x^2) + (B_z^2 + B_x^2 - E_y^2) + (B_y^2 + B_x^2 - E_z^2) = 2(B^2 - E^2) \ ,$$

and we thus conclude that

$$T^{00}_{EM} = \frac{1}{2}(E^2 + B^2) \ ,$$

which may be familiar as the energy density of the electromagnetic field. We see that the interpretation of the 00-component of the stress-energy tensor as an energy density makes sense in the electromagnetic case.

4.5 Einstein-Hilbert action in Cartan formalism

In the last section we discussed the Einstein-Hilbert action, from which we derived Einstein's field equations. Armed with knowledge of Cartan's formulation of general relativity we are now ready to translate the action in terms of vielbeins and spin connections, i.e. $\mathcal{S}[g_{\mu\nu}] \to \mathcal{S}[e,\omega]$. and derive the corresponding equations of motion, the analogue to Einstein's equations. For convenience we provide the Einstein-Hilbert action once again:

$$\mathcal{S}_{EH}[g_{\mu\nu}] = \frac{1}{2\kappa}\int (R - 2\Lambda) e d^3x \ , \tag{4.41}$$

where we have used the relation (4.9) to express the determinant of the metric in terms of the determinant of the veilbeins, denoted by e.

For reasons which will soon become apparent we consider the quantity $e^a \wedge R_a$. Using (4.27) we find

$$e^a \wedge R_a = \frac{1}{2}\epsilon_{abc} e^a \wedge R^{bc} \ .$$

Now we expand the one-form e^a and the two-form R^{bc} in terms of the basis one-forms dx^μ:

$$\frac{1}{2}\epsilon_{abc}e^a \wedge R^{bc} = \frac{1}{4}\epsilon_{abc}e^a{}_\rho e^b{}_\alpha e^c{}_\beta R^{\alpha\beta}{}_{\mu\nu} dx^\mu \wedge dx^\nu \wedge dx^\rho \ . \tag{4.42}$$

The quantity $dx^\mu \wedge dx^\nu \wedge dx^\rho$ is related to the three-dimensional volume measure d^3x as follows:

$$dx^\mu \wedge dx^\nu \wedge dx^\rho = \varepsilon^{\mu\nu\rho} d^3x \ ,$$

as stated in equation (C.12). Inserting this identity into equation (4.42) yields

$$\frac{1}{4}\epsilon_{abc}\varepsilon^{\mu\nu\rho}e^a{}_\rho e^b{}_\alpha e^c{}_\beta R^{\alpha\beta}{}_{\mu\nu} d^3x \ . \tag{4.43}$$

Now we consider the factor $\epsilon_{abc}\varepsilon^{\mu\nu\rho}e^a{}_\rho e^b{}_\alpha e^c{}_\beta$. We find:

$$\epsilon_{abc}\varepsilon^{\mu\nu\rho}e^a{}_\rho e^b{}_\alpha e^c{}_\beta = -e(\delta^\mu{}_\alpha \delta^\nu{}_\beta - \delta^\nu{}_\alpha \delta^\mu{}_\beta) \ .$$

Plugging this into equation (4.43) leading to

$$-\frac{1}{4}(\delta^\mu{}_\alpha \delta^\nu{}_\beta - \delta^\nu{}_\alpha \delta^\mu{}_\beta) R^{\alpha\beta}{}_{\mu\nu} e d^3x = -\frac{1}{4}e(R^{\mu\nu}{}_{\mu\nu} - R^{\nu\mu}{}_{\mu\nu})d^3x = -\frac{eR}{2}d^3x \ ,$$

where we used the fact that $R^{\nu\mu}{}_{\mu\nu} = -R^{\mu\nu}{}_{\mu\nu} = -R$, in the last step. We have thus found that $e^a \wedge R_a = -\frac{eR}{2}$ and can rewrite the first term in our Einstein-Hilbert action in equation (4.41). The term involving the cosmological constant can be found as well. Consider the quantity $\Lambda\epsilon_{abc}e^a \wedge e^b \wedge e^c$. If we rewrite the quantity by expand it in terms of its basis vectors, the calculation is straightforward:

$$\Lambda\epsilon_{abc}e^a \wedge e^b \wedge e^c = \epsilon_{abc}\Lambda e^a{}_\mu e^b{}_\nu e^c{}_\rho dx^\mu \wedge dx^\nu \wedge dx^\rho = e\epsilon_{abc}\epsilon^{abc}\Lambda d^3x \ ,$$

where we once again used the relationship between the wedge product and volume element. Finally we make use of the identity $\epsilon_{abc}\epsilon^{abc} = -6$ where a,b and c are Lorentzian flat indices:

$$e\epsilon_{abc}\epsilon^{abc}\Lambda d^3x = -6e\Lambda d^3x \ .$$

Thus we can rewrite the Einstein-Hilbert action in absence of matter as follows using Cartan formalism:

$$\mathcal{S}_{EH}[e,\omega] = -\frac{1}{2\kappa}\int \left[2e^a \wedge (\mathrm{d}\omega_a + \frac{1}{2}\epsilon_{abc}\omega^b \wedge \omega^c) - \frac{1}{3}\Lambda\epsilon_{abc}e^a \wedge e^b \wedge e^c\right] \ . \tag{4.44}$$

Now we conclude that we have found an equivalent action such that $\mathcal{S}_{EH}[g_{\mu\nu}] = \mathcal{S}_{EH}[e,\omega]$. As usual when having found the proper action we turn our attention to the equations of motion. These are conventionally obtained from the principle of least action, i.e. by computing the variation and setting $\delta\mathcal{S}_{EH} = 0$. However, in this case considering the Cartan form of $\mathcal{S}_{EH}$ it is possible to do the variation on both the spin connection fields ω and the frame fields connections e. This will yield two different equations of motion. As a result of the equivalence between the Cartan action and the metric action the equations of motion will of course be independent of the fashion they are derived. The Cartan equations can therefore be translated into the original formulation and vice versa. Taking the variation of the action with respect to the frame fields we deduce

$$\delta\mathcal{S}_{EH} = \int 2\delta e^a \wedge (\mathrm{d}\omega_a + \frac{1}{2}\epsilon_{abc}\omega^b \wedge \omega^c) - \frac{1}{3}\Lambda\epsilon_{abc}\left[\delta e^a \wedge e^b \wedge e^c + e^a \wedge \delta e^b \wedge e^c + e^a \wedge e^b \wedge \delta e^c\right] = 0 \ .$$

The first and second term can be rewritten and we get

$$\delta\mathcal{S}_{EH}[e] = \int (\mathrm{d}\omega_a + \frac{1}{2}\epsilon_{abc}\omega^b \wedge \omega^c) \wedge 2\delta e^a - \Lambda\epsilon_{abc}e^b \wedge e^c \wedge \delta e^a = 0 \ ,$$

resulting in the following equations of motion

$$\mathrm{d}\omega_a + \frac{1}{2}\epsilon_{abc}\omega^b \wedge \omega^c = \frac{\Lambda}{2}\epsilon_{abc}e^b \wedge e^c \implies R_a = \frac{\Lambda}{2}\epsilon_{abc}e^b \wedge e^c \ , \tag{4.45}$$

where we used the identity $\mathrm{d}\omega_a + \frac{1}{2}\epsilon_{abc}\omega^b \wedge \omega^c = R_a$. Continuing to the variation of the spin connections ω^a we have

$$\delta\mathcal{S}_{EH}[\omega] = \int e^a \wedge (\delta\,\mathrm{d}\omega_a + \frac{1}{2}\epsilon_{abc}(\delta\omega^b \wedge \omega^c + \omega^b \wedge \delta\omega^c)) \ .$$

We may rewrite the action further by using the fact that total derivatives are vanishing boundary terms when performing the action integral, i.e.:

$$\mathrm{d}(e^a \wedge \delta\omega_a) = de^a \wedge \delta\omega_a - e^a \wedge \mathrm{d}\delta\omega_a = 0 \ ,$$

leading to $\mathrm{d}e^a \wedge \delta\omega_a = e^a \wedge \mathrm{d}\delta\omega_a$ and the action

$$\delta\mathcal{S}_{EH}[\omega] = \int \mathrm{d}e^a \wedge \delta\omega_a + \frac{1}{2}\epsilon_{abc}e^a \wedge (\delta\omega^b \wedge \omega^c + \omega^b \wedge \delta\omega^c) = \int (\mathrm{d}e^c + \frac{1}{2}\epsilon^c{}_{ab}e^a \wedge \omega^b)\delta\omega_c \ .$$

Thus we have found our second equation of motion as follows

$$\mathrm{d}e^c + \frac{1}{2}\epsilon^c{}_{ab}e^a \wedge \omega^b = 0 \ , \tag{4.46}$$

which is precisely Cartan's first structure equation with vanishing torsion, i.e. $T^c = 0$, see (4.22). Having derived Einstein's equations and recast them into the language of Cartan we are now ready to find the solutions of the equations. This will be the main subject for next chapter.

Chapter 5

Solutions of Einstein's Field Equations

In the previous chapter we found Einstein's equations to which there are no known general solution. Exact solutions to these equations only exist under simplifying assumptions such as radial symmetry or very specific time dependencies [7].

We begin this chapter with a short discussion of Minkowski and Anti-de Sitter spacetime. These are what is known as maximally symmetric spacetimes, meaning that the geometry of spacetime looks identical from every point. We will see that more complicated solutions often approach these spacetimes asymptotically.

We then discuss the first exact non-trivial solution to the Einstein field equations, namely the Schwarzschild black hole. The Schwarzschild black hole contains geometrical peculiarities such as an *event horizon* and a *singularity*. Remarkably, we find that the Schwarzschild solution can be interpreted as a wormhole solution, meaning that it connects two separate regions of spacetime. This discovery spurred the interest in *traversable* wormholes. Since we want to connect gravity to gauge theory we primarily study solutions in 2+1 dimensions, leading us to investigate the *BTZ black hole* and eventually the traversable *Morris-Thorne wormhole*. We study properties of the Morris-Thorne solution and find that we are required to introduce *exotic matter*, i.e. matter with negative energy density, in order to achieve a traversable wormhole. To find a theory that admits a traversable wormhole solution without a necessity for exotic matter we express gravity as a Chern-Simons gauge theory, and then generalize it by extending the symmetry group while preserving the original symmetry. This is the main subject of chapters 6 and 7 respectively.

5.1 Minkowski and Anti-de Sitter space

Einstein's equivalence principle states that the world is locally Minkowski, i.e. that it is described by the Minkowski metric $\eta_{\mu\nu} = \mathrm{diag}(-1,1,1,1)$. This better then be a solution to the vacuum equations (4.46) with $\Lambda = 0$. To verify this we note that the curvature two-form R_a is proportional to the derivative of the vielbein. Since the vielbeins are constant for the Minkowski-metric the curvature two-form must vanish and the vacuum equations are satisfied. However, if $\Lambda \neq 0$ Minkowski spacetime is no longer a solution and we need to find an analogue. This space is known as de Sitter space, for $\Lambda > 0$, or Anti-de Sitter space for $\Lambda < 0$. Both of these spaces have a constant curvature, positive for de Sitter and negative for Anti-de Sitter. This follows immediately from equation (4.35). They are also, just as Minkowski spacetime, maximally symmetric. This implies that we cannot distinguish any point in spacetime from another. We will in this thesis primarily discuss Anti-de Sitter space and will provide a short description below, a more thorough discussion can be found in Appendix F.2.

As mentioned earlier we will work primarily in 2+1 dimensions and hence our main interest is three-dimensional Anti-de Sitter space, AdS_3 for short. AdS_3 is a spacetime with two spatial dimensions and one time-like. However, its structure is most easily written down as a hypersurface in an embedding four-dimensional space with two spatial dimensions and two time-like. The embedding space has a line element

$$ds^2 = -dV^2 - dU^2 + dX^2 + dY^2 \ ,$$

where U, V, X and Y are coordinates. In this embedding space AdS_3 is the hypersurface satisfying

$$-V^2 - U^2 + X^2 + Y^2 = -l^2 \ , \tag{5.1}$$

where l is a length related to the cosmological constant by $\Lambda = -\frac{1}{l^2}$. There exist several parametrizations of AdS_3, but one we will find particularly useful is a parametrization by hyperbolic coordinates t,ρ and

ϕ given by

$$U = l\cosh\rho\cos t\ , \qquad V = l\cosh\rho\sin t\ ,$$
$$X = l\sinh\rho\cos\phi\ , \qquad Y = l\sinh\rho\sin\phi\ ,$$

where $\rho, t \in (0,\infty)$ and $\phi \in (0, 2\pi)$. These coordinates satisfy (5.1) and with them we can express the line element of AdS_3 as

$$ds^2 = -l^2\cosh^2\rho dt^2 + l^2 d\rho^2 + l^2\sinh^2\rho d\phi^2\ . \tag{5.2}$$

It is natural to define yet another coordinate $r = l\sinh\rho$ by which the line element becomes

$$ds^2 = -(l^2 + r^2)dt^2 + \frac{dr^2}{1+\frac{r^2}{l^2}} + r^2 d\phi^2\ . \tag{5.3}$$

We have stated that AdS_3 is a solution of Einstein's vacuum equations with $\Lambda < 0$, but we have not shown it. The explicit calculation may be found in Appendix F.2.

5.2 The Schwarzschild black hole

The theory of black holes and wormholes has its roots in a fundamental solution to Einstein's equations. The well-known solution was found by the German physicist Karl Schwarzschild (1873-1916) over a century ago (1915) and is considered to be the first exact solution to Einstein's field equations. What was not at all evident to Schwarzschild and the physics community back then was that the solution possessed the properties of a black hole. Schwarzschild proposed the following four-dimensional metric as a solution, here expressed in spherical coordinates:

$$ds^2 = -\left(1 - \frac{2MG}{r}\right)dt^2 + \left(1 - \frac{2MG}{r}\right)^{-1} dr^2 + r^2 d\Omega^2\ , \tag{5.4}$$

where $d\Omega^2 = d\theta^2 + \sin^2\theta d\phi^2$, M is the mass of a gravitational source and G Newtons's gravitational constant. The solution is a spherically symmetric vacuum solution to Einstein's equations. Schwarzschild assumed the mass M had no electric charge or angular momentum and existed in a flat space, i.e. a space with the cosmological constant equal to zero.

From the metric given in equation (5.4) we note that the solution becomes singular when $r = 0$. This is a characterizing property of a black hole in 3+1 dimensions.[1] An even more important property which characterizes a black hole can be found by noting that the geometry of the Schwarzschild solution has a surrounding spherical boundary situated at the radius $r = 2MG$, also called the *Schwarzschild radius*. This spherical boundary is referred to as the *event horizon*. After passing the event horizon, towards the origin at $r = 0$, nothing can escape the gravitational field. The Schwarzschild solution does indeed posses the properties of a black hole in 3+1 dimensions [5]. Moreover, to make a connection to the previous section, we note from the metric above that the Schwarzschild solution approaches the Minkowski metric as $r \to \infty$, with a spacetime manifold resembling that of Minkowski space.

Not only can Schwarzschild's solution be interpreted as a black hole, it can also be interpreted as a wormhole after a proper extension of the spacetime. The theory of wormholes originates from Einstein and Rosens' idea of a geometric particle model which avoided point singularities caused by particles with infinite mass or charge distributions. Fundamentally, Einstein and Rosen proposed a theory in which one could connect two separated points in spacetime through a topological feature - or in other words: a bridge. Today we call these bridges wormholes. By performing a coordinate transformation in Schwarzschild's metric, Einstein and Rosen were able to show that Schwarzschild's solution could also describe an Einstein-Rosen bridge, connecting two black holes. To see this, consider the coordinate transformation

$$u^2 = r - 2MG \Rightarrow 2udu = dr\ ,$$

under which the Schwarzschild metric transforms to

$$\begin{aligned} ds^2 &= -\frac{u^2}{u^2+2MG}dt^2 + \frac{u^2+2MG}{u^2}4u^2du^2 + (u^2+2MG)^2d\Omega^2 \\ &= -\frac{1}{1+\frac{2MG}{u^2}}dt^2 + 4u^2(1+\frac{2MG}{u^2})du^2 + (u^2+2MG)^2d\Omega^2\ . \end{aligned}$$

[1] In 2+1 dimensions a curvature singularity is no longer characteristic for black holes, as we will see later when discussing the BTZ black hole solution.

Letting $u \in (-\infty,\infty)$, we note from this substitution that r varies from $-\infty$ to $-2MG$ and from $+2MG$ to $+\infty$. From this Einstein and Rosen concluded that the four-dimensional space could be mathematically expressed by two congruent sheets, one sheet corresponding to $u > 0$ and another sheet corresponding to $u < 0$. These sheets are joined by a hyperplane, $r = 2MG$ (or $u = 0$), creating a bridge between the sheets. In their paper from 1935, Einstein and Rosen also took it one step further and associated their bridge with the presence of an elementary particle. Furthermore they investigated the possibility of particles with negative mass by substituting the mass M in Schwarzschild's metric for a negative one. However, Einstein and Rosen argued that it was impossible to perform a proper change of variables in order to achieve a bridge in the case of particles with negative mass, explaining why there were no neutral particles with negative mass to be found in the universe [18]. Einstein and Rosens' bridges were later proven to be unsuccessful for describing particles. They are, however, yet of today seen as prototype wormholes and are of great importance in gravitational physics. Moreover, ER-bridges triggered the idea of traversable wormholes, which theoretical existence remained unclear until very late into the 20th century [5].

5.3 Black holes in 2+1 dimensions

Let us now turn to the case of a $2+1$-dimensional spacetime and derive a metric which we will later interpret as a black hole. To do this we assume a radially symmetric metric and make the symmetry obvious by using a time coordinate t, a radial coordinate r and an angular coordinate ϕ with $0 \leq \phi < 2\pi$. We start by writing down a quite general radially symmetric metric:

$$ds^2 = -B^2(r)dt^2 + A^2(r)dr^2 + r^2 d\phi^2 \ . \tag{5.5}$$

If this metric is to describe spacetime it must satisfy Einstein's equations in three dimensions (equation (4.45)), potentially with a cosmological constant. Using (4.8), we write down our vielbeins:

$$e^0 = B(r)dt \ , \qquad e^1 = A(r)dr \ , \qquad e^2 = r d\phi \ . \tag{5.6}$$

We also require the spin connection to solve Einstein's equations. We may find the components of this by using Cartan's first structure equation, $\mathrm{d}e^a = -\epsilon^a{}_{bc}\omega^b \wedge e^c$. This gives us three equations:

$$\mathrm{d}e^0 = B'(r)dr \wedge dt = -\omega^1_t dt \wedge e^2 - \omega^1_r dr \wedge e^2 + \omega^2_t dt \wedge e^1 + \omega^2_\phi d\phi \wedge e^1 \ , \tag{5.7}$$

$$\mathrm{d}e^1 = 0 = \omega^2_r dr \wedge e^0 + \omega^2_\phi d\phi \wedge e^0 - \omega^0_t dt \wedge e^2 + \omega^0_r dr \wedge e^2 \ , \tag{5.8}$$

$$\mathrm{d}e^2 = dr \wedge d\phi = \omega^0_t dt \wedge e^1 + \omega^0_\phi d\phi \wedge e^1 - \omega^1_r dr \wedge e^0 + \omega^1_\phi d\phi \wedge e^0 \ . \tag{5.9}$$

We start by noticing that in (5.7) the only term with $dr \wedge dt$ on the right hand side is $w^2_t dt \wedge e^1$. Similarly, the only term on the right hand side in (5.9) with $dr \wedge d\phi$ is $\omega^0_\phi d\phi \wedge e^1$. Using this we may conclude that

$$\omega^0_\phi = -\frac{1}{A(r)} \ , \qquad \omega^2_t = -\frac{B'(r)}{A(r)} \ . \tag{5.10}$$

Returning to equation (5.9) we see that there is only one term with $dt \wedge d\phi$, more precisely $\omega^1_t dt \wedge e^2$, so this has to be zero. The other terms, however, may cancel each other. The same analysis holds for the other two equations. We therefore have

$$\omega^1_t = \omega^2_r = \omega^0_r = \omega^1_\phi \equiv 0 \ .$$

The other terms form a new system of equations:

$$r\omega^1_r + A\omega^2_\phi = 0 \ , \tag{5.11}$$

$$B\omega^2_\phi + r\omega^0_t = 0 \ , \tag{5.12}$$

$$A\omega^0_t + B\omega^1_r = 0 \ . \tag{5.13}$$

Substituting (5.11) and (5.12) into (5.13) we find that

$$A\omega^0_t + B(\frac{A}{r}\frac{r}{B}\omega^0_t) = 2A\omega^0_t = 0 \implies \omega^0_t = 0 \ .$$

The same is true for the rest of the spin connections. Thus, the only non-zero terms were the first ones we found. To summarize, we have

$$\omega^0 = -\frac{1}{A(r)}d\phi \ , \qquad \omega^1 = 0 \ , \qquad \omega^2 = -\frac{B'(r)}{A(r)}dt \ . \tag{5.14}$$

We can now form the curvature two-form $R^a = \mathrm{d}\omega^a + \frac{1}{2}\epsilon^a{}_{bc}\omega^b \wedge \omega^c$, and attempt to solve equation (4.45), restated below for convenience:

$$R^a = \frac{\Lambda}{2}\epsilon^a{}_{bc}e^b \wedge e^c \ .$$

Since our indices run from 0 to 2 we have three equations to solve:

$$\begin{aligned}
R^0 &= \frac{A'(r)}{A^2(r)}dr \wedge d\phi = \Lambda A(r) r dr \wedge d\phi \ , \\
R^1 &= -\frac{B'(r)}{A^2(r)}dt \wedge d\phi = -\Lambda B(r) r d\phi \wedge dt \ , \\
R^2 &= \Big(-\frac{B''(r)}{A(r)} + \frac{B'(r)A'(r)}{A^2(r)} \Big) dr \wedge dt = -\Lambda B(r) A(r) dt \wedge dr \ .
\end{aligned}$$

These are three differential equations for A and B. By dropping the differentials we find them to be

$$A'(r) = \Lambda A^3(r) r \ , \tag{5.15}$$

$$B'(r) = -\Lambda A^2(r) B(r) r \ , \tag{5.16}$$

$$B''(r)A(r) - B'(r)A'(r) = -\Lambda B(r) A^3(r) \ . \tag{5.17}$$

The first equation is a separable differential equation. Dividing by A^3 on both sides and integrating yields

$$-\frac{1}{A^2(r)} = \Lambda r^2 + M \implies A = \frac{1}{\sqrt{-\Lambda r^2 - M}} \ ,$$

where M is a constant of integration. We may then attack the second equation, writing

$$\frac{B'(r)}{B(r)} = \frac{-\Lambda r}{-\Lambda r^2 - M} \ .$$

Integrating this gives us

$$\ln B(r) = \ln \sqrt{-\Lambda r^2 - M} \implies B(r) = \sqrt{-\Lambda r^2 - M} \ .$$

The sharp eyed reader will notice that we have set the integration factor to one in the previous equation. We have now found the radial functions $A(r)$ and $B(r)$, and we may now write down the metric (5.5) as

$$ds^2 = -\Big(\frac{r^2}{l^2} - M\Big)dt^2 + \Big(\frac{r^2}{l^2} - M\Big)^{-1} dr^2 + r^2 d\phi^2 \ , \tag{5.18}$$

where we substituted the cosmological constant Λ with $\Lambda = -\frac{1}{l^2}$ for AdS space. We clearly see that if $M > 0$ we have an event horizon in the metric and therefore a potential black hole. However, if $M < 0$ there is no obvious singularity. The case $M > 0$ is the *BTZ black hole* [19]. Amazingly these results were discovered very recently (1990s) by Bañados, Teitelbom and Zanelli, contradicting the old hypothesis that black holes cannot exist in three dimensions due to the lack of local gravitational attraction. Hence, black hole solutions to the Einstein equations in three dimensions are referred to as BTZ solutions or BTZ black holes, named after the three brilliant physicists mentioned above.

Let us first make a quick comment about the asympotic behaviour of the metric as $r \to \infty$. In this limit it is clear that the metric approaches

$$ds^2_{r\to\infty} = -\frac{r^2}{l^2}dt^2 + r^2 d\phi^2 \ ,$$

which is not Minkowski spacetime. Instead we have an asymptotic AdS_3 spacetime, as can be seen by comparison with the metric (5.3).

We now discuss the parameter M more in-depth. First and foremost, let us go from AdS to Minkowski. We will recover Minkowski spacetime if we let $\Lambda \to 0 \implies l \to \infty$. In this limit the BTZ metric (5.18) approaches

$$ds^2 = Mdt^2 - \frac{dr^2}{M} + r^2 d\phi^2 \ .$$

This does not make much sense physically unless we set $M < 0$ becouse our spacetime must be Lorentzian. We may then rescale the time coordinate $t \to \frac{t}{\sqrt{-M}}$ and the radial coordinate $r \to r\sqrt{-M}$. This gives us a new metric:

$$ds^2 = -dt^2 + dr^2 + |M|r^2 d\phi^2 \ .$$

Now, if $M = -1$ this is simply Minkowski space and not of much interest. If we instead set $M \in (-1{,}0)$ we get the metric of a cone. This is seen by rescaling the angular variable ϕ by $\phi \to \frac{\phi}{\sqrt{|M|}}$. While the explicit dependence of M disappears from the metric we now have that $\phi \in (0{,}2\pi\sqrt{|M|})$ and our spacetime is a cone. We conclude that when we let $\Lambda \to 0$, our black hole disappears. We therefore claim that there are no three-dimensional black holes in Minkowski space.

Let us now return to AdS$_3$. We will first consider the case $M < 0$ where we will once again find conical solutions. One way to find out if we have a conical space around a point is to take the length of a circle around the point and divide it by the length of a closed curved with a fixed distance from the point. Then if we let the distance to the curve go to zero we will have a conical singularity if the ratio is between zero and one. We will do this explicitly. If our space where flat the length of a circle a coordinate-distance r_d from the origin would be $2\pi r_d$. However, the distance measured by an observer travelling from the origin ($r = 0$) to $r = r_d$ is

$$\int_0^{r_d} \frac{dr}{\sqrt{\frac{r^2}{l^2} - M}} = l\log\left(l\sqrt{\frac{r_d^2}{l^2} - M} - r_d\right) - l\log\left(l\sqrt{-M}\right) \ .$$

Hence, we consider the limit

$$\lim_{r_d \to 0} \frac{2\pi r_d}{2\pi l\log\left(l\sqrt{\frac{r_d^2}{l^2} - M} - r_d\right) - 2\pi l\log\left(l\sqrt{-M}\right)} \approx \lim_{r_d \to 0} \frac{r_d}{r_d\frac{1}{\sqrt{-M}}} = \sqrt{-M} \ , \tag{5.19}$$

where we in the second step performed a Taylor expansion of the logarithm around $r_d = 0$. From this we see that for all values $M \in (0, -1)$ we have a conical singularity. However, if $M = -1$ there is no conical singularity. There is no mystery here, the case $M = -1$ is precisely AdS$_3$! Because the conical singularities separate empty AdS$_3$ ($M = -1$) and the BTZ black hole ($M > 0$) we speak of a mass gap. However, these singularities turns out to be very interesting because they correspond to a point like particle in the origin. To see this one considers a stress-energy tensor with a delta function in the origin, $T = m\delta(x^\mu)$. From this it is possible to deduce that the geometry is indeed a cone and that the deficit angle of the cone, α, is related to the mass, m, according to $\alpha = 8\pi Gm$ [20]. Note that our result also relates the deficit angle to M by $(1 - \frac{\alpha}{2\pi}) = \sqrt{-M}$, so it is natural to interpret M as a mass parameter. However, since $\alpha \in (0{,}2\pi)$ we have a natural upper bound on the mass of a particle before it collapses into a black hole. In recent work Jonathan Lindgren has successfully used this geometrical approach to study particle collisions and black hole formations [20].

Finally we turn to the case $M > 0$ where we will indeed have a black hole, the BTZ black hole. The dr^2-component of the metric (5.18) is clearly divergent at $r = l\sqrt{M}$. This is the horizon. To see that it is indeed not a curvature singularity we can compute the *Kretschmann scalar* defined as $K = R_{\mu\nu\rho\sigma}R^{\mu\nu\rho\sigma}$. The Kretschmann scalar is the standard tool to show that there is a singularity at the origin for a Schwarzschild black hole. For the BTZ-hole the result is $K = \frac{12}{l^4}$, constant everywhere. While this does imply that there is no singularity at the horizon it also means that we do not have a curvature singularity at the origin! However, there is another kind of singularity at $r = 0$, a causality singularity, although we will not show this and instead refer the reader to [21].

We show in Appendix K that the parameter M will indeed depend on the energy of the system, but we will here find a geometrical interpretation of M following a paper by Dieter Brill [22]. To do so we first need a useful parametrization of the line element of AdS$_3$. Remember that given four coordinates X, Y, U and V that satisfy

$$-V^2 - U^2 + X^2 + Y^2 = -l^2 \ , \tag{5.20}$$

the line element for AdS$_3$ can be written as

$$ds^2 = -dV^2 - dU^2 + dX^2 + dY^2 \ .$$

We may now introduce new parameters according to

$$U = l\sinh\xi\sinh t\ , \qquad V = l\cosh\xi\cosh\varphi\ ,$$
$$X = l\cosh\xi\sinh\varphi\ , \qquad Y = l\sinh\xi\cosh t\ .$$

Notice that all our variables takes values in $\mathbb{R}$ and they will still satisfy (5.20). Let us now define the radial component $\tilde{r} = l\cosh\xi$. This implies that $\tilde{r} \in (l,\infty)$. The new line-element becomes, after some manipulations,

$$ds^2 = -\Big(\frac{\tilde{r}^2}{l^2} - 1\Big)dt^2 + \Big(\frac{\tilde{r}^2}{l^2} - 1\Big)^{-1}d\tilde{r}^2 + \tilde{r}^2 d\varphi^2\ .$$

This is indeed very similar to the metric for the BTZ black hole, but let us not forget that $\phi \in (0, 2\pi)$ while $\varphi \in (-\infty,\infty)$. How do we make these equal? We will identify different values of φ! While the identification $\varphi \sim \varphi + 2\pi$ certainly is tempting we can do even better. Let us set $\varphi \sim \varphi + 2\pi a$, and rescale all our coordinates according to

$$\varphi \mapsto a\varphi\ , \qquad \tilde{r} \mapsto \frac{\tilde{r}}{a}\ , \qquad t \mapsto ta\ .$$

We then find the metric

$$ds^2 = -\Big(\frac{\tilde{r}^2}{l^2} - M\Big)dt^2 + \Big(\frac{\tilde{r}^2}{l^2} - M\Big)^{-1}dr^2 + \tilde{r}^2 d\varphi^2\ ,$$

where $M = a^2$. Remember that with $\tilde{r} \in (l\sqrt{M},\infty)$ we can identify $l\sqrt{M}$ as the horizon and furthermore compute the minimal distance around the black hole to be $2\pi l\sqrt{M}$. The constant M introduced in the black hole metric (5.18) has thus been given a geometrical interpretation.

5.4 The construction of traversable wormholes

Having made a quick detour through the theory of black holes in AdS_3 in the previous section, we now continue where we left off in section 5.2 and ask ourselves if traversable wormholes exist in general relativity. In 1988, American professor of theoretical physics Kip Thorne (1940-)[2] and his graduate student Mike Morris constructed a traversable wormhole,[3] purely as a tool for learning general relativity [23]. In order to construct a traversable wormhole, the time component of the metric was required to always be non-zero, while having a divergent radial component of constant sign around $r = 0$. Based on this, Morris and Thorne suggested the radially symmetric and time-independent metric

$$ds^2 = -e^{2\Phi(r)}dt^2 + \frac{dr^2}{1 - \frac{b(r)}{r}} + r^2 d\varphi^2\ , \tag{5.21}$$

where $\Phi(r)$ is referred to as the *redshift function* and is everywhere finite in order to prevent an event horizon. The second radial dependent function, $b(r)$, is called the *shape function* as it determines the shape of the wormhole when observed in an embedding diagram [24].

This is really a special case of the ansatz considered earlier in the section 5.3, corresponding to $B^2(r) = e^{2\Phi(r)}$ and $A^2(r) = \frac{1}{1-\frac{b(r)}{r}}$. For convenience we restate the Riemann tensor components found in section 5.3 in terms of the vielbeins e^0, e^1 and e^2:

$$R^0 = \frac{A'(r)}{rA^3(R)}e^1 \wedge e^2\ , \qquad R^1 = -\frac{B'(r)}{A^2(r)rB(r)}e^0 \wedge e^2\ , \qquad R^2 = \Big(-\frac{B''(r)}{A^2(r)B(r)} + \frac{B'(r)A'(r)}{A^3(r)B(r)}\Big)e^1 \wedge e^0\ .$$

By using the identities $R^a = \frac{1}{2}\epsilon^{abc}R_{bc}$ and $R_{ab} = R^c{}_{acb}$, we find the non-zero components of the Riemann tensor of the Morris-Thorne metric in an orthonormal basis to be:

$$R^1{}_{212} = \frac{b'r - b}{2r^3}\ , \qquad R^2{}_{020} = (1 - \frac{b}{r})\frac{\Phi'}{r}\ , \qquad R^1{}_{010} = (1 - \frac{b}{r})(\Phi'^2 + \Phi'') - \frac{b'r - b}{2r^2}\Phi'\ .$$

[2]For those readers who are also into science fiction movies we may reveal that Thorne did scientific consulting for Christopher Nolan's film *Interstellar* in 2012.

[3]Unfortunately the wormhole in question was purely theoretical.

We now compute the Einstein tensor. Using its definition: $G_{ab} = R_{ab} - \frac{1}{2}\eta_{ab}R$, we find that

$$\begin{aligned} G_{00} &= R_{00} - \frac{1}{2}\eta_{00}R = R_{00} + \frac{1}{2}(-R_{00} + R_{11} + R_{22}) \\ &= \frac{1}{2}({R^1}_{010} + {R^2}_{020} - {R^1}_{010} + {R^2}_{121} - {R^2}_{020} + {R^2}_{121}) = {R^2}_{121} \ , \end{aligned}$$

$$G_{11} = ... = {R^2}_{020} \ , \quad G_{22} = ... = {R^1}_{010} \ .$$

We start our investigation by considering the *Ellis wormhole* [25] which, as we will see, is a special case of the Morris-Thorne wormhole[4]. The metric of the Ellis wormhole is given by

$$ds^2 = -dt^2 + dl^2 + (l^2 + b_0^2)d\varphi^2 \ ,$$

where $l \in (0,\infty)$ and $\varphi \in (0,2\pi)$. By setting $r = \sqrt{l^2 + b_0^2}$ we may rewrite the metric as

$$ds^2 = -dt^2 + \frac{dr^2}{1 - \frac{b_0^2}{r^2}} + r^2 d\varphi^2 \ .$$

This is precisely the metric we get if we set $\Phi(r) = 0$ and $b(r) = \frac{b_0^2}{r}$ in the Morris-Thorne metric presented above in equation (5.21). Using the previous results we can write down G_{00} as

$$G_{00} = -\frac{b_0^2}{2}\left(\frac{1}{r^4} + \frac{1}{r^2}\right) = -\frac{b_0^2}{r^4} \ .$$

It is interesting to note that G_{00} is negative, which implies that the 00-component of the stress-energy tensor has to be negative as well. Let us return to our Morris-Thorne wormhole. Assuming we have a stress-energy tensor with positive energy density (that is $T_{00} > 0$), then the Einstein tensor component, G_{00}, has to satisfy

$$\frac{b'(r)}{2r^2} - \frac{b(r)}{2r^3} > 0 \ .$$

It is natural to consider the ansatz $b(r) = r^\alpha$ which gives us the restriction $\alpha > 1$. But if we allow the radial component of our metric to change sign for a large enough value of r this is precisely the kind of behaviour we want to avoid if we are to construct a traversable wormhole. So creating a traversable wormhole will require us to construct a stress-energy tensor with negative energy density. Matter with this property is called *exotic matter*.

To see whether this is possible we investigate two different possible stress-energy tensors. One from the electromagnetic field and one from a scalar field. In this section we will work in curvilinear coordinates and we will therefore have to convert G_{00} using the vielbeins. To avoid confusion we will denote the 00-component of the Einstein tensor in curvilinear coordinates with G_{tt} and we find it to be

$$G_{tt} = e_t^a e_t^b G_{ab} = e_t^0 e_t^0 G_{00} = e^{2\Phi(r)} G_{00} = e^{2\Phi(r)} \frac{b'r - b}{2r^3} \ .$$

We start with the electromagnetic field. According to (4.40) the electromagnetic stress-energy tensor is

$$T_{EM}^{\mu\nu} = F^{\mu\alpha}{F^\nu}_\alpha - \frac{g^{\mu\nu}}{4}F^{\alpha\beta}F_{\alpha\beta} \ .$$

To tell if T_{EM}^{tt} is positive, negative or may change sign we rewrite the terms according to

$$\begin{cases} F^{t\alpha}{F^t}_\alpha = g_{rr}F^{tr}F^{tr} + g_{\varphi\varphi}F^{t\varphi}F^{t\varphi} \\ F^{\alpha\beta}F_{\alpha\beta} = 2(g_{tt}g_{rr}F^{tr}F^{tr} + g_{tt}g_{\varphi\varphi}F^{t\varphi}F^{t\varphi} + g_{rr}g_{\varphi\varphi}F^{r\varphi}F^{r\varphi}) \end{cases} .$$

Using this expression we find that the tt component of the stress-energy tensor may be written as

$$T^{tt} = F^{tr}F^{tr}\left(g_{rr} - \frac{1}{2}g_{rr}\right) + F^{t\varphi}F^{t\varphi}\left(g_{\varphi\varphi} - \frac{1}{2}g_{\varphi\varphi}\right) - \frac{1}{2}F^{r\varphi}F^{r\varphi}\left(g^{tt}g_{rr}g_{\varphi\varphi}\right) > 0 \ .$$

[4]Ellis wormhole was in fact constructed 1969 by H.G Ellis, presented in his paper "Ether flow through a drainhole: A particle model in general relativity" [25], more than a decade before Morris and Thorne proposed their solution. Back then Ellis solution was referred to as a *drainhole*, but nevertheless making it the earliest-known model of a traversable wormhole.

As indicated this expression is positive since g_{rr},$g_{\varphi\varphi} > 0$ and $g^{tt} < 0$, otherwise we would not have a traversable wormhole! Thus we cannot use the electromagnetic field to create negative energy density. For a recent, more detailed attempt to use a modified electromagnetic field in AdS_3 to create a wormhole we refer the reader to [26]. The construction in this paper demands a divergent electromagnetic field in the origin.

We now investigate if it is possible to create negative energy density using a scalar field. The scalar field has the Lagrangian

$$\mathcal{L}_{\mathrm{S}} = -\frac{1}{2}g^{\mu\nu}(\partial_\mu\phi\partial_\nu\phi) - \frac{1}{2}m^2\phi^2 - \frac{\xi}{2}R\phi^2 - \xi'\phi^6 \ .$$

We will consider two different values for ξ. Either we set $\xi = 0$ which is known as a *minimal coupling.* We will also consider the case $\xi = \frac{1}{8}$ which is referred to as *conformal coupling.* The specific value of $\frac{1}{8}$ has been determined by demanding invariance under a *Weyl transformation*, $g_{\mu\nu} \to g'_{\mu\nu} = \Omega^2 g_{\mu\nu}$, where Ω is a scalar function. We show the procedure explicitly in Appendix H. The term ϕ^6 is also picked to preserve the invariance under a Weyl transformation. To find the stress tensor several identities have to be used and even then the full calculation is rather long. We will thus only quote the result and refer the interested reader to Appendix H where all the calculation is written out and all necessary identities are proven. The energy-momentum tensor for the scalar field is found to be

$$T_S^{\mu\nu} = \xi G^{\mu\nu}\phi^2 + \xi(g^{\mu\nu}\Box(\phi^2) - D^\mu D^\nu(\phi^2)) - g^{\mu\nu}\xi'\phi^6 - \frac{g^{\mu\nu}}{2}(\partial^\rho\phi\partial_\rho\phi + m^2\phi^2) + \partial^\mu\phi\partial^\nu\phi \ ,$$

where $\Box = D^\mu D_\mu$ is the *D' Alembert operator* in curved spacetime. We will henceforth drop the subscript S to avoid clutter. This result is a generalization of the stress-energy tensor presented in Zelnikov and Frolovs' *Introduction to Black Hole Physics* [27]. It is also shown in Appendix H that the equation of motion for the scalar field is

$$\Big(\Box - m^2 - \frac{1}{8}R\Big)\phi - 6\xi'\phi^5 = 0 \ . \tag{5.22}$$

The expression for the energy-stress tensor is rather complicated and to be able to deal with it we first make the natural assumption that our field only depends on the radius, $\phi = \phi(r)$, precisely like our wormhole metric. Now we can start to evaluate terms more explicitly. Below we give the time-component of the stress-energy tensor:

$$T_{tt} = \xi G_{tt}\phi^2 + e^{2\Phi}\Big(m^2(\frac{1}{2} - 2\xi) - 2\xi^2 R\Big)\phi^2 + e^{2\Phi}(\frac{1}{2} - 2\xi)(1 - \frac{b}{r})(\phi')^2 + e^{2\Phi(r)}(\xi' - 12\xi\xi')\phi^6 \ .$$

This component and the rest of the components (T_{rr} and $T_{\phi\phi}$) are evaluated explicitly in Appendix I for the conformal case, i.e. $\xi = \frac{1}{8}$. As should be expected the minimally coupled case, $\xi = 0$ is the simplest. We then find that (5.4) reduces to:

$$T_{tt}(\xi = 0) = \frac{1}{2}e^{2\Phi}m^2\phi^2 + \frac{1}{2}e^{2\Phi}(1 - \frac{b}{r})(\phi')^2 + e^{2\Phi}\xi'\phi^6 > 0 \ ,$$

which is strictly positive. Notice that this is true since we must have $\xi' > 0$ due to the fact that it is identified as a potential energy term and that $(1 - \frac{b}{r}) > 0$ since this is precisely g_{11} which must be positive for our wormhole.

The conformally coupled system is more complicated, we find that (5.4) now becomes

$$T_{tt}(\xi = \frac{1}{8}) = \frac{1}{8}G_{tt}\phi^2 - \frac{e^{2\Phi}}{32}R\phi^2 + \frac{e^{2\Phi}}{4}(1 - \frac{b}{r})(\phi')^2 - \frac{e^{2\Phi(r)}}{2}\xi'\phi^6 \ .$$

which is very difficult to analyze when it comes to signature. We may instead investigate whether there exist Morris-Thorne wormhole solutions which arise from a conformally coupled scalar field. We have from Einstein's equations (without the cosmological constant) that $\frac{1}{\kappa}G_{\mu\nu} = T_{\mu\nu}$. With our stress-tensor for a conformally coupled scalar field and our Einstein tensor components for the Morris and Thorne-solution we thus have a system of coupled differential equations for the redshift function $\Phi(r)$, the shape function $b(r)$ and the scalar field $\phi(r)$. Additionally to these equations we have the equation of motion of the scalar field, equation (5.22). Due to their complexity we will not bother to write the equations down. Under certain simplifications (i.e. massless scalar field, $\xi' = 0$) we may actually draw conclusions about the existence of solutions. In Appendix I we provide a discussion on the differential equations. In particular we find that there are none but trivial solutions in the case of a massless scalar field in

2+1 dimensions. Interestingly, the situation is radically different in 3+1 dimensions. This is stressed in e.g. "Traversable wormholes from massless conformally coupled scalar fields", in which Barcelo and Visser state that they, in four dimensions, have found a conformally coupled scalar field stress-tensor which violates the classical energy conditions, allowing for traversable wormhole solutions [28]. We omit a complete treatment of the more general case, i.e. a scalar field with mass and $\xi' \neq 0$, and refer to Appendix I for a somewhat more detailed discussion on the subject.

Chapter 6

Gravity From Chern-Simons Gauge Theory

In this chapter we relate the Chern-Simons theory with the gauge group $SO(2,2)$ to general relativity with negative cosmological constant. To achieve this, we couple the frame field e^a to the generalized momentum generators P_a of $SO(2,2)$, and the spin connections, ω^a, to the Lorentz transforms generators, M_a. This coupling is well justified, because the Lorentz generators correspond to an infinitesimal change of coordinate systems, which is exactly what the spin connection ω^a measures. Since the frame fields are just the flat local basis of the manifold they measure infinitesimal translation, and the P_a generators are related to the translation generators of the Poincaré group via the *Inönu-Wigner contraction* performed in Appendix F.1.2.

We then split the symmetry group $SO(2,2)$ up into $SL(2) \times SL(2)$ via the local isomorphism $so(2,2) \cong sl(2) \oplus sl(2)$, resulting in two independent connections A and $\bar{A}$. Combining the actions from each of these two connections we obtain a Chern-Simons action that is equivalent to the Einstein-Hilbert action with an appropriate choice of scale factor.

After showing that the Chern-Simons gauge theory of $SO(2,2)$ is equivalent to general relativity in 2+1 dimensions we will investigate the possibility of extending the symmetry group in chapter 7. This will, as we shall see in the next chapter, give us a higher spin theory of gravity.

6.1 Chern-Simons gravity on AdS_3

As mentioned before Chern-Simons theory can be seen as a theory of gravity. Our goal with this section is to give a motivation to this statement by showing that Chern-Simons action is in fact equivalent to the Einstein-Hilbert action up to boundary terms and with a certain choice of cosmological constant. That choice of cosmological constant happens to be $-\frac{1}{l^2}$, which coincides well with our conclusions in the last section, where we stated that AdS_3 with $\Lambda = -\frac{1}{l^2}$ is a solution to Einstein's equations. We therefore initially show that Chern-Simons action in AdS_3 is indeed equivalent to the Einstein-Hilbert action in 2+1 dimensions.

Having retrieved Einstein's equations in the Cartan formalism and discussed Chern-Simons theory we are now going to unite them. We are ready to investigate a Chern-Simons theory on AdS_3. Recall that the Chern-Simons action is written as

$$\mathcal{S}_{CS}[A] = \frac{k}{4\pi}\int \mathrm{tr}[A \wedge \mathrm{d}A + \frac{2}{3}A \wedge A \wedge A] \ . \tag{6.1}$$

The trace reminds us that we are working with a Lie algebra-valued potential A. To find the Lie algebra of the isometry group of AdS_3 we turn to the metric of the embedding space and the restriction, equation (5.1). Since both the metric and the restriction is unchanged under the $SO(2,2)$ group of transformations we recognise this as the isometry group of AdS_3. The Lie algebra $so(2,2)$ can be summarized with two sets of generators, M^a and P^b. They form the Lie algebra

$$\begin{aligned}
[M^a, M^b] &= \epsilon^{ab}{}_c M^c \ , \\
[P^a, M^b] &= \epsilon^{ab}{}_c P^c \ , \\
[P^a, P^b] &= \epsilon^{ab}{}_c M^c \ .
\end{aligned}$$

A possible interpretation is that M^a is the generator of Lorentz transformations and P^a the generator of generalized momentum. A derivation of these generators and their explicit form is found in Appendix F.2.2. In the vielbein formalism we construct the Lie algebra-valued connection as

$$A \equiv \frac{1}{l} e^a_\mu P_a dx^\mu + \omega^a_\mu M_a dx^\mu \ .$$

The identification of the vielbein indices with the generator indices is of fundamental importance for the gauge-gravity equivalence. In the same sense as the vielbeins are a local approximation of a Riemannian manifold, the Lie algebra generators are a local approximation of the Lie group manifold. Writing the connection like this can be seen as an association of the six dimensions of the group manifold with the three-dimensional local frames e^a_μ plus the three-dimensional spin connections ω^a_μ.

A remarkable feature of the Lie algebra of $SO(2.2)$ is that we can decompose it according to $so(2,2) \simeq sl(2,\mathbb{R}) \oplus sl(2,\mathbb{R})$. The six generators of $so(2,2)$ then split into two sets, $T^a = \frac{1}{2}(M^a + P^a)$ and $\bar{T}^a = \frac{1}{2}(M^a - P^a)$ satisfying

$$[T^a,T^b] = \epsilon^{ab}{}_c T^c \ ,$$
$$[T^a,\bar{T}^b] = 0 \ ,$$
$$[\bar{T}^a,\bar{T}^b] = \epsilon^{ab}{}_c \bar{T}^c \ .$$

We pick our representation to satisfy the trace relationship

$$\mathrm{tr}[T^a T^b] = \mathrm{tr}[\bar{T}^a \bar{T}^b] = \frac{1}{2}\eta^{ab} \ .$$

This is furthered motivated in Appendix E.2.1 with use of *Killing forms*. Since these generators split into two distinct sets we can use them to split the Chern-Simons action according to

$$\mathcal{S}_{CS}[A,\bar{A}] = \mathcal{S}_{CS}[A] - \mathcal{S}_{CS}[\bar{A}] \ . \tag{6.2}$$

Having split the gauge group in this fashion, the new connections A and $\bar{A}$ are written as

$$A = \left(\omega^a + \frac{e^a}{l}\right)T_a \ , \qquad \bar{A} = \left(\omega^a - \frac{e^a}{l}\right)\bar{T}_a \ .$$

The difference in sign being attached to e^a instead of ω^a is a result of the sign convention used when we split up our action. Now, let us consider the first term in the Chern-Simons action, $\mathrm{tr}[A \wedge dA]$. Using our stated definitions we find

$$A \wedge dA = (\omega^a + \frac{e^a}{l})T_a \wedge (\mathrm{d}\omega^b + \frac{\mathrm{d}e^b}{l})T_b = (\omega^a \wedge \mathrm{d}\omega^b + \frac{e^a \wedge \mathrm{d}\omega^b}{l} + \frac{\omega^a \wedge \mathrm{d}e^b}{l} + \frac{e^a \wedge \mathrm{d}e^b}{l^2})T_a T_b \ .$$

Remember that we are dealing with an expression under an integral sign, thus by performing a partial integration on the third term and dropping the boundary term the expression can be simplified as

$$A \wedge \mathrm{d}A = (\omega^a \wedge \mathrm{d}\omega^b + 2\frac{e^a \wedge \mathrm{d}\omega^b}{l} + \frac{e^a \wedge \mathrm{d}e^b}{l^2})T_a T_b$$

Taking the trace of both sides gives us the first term in the Chern-Simons action:

$$\mathrm{tr}[A \wedge \mathrm{d}A] = (\omega^a \wedge \mathrm{d}\omega^b + 2\frac{e^a \wedge \mathrm{d}\omega^b}{l} + \frac{e^a \wedge \mathrm{d}e^b}{l^2})\frac{\eta_{ab}}{2} = \frac{1}{2}(\omega^a \wedge \mathrm{d}\omega_a + 2\frac{e^a \wedge \mathrm{d}\omega_a}{l} + \frac{e^a \wedge \mathrm{d}e_a}{l^2}) \ ,$$

where we used the invariant bilinear form condition $\mathrm{tr}\,[T_a T_b] = \eta_{ab}/2$. Now we consider the second term in the Chern-Simon action, $\mathrm{tr}\,[A \wedge A \wedge A]$. Making the Lie algebra nature of A explicit we find

$$A^a \wedge A^b \wedge A^c T_a T_b T_c = \frac{1}{2}A^a \wedge A^b \wedge A^c T_a T_b T_c - \frac{1}{2}A^a \wedge A^c \wedge A^b T_a T_b T_c = \frac{1}{2}A^a \wedge A^b \wedge A^c T_a [T_b,T_c] \ ,$$

where we in the last step simply renamed the dummy indices in the second expression. Now, since $[T_b,T_c] = \epsilon_{bc}{}^d T_d$ we see

$$\mathrm{tr}[A^a \wedge A^b \wedge A^c T_a T_b T_c] = \frac{1}{2}A^a \wedge A^b \wedge A^c \epsilon_{bc}{}^d\, \mathrm{tr}[T_a T_d] = \frac{1}{4}A^a \wedge A^b \wedge A^c \epsilon_{abc} \ .$$

Writing out all the vielbeins and spin connections explicitly we finally reach

$$\mathrm{tr}\,[A \wedge A \wedge A] = \frac{\epsilon_{abc}}{4}(\omega^a \wedge \omega^b \wedge \omega^c + \frac{e^a \wedge e^b \wedge e^c}{l^3} + \frac{3}{l}e^a \wedge \omega^b \wedge \omega^c + \frac{3}{l^2}e^a \wedge e^b \wedge \omega^c) \ .$$

Analogously we find the corresponding Chern-Simons action for $\bar{A}$, resulting in

$$\operatorname{tr}[\bar{A}\wedge \mathrm{d}\bar{A}] = \frac{1}{2}(\omega^a \wedge \mathrm{d}\omega_a - 2\frac{e^a\wedge \mathrm{d}\omega_a}{l} + \frac{e^a \wedge \mathrm{d}e_a}{l^2}) \ , \tag{6.3}$$

$$\operatorname{tr}[\bar{A}\wedge\bar{A}\wedge\bar{A}] = \frac{\epsilon_{abc}}{4}(\omega^a\wedge\omega^b\wedge\omega^c - \frac{e^a\wedge e^b\wedge e^c}{l^3} - \frac{3}{l}e^a\wedge\omega^b\wedge\omega^c + \frac{3}{l^2}e^a\wedge e^b\wedge\omega^c) \ . \tag{6.4}$$

To get our Chern-Simons action of AdS_3 we need to subtract $\operatorname{tr}[A\wedge dA] - \operatorname{tr}[\bar{A}\wedge d\bar{A}]$ and $\operatorname{tr}[A\wedge A\wedge A] - \operatorname{tr}[\bar{A}\wedge\bar{A}\wedge\bar{A}]$:

$$\operatorname{tr}[A\wedge \mathrm{d}A] - \operatorname{tr}[\bar{A}\wedge\mathrm{d}\bar{A}] = \frac{2e^a\wedge\mathrm{d}\omega_a}{l} \ , \tag{6.5}$$

$$\operatorname{tr}[A\wedge A\wedge A] - \operatorname{tr}[\bar{A}\wedge\bar{A}\wedge\bar{A}] = \frac{\epsilon_{abc}}{2l^3}e^a\wedge e^b\wedge e^c + \frac{3\epsilon_{abc}}{2l}e^a\wedge\omega^b\wedge\omega^c \ , \tag{6.6}$$

and if we plug these results into our original Chern-Simons action, equation 6.1, we find

$$\mathcal{S}_{CS}[A,\bar{A}] = \frac{k}{4\pi}\int_M \frac{2e^a\wedge\mathrm{d}\omega_a}{l} + \frac{2}{3}\left[\frac{\epsilon_{abc}}{2l^3}e^a\wedge e^b\wedge e^c + \frac{3\epsilon_{abc}}{2l}e^a\wedge\omega^b\wedge\omega^c\right] \ .$$

In order to compare this expression with Einstein-Hilbert action we extract a factor of $\frac{2}{l}$ outside the integral and collect terms:

$$\mathcal{S}_{CS}[A,\bar{A}] = \frac{k}{2\pi l}\int_M e^a\wedge\left[\mathrm{d}\omega_a + \frac{\epsilon_{abc}}{2}\omega^b\wedge\omega^c\right] + \frac{\epsilon_{abc}}{6l^2}e^a\wedge e^b\wedge e^c \ . \tag{6.7}$$

For convenience we restate the Einstein-Hilbert action in 2+1 dimensions

$$\mathcal{S}_{EH}[e,\omega] = -\frac{1}{\kappa}\int e^a\wedge\left[\mathrm{d}\omega_a + \frac{1}{2}\epsilon_{abc}\omega^b\wedge\omega^c\right] - \frac{1}{6}\Lambda\epsilon_{abc}e^a\wedge e^b\wedge e^c \ , \tag{6.8}$$

and we see from our expression above that we do indeed get the Chern-Simons action from Einstein-Hilbert action if we set $-\frac{1}{\kappa} = \frac{k}{2\pi l}$ (leading to $k = -\frac{l}{4G}$) and $\Lambda = -\frac{1}{l^2}$). The negative cosmological constant, i.e. $\Lambda = -\frac{1}{l^2}$, corresponds to an Anti-de Sitter gravitational theory. One may also consider the case of an imaginary radius of curvature, leading to a positive cosmological constant. Then we have to perform the same calculations as before but with the gauge group $SL(2,\mathbb{C})$. This leads to the concept of de Sitter gravity which will not be treated in this thesis [29]. We have now shown that we can translate the Chern-Simons action in 2+1 dimensions in AdS_3 space into an Einstein-Hilbert action. The same equivalence holds for Minkowski space. This can be shown by using that the isometry group of Minkowski space is $SO(1,2)$ (or the *Poincaré group*), where the Lie algebra can be derived by performing a Inönu-Wigner contraction on the AdS_3 Lie algebra. The contraction and calculations are performed in its entirety in Appendix F.2.3 and Appendix F.1.2 respectively.

Chapter 7

Higher Spin Gravity

In section 5.3 we found that if we tried to force a wormhole solution to Einstein's equations in 2+1 dimensions we had to introduce exotic matter. To resolve this we will attempt to generalize $SL(2)\times SL(2)$ gravity by using a symmetry group of higher order. Since $SL(2)$ is a subgroup of $SL(3)$ a natural such generalization is $SL(2) \times SL(2) \rightarrow SL(3) \times SL(3)$. In performing this extension we have obtained a symmetry group that preserves the symmetries of $SL(2)$ theory while adding new gauge degrees of freedom. The theory obtained is a particularly simple *higher spin gravity theory.* The term *higher spin* refers to the fact that conventional gravity is mediated by spin-2 particles, gravitons, and when extending the gauge group we couple these particles to a massless higher spin field, in our case a spin-3 field [10].

The formulation of gravity as a $SL(3)\times SL(3)$ gauge theory introduces some problems. In particular, we show that the previously central metric tensor is no longer a gauge invariant quantity since the spin-3 field act non-trivially on the metric. This is shown explicitly by performing a *trivial gauge transformation* of empty AdS$_3$, yielding a singular geometry. This shows that the connection between a solution to the equations of motion and the geometry is no longer clear.

To resolve the problem of classifying solutions we then turn to a new fundamental property, namely the *holonomy* around a closed loop. The holonomy measures the extent to which a coordinate system is changed by parallel transport on a manifold, and we show that it is gauge invariant under a trivial gauge transformation.

7.1 Chern-Simons as a higher spin theory

In the last section we formulated gravity as a Chern-Simons gauge theory on AdS$_3$ with the gauge group $SL(2) \times SL(2)$. To this regular theory of 2+1-dimensional gravity we couple a massless *higher spin field* to gravity by promoting the group $SL(2)$ to $SL(3)$. As we promote our group we keep the old generators from $sl(2)$, which is possible because $SL(2)$ is a subgroup of $SL(3)$. We also introduce five new symmetric traceless generators, T_{ab}. The purpose of this is that our Lagrangian will end up with terms that look exactly like the equations for spin-2 gravity, and we will have additional spin-3 terms that act as our additional degrees of freedom. Together with our previous three generators they form the Lie algebra $sl(3,\mathbb{R})$:

$$[T_a,T_b] = \epsilon_{ab}{}^{c}T_c \ , \tag{7.1}$$

$$[T_a,T_{bc}] = 2\epsilon^{d}{}_{a(b}T_{c)d} \ , \tag{7.2}$$

$$[T_{ab},T_{cd}] = -2\left(\eta_{a(c}\epsilon_{d)b}{}^{e} + \eta_{b(c}\epsilon_{d)a}{}^{e}\right)T_e \ . \tag{7.3}$$

With our gauge group being $SL(3) \times SL(3)$ we want to form the gauge connections A and $\bar{A}$ in analogy with section 6.1, which we restate here for convenience:

$$A = \omega + \frac{e}{l} \ , \qquad \bar{A} = \omega - \frac{e}{l} \ .$$

where under a gauge transformation with g in our gauge group, we have

$$A \rightarrow A' = g^{-1}Ag + g^{-1}\,\mathrm{d}g \ , \qquad \overline{A} \rightarrow \overline{A}' = gAg^{-1} + g\,\mathrm{d}g^{-1} \ .$$

To generalize the gauge connections we need to express the additional degrees of freedom gained from the promotion of $SL(2)$ to $SL(3)$ through generalized frame fields $e_\mu{}^{ab}$ and spin connections $\omega_\mu{}^{ab}$. We define

$$A = \mathrm{d}x^\mu\left(\omega^a_\mu + \frac{e^a_\mu}{l}\right)T_a + \mathrm{d}x^\mu\left(\omega_\mu{}^{bc} + \frac{e_\mu{}^{bc}}{l}\right)T_{bc} \ , \tag{7.4}$$

$$\tilde{A} = \mathrm{d}x^\mu\left(\omega^a_\mu - \frac{e^a_\mu}{l}\right)\bar{T}_a + \mathrm{d}x^\mu\left(\omega_\mu{}^{bc} - \frac{e_\mu{}^{bc}}{l}\right)\bar{T}_{bc} \ . \tag{7.5}$$

Our action is as usual

$$\mathcal{S}_{CS}[A,\overline{A}] = \mathcal{S}_{CS}[A] - \mathcal{S}_{CS}[\overline{A}] \ , \tag{7.6}$$

and the Chern-Simons equations of motion are given by:

$$F = \mathrm{d}A + A \wedge A = 0 \ .$$

We can, using the connections of equation (7.4), find the equations for the $SL(3)$ connection. We begin by writing down $F(A)$ for the A-connection in an appropriate form and can then very easily see what terms change sign for $\bar{A}$. We then add/subtract the equations of motion for A and $\bar{A}$ to obtain four independent equations of motion. The method described is performed explicitly in Appendix J.1. We state the resulting spin-3 equations of motion below:

$$\begin{aligned}
&\mathrm{d}e^a + \epsilon^a{}_{bc} e^b \wedge w^c - 4\epsilon^a{}_{fc} e^{bc} \wedge w^f{}_b = 0 \ , \\
&\mathrm{d}w^a + \frac{1}{2}\epsilon^a_{bc}\left(w^b \wedge w^c + \frac{e^b \wedge e^c}{l^2}\right) - 2\epsilon^a{}_{fc}\left(w^{bc} \wedge w^f{}_b + \frac{e^{bc} \wedge e^f{}_b}{l^2}\right) = 0 \ , \\
&\mathrm{d}e^{ab} + 2\epsilon^{ga(a|} w_a \wedge e_e{}^{|b)} + 2\epsilon^{ga(e|} e_a \wedge w_e{}^{|f)} = 0 \ , \\
&\mathrm{d}w^{ef} + 2\epsilon^{ag(e|} e_a \wedge e_g{}^{|f)} + 2\epsilon^{ga(e|} w_a \wedge w_e{}^{|f)} = 0 \ .
\end{aligned} \tag{7.7}$$

We could also have derived the spin-3 equations of motion from a corresponding action, similar to the discussion in the spin-2 case where we derived our structure equations by taking the variation of the Einstein-Hilbert action, see section 4.5. However, having already found the equations of motion for spin-3, we settle for finding the corresponding action instead.

From [29] we find that the action (7.6) can be rewritten according to

$$\mathcal{S}_{CS[A,\tilde{A}]} = \frac{k}{2\pi}\int_M \mathrm{tr}\left[e \wedge R + \frac{1}{3l^2} e \wedge e \wedge e\right] \ , \tag{7.8}$$

where l is the radius of curvature. As in the spin-2 case we wish to translate this action into the language of Cartan, i.e. in terms of vielbeins and spin connections, in order to get a modified Einstein-Hilbert action. After a rather cumbersome calculation we reach the following result:

$$\begin{aligned}
\mathcal{S}_{CS} = \frac{k}{\pi l}\int_M e^a \wedge R_a + \frac{1}{6l^2} e^a \wedge e^b \wedge e^c \epsilon_{abc} - 2e^a \wedge \omega^{bc} \wedge \omega_b{}^e \epsilon_{eca} + 2e^{ab} \wedge d\omega_{ab} \\
+ e^{ab}\epsilon_{ih(a|}\omega^h \wedge \omega^i_{|b)} - \frac{2}{l^2} e^a \wedge e^{bc} \wedge e_b{}^e \epsilon_{aec} \ .
\end{aligned}$$

The full derivation can be found in Appendix J. Note, when we set $k = -\frac{1}{8G}$ and $\frac{1}{l^2} = -\Lambda$ we partly recover our Einstein-Hilbert action. The rest of the terms are due to the extension to a higher spin field. Our derived equations of motion and the corresponding action are written in the same fashion as the e.o.m. and action as presented in Campoleoni et al' [30].

In the Cartan formalism of gravity we have Lorentz transformations and spatial translations that transform our coordinates, vielbeins and spin connections. We should now expect these transformations to be generated by gauge transformations. This is the case for $SL(2) \times SL(2)$ as was shown by James H. Horne (1964-2012) and Edward Witten in 1989 [31]. We will instead treat the $SL(3) \times SL(3)$ case. A vector is a tensor, thus it transforms according to $v'_\mu(x') = \frac{dx^\mu}{dx'^\mu} v_\mu(x)$ under a coordinate transformation $x^\mu \to x'^\mu$. By simply interchanging x and x' we find that

$$v'^a_\mu(x) = \frac{dx'^\nu}{dx^\mu} v^a_\nu(x') = (\delta^\nu_\mu + \partial_\mu \xi^\nu)(v^a_\nu(x) + \xi^\rho \partial_\rho v^a_\nu(x)) = v^a_\mu(x) + \xi^\rho \partial_\rho v^a_\mu(x) + (\partial_\mu \xi^\nu) v^a_\nu(x) \ ,$$

to first order in ξ. The variation becomes

$$\delta_\xi v^a_\mu(x) = \partial_\mu(\xi^\nu v^a_\nu(x)) + \xi^\nu(\partial_\nu v^a_\mu(x) - \partial_\mu v^a_\nu(x)) \ .$$

Let us now turn to the effect of a gauge transformation with gauge group element $g = \exp(\Lambda)$, where Λ is a gauge parameter. The variation of the connection due to an infinitesimal gauge transformation is

$$\delta A = A' - A = g^{-1}Ag + g^{-1}\,\mathrm{d}g - A = \mathrm{d}\Lambda + [A,\Lambda] \ .$$

$\overline{A}$ transform almost identically but with a different transformation $\bar{g} = \exp(\overline{\Lambda})$. The variation of the vielbeins and spin connections are

$$\delta e = \frac{\delta A - \delta \overline{A}}{2} = \mathrm{d}\Lambda_- + [e,\Lambda_+] + [\omega,\Lambda_-] \ , \qquad \delta\omega = \frac{\delta A + \delta\overline{A}}{2} = \mathrm{d}\Lambda_+ + [e,\Lambda_-] + [\omega,\Lambda_+] \ ,$$

where $\Lambda_\pm = \frac{\Lambda \pm \overline{\Lambda}}{2}$. This is really four equations since we have that $e_\mu = e_\mu{}^a + e_\mu{}^{bc}$, and similarly for ω_μ. The complete explicit expressions are rather long but the full calculation can be found in Appendix J.3. We will be content with stating the fact that if we pick $\Lambda_\pm = \tau^a_\pm T^a + \tau^{bc}_\pm T^{bc}$ with $\tau^a_+ = \xi^\rho \omega_\rho{}^a$ and $\tau^a_- = \xi^\rho e_\rho{}^a$, then all differences between coordinate transformations and gauge transformations, i.e. $\delta e^a - \delta_\xi e^a$ vanishes precisely when the e.o.m. (7.7) are satisfied.

7.2 Gauge transformations in higher spin gravity

A key ingredient when it comes to describing gravity as a Chern-Simons theory is that the theory should possess gauge symmetries, i.e. be invariant under gauge transformations. However, in higher spin gravity we are describing a gravitational theory coupled to massless higher spin fields, in this case a spin-3 field, and in general performing a gauge transformation will mix the ordinary metric and the spin-3 field in a complicated way. Thus standard invariant notions, e.g. curvature invariants and causality associated with the metric, are no longer gauge invariant quantities. In this way we may construct a singular metric from a smooth and regular metric by simply performing a gauge transformation, and the other way around. This is done explicitly in this section. The discussion below follows closely the discussion given in Castro et als' "Black Holes and Singularity Resolution in Higher Spin Gravity" [32].

The $sl(3,\mathbb{R})$ gauge connections A and $\overline{A}$, as defined in equation (7.4), can be rewritten by introducing a spin-2 connection $A_{(2)}$, according to

$$A = A_{(2)} + \mathrm{d}x^\mu \left(\omega_\mu{}^{bc} + \frac{e_\mu{}^{bc}}{l}\right) T_{bc} \ , \qquad \overline{A} = \overline{A}_{(2)} + \mathrm{d}x^\mu \left(\omega_\mu{}^{bc} - \frac{e_\mu{}^{bc}}{l}\right) T_{bc} \ , \tag{7.9}$$

where $A_{(2)} = \mathrm{d}x^\mu \left(\omega^a_\mu + \frac{e^a_\mu}{l}\right) T_a$ and $\overline{A}_{(2)} = \mathrm{d}x^\mu \left(\omega^a_\mu - \frac{e^a_\mu}{l}\right) \bar{T}_a$. Having defined these gauge fields we may construct the spacetime metric $g_{\mu\nu}$ and the corresponding spin-3 field $\psi_{\mu\nu\rho}$ as follows

$$g_{\mu\nu} = \frac{1}{2}\,\mathrm{tr}[e_\mu e_\nu] \ , \qquad \psi_{\mu\nu\rho} = \frac{1}{9}\,\mathrm{tr}\left[e_\mu e_\nu e_\rho\right] \ , \tag{7.10}$$

where the vielbein is given by $e = \frac{1}{2}(A - \overline{A})$.

Consider the purely spin-2 gauge connections[1]

$$A_{(2)} = (\mathrm{e}^\rho L_1 - \mathcal{L}\mathrm{e}^{-\rho} L_{-1})dx^+ + L_0 d\rho \ , \tag{7.11}$$

$$\overline{A}_{(2)} = -(\mathrm{e}^\rho L_{-1} - \mathcal{L}\mathrm{e}^{-\rho} L_1)dx^- - L_0 d\rho \ , \tag{7.12}$$

where $x^\pm = t \pm \phi$ and $\phi \sim \phi + 2\pi$. Here L_1, L_0 and L_{-1} are generators formed by linear combinations of the regular generators T_a, T_b etc. to $sl(2,R)$, see Appendix E.5.2. Given these gauge connections we may construct a metric by using equations (7.10). Thus we need to compute the vielbeins:

$$e = \frac{1}{2}(A_{(2)} - \overline{A}_{(2)}) = L_0 d\rho + \frac{1}{2}\Big((\mathrm{e}^\rho - \mathcal{L}\mathrm{e}^{-\rho})(L_1 + L_{-1})dt + (\mathrm{e}^\rho + \mathcal{L}\mathrm{e}^{-\rho})(L_1 - L_{-1})d\phi\Big) \ . \tag{7.13}$$

Before taking the trace we also need to compute the product e^2:

$$e^2 = (L_0)^2 d\rho^2 + \frac{1}{4}\Big((\mathrm{e}^\rho - \mathcal{L}\mathrm{e}^{-\rho})^2(L_1 + L_{-1})^2 dt^2 + (\mathrm{e}^\rho + \mathcal{L}\mathrm{e}^{-\rho})^2(L_1 - L_{-1})^2 d\phi^2\Big) + K_{mixed} \ , \tag{7.14}$$

where K_{mixed} are terms containing mixed differentials ($dtd\phi$ etc.). The only non-zero traces are

$$\mathrm{tr}\left[L_0 L_0\right] = 2 \ , \qquad \mathrm{tr}\left[L_1 L_{-1}\right] = \mathrm{tr}\left[L_{-1} L_1\right] = -4 \ , \tag{7.15}$$

[1]The stated gauge connections are supplied with a subscript "(2)", indicating they are only coupled to spin-2 generators. Thus we consider the special case $A = A_{(2)}$.

as seen in Appendix E.5.2. Hence $\mathrm{tr}[K_{mixed}]$ vanishes and we may finally write down the metric:

$$ds^2 = g_{\mu\nu}dx^\mu dx^\nu = \frac{1}{2}\,\mathrm{tr}[e_\mu e_\nu]dx^\mu dx^\nu = d\rho^2 - (\mathrm{e}^\rho - \mathcal{L}\mathrm{e}^{-\rho})^2dt^2 + (\mathrm{e}^\rho + \mathcal{L}\mathrm{e}^{-\rho})^2d\phi^2 \ . \tag{7.16}$$

The real parameter $\mathcal{L}$ does actually determine whether the metric is that of a black hole or contain conical singularities. In particular, we shall see that a certain choice of $\mathcal{L}$ leads to the metric being equivalent to global AdS$_3$. In order to be able to compare this metric with our BTZ black hole metric as stated in the Chapter 5, equation (5.18), we perform a change of variables. Let

$$(e^\rho + \mathcal{L}\mathrm{e}^{-\rho})^2 = r^2 \Rightarrow d\rho = \frac{dr}{(e^\rho - \mathcal{L}e^{-\rho})} \ .$$

Since $(e^\rho + \mathcal{L}\mathrm{e}^{-\rho})^2 = e^{2\rho} + 2\mathcal{L} + \mathcal{L}^2\mathrm{e}^{-2\rho} = r^2$, it follows that $(e^\rho - \mathcal{L}\mathrm{e}^{-\rho})^2 = e^{2\rho} - 2\mathcal{L} + \mathcal{L}^2\mathrm{e}^{-2\rho} = r^2 - 4\mathcal{L}$. Thus the metric in our new radial variable is

$$\tilde{ds}^2 = -dt^2(r^2 - 4\mathcal{L}) + dr^2(r^2 - 4\mathcal{L})^{-1} + r^2d\phi^2 \ .$$

Upon comparison with our original BTZ metric, equation (5.18), we note that whenever $\mathcal{L} \geq 0$ we have a BTZ black hole. Moreover we conclude that the case $\mathcal{L} = -\frac{1}{4}$ corresponds to global AdS$_3$. In analogue with the discussion in Section 5.3, we have a conical singularity in our metric whenever $\mathcal{L} \in (-\frac{1}{4},0)$.
Having found and analyzed our metric we are now ready to perform a gauge transformation of our gauge connections A and $\overline{A}$. In the spin-2 case ($SL(2,\mathbb{R})$) performing a gauge transformation will just lead to a *diffeomorphism*[2] of the regular metric $g_{\mu\nu}$. However, in higher spin (here $n = 3$ and $SL(3,\mathbb{R})$) we will see that the properties of the metric will change dramatically and we will also get a non-zero spin-3 field. When performing a (trivial) gauge transformation we want to be able to write our modified gauge connection A' as

$$A' = g_{new}^{-1}\,\mathrm{d}g_{new} \ ,$$

where $g_{new} = g_{old}g_\Lambda$ and g_Λ is the gauge transformation matrix. Remember that our original gauge connection A where constructed from the condition $A = g_{old}^{-1}\,\mathrm{d}g_{old}$. We now let A' and $\overline{A}'$ be of the form

$$A' = b^{-1}(L_1 - \mathcal{L}L_{-1} + \alpha W_{-1})dx^+b + b^{-1}\,\mathrm{d}b \ ,$$

$$\overline{A}' = -b(L_{-1} - \mathcal{L}L_1 + \overline{\alpha}W_1)dx^-b^{-1} + b\,\mathrm{d}b^{-1} \ ,$$

with $b = \mathrm{e}^{\rho L_0}$ and α constant. W_1 and W_{-1} are generators formed from linear combinations of the generators of $SL(3,R)$, see Appendix E.5.2. Given the explicit representations of A and A', the group elements g_{old} and g_{new} are known. In particular, they are

$$\begin{aligned} g_{old} &= \mathrm{e}^{\rho L_0}\mathrm{e}^{x^+(L_1 - \mathcal{L}L_{-1})} \ , \\ g_{new} &= \mathrm{e}^{x^+(L_1 - \mathcal{L}L_{-1} + \alpha W_{-1})}\mathrm{e}^{\rho L_0} \ , \end{aligned}$$

which are seen to hold from the conditions $A' = g_{new}^{-1}\,\mathrm{d}g_{new}$ and $A = g_{old}^{-1}\,\mathrm{d}g_{old}$. From this we can now solve for g_Λ which relate the group elements g_{old} and g_{new}:

$$g_\Lambda = g_{old}^{-1}g_{new} = \mathrm{e}^{-\rho L_0}\mathrm{e}^{-x^+(L_1 - \mathcal{L}L_{-1})}\mathrm{e}^{x^+(L_1 - \mathcal{L}L_{-1} + \alpha W_{-1}))}\mathrm{e}^{\rho L_0} \ . \tag{7.17}$$

We may, by expanding the exponential in a Taylor series, compute the matrix g_Λ explicitly.[3] One finds that all off-diagonal terms are zero, why we only state the non-zero diagonal terms below:

$$\begin{cases} g_\Lambda^{00} = \frac{1}{8}\mathrm{e}^{-ix^+}\left(1 + \mathrm{e}^{ix^+}(2 - 8\alpha) + 4\alpha + \mathrm{e}^{2ix^+}(1 + 4\alpha)\right)(1 + \cos x^+) \\ g_\Lambda^{11} = \frac{1}{2}\mathrm{e}^{-ix^+}(1 + \mathrm{e}^{2ix^+})\cos x^+ \\ g_\Lambda^{22} = -\frac{1}{8}\mathrm{e}^{-ix^+}\left(-1 + 4\alpha + \mathrm{e}^{2ix^+}(-1 + 4\alpha) - 2\mathrm{e}^{ix^+}(1 + 4\alpha)\right)(1 + \cos x^+) \end{cases} ,$$

[2]A diffeomorphism is an isomorphism of a smooth manifold. It can be interpreted as an invertible function mapping one differential manifold to another in such a way that the function and its inverse are non-singular. In other words, the metric remains regular when performing a gauge transformation in the spin-2 case.

[3]These calculations were also performed with the use of the symbolic computation program *Mathematica*, which moreover was used to greatly simplify the result.

where we recall that $x^+ = t+\phi$. From the stated terms above it is evident that the matrix g_Λ is invariant under the identification $\phi \sim \phi + 2\pi$ (the complex exponential and the cosine are both 2π-periodic), and thus g_Λ constitutes a trivial gauge transformation.

In the same fashion as before we may construct a corresponding metric to A'. We simply state the result here:

$$ds^2 = d\rho^2 - \left((\mathrm{e}^\rho - \mathcal{L}\mathrm{e}^{-\rho})^2 - \alpha^2\mathrm{e}^{-2\rho}\right)dt^2 + \left((\mathrm{e}^\rho + \mathcal{L}\mathrm{e}^{-\rho})^2 - \alpha^2\mathrm{e}^{-2\rho}\right)d\phi^2 \ , \tag{7.18}$$

where we have chosen $\alpha = -\overline{\alpha}$. With the connections A' and $\overline{A}'$ we will, on the contrary to our former connections, have a non-zero spin-3 field:

$$\psi_{\mu\nu\rho} = -8\alpha d\rho d\phi dt \ , \tag{7.19}$$

which can be computed from the relation $\psi_{\mu\nu\rho} = \frac{1}{9}\,\mathrm{tr}\,[e_\mu e_\nu e_\rho]$, but this is a rather tedious process why we simply rely on [32]. It may not be obvious that the metric (7.18), derived from the gauge connection A', possesses very different properties compared to our original metric as given in equation (7.16). However, if we restrict to the case $\mathcal{L} = -\frac{1}{4}$ i.e. AdS$_3$, this will become evident. For convenience we restate the original metric in the case $\mathcal{L}$ (here denoted with subscript "AdS"):

$$ds^2_{\mathrm{AdS}} = d\rho^2 - (\mathrm{e}^\rho + \frac{1}{4}\mathrm{e}^{-\rho})^2dt^2 + (\mathrm{e}^\rho - \frac{1}{4}\mathrm{e}^{-\rho})^2d\phi^2 \ .$$

This metric is completely smooth and its individual components ($g_{tt}, g_{\phi\phi}$ etc.) are non-zero for all values of ρ, i.e. it has no singularities. If we instead consider our new metric derived from the gauge equivalent connections A' and $\overline{A}'$, we eventually notice something non regular:

$$ds'^2_{\mathrm{AdS}} = d\rho^2 - \left((\mathrm{e}^\rho + \frac{1}{4}\mathrm{e}^{-\rho})^2 - \alpha^2\mathrm{e}^{-2\rho}\right)dt^2 + \left((\mathrm{e}^\rho - \frac{1}{4}\mathrm{e}^{-\rho})^2 - \alpha^2\mathrm{e}^{-2\rho}\right)d\phi^2 \ .$$

For a particular value of $\rho > 0$ the components g'_{tt} and $g'_{\phi\phi}$ may vanish, leading to a singular metric.[4] Thus to wrap this discussion up, we started with a completely smooth spacetime (and a vanishing spin-3 field) and ended up with a singular spacetime (and a non-zero spin-3 field) simply by performing a trivial gauge transformation.

It may not seem so desirable to get a singular spacetime from a regular one. We would probably be more delighted if we were able to deform a singular spacetime into a smooth spacetime. Amazingly this is possible for the singularity of the Milne universe as discussed in [29].

7.3 Holonomies

In the last section we found that when we introduced a higher spin field we where able to deform or resolve singularities in the metric using gauge transformations. This implies that we cannot use the metric to label different solutions, instead we need a better tool: *holonomies*. To describe these we will make use of parallel transport, introduced in 4.1. Parallel transport can be explained through a transport of a vector $U(t)$, with $t \in (0,T)$, from an original position x_0 along a path $\gamma(t)$, while keeping the vector constant in a local frame. The requirement that the vector is constant locally may be realized by the differential equation

$$D_\gamma U(t) = 0 \implies \frac{d}{dt}U(t) = A(\gamma'(t))U(t) \ . \tag{7.20}$$

With the initial condition $U(0) = 1$, we can solve this equation implicitly by:

$$U(t) = 1 + \int_0^t A(\gamma'(t_1))U(t_1)dt_1 \ .$$

Now this equation does not help us to find $U(t)$ since it appears both in the left-hand side and the right-hand side. However, we can plug the right-hand side into the left-hand side. Doing this gives us

$$U(t) = 1 + \int_0^t A(\gamma'(t_1))dt_1 + \int_0^t\int_0^t A(\gamma'(t_1))A(\gamma'(t_2))U(t_2)dt_1dt_2 \ .$$

[4]The particular value of ρ can be computed as a function of α. This can be done by performing the change of variables $\mathrm{e}^\rho = x$ which leads to a fourth degree polynomial equation with only even powers of x, solvable by algebraic means!

At first this might seem pointless. But if we continue on and do this an infinite amount of times we will find that the series converges! Thus the last term on the right-hand side containing the function U must vanish in the limit. We may therefore write

$$U(t) = \sum_{0}^{\infty} \int_{t>t_1...>t_n} A(\gamma'(t_1))...A(\gamma'(t_n))dtdt_1...dt_n \ .$$

By introducing the path ordering symbol, $\mathcal{P}$, which acts according to

$$\int_{t>t_1...>t_n} A(\gamma'(t_1))...A(\gamma'(t_n))dtdt_1...dt_n = \frac{1}{n!}\mathcal{P}\Big(\int A(\gamma'(s))ds\Big)^n \ ,$$

we can rewrite the above expression as

$$U(t) = \mathcal{P}\exp\Big(\int_\gamma A\Big) \ .$$

The use of an exponential is furthered motivated by the fact that if we let A be independent of position, and thus abelian, we may remove the path ordering prescription. This follows since then (7.20) can easily be solved according to

$$\frac{d}{dt}U(t) = AU(t) \implies U(t) = \exp(tA) \ .$$

If γ is a smooth closed curve this is the holonomy around that curve, which we will denote by

$$\mathrm{Hol}_\gamma(A) = \mathcal{P}\exp\Big(\oint_\gamma A\Big) \ .$$

Having introduced holonomies we will now specialize to our specific case of a Chern-Simons theory. Since the equations of motion implies that $F = 0$ we have a flat connection. This fact is essential because if the connection is flat two homotopic curves have the same holonomy. This result immediately tells us that the holonomy of all contractible curves is the identity since they have the same holonomy as a point. However, if the curve is non-contractible we may have a non-trivial holonomy. This fact is crucial when we construct our connection. Since $F = 0$ it is tempting to believe that our connection will be pure gauge, i.e $A = g^{-1}\,\mathrm{d}g$. However, if we transport this around a non-contractible curve our connection will pick up a factor of the holonomy and thus g must be a multivalued function. This fact allows us to classify connections depending on their holonomies around a non-contractible curve. This will be the way we label solutions from now on, not with the metric. We will in the rest of this thesis only consider connections with one non-contractible curve. As an example, let us return to the connection (7.11) and study the holonomy around the ϕ-cycle. To simplify our calculations we want to use a gauge in which we eliminate all explicit dependence on ρ. This is possible if we set

$$A = b^{-1}(L_1 - \mathcal{L}L_{-1})dx^+ b + b^{-1}\,\mathrm{d}b \ , \tag{7.21}$$

with $b = \exp(\rho L_0)$. If this is to be the same connection we must have

$$b^{-1}L_1 b = e^\rho L_1 \ , \qquad b^{-1}L_{-1}b = e^{-\rho}L_{-1} \ . \tag{7.22}$$

We will calculate the first term explicitly. The goal is to move the exponential b past L_1 and cancel it with b^{-1}. The definition of an exponentiated matrix is $e^X = \sum \frac{X^n}{n!}$, so it follows that $L_1 b = L_1 \sum \frac{(\rho L_0)^n}{n!}$. Acting with L_1 termwise and using the commutator of L_0 and L_1 we see that

$$L_1 L_0^n = (L_0 L_1 + [L_1{,}L_0])L_0^{n-1} = (L_0+1)L_1 L_0^{n-1} = ... = (L_0+1)^n L_1 \ ,$$

and accordingly we must have

$$L_1 b = L_1 \sum \frac{(\rho L_0)^n}{n!} = \Big(\sum \frac{(\rho(L_0+1))^n}{n!}\Big)L_1 = be^\rho L_1 \ .$$

The calculation for L_{-1} is identical up to a sign, $L_{-1}b = be^{-\rho}L_{-1}$. From this the relations (7.22) follows immediately. Let us now finally consider the holonomy of (7.21) around the ϕ-cycle, that is

$$\mathrm{Hol}_\phi(A) = \mathcal{P}\exp(A_\phi d\phi) = \exp(2\pi(L_1 - \mathcal{L}L_{-1})) \ .$$

Since we have not yet involved higher spin fields we can pick the 2×2 matrix representation of $SL(2,R)$ and get

$$\exp(2\pi(L_1-\mathcal{L}L_{-1}))=\exp\left(\underbrace{2\pi\begin{bmatrix}0 & \mathcal{L}\\ 1 & 0\end{bmatrix}}_{X}\right) .$$

Writing out the first four terms in the expansion $e^X=\sum\frac{X^n}{n!}$, we find

$$X=2\pi\begin{bmatrix}0 & \mathcal{L}\\ 1 & 0\end{bmatrix} ,\qquad X^2=(2\pi)^2\begin{bmatrix}\mathcal{L} & 0\\ 0 & \mathcal{L}\end{bmatrix} ,\qquad X^3=(2\pi)^3\begin{bmatrix}0 & \mathcal{L}^2\\ \mathcal{L} & 0\end{bmatrix} ,\qquad X^4=(2\pi)^4\mathcal{L}^2\begin{bmatrix}1 & 0\\ 0 & 1\end{bmatrix} .$$

We can see that because of the identity matrices we get a repeating sequence. Using this we find that

$$\exp(X)=\sum_{n=0}\frac{X^n}{n!}=\begin{bmatrix}\sum\frac{\left(2\pi\sqrt{\mathcal{L}}\right)^{2n}}{(2n)!} & \sqrt{\mathcal{L}}\sum\frac{\left(2\pi\sqrt{\mathcal{L}}\right)^{2n+1}}{(2n+1)!}\\ \frac{1}{\sqrt{\mathcal{L}}}\sum\frac{\left(2\pi\sqrt{\mathcal{L}}\right)^{2n+1}}{(2n+1)!} & \sum\frac{\left(2\pi\sqrt{\mathcal{L}}\right)^{2n}}{(2n)!}\end{bmatrix}=\begin{bmatrix}\cosh 2\pi\sqrt{\mathcal{L}} & \sqrt{\mathcal{L}}\sinh 2\pi\sqrt{\mathcal{L}}\\ \frac{1}{\sqrt{\mathcal{L}}}\sinh 2\pi\sqrt{\mathcal{L}} & \cosh 2\pi\sqrt{\mathcal{L}}\end{bmatrix} , \tag{7.23}$$

which is the same result as reached by Bañados, Castro et al [33]. We showed earlier that we get standard AdS_3 if we set $\mathcal{L}=-\frac{1}{4}$. The holonomy becomes $\mathrm{Hol}_\phi(A_{\mathrm{AdS}})=-\mathbf{1}$. Now this is minus the identity, and not the identity but it is still in the center of the group. Thus it is possible to interpret our result as a trivial holonomy, exactly what is expected from empty AdS_3 [32].

It is now possible to classify solutions with one non-contractible cycle using the holonomy. Remember that a holonomy is only defined up to conjugation. However, all matrices that are related by conjugation have the same eigenvalues. The characteristic equation for the holonomy matrix (7.23) is

$$\det(\exp(X)-\lambda I)=1-2\lambda\cosh 2\pi\sqrt{\mathcal{L}}+\lambda^2=0 \implies \lambda=\begin{cases}e^{2\pi\sqrt{\mathcal{L}}}\\ e^{-2\pi\sqrt{\mathcal{L}}}\end{cases} .$$

We earlier discussed how different values of $\mathcal{L}$ gave rise to different solutions. Using this we may identify BTZ solutions as those with real, non-degenerate eigenvalues ($\mathcal{L}>0$). Conical singularities have imaginary non-degenerate eigenvalues ($\mathcal{L}<0$). The special case of $\mathcal{L}=0$ led to degenerate eigenvalues and is usually interpreted as an extremal black hole [33]. In many cases it might be cumbersome or even impossible to evaluate the holonomy matrix exactly, as was done above. But often we do not need the full matrix and will be satisfied with knowledge of the eigenvalues of the exponentiated matrix, that is $\mathcal{P}\left(\oint A\right)$. To easily classify these eigenvalues we use the *Cayley-Hamilton theorem* which states that every matrix satisfies its own characteristic equation. This means that we may express X, a general 3×3 matrix, as

$$X^3=\alpha+\beta X+\gamma X^2 , \tag{7.24}$$

where α,β and γ are (in general) complex coefficients. To find the coefficients we consider the general characteristic equation of X, explicitly $(\lambda-\lambda_1)(\lambda-\lambda_2)(\lambda-\lambda_3)=0$, where λ_i is the i:th eigenvalue of X. Expanding and rearranging in powers of λ, we find

$$\lambda^3=\lambda_1\lambda_2\lambda_3-(\lambda_1\lambda_2+\lambda_2\lambda_3+\lambda_1\lambda_3)\lambda+(\lambda_1+\lambda_2+\lambda_3)\lambda^2 .$$

We can now use the identities

$$\mathrm{tr}[X]=\sum_i\lambda_i ,\qquad \det(X)=\Pi_i\lambda_i , \tag{7.25}$$

to identify the coefficients in (7.24) as

$$\alpha=\det(X) ,\qquad \beta=-\frac{1}{2}\Big(\mathrm{tr}^2[X]-\mathrm{tr}[X^2]\Big) ,\qquad \gamma=\mathrm{tr}[X] .$$

If we let $X\in sl(3)$ we have that $\mathrm{tr}[X]=0$ due to the fact that the generators of $sl(3)$ are traceless. The defining equation then becomes

$$X^3=\det(X)+\frac{1}{2}\,\mathrm{tr}[X^2]X . \tag{7.26}$$

This implies that instead of finding the explicit holonomy we can compute $\det(X)$ and $\frac{1}{2}\operatorname{tr}[X^2]$. These coefficients are referred to as the *holonomy invariants* and are often denoted with Θ_i, where $\Theta_0 = \det(X)$ and $\Theta_1 = \frac{1}{2}\operatorname{tr}[X^2]$. We will in the next chapter often consider the case of a trivial holonomy around a contractible cycle. If a holonomy is trivial the exponentiated matrix, $X = \mathcal{P}\left(\oint A\right)$, must have eigenvalues $0, 2\pi i, -2\pi i$. Hence, using (7.25), it follows that

$$\det(X) = 0 \ , \qquad \operatorname{tr}[X^2] + 8\pi^2 = 0 \ . \tag{7.27}$$

Chapter 8

Black Holes and Wormholes in Higher Spin Gravity

Having developed the theory of spin-2 and spin-3 gravity, and having introduced holonomies, we are now ready to start investigating spin-3 black hole and wormhole solutions in 2+1 dimensions. First, we find a higher spin black hole, that is a black hole that carries higher spin charge. In the limit where the spin-3 field is turned off it is a spin-2 BTZ solution. We then use a trivial gauge transformation to resolve a conical singularity resulting in a wormhole geometry. Similarly we can transition between a black hole and a wormhole using the same gauge transformation. Finally, we study a black hole in a wormhole gauge, which may also be interpreted as a traversable wormhole.

We have already seen how Chern-Simons gauge theory with gauge group $SL(2) \times SL(2)$ is classically equivalent to 2+1-dimensional Einstein gravity formulated on AdS_3.[1] When generalizing this spin-2 gravity theory to a spin-3 gravity theory we extend the gauge group of our Chern-Simons gauge theory to $SL(3) \times SL(3)$. This will be equivalent to AdS_3 Einstein gravity coupled to two additional higher spin degrees of freedom, which we will refer to as higher spin charges. However, depending on how the spin-2 subgroup $SL(2) \times SL(2)$ is embedded into the full spin-3 $SL(3) \times SL(3)$ gauge group, we can interpret a variety of different higher spin theories from our Chern-Simons gauge theory. Hence, a black hole solution may be interpreted differently depending on this embedding [32]. This raises difficult questions about the physical meaning of higher spin gravity solutions. Furthermore, as mentioned in the last chapter, the metric is not an adequate quantity to classify solutions as a trivial gauge transformation may drastically alter its characteristic properties. For example, a black hole metric can be transformed into a metric which cannot be interpreted as a black hole. Thus, the metric is not an observable physical quantity in higher spin theories, as such physical quantities are necessarily gauge invariant. Instead, the observable physical quantity that will specify a black hole or wormhole solution is the holonomy around a non-contractible cycle, i.e. a cycle around the singularity. Such an holonomy will be non-trivial. In this thesis we will restrict ourselves to one of the possible embeddings of $SL(2) \times SL(2)$ into $SL(3) \times SL(3)$, namely the *principal embedding*.

8.1 Black holes in spin-3 gravity

Before looking for black hole solutions in our spin-3 gravity theory, we need to be more precise about what we mean by a spin-3 black hole. We may expect the metric of a black hole to have an event horizon. The only black hole solution in 2+1 dimensions we have encountered so far, the BTZ black hole, did not have a curvature singularity, see section 5.3. We therefore have no reason to expect a black hole solution in spin-3 gravity in 2+1 dimensions to have a curvature singularity.

Since the metric is not a gauge invariant quantity, a given connection may have multiple geometrical interpretations. This gauge symmetry may be exploited to put the metric in a form which allows it to be interpreted as a black hole. Still, we need to ask ourselves what the properties are of an interesting black hole solution in higher spin theory. The black hole solutions we will be interested in are required to satisfy the following properties:

- The black hole should have a smooth BTZ limit.
- It should have a Lorentzian horizon and a regular Euclidean continuation.
- It must allow for a thermodynamical interpretation.

These conditions are the same as the ones stated in [34]. The first property is perhaps the most natural one. We already know that the BTZ black hole is a solution to spin-2 gravity theory in 2+1 dimensions.

[1] Technically, the gauge group corresponding to a negative cosmological constant is $SL(2) \times SL(2)/\mathbb{Z}_2$. For our purposes it makes no difference in omitting the $\mathbb{Z}_2$ factor.

The two extra higher spin charges arose from two additional degrees of freedom, and should therefore be independent. Hence, we should be able to smoothly set each of them to zero while keeping the other degrees of freedom finite, i.e., the mass and angular momentum of the black hole. The second and third conditions are more subtle, and we will completely omit a treatment and explanation of the latter.

A defining property of a black hole in general is that it has a smooth horizon. We will choose a set of coordinates (t,ρ,ϕ). The ρ coordinate is a radial coordinate, i.e., $\rho \in [0,\infty)$. The ϕ coordinate is an angular coordinate periodic with period 2π. The horizon is located at some radial distance ρ_h in this coordinate system, and at the horizon the time component of the metric vanishes. A Lorentzian signature can conveniently be turned into a Euclidean signature by the Euclidean time, t_E, defined by

$$t = it_E \ , \tag{8.1}$$

where t is the Lorentzian time coordinate.[2] Our condition above about a regular Euclidean continuation requires that the black hole solution has a smooth geometry in Euclidean signature at ρ_h. This requires the Euclidean time and angular coordinates to have a definite periodicity,

$$(t_E,\phi) \sim (t_E,\phi) + 2\pi(\beta,\beta\Omega) \ ,$$

where β[3] and Ω are the temperature and angular potential, respectively [35]. These periodicities can be conveniently formulated by introducing a new variable $z = \phi + it_E$. Then we can write

$$z \sim z + 2\pi\tau \ ,$$

where $\tau = \beta\Omega + i\beta$.

The condition about a regular Euclidean geometry at the horizon is realized by statements involving the coordinates, and hence the metric. But we know that the metric is not gauge invariant and hence the above conditions about the periodicities of t_E and ϕ are not necessarily gauge invariant. However, it is possible to translate the conditions about the periodicities of t_E and ϕ into a gauge invariant condition on the holonomy. The Euclidean geometry has two different cycles; the spatial cycle $\phi \sim \phi + 2\pi$ and the thermal cycle $z \sim z + 2\pi\tau$. The non-contractible spatial cycle has to do with the size of the horizon and the mass, angular momentum and the two higher spin charges defined by the holonomy around the spatial cycle. The requirement of a smooth Euclidean horizon implies that the holonomy around the thermal cycle is trivial. More explicitly, this condition can be formulated as

$$\mathrm{Hol}_\tau(A) = \pm\mathbf{1} \ , \qquad \mathrm{Hol}_\tau(\bar{A}) = \pm\mathbf{1} \ , \tag{8.2}$$

where $\mathbf{1}$ is the identity element of the gauge group and the holonomies are taken over the thermal cycle, as denoted by the subscript. These conditions, which will be referred to as the *trivial holonomy constraint*, can be seen as equations which define τ and constrain the connections A and $\bar{A}$. One remark about the sign of the identity element in (8.2) is in order. Minus the identity also acts trivially on the field and simply reflects the fact that the we have omitted the factor of $\mathbb{Z}_2$ in the subgroup $SL(2) \times SL(2)$. In doing so we have effectively obtained a gravity theory whose solutions are manifolds equipped with a spin structure. Having a holonomy around the thermal cycle equal to minus the identity element means that half integer spin particles will pick up a minus sign when translated around the contractible thermal cycle, which is equivalent to a rotation by 2π around the horizon. This is a characteristic property of a spin structure. However, for the principal embedding, the corresponding spin-3 theory does not have a spin structure. The trivial holonomy constraints (8.2) then simply states that the holonomies around the thermal cycle is *exactly* equal to the identity element of the gauge group [32].

Implementing the trivial holonomy constraint or calculating the holonomy around the spatial cycle may be very difficult in practice. A way around this difficulty is to express the holonomy matrix by its characteristic polynomial as in equation (7.26). First however, the connections we will consider allows us to simplify the holonomy matrix a bit.

We will exclusively work with connections of the form

$$A = b^{-1}ab + b^{-1}\mathrm{d}b \ , \qquad \bar{A} = b\bar{a}b^{-1} + b\mathrm{d}b^{-1} \ , \tag{8.3}$$

[2] This transformation was used (and "invented") by Gian-Carlo Wick (1909-1992) in his method of finding a solution in Minkowski space from a solution in Euclidean space and vice versa. Thus the transformation $t \to it$ in the Minkowski metric leading to an Euclidean metric (and the other way around) is referred to as *Wick rotation*.

[3] This is the inverse of the Boltzmann constant times the temperature, as usually seen in statistical physics.

where a, $\bar{a}$ and $b \in sl(3)$, and "d" is the exterior derivative. The holonomy of A around the spatial cycle is defined as

$$\mathrm{Hol}_\phi(A) = \mathcal{P}\exp\left(\oint A_\phi d\phi\right) , \tag{8.4}$$

where A_ϕ is the ϕ component of A, i.e. the part of A proportional to $d\phi$. Of course, an analogous expression holds for the holonomy of $\bar{A}$ around the spatial cycle. If the connection A does not depend on ϕ, then the integral may be evaluated directly:

$$\mathcal{P}\exp\left(\oint A_\phi d\phi\right) = \exp(2\pi A_\phi) . \tag{8.5}$$

For the particular connections (8.3) we will be considering, $A_\phi = b^{-1}a_\phi b$, where a_ϕ is the ϕ component of a, and by expanding the exponential (8.5) in its Taylor series expansion we find

$$\exp(2\pi A_\phi) = \exp\big(2\pi b^{-1}a_\phi b\big) = \sum_{n=0}^{\infty}\frac{(2\pi b^{-1}a_\phi b)^n}{n!} = b^{-1}\sum_{n=0}^{\infty}\frac{(2\pi a_\phi)^n}{n!}b = b^{-1}\exp(2\pi a_\phi)b .$$

Furthermore, flat connections are only uniquely specified by their holonomies around the non-contractible cycles of spacetime up to an overall gauge transformation, see section 7.3. If the eigenvalues of a and $\bar{a}$ are non-degenerate, the holonomy of A and $\bar{A}$ around the spatial cycle is specified by the invariants

$$\Theta_{0,A} = 2\pi\mathrm{det}[a_\phi] , \qquad \Theta_{1,A} = 2\pi^2\mathrm{tr}[a_\phi^2] , \tag{8.6}$$

and

$$\Theta_{0,\bar{A}} = 2\pi\mathrm{det}(\bar{a}_\phi) , \qquad \Theta_{1,\bar{A}} = 2\pi^2\mathrm{tr}(\bar{a}_\phi^2) , \tag{8.7}$$

respecively, compare with (7.26). The eigenvalues of the connections we will be considering in this chapter are in fact non-degenerate. Hence, these holonomy invariants, as opposed to the actual holonomy matrix are convenient for specifying the physical invariants corresponding to a black hole (or wormhole) solution.

As for the trivial holonomy constraint, it will suffice to only consider the holonomy of the Euclidean time component of the connection around the thermal cycle. And by the Euclidean time component A_{t_E} of a connection A, we mean the part of the connection proportional to $dt_E = -idt$, i.e. $A_{t_E} = iA_t$ where A_t is the regular time-component of the connection. Since the connections we will consider does not depend on time, we get

$$\mathrm{Hol}_\tau(A_{t_E}) = \mathcal{P}\exp\left(\oint_\tau A_{t_E}dt_E\right) = \exp\left(\int_0^{2\pi\beta} A_{t_E}dt_E\right) = \exp(-2\pi\beta i A_t) .$$

Once again, the holonomy matrix is only uniquely specified up to an overall gauge transformation, i.e. an overall conjugation by an element of the gauge group. From this it follows that

$$\exp(-2\pi\beta i A_t) = \exp(-2\pi\beta i a_t) ,$$

where a_t is the time component of a. The eigenvalues of a_t will be non-degenerate for the connections considered here. Hence, we can use (7.27) with $X = -2\pi\beta i a_t$ to write the trivial holonomy constraint in a more practical form. After simplifying the resulting expression a bit, we arrive at

$$\det(a_t) = 0 , \qquad \frac{1}{2}\beta^2\mathrm{tr}[a_t^2] = 1 . \tag{8.8}$$

Analogous equations hold for $\bar{A}$, although it will suffice to consider only the constraints on A due to the symmetry of the connections we are considering in this thesis.

We have now seen how the holonomy of an $SL(3)$ connection around a non-contractible cycle is parametrized by two gauge invariant variables, see (8.6). In general our solution will contain four independent charges, since we have two connections A and $\bar{A}$. Two of these charges will be related to the mass and angular momentum of the black hole while the other two will be non-trivial higher spin charges. However, for simplicity, we will only consider a non-rotating black hole solution for which $\mathrm{Hol}(A) = \mathrm{Hol}(\bar{A})$. In that case the angular momentum of the black hole is zero, and since we have reduced the total number of degrees of freedom from four to two, one of the higher spin charges also vanishes. The black hole will then only carry mass and one higher spin charge.

8.1.1 A black hole solution

We are now ready to investigate a non-rotating black hole solution in spin-3 gravity with higher spin charge. This is done by constructing $SL(3)$ connections which leads to a black hole metric. We will consider connections of the form (8.3) with $b = e^{\rho L_0}$ and

$$a = a_+ dx^+ + a_- dx^- \ , \qquad \bar{a} = \bar{a}_+ dx^+ + \bar{a}_- dx^- \ ,$$

where $x^{\pm} = t \pm \phi$. Here L_0 is a generator of $SL(3)$ (see Appendix E.5.2) and $a_{\pm}$ and $\bar{a}_{\pm}$ are matrix representations of the Lie algebra $sl(3)$. We will make a particular ansatz for the $a_{\pm}$ and the bared counterparts, proposed by [32]:

$$a = [l_D W_2 + \mathcal{W} W_{-2} - Q W_0] dx^+ + [l_P L_1 - \mathcal{L} L_{-1} + \Phi W_0] dx^- \ , \tag{8.9}$$

and

$$\bar{a} = [l_D W_{-2} + \mathcal{W} W_2 - Q W_0] dx^- - [l_P L_{-1} - \mathcal{L} L_1 - \Phi W_0] dx^+ \ . \tag{8.10}$$

Here L_i and W_i are generators of $SL(3)$ given explicitly in Appendix E.5.2, and l_D, l_P, $\mathcal{W}$, $\mathcal{L}$, Q and Φ are parameters specifying the charges of the black hole. However, as mentioned earlier a non-rotating black hole in spin-3 gravity has only got two independent charges which are completely specified by two independent parameters. Hence, the six parameters in our connections are not all independent of each other. The equations of motion of Chern-Simons gauge theories state that the connections are flat, i.e. $\mathrm{d}A + A \wedge A = 0$ and similarly for $\bar{A}$. As a consequence we obtain the following restrictions:

$$Q = \frac{2\mathcal{W} l_P}{\mathcal{L}} \ , \qquad \frac{\mathcal{L}^2}{l_P^2} = \frac{\mathcal{W}}{l_D} \ . \tag{8.11}$$

For a complete derivation of this result see Appendix J.4.1.

By first calculating the frame field by $e = \frac{1}{2}(A - \bar{A})$, the spacetime interval by $ds^2 = \frac{1}{2}\mathrm{tr}[e^2]$ and using (8.11), we find the metric in the principal embedding which corresponds to our connections:

$$\begin{aligned} ds^2 = &- \left[4 l_D^2 \left(e^{2\rho} - e^{-2\rho} \frac{\mathcal{L}^2}{l_P^2} \right)^2 + l_P^2 \left(e^{\rho} - e^{-\rho} \frac{\mathcal{L}}{l_P} \right)^2 \right] dt^2 + d\rho^2 \\ &+ \left[4 l_D^2 \left(e^{2\rho} + e^{-2\rho} \frac{\mathcal{L}^2}{l_P^2} \right)^2 + l_P^2 \left(e^{\rho} + e^{-\rho} \frac{\mathcal{L}}{l_P} \right)^2 + \frac{4}{3}(Q + \Phi)^2 \right] d\phi^2 \ . \end{aligned} \tag{8.12}$$

This calculation is provided in its entirety in Appendix J.4.1.

We see that the time component of the metric (8.12) vanishes when

$$e^{\rho} - e^{-\rho} \frac{\mathcal{L}}{l_P} = 0 \ .$$

It follows that there is a horizon at

$$e^{\rho_h} = \sqrt{\frac{\mathcal{L}}{l_P}} \ . \tag{8.13}$$

Furthermore, the black hole approaches AdS_3 with radius of curvature $\frac{l}{2}$ in the limit $\rho \to \infty$ and there is no curvature singularity, which is consistent with having a smooth BTZ limit.

It is still possible to restrict the parameters of the connections further by invoking the trivial holonomy constraint (8.8). From (8.9) we can read off the a_t-component as

$$a_t = [l_D W_2 + \mathcal{W} W_{-2} - Q W_0] + [l_P L_1 - \mathcal{L} L_{-1} + \Phi W_0] \ .$$

Using the explicit form of the generators L_i and W_i given in Appendix E.5.2, it is straightforward to show that (8.8) implies

$$\frac{4}{27}(Q - \Phi)^3 + \frac{2}{3}(Q - \Phi)(l_P \mathcal{L} - 8 l_D \mathcal{W}) + 4 l_D \mathcal{L}^2 = 0 \ ,$$

and

$$\frac{4}{3}\beta^2 (Q - \Phi)^2 + 4\beta^2 (l_P \mathcal{L} + 4 l_D \mathcal{W}) = 1 \ .$$

Solving these equations simultaneously for β^2 and Φ yields

$$\beta^2 = \frac{\mathcal{L}^2}{4l_P}(16\mathcal{W}^2 l_P + \mathcal{L}^3)^{-1} , \qquad \Phi = 8\frac{\mathcal{W} l_P}{\mathcal{L}} . \tag{8.14}$$

In Appendix J.4.1 we show that β is exactly the periodicity of Euclidean time that removes the conical singularity from the horizon of the black hole metric (8.12). Thus, we have shown explicitly that the trivial holonomy constraint is sufficient to assure a regular Euclidean horizon.

This rather simple black hole solution satisfies all of the three desired properties of black hole solutions stated above. In particular, the BTZ black hole can be obtained smoothly by taking either the limit $l_D = \mathcal{W} = 0$ or $l_P = \mathcal{L} = 0$.

The holonomy invariants around the spatial cycles are

$$\Theta_{0,A} = -\Theta_{0,\bar{A}} = \frac{32\pi}{27}(Q + \Phi)^3 + \frac{16\pi}{3}(Q + \Phi)(l_P\mathcal{L} - 8l_D\mathcal{W}) + 32\pi\mathcal{L}^2 l_P , \tag{8.15}$$

and

$$\Theta_{1,A} = \Theta_{1,\bar{A}} = \frac{48\pi^2}{9}(Q + \Phi)^2 + 16\pi^2(l_P\mathcal{L} + 4l_D\mathcal{W}) . \tag{8.16}$$

These are the gauge invariant physical quantities that classify the black hole solution, and they are related to the mass of the black hole and one additional higher spin charge.

In addition to the metric there is also an associated spin-3 field to a given set of connections A and $\bar{A}$. This spin-3 field is specified by $\psi = \frac{1}{9}\mathrm{tr}[e^3]$, with the frame field e given by $e = \frac{1}{2}(A - \bar{A})$. The spin-3 field for the connections of our black hole solution is given explicitly in [32]. This concludes our analysis of spin-3 black hole solutions.

8.2 Wormholes in spin-3 gravity

The theory of wormholes in higher spin gravity is, as of today, still a very unexplored area. In fact, there have yet to be any serious attempts to properly define a wormhole solution in higher spin gravity. Nor have the qualities of a physically interesting wormhole solution been treated to any greater extent. Although, our aim is not to fully resolve these matters, we hope to give at least some insight regarding these questions. Of course it is not by any means obvious that any kind of interesting wormhole solution should even exist in spin-3 gravity theory, and in particular in 2+1 dimensions. However, previous work by [34] has found that a black hole solution in 2+1 dimensions expressed in a particular gauge (often referred to as the *wormhole gauge*) allows it to be interpreted as a wormhole. We will investigate further this black hole solution in the wormhole gauge, as it may actually be a natural way of describing a wormhole in spin-3 theory. Before doing so, it will be worth discussing the physical interpretation of spin-3 gravity solutions.

As we have already seen, the metric is not a gauge invariant quantity in spin-3 gravity and hence can not represent a physically observable quantity. Instead, the observable quantity is the holonomy around a non-contractible cycle of spacetime. Yet, in the case of finding a black hole solution earlier, we choose to represent a solution to our theory in a particular gauge which made it possible to interpret the corresponding metric as one describing a black hole. A trivial gauge transformation could change the metric to a form which does not allow for a black hole interpretation while keeping the physical observable fixed. A solution may then have multiple geometrical interpretations. Thus, to find a wormhole solution, we could look for a particular gauge which allows a solution to be geometrically interpreted as a wormhole by its metric. This is perhaps the best way of defining a wormhole solution in higher spin gravity, although the same solution may also be a black hole or something else depending on which gauge it is represented in.

Our interpretation of the theory of higher spin gravity relies heavily on our previous knowledge of the role the metric plays in spin-2 gravity, i.e. conventional Einstein gravity theory. There the metric is a physical observable and describes the geometry of spacetime. In spin-3 gravity, however, the metric is no longer a physical observable and there is also an additional spin-3 field present. Gauge transformations may alter the form of the metric and the spin-3 field; for example a singularity may be removed from the metric and incorporated in the spin-3 field by a trivial gauge transformation. This provides a possible way of resolving singular metrics. However, for a better understanding of higher spin gravity theories, we need to better understand the role of the higher spin fields and how to interpret them physically. This is an active topic in current research of higher spin gravity and will most likely have to be resolved before we will see any physical applications of the theory.

With these remarks in mind, we will continue by investigating a simple wormhole solution obtained by gauge transforming a conical singularity or a BTZ black hole to a wormhole. Then we finish our investigation of wormholes by considering a more general ansatz which will lead to a solution which may be interpreted as a traversable wormhole in our particular choice of gauge, although it is perhaps best interpreted as a black hole expressed in a wormhole gauge [34].

8.2.1 A simple spin-3 wormhole

Here we will show how a trivial gauge transformation can resolve a conical singularity and also bring a BTZ black hole to a wormhole. The connection is given by

$$A = b^{-1}(L_1 - \mathcal{L}L_{-1})b dx^+ + b^{-1}\,\mathrm{d}b\ , \qquad \overline{A} = b(L_{-1} - \mathcal{L}L_1)b^{-1}dx^- + b\,\mathrm{d}b^{-1}\ .$$

where we have a conical singularity if $\mathcal{L} \in (-\frac{1}{4},0)$ or a BTZ black hole if $\mathcal{L} > 0$. We now perform the trivial gauge transformation (7.17) with parameters $\alpha = \overline{\alpha} = \sqrt{\gamma} > 0$. The result is the new connections

$$A' = b^{-1}(L_1 - \mathcal{L}L_{-1} + \sqrt{\alpha}W_{-1})b dx^+ + b^{-1}\,\mathrm{d}b\ , \qquad \overline{A}' = b(L_{-1} - \mathcal{L}L_1 + \sqrt{\alpha}W_1)b^{-1}dx^- + b\,\mathrm{d}b^{-1}\ .$$

which gives us a metric

$$ds^2 = d\rho^2 - \Big((e^\rho - \mathcal{L}e^{-\rho})^2 + \gamma e^{-2\rho}\Big)dt^2 + \Big((e^\rho + \mathcal{L}e^{-\rho})^2 + \gamma e^{-2\rho}\Big)d\phi^2\ . \tag{8.17}$$

It is straightforward to check that these new connections have the same holonomy invariants as the old ones. The metric (8.17) has the appearance of a wormhole since none the components of the metric ever vanish. We can thus let $\rho \in (-\infty,\infty)$. To obtain a more familiar form we perform a change of variables:

$$r^2 = (\mathrm{e}^\rho + \sqrt{\mathcal{L}^2+\gamma}\mathrm{e}^{-\rho})^2\ , \qquad b_0^2 = 4(\sqrt{\mathcal{L}^2+\gamma})\ ,$$

which brings the metric to the form

$$ds^2 = -(r^2 - \frac{1}{2}b_0^2 - 2\mathcal{L})dt^2 + \frac{dr^2}{r^2(1-\frac{b_0^2}{r^2})} + (r^2 - \frac{1}{2}b_0^2 + 2\mathcal{L})d\phi^2\ . \tag{8.18}$$

We can clearly see a resemblance to the Morris-Thorne wormhole, see (5.21). The main difference is the factor of r^2 appearing in the denominator of dr^2 and before dt^2. This is simply because this wormhole is asymptotically AdS_3 as opposed to Minkowski, as can easily be seen by comparing to the AdS_3 metric. However, while not strictly a Morris-Thorne wormhole it still requires exotic mass if it is to be constructed without higher spin as can be determined by computing the Einstein tensor from the metric. In contrast to this, interpreting w^a and e^a classically and rewriting the second equation of motion(7.7) of the theory,

$$\mathrm{d}w^a + \frac{1}{2}\epsilon^a{}_{bc}\left(w^b \wedge w^c + \frac{e^b \wedge e^c}{l^2}\right) = 2\epsilon^a{}_{fc}\left(w^{bc} \wedge w^f_{\ b}\frac{e^{bc} \wedge e^f_{\ b}}{l^2}\right)\ ,$$

we have a candidate for an equivalent of the stress-energy tensor on the right-hand side. The right-hand side evaluates to 0, so our equations of motion indicate that there is no equivalent to the stress-energy tensor present in this solution. Thus, attempting to interpret the geometry in two classically equivalent ways we obtain opposite results, demonstrating that a classical geometric interpretation requires more work.

It is interesting to note that both the conical singularity and the BTZ black hole become wormholes in this gauge. Notice that it does not matter if $\mathcal{L}$ is positive or negative, the dt^2 and $d\phi^2$ terms never vanish since $r \in (-\infty, -b_0) \cup (b_0,\infty)$.

8.2.2 A black hole in a wormhole gauge

We start by making a semi-general ansatz for the connections A and $\bar{A}$ of the form (8.3), with $b = e^{\rho L_0}$,

$$a = (l_P L_1 - \mathcal{L}L_{-1} - \mathcal{W}W_{-2})dx^+ + (l_D W_2 + \mathcal{A}L_{-1} + \mathcal{B}W_{-2} - \mathcal{C}W_0)dx^-\ ,$$

and

$$\bar{a} = -(l_P L_{-1} - \mathcal{L}L_1 + \mathcal{W}W_2)dx^- + (l_D W_{-2} - \mathcal{A}L_1 + \mathcal{B}W_2 - \mathcal{C}W_0)dx^+\ .$$

Here L_i and W_i are the usual generators of $SL(3)$ given explicitly in Appendix E.5.2, and l_P, l_D, $\mathcal{L}$, $\mathcal{W}$, $\mathcal{A}$, $\mathcal{B}$ and $\mathcal{C}$ are parameters which specify the properties of the wormhole. This ansatz is highly inspired by our knowledge about solutions from [34] and [35], although, it is probably more natural to make an ansatz for a solution by considering the physical meanings of the parameters. We have instead written down an ansatz that we know will give us the desired solution, without regarding its physical interpretation. The purpose of making such an ansatz in the first place, as opposed to just writing down the correct solution, is solely for illustrating how to find a solution from a fairly general ansatz.

Since a pair of general spin-3 connections A and $\bar{A}$ has a maximum of four independent charges, only four independent parameters are needed to specify the solution. The seven parameters used in our ansatz can therefore not be independent.

To keep the equations a bit simpler we completely eliminate one of the parameters by choosing units in which $l_P = 1$. Introducing new parameters $l_0 = \mathcal{L}/l_P$, $w_0 = \mathcal{W}/l_P$, $\mu = l_D/l_P$, $\alpha = \mathcal{A}/l_P$, $\beta = \mathcal{B}/l_P$ and $\gamma = \mathcal{C}/l_P$, we write

$$a = (L_1 - l_0 L_{-1} - w_0 W_{-2})dx^+ + (\mu W_2 + \alpha L_{-1} + \beta W_{-2} - \gamma W_0)dx^- \ ,$$

and

$$\bar{a} = -(L_{-1} - l_0 L_1 + w_0 W_2)dx^- + (\mu W_{-2} - \alpha L_1 + \beta W_2 - \gamma W_0)dx^+ \ .$$

Furthermore, the equations of motion of our Chern-Simons gauge theory requires that the connections be flat. Solving $\mathrm{d}A + A \wedge A = 0$ we find that

$$\alpha = 8\mu w_0 \ , \qquad \beta = \mu l_0^2 \ , \qquad \gamma = 2\mu l_0 \ . \tag{8.19}$$

A derivation of this is provided in Appendix J.4.2. The flatness condition applied to $\bar{A}$ will also result in (8.19), and no further restrictions, as a consequence of the symmetry between a and $\bar{a}$ made in our ansatz.

Using (8.19), we turn our ansatz to a solution of our spin-3 gravity theory:

$$a = (L_1 - l_0 L_{-1} - w_0 W_{-2})dx^+ + \mu(W_2 + 8w_0 L_{-1} + l_0^2 W_{-2} - 2l_0 W_0)dx^- \ , \tag{8.20}$$

and

$$\bar{a} = -(L_{-1} - l_0 L_1 + w_0 W_2)dx^- + \mu(W_{-2} - 8w_0 L_1 + l_0^2 W_2 - 2l_0 W_0)dx^+ \ . \tag{8.21}$$

The solution is written in the *highest weight gauge*, meaning that the sources and charges are explicitly decoupled in the Chern-Simons connection.

As usual the frame field is given by $e = \frac{1}{2}(A - \bar{A})$, and the spacetime interval by $ds^2 = \frac{1}{2}\mathrm{tr}[e^2]$. In Appendix J.4.2 we perform these calculations in greater detail, here we simply state the resulting metric:

$$\begin{aligned} ds^2 = &-\left([e^\rho + (8\mu w_0 - l_0)e^{-\rho}]^2 + 4[\mu e^{2\rho} + (w_0 - \mu l_0^2)e^{-2\rho}]^2\right) dt^2 + d\rho^2 \\ &+ \left([e^\rho + (8\mu w_0 + l_0)e^{-\rho}]^2 + 4[\mu e^{2\rho} + (w_0 + \mu l_0^2)e^{-2\rho}]^2 + \frac{16}{3} l_0^2\right) d\phi^2 \ . \end{aligned} \tag{8.22}$$

Performing the substitutions $l_0 = \frac{2\pi}{k}\mathcal{L}$ and $w_0 = \frac{\pi}{2k}\mathcal{W}$, we reach exactly the metric stated in Kraus and Gutperles' "Higher Spin Black Holes" (equation (5.8)) [34].

We see that the time component of the metric never vanishes, i.e. there is no event horizon. In fact, the metric describes a traversable wormhole connecting two asymptotic regions at $\rho \to \pm\infty$. However, by performing a complicated spin-3 gauge transformation, the metric of the solution can be transformed to describe a black hole with a smooth horizon. Such a gauge transformation is performed explicitly in [35], where they also argued that the solution is that of a black hole since it satisfies the stated conditions of a higher spin black hole. Without first giving a more complete definition of a wormhole in a higher spin gravity theory, we can not say for sure whether or not our solutions can be properly interpreted as wormholes. Although, we could settle for defining a wormhole in terms of the metric and then we would have found an honest wormhole solution. However, in doing so the notion of geometry in higher spin gravity would become even more ambiguous than it already is.

Chapter 9

Conclusions

In this thesis we successfully managed to express Einstein's theory of gravity as a gauge theory in 2+1 dimensions, eventually allowing us to study interesting phenomena such as black holes and wormholes in a higher spin toy model. Before performing this unification of gauge theory and general relativity an essential knowledge of both theories was required, and thus we initially provided proper introductions to these subjects.

In conventional 2+1-dimensional Einsteinian gravity we constructed a general radial symmetric solution. From this we obtained both the BTZ black hole and a conical singularity, which could be interpreted as a free particle. Another radial symmetric solution of interest is Morris and Thornes' traversable wormhole solution, which we found out required exotic matter, i.e. having negative energy density, to exist. This fact is well-known[23] and numerous attempts have been made to resolve the issue, see e.g. [26], [28], [36]. We investigated the possibility of using the electromagnetic field or a conformally coupled scalar field to create the wormhole. These attempts were, as expected, unsuccessful.

Wormholes have also recently had their interest revived in the context of string theory [37], [38]. However, the complexity of string theory may be prohibitive, and as such we instead studied a higher spin theory. Higher spin theory is considered to be the tensionless limit of string theory, and serves as a simpler toy model [32]. To extend 2+1 dimensional Einsteinian gravity to a higher spin theory we expressed it as a Chern-Simons gauge theory of $SL(2) \times SL(2)$, and then extended its gauge group to $SL(3) \times SL(3)$ thus yielding a spin-3 formulation of gravity.

Exploring the consequences of our spin-3 gravity theory we found that the conventional geometry of the spacetime described by the metric, is no longer gauge invariant. For instance we performed a gauge transformation that deformed empty space into a geometry containing a singularity. Thus new gauge invariant tools with which to classify our solutions were needed, leading to the concept of holonomies. Relating the holonomy invariants of known solutions in spin-2 gravity, and requiring our gauge transformations to preserve the holonomy, we were able to interpret the new solutions in terms of classical objects. Specifically, we investigated the higher spin BTZ black hole, and its definition through a trivial holonomy around the thermal cycle.

In this thesis we have also regularized the singular spacetimes of a cone yielding a wormhole. The procedure was also shown to transform a black hole into a wormhole. Thus we have side-stepped the problem of exotic matter and created wormholes through gauge transformations. However, it is difficult to interpret these results since we in the process have been forced to give up the notion of gauge invariant geometry, at least in the conventional sense. In particular, a better understanding of the higher spin fields and its interaction with the spin-2 metric is required if we are to obtain a geometrical interpretation of higher spin gravity theories. To this end, a metric formulation may be useful. Attempts at such a formulation have been made, but the theory obtained is too complicated to constitute a useful toy model [39], [40]. Another approach could be the AdS/CFT correspondence, a conjecture proposed by Maldacena which states that string theory formulated on AdS is equivalent to a conformal field theory on its boundary [41]. For example, it has been shown that the general (traversable) wormhole considered in section 8.2.2 is actually an honest black hole solution using tools from the AdS/CFT correspondence. In short, it could be shown by coupling dynamical matter to the higher spin gravity theory [42].

Another useful aspect of the AdS/CFT correspondence is that it is well known how to quantize conformal field theories. Therefore, it provides a way of quantizing gravity theories in the form of string theories on AdS space, by quantizing its dual CFT. Higher spin gravity theories lies somewhere in the borderline of string theory and supersymmetry, and there is a dual CFT to higher spin gravity theories on AdS. The CFT corresponding to our spin-3 gravity theory on AdS_3 is the W_n minimal model [43], and a natural continuation of our work would be to investigate and quantize this CFT. Undoubtedly, there is more to be said about our black holes and wormholes solutions by analysing them in the dual CFT.

Appendices

Appendix A

Tensors

A.1 Definition and transformation properties

Tensors are probably the most ubitiqous objects in this entire paper and are of immensely great importance in different fields of physics, not least in relativity and quantum field theory. In short tensors are geometric objects which are used to describe linear relations of vectors, matrices or even other tensors and that allows one to express a physical theory independent of coordinate basis. To start off simple, consider a vector $\mathbf{v}$. We may express $\mathbf{v}$ in a basis $\mathbf{e_i}$, $i = 1...N$ according to

$$\mathbf{v} = \sum_{i=1}^{N} v^i \cdot \mathbf{e_i} \ , \tag{A.1}$$

where N is the dimension of the vector space spanned by the basis vectors $\mathbf{e_i}$. However, this can be written in a more elegant manner by introducing the *Einstein summation convention.* Einstein's summation convention states that terms with the same indices imply that the terms should be summed over for all values of the indices. In this case we may rewrite the equation above as follows

$$\mathbf{v} = v^i \mathbf{e_i} \ . \tag{A.2}$$

Thus, essentially we just omit the summation sign when using Einstein convention.

As we stated earlier tensors are very handy when it comes to choosing coordinate systems due to its independence of coordinate basis. A coordinate system can therefore be chosen completely arbitrarily! However, once a tensor is attached to a given coordinate basis a transformation matrix is needed to switch the basis. From linear algebra a coordinate basis $\mathbf{e}_i$ is related to another basis $\mathbf{e}'_i$ according to

$$\mathbf{e}'_i = T_i^{\ j} \mathbf{e_j} \ ,$$

where T is the transformation matrix between the bases. Using v from our previous example we find that its components in the basis $\mathbf{e_i}'$ is given by

$$v'^i = (T^{-1})_j^{\ i} v^j \ .$$

Vectors transformed in this way are called *contravariant* vectors. If there are contravariant vectors then there are of course *covariant* vectors as well. If $\mathbf{u}$ is another vector given in the basis $\mathbf{e}_i$, whose basis vectors transform using the inverse transformation matrix, $(T^{-1})_j^i$, then its components in another basis e'_i are given by

$$u'^i = T_j^{\ i} u^j \ ,$$

and $\mathbf{u}$ is a covariant vector. From transformations of vectors it is not hard to generalize the transformation properties to general tensors. In general a tensor can be seen as a tensor product of vector spaces according to

$$A = A^{\mu_1 \dots \nu_n}{}_{\nu_1, \dots, \nu_m} \mathbf{e}_{\mu 1} \otimes \dots \otimes \mathbf{e}_{\mu n} \otimes \mathbf{e}_{\nu 1} \otimes \dots \otimes \mathbf{e}_{\nu m} \ .$$

In order to express A in another basis we transform each basis vector with free indices according to

$$A'^{\mu 1 \dots \mu n}{}_{\nu 1 \dots \nu m} = T^{\mu 1}{}_{\rho 1} \dots T^{\mu n}{}_{\rho n} (T^{-1})^{\nu 1}{}_{\sigma 1} \dots (T^{-1})^{\nu m}{}_{\sigma 1} A^{\rho_1 \dots \rho_n}{}_{\sigma_1 \dots \sigma_m} \ ,$$

in analogy to ordinary vectors. Using the tensor product one can construct new tensors. For example a tensor $A^\rho_{\mu\nu}$ can be constructed from tensors B^ρ_μ and C_ν:

$$A^\rho{}_{\mu\nu} = B^\rho{}_\mu C_\nu \ .$$

Moreover tensors can be constructed by *contracting* indices. Consider the tensor $T^{ab}{}_{bc}$. This is a tensor of type (2,2) with a repeated index b, why we may use Einstein's summation convention:

$$T^{ab}{}_{bc} = T^{a1}{}_{1c} + ... + T^{an}{}_{nc} = P^{a}{}_{c} \ ,$$

where we introduced the tensor $P^a{}_c$ as the contracted version of $T^{ab}{}_{bc}$. Note that the contracted tensor $P^a{}_c$ now is of rank (1,1), i.e of type (1,1). If we contract a type (1,1) tensor we retrieve a generalized trace:

$$P^a a = P^1{}_1 + P^2{}_2 + ... + P^n{}_n = \mathrm{tr}\,[P] \ .$$

A.2 Symmetries and famous tensors

In this thesis several calculations are performed using different symmetry properties of tensors. For example, consider a tensor $A_{\mu\nu}$. If $A_{\mu\nu}$ is invariant under a change of indices the μ and ν, i.e $A_{\mu\nu} = A_{\nu\mu}$, the tensor $A_{\mu\nu}$ is called *symmetric*. On the other hand, if $A_{\mu\nu} = -A_{\nu\mu}$, the tensor $A_{\mu\nu}$ is *antisymmetric*. A very important antisymmetric tensor is the electromagnetic tensor, $F_{\mu\nu}$, used to derive Maxwell's equations in tensor formalism (see section 2.1). However, it is also used as a gauge invariant quantity in Yang-Mills theory and Chern-Simons theory.
Many calculations are simplified using the fact that a general tensor can be divided into its symmetric and its anti-symmetric parts:

$$A_{[\mu\nu]} = \frac{1}{2}(A_{\mu\nu} - A_{\nu\mu}) \ ,$$
$$A_{(\mu\nu)} = \frac{1}{2}(A_{\mu\nu} + A_{\nu\mu}) \ ,$$

where $A_{[\mu\nu]}$ is the anti-symmetric part of the tensor and $A_{(\mu\nu)}$ is the symmetric part. In particular we find

$$A_{\mu\nu} = A_{[\mu\nu]} + A_{(\mu\nu)} \ .$$

Returning to the electromagnetic tensor $F_{\mu\nu}$ usually defined (in an abelian theory) as

$$F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu \ ,$$

and using the fact that $F_{\mu\nu}$ is anti-symmetric and using the definitions above we deduce

$$F_{\mu\nu} = \partial_\mu A_\nu - A_\mu \partial_\nu = 2\partial_{[\mu} A_{\nu]} \ .$$

This relation is used extensively in the paper. Another very useful property is that the contraction of a symmetric and an antisymmetric tensor is identically zero. To prove this let $S^{\mu\nu}$ be a symmetric tensor and $A^{\mu\nu}$ an anti-symmetric tensor, then

$$S^{\mu\nu} A_{\mu\nu} = -S^{\mu\nu} A_{\nu\mu} = -S^{\nu\mu} A_{\nu\mu} = -S^{\mu\nu} A_{\mu\nu} \ ,$$

and since the quantity is equal to itself negative we conclude it has to be zero.
In addition to contraction of tensors it is often useful in calculations to be able to raise and lower indices on tensors. This can be done using the metric tensor, $g_{\mu\nu}$ (we often tend to work in Minkowski space with $g_{\mu\nu} = \eta_{\mu\nu}$). Consider a general tensor $T^{\alpha\beta}{}_{\gamma\delta}$. Using this tensor we may create new tensors by applying the metric:

$$g^{\mu\delta} T^{\alpha\beta}{}_{\gamma\delta} = T^{\alpha\beta\mu}_{\gamma} \ ,$$
$$g_{\mu\beta} T^{\alpha\beta}{}_{\gamma\delta} = T^{\alpha}_{\gamma\delta\mu} \ ,$$
$$g_{\mu\alpha} g_{\nu\beta} g^{\rho\gamma} g^{\sigma\delta} T^{\alpha\beta}{}_{\gamma\delta} = T^{\rho\sigma}{}_{\mu\nu} \ .$$

We continue our venture by turning dual vectors into vectors and vice versa using the metric:

$$V^\mu = g^{\mu\nu} V_\nu \ ,$$

and

$$\omega_\mu = g_{\mu\nu} \omega^\nu \ .$$

From these manipulations and the fact that $g_{\mu\nu} = \delta_{\mu\nu}$ in Euclidean space we conclude that the components of a dual vector which is transformed to a vector are the same. However, in Minkowski spacetime this is not the case (due to the component $\eta_{00} = -1$).
Not only the symmetry properties of tensors are helpful when it comes to cumbersome and lengthy tensor calculations. Some calculations can be very reduced with the use of other tensors. The *Levi-Civita tensor* is an example of such a tensor.

A.2.1 The Levi-Civita tensor

Our treatment of the Levi-Civita tensor follows closely the material of [44].

We start by defining the totally-antisymmetric Levi-Civita symbol in n dimensions

$$\varepsilon^{\mu_1 \dots \mu_n} = \begin{cases} +1, & \text{if } \mu_1 ... \mu_n \text{ is an even permutation of } 0\ 1\ ...\ n-1 \\ -1, & \text{if } \mu_1 ... \mu_n \text{ is an odd permutation of } 0\ 1\ ...\ n-1 \\ 0, & \text{else} \end{cases} \tag{A.3}$$

Furthermore we define the Levi-Civita symbol to have the same value in all coordinate systems, i.e. we require it to be invariant under a general coordinate transformation;

$$\varepsilon'^{\mu_1 \dots \mu_n} = \varepsilon^{\mu_1 \dots \mu_n} \ . \tag{A.4}$$

Consider a general coordinate transformation from coordinates x^μ to coordinates x'^μ, denoted by $x^\mu \to x'^\mu$. If the Levi-Civita symbol were to transform as a tensor under such a coordinate transformation, we then would have

$$\tilde{\varepsilon}'^{\mu_1 \dots \mu_n} = \frac{\partial x'^{\mu_1}}{\partial x^{\nu_1}} ... \frac{\partial x'^{\mu_n}}{\partial x^{\nu_n}} \varepsilon^{\nu_1 \dots \nu_n} = \left| \frac{\partial x'}{\partial x} \right| \varepsilon^{\mu_1 \dots \mu_n} \ , \tag{A.5}$$

where $\left| \frac{\partial x'}{\partial x} \right|$ is the *Jacobian* of the transformation. However, the Jacobian of the transformation does not necessarily equals one for a general coordinate transformation so $\tilde{\varepsilon}'^{\mu_1 \dots \mu_n} \neq \varepsilon^{\mu_1 \dots \mu_n}$ in general. By the defining property (A.4) it follows that the Levi-Civita symbol does not transform as a tensor under a general coordinate transformation, and hence it does not consititute a proper tensor. It does however constitute a tensor density.

Definition A.2.1. A tensor density of weight w is a quantity with components $B^{\mu_1 \dots \mu_p}$ which transforms as

$$B'^{\mu_1 \dots \mu_p} = \left| \frac{\partial x}{\partial x'} \right|^{-w} \frac{\partial x'^{\mu_1}}{\partial x^{\nu_1}} ... \frac{\partial x'^{\mu_p}}{\partial x^{\nu_p}} B^{\nu_1 \dots \nu_p} \ , \tag{A.6}$$

under a general coordinate transformation $x^\mu \to x'^\mu$.

Note how a tensor is actually a special case of a tensor density; it is a tensor density of weight zero. Furthermore, since $\left| \frac{\partial x}{\partial x'} \right| = \left| \frac{\partial x'}{\partial x} \right|^{-1}$, the Levi-Civita symbol transforms as a tensor density of weight -1;

$$\varepsilon'^{\mu_1 \dots \mu_n} = \left| \frac{\partial x}{\partial x'} \right|^{-(-1)} \frac{\partial x'^{\mu_1}}{\partial x^{\nu_1}} ... \frac{\partial x'^{\mu_n}}{\partial x^{\nu_n}} \varepsilon^{\nu_1 \dots \nu_n} = \left| \frac{\partial x}{\partial x'} \right| \left| \frac{\partial x'}{\partial x} \right| \varepsilon^{\mu_1 \dots \mu_n} = \varepsilon^{\mu_1 \dots \mu_n} \ ,$$

as required by the defining coordinate invariant property (A.4).

By multiplying the Levi-Civita symbol by a scalar density of weight 1 we can construct a tensor. This scalar density of weight 1 can be built from the metric tensor $g_{\mu\nu}$. Consider the determinant of the metric tensor, which we will frequently denote by g, i.e. $g = \det(g_{\mu\nu})$. The determinant of any $n \times n$ matrix (or rank-2 tensor for that matter) can be expressed in index notation by the use of the Levi-Civita symbol as

$$\det(M) = \frac{1}{n!} M_{i_1 j_1} ... \ M_{i_n j_n} \varepsilon^{i_1 \dots i_n} \varepsilon^{j_1 \dots j_n} \ ,$$

where M_{ij} are the components of the matrix. Using this result we can write down an expression for the determinant of the metric tensor in terms of the Levi-Civita symbol as following

$$g = \frac{1}{n!} g_{\mu_1 \nu_1} ... \ g_{\mu_n \nu_n} \varepsilon^{\mu_1 \dots \mu_n} \varepsilon^{\nu_1 \dots \nu_n} \ .$$

Now we perform a general coordinate transformation $x^\mu \to x'^\mu$, while remembering that the Levi-Civita symbol is an invariant under such a transformation. The determinant of the metric tensor transforms as

$$\begin{aligned} g' &= \frac{1}{n!} g'_{\mu_1 \nu_1} ... \ g'_{\mu_n \nu_n} \varepsilon^{\mu_1 \dots \mu_n} \varepsilon^{\nu_1 \dots \nu_n} \\ &= \frac{1}{n!} g_{\mu_1 \nu_1} ... \ g_{\mu_n \nu_n} \frac{\partial x^{\mu_1}}{\partial x'^{\rho_1}} ... \frac{\partial x^{\mu_n}}{\partial x^{\rho_n}} \frac{\partial x^{\nu_1}}{\partial x'^{\sigma_1}} ... \frac{\partial x^{\nu_n}}{\partial x'^{\sigma_n}} \varepsilon^{\rho_1 \dots \rho_n} \varepsilon^{\sigma_1 \dots \sigma_n} \\ &= \frac{1}{n!} g_{\mu_1 \nu_1} ... \ g_{\mu_n \nu_n} \left| \frac{\partial x}{\partial x'} \right|^2 \varepsilon^{\mu_1 \dots \mu_n} \varepsilon^{\nu_1 \dots \nu_n} \\ &= \left| \frac{\partial x}{\partial x'} \right|^2 g \ , \end{aligned} \tag{A.7}$$

under this transformation. By (A.6) this directly implies that g is a scalar density of weight -2, and hence $\sqrt{|g|}$ is a scalar density of weight -1. Now the determinant of the inverse of the metric tensor is $\frac{1}{g}$, and by inverting (A.7), this is a scalar density of weight 1. The totally-antisymmetric Levi-Civita tensor may may now be defined as

$$\epsilon^{\mu_1\dots\mu_n} = \frac{1}{\sqrt{|g|}}\varepsilon^{\mu_1\dots\mu_n} \ . \tag{A.8}$$

We will consistently use the notation ε for the Levi-Civita *tensor* and ϵ for the Levi-Civita *symbol*.

We define the Levi-Civita symbol with downstairs indices to be numerically given by

$$\varepsilon_{\mu_1\dots\mu_n} = (-1)^t\varepsilon^{\mu_1\dots\mu_n} \ , \tag{A.9}$$

where t is the number of negative eigenvalues of the metric tensor, (a Euclidian metric will have $t = 0$ and a Lorentzian metric will have $t = 1$ or $t = 3$ depending on the choice of time signature). It follows directly that $\varepsilon_{\mu_1\dots\mu_n}$ is invariant under a general coordinate transformation since this is true for $\varepsilon^{\mu_1\dots\mu_n}$. Performing a general coordinate transformation $x^\mu \to x'^\mu$ we show that $\varepsilon_{\mu_1\dots\mu_n}$ must be a tensor density of weight 1 in order to be invariant under the transformation;

$$\varepsilon'_{\mu_1\dots\mu_n} = \left|\frac{\partial x}{\partial x'}\right|^{-1}\frac{\partial x^{\nu_1}}{\partial x'^{\mu_1}}\dots\frac{\partial x^{\nu_n}}{\partial x'^{\mu_n}}\varepsilon_{\nu_1\dots\nu_n} = \left|\frac{\partial x}{\partial x'}\right|^{-1}\left|\frac{\partial x}{\partial x'}\right|\varepsilon_{\mu_1\dots\mu_n} = \varepsilon_{\mu_1\dots\mu_n} \ .$$

Hence we can make a tensor with downstairs indices from $\varepsilon_{\mu_1\dots\mu_n}$ in complete analogy to the making of the Levi-Civita tensor with upstairs indices above; we simply multiply $\varepsilon_{\mu_1\dots\mu_n}$ by $\sqrt{|g|}$, which is indeed a scalar density of weight 1 by (A.7). We write

$$\epsilon_{\mu_1\dots\mu_n} = \sqrt{|g|}\varepsilon_{\mu_1\dots\mu_n} \ . \tag{A.10}$$

For consistency we should check whether or not $\epsilon_{\mu_1\dots\mu_n}$ as given by (A.10) is the same tensor as $\epsilon^{\mu_1\dots\mu_n}$, as defined by (A.8), with all indices lowered. Indeed, since $\epsilon_{\mu_1\dots\mu_n}$ is a proper tensor we can raise its indices by using the metric tensor. Using (A.9) and (A.8) we find that

$$\begin{aligned}\epsilon_{\mu_1\dots\mu_n} &= g_{\mu_1\nu_1}\dots\, g_{\mu_n\nu_n}\epsilon^{\nu_1\dots\nu_n} = g_{\mu_1\nu_1}\dots\, g_{\mu_n\nu_n}\frac{1}{\sqrt{|g|}}\varepsilon^{\nu_1\dots\nu_n}\\ &= g\frac{1}{\sqrt{|g|}}\varepsilon^{\mu_1\dots\mu_n} = (-1)^t\sqrt{|g|}\varepsilon^{\mu_1\dots\mu_n} = (-1)^{2t}\sqrt{|g|}\varepsilon_{\mu_1\dots\mu_n}\\ &= \sqrt{|g|}\varepsilon_{\mu_1\dots\mu_n} \ ,\end{aligned}$$

which is consistent with (A.10), thereby justifying our use of notation.

A useful identity for contracting $p = n - q$ indices on a pair of Levi-Civita tensors is

$$\epsilon^{\mu_1\dots\mu_q\lambda_1\dots\lambda_p}\epsilon_{\nu_1\dots\nu_q\lambda_1\dots\lambda_p} = (-1)^t p!q!\delta^{\mu_1\dots\mu_q}_{\nu_1\dots\nu_q} \ , \tag{A.11}$$

where $\delta^{\mu_1\dots\mu_k}_{\nu_1\dots\nu_k}$ is the generalized Kronecker-delta, defined as

$$\delta^{\mu_1\dots\mu_k}_{\nu_1\dots\nu_k} = \delta^{[\mu_1}_{[\nu_1}\delta^{\mu_2}_{\nu_2}\dots\,\delta^{\mu_k]}_{\nu_k]} \ . \tag{A.12}$$

By (A.8) and (A.10) it follows that

$$\epsilon^{\mu_1\dots\mu_q\lambda_1\dots\lambda_p}\epsilon_{\nu_1\dots\nu_q\lambda_1\dots\lambda_p} = \varepsilon^{\mu_1\dots\mu_q\lambda_1\dots\lambda_p}\varepsilon_{\nu_1\dots\nu_q\lambda_1\dots\lambda_p} \ ,$$

so (A.2.1) actually holds for both the Levi-Civita tensor and the Levi-Civita symbol, although contracting one kind with the other will produce a factor proportional to $\sqrt{|g|}$. Two interesting special cases of are the contraction of all of the indices,

$$\epsilon^{\lambda_1\dots\lambda_n}\epsilon_{\lambda_1\dots\lambda_n} = (-1)^t n! \ , \tag{A.13}$$

and the contraction of none of the indices,

$$\epsilon^{\mu_1\dots\mu_n}\epsilon_{\nu_1\dots\nu_n} = (-1)^t n!\delta^{\mu_1\dots\mu_n}_{\nu_1\dots\nu_n} \ . \tag{A.14}$$

The identity (A.2.1) is perhaps best verified by enumerating both sides for a particular choice of values for the indices $\mu_1, \dots, \mu_p$ and $\nu_1, \dots, \nu_p$. Since both sides are manifestly totally-antisymmetric in these indices, it follows that if they agree for one particular choice of index values, they must agree for all possible choices of index values.

Appendix B

Metric and Metric Tensors

Throughout this thesis the term *metric* is used extensively. In this appendix we give a brief introduction to the metric, assuming the reader is familiar with changes of variables under the integral sign in multiple dimensions. Through the metric, or the *spacetime interval*, the geometric and causal structure of a spacetime can be determined, and thus the metric can help us define curvature, volume, distance and other geometrical quantities. As such the spacetime interval is the fundamental object which classifies our solutions in regular Einstein gravity.

The reader should be familiar with an object very similar to the metric from vector calculus, namely the *Jacobian integration measure*. The Jacobian integration measure J is related to the metric as $g = J^T J$. In general the diagonal elements of g will be positive definite in a *Euclidean space* unless the Jacobian contains complex elements. In general relativity one instead assumes that the base space is a *Minkowski space*. Minkowski space has the metric $\eta_{\mu\nu} = \mathrm{diag}(-1,1,1,1)$ in 3+1 dimensions.[1] The metric in Minkowski space is related to the Jacobian as $\eta J^T J$. Thus, the Jacobian can be seen as measuring the difference between a choice of coordinates and the base space on which a differential volume element is defined.

By introducing metric components with negative sign, space and time can be combined into a single space, called *spacetime*. In spacetime, the metric measures the infinitesimal distance between *events*, that is a point that has a location both in space and in time. A time-like component of the metric tensor conventionally has a negative sign, opposite to the space-like components. This allows for the infinitesimal displacement ds^2 to be 0 in multiple points, introducing the notion of *simultaneity*. Two events are considered simultaneous if they are separated by $ds^2 = 0$. In special relativity this says that to an observer an event happens when the light it generates reaches the observer. By fixing $dt^2 = 0$ one recovers the usual notion of space and distance, where no two points are simultaneous.

Mathematically, we may define the metric in some coordinates x^μ through an infinitesimal displacement ds^2 according to

$$ds^2 = g_{\mu\nu} dx^\mu dx^\nu \ , \tag{B.1}$$

where $g_{\mu\nu}$ is the metric tensor. In Euclidean space in three dimensions this is simply

$$g_{\mu\nu} = \delta_{\mu\nu} \ , \tag{B.2}$$

where $\delta_{\mu\nu}$ is the identity matrix. However, in Minkowski space the metric tensor has a *time-like* component, i.e. a component $g_{\mu\nu} < 0$[2]:

$$g_{\mu\nu} = \eta_{\mu\nu} = \begin{bmatrix} -1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \ . \tag{B.3}$$

One may also define the Minkowski metric as $g_{\mu\nu} = \mathrm{diag}(1, -1, -1, -1)$, it is just a matter of conventions. Throughout our thesis we use the signature $(-, +, +, + ...)$. The metric tensor $g_{\mu\nu}$ is commonly denoted $\eta_{\mu\nu}$ in Minkowski space. From now on follows a pair of examples of metrics in different coordinate systems.

In spherical coordinates we may write down the metric directly as

$$ds^2 = dr^2 + r^2 d\theta^2 + r^2 \sin^2\theta d\phi^2 \ , \tag{B.4}$$

and from equation (B.4) we deduce that $g_{rr} = 1, g_{\theta\theta} = r^2$ and $g_{\phi\phi} = r^2 \sin^2\theta$, and the rest of the elements in $g_{\mu\nu}$ are zero.

[1] The signature of the metric is just a matter of conventions, the signature (1,-1,-1,-1) is also used extensively in literature. Throughout this thesis we use the signature (-1,1,1,1).

[2] We tend to be working in 2+1 dimensions in this thesis, with two spatial dimensions and one time dimension, and thus naturally the Minkowski metric reduces to a 3 by 3 matrix, $g_{\mu\nu} = \mathrm{diag}(-1,1,1)$

Consider the equation

$$x^2 + y^2 - z^2 = \pm R^2 \ . \tag{B.5}$$

This equation describes a hyperboloid with its center rotated around either the x-axis or the z-axis depending on the sign of R. We begin with the equation $x^2 + y^2 - z^2 = +R^2$. We may write this as

$$x^2 + y^2 - z^2 = r^2 - z^2 = R^2 \ , \tag{B.6}$$

where we used that $x^2 + y^2$ describes a circle with the polar radius r. Furthermore we can write down the expression for the metric in polar coordinates (in Minkowski space) according to

$$ds^2 = dx^2 + dy^2 - dz^2 = dr^2 + r^2 d\theta^2 - dz^2 \ . \tag{B.7}$$

Differentiating equation B.6 yields

$$2rdr - 2zdz = 0 \Rightarrow dz = \frac{rdr}{z} \ . \tag{B.8}$$

Using $r^2 - z^2 = R^2$ we find the metric as

$$ds^2 = dr^2 + r^2 d\theta^2 - dz^2 = dr^2 (1 - \frac{r^2}{r^2 - R^2}) + r^2 d\theta^2 = \frac{1}{1 - (r/R)^2} dr^2 + r^2 d\theta^2 \ , \tag{B.9}$$

from which we note that $g_{rr} = \frac{1}{1-(r/R)^2}$ and $g_{\theta\theta} = r^2$. We may also note that $g_{rr} < 0$ for $\forall r$ with $z \neq 0$, thus g has a time-like component. To conclude we find the metric tensor as

$$g_{r\theta} = \begin{bmatrix} \frac{1}{1-\frac{r^2}{R^2}^2} & 0 \\ 0 & r^2 \ . \end{bmatrix} \tag{B.10}$$

The case $x^2 + y^2 - z^2 = -R^2$ yields, after an analogous calculation, a metric tensor g:

$$g_{r\theta} = \begin{bmatrix} \frac{1}{1+\frac{r^2}{R^2}} & 0 \\ 0 & r^2 \ . \end{bmatrix} \tag{B.11}$$

All elements in g are obviously ≥ 0, to compare with the other metric tensor in equation $B.10$ which has a time-like component.

Appendix C

Differential Geometry

Since Einstein's discovery of general relativity, differential geometry has been of fundamental importance to physics as it is the mathematical framework upon which the concept of spacetime is built upon. As a field of mathematics, it is the study of calculus on *differential manifolds.* A manifold is a special type of a topological space and a differential manifold is in turn a smooth manifold. The precise mathematical definitions of these concepts is rather abstract and since we will not be needing any deeper knowledge of this for our purposes. We refer the interested reader to [16] and [1] for a more rigorous treatment of the field. Although, a somewhat less rigorous explanation of the concept of a differential manifold is in order.

We can think of a differential manifold as a smooth space that may or may not be curved. Three concrete examples of differential manifolds are Euclidean space $\mathbb{R}^n$, the n-dimensional sphere $\mathbb{S}^n$ and the four-dimensional Minkowski space which the theory of special relativity is founded upon. Introducing curved spaces complicates the notion of vectors, and more general tensors, on the space. In particular we can not think of a vector as an object connecting two different points. A resolution to this problem is the introduction of a *tangent space* at each point, which is a vector space consisting of every vector at a certain point on the manifold. This is in complete analogy to the notion of a tangent plane as a local approximation to a curved surface. A natural basis for this tangent space is ∂_μ, and a vector at a specific tangent space can then be written as $V = V^\mu \partial_\mu$. In the language of tensors we see that the vectors of a tangent space are contravariant vectors. The dimension of the tangent spaces of a manifold is the same as the dimension of the manifold itself. There is an associated vector space to each tangent space known as the cotangent space. They consist of the dual vectors (or covariant vectors) corresponding to the vectors (or contravariant vectors) of the tangent spaces. A natural set of bases for the cotangent space is dx^μ, and a dual vector in a cotangent space at a certain point can be written as $V* = V_\mu dx^\mu$.[1] More general tensors can now be constructed by the use of the tensor product as usual. The properties and operations of tensors that we developed earlier still hold but one has to be careful and remember that each vector or tensor is defined only in one point on the manifold. In physics we are mainly interested in vector or tensor fields, which is a collection of tensors, one for every point on a manifold.

An important mathematical formalism we will have to introduce is the formalism of *differential forms.* A great advantage of differential forms is that they allow for differentiation and integration to be generalized from a Euclidean space to a more general differential manifold in a natural way. The next section is devoted to developing the most important definitions and results we will be needing concerning differential forms.

C.1 Differential forms

Differential forms, (or in short "forms"), are a special class of tensors. More specifically; a scalar-valued k-form is a completely antisymmetric $(0,k)$-tensor, thus zero-forms are scalars and one-forms are dual vectors (or covariant vectors). More generally a completely antisymmetric (m,k)-tensor is a tensor valued k-form with m upper indices. In this section we will almost exclusively treat scalar valued differential forms, simply because there will be somewhat fewer indices to keep track of, although every definition and result applies equally well to more general tensor-valued differential forms.

In terms of the basis vectors dx^μ for the cotangent space a general one-form can be written as

$$\omega = \omega_\mu dx^\mu \ . \tag{C.1}$$

The set of all k-forms form a vector space which we will denote Λ^k. Moreover, the space of all k-form fields over a manifold M is denoted by $\Lambda^k(M)$, and this also constitutes a vector space.

We would like to be able to evaluate the products of differential forms. For this purpose we define the wedge product, which is an antisymmetric tensor product.

[1] The asterisk * is a common way of denoting a dual vector. We will not be working with these kinds of dual vectors in this thesis and the asterisk will be reserved for the Hodge dual operator which we will define later in this appendix.

Definition C.1.1. The wedge product is an operator $\wedge : \Lambda^k \times \Lambda^l \to \Lambda^{k+l}$. If $A \in \Lambda^k$ and $B \in \Lambda^l$ then the wedge product $A \wedge B$ is defined as the antisymmetrized tensor product of A and B,

$$(A \wedge B)_{\mu_1\mu_2\ldots\mu_{k+l}} = \frac{(k+l)!}{k!l!} A_{[\mu_1\mu_2\ldots\mu_k} B_{\mu_{k+1}\mu_{k+2}\ldots\mu_{k+l}]} \ . \tag{C.2}$$

As a consequence of the definition above we get the following important property of the wedge product:

$$A \wedge B = (-1)^{kl} B \wedge A, \quad A \in \Lambda^k, \quad B \in \Lambda^l \ . \tag{C.3}$$

This directly implies that the wedge product of a k-form with it self is identically zero whenever k is an odd number. We can now express a general k-form A in terms of the basis vectors dx^μ as

$$A = A_{\mu_1\ldots\mu_k} dx^{\mu_1} \otimes \ldots \otimes dx^{\mu_k} = \frac{1}{k!} A_{\mu_1\ldots\mu_k} dx^{\mu_1} \wedge \ldots \wedge dx^{\mu_k} \ . \tag{C.4}$$

The basis vectors dx^μ are one-forms so from property (C.3) it follows that $dx^{\mu_1} \wedge \ldots \wedge dx^{\mu_k} = 0$ if any of the wedged basis vectors are identical. Constructing linearly independent k-form fields on a n-dimensional manifold M is therefore equivalent to picking k of the n different basis vectors. It follows that the dimension of $\Lambda^k(M)$ is $n!/(k!(n-k)!)$. Note that if $k > n$ then at least two of the wedged basis vectors in (C.4) above must equal, since there are only n different basis vectors, and such a k-form is therefore identically zero.

One final remark; since the quantity $dx^{\mu_1} \wedge \ldots \wedge dx^{\mu_k}$ is completely antisymmetric it extracts the antisymmetric part of the coefficients $A_{\mu_1\ldots\mu_k}$ in (C.4). Thus, in general, even if the coefficients $A_{\mu_1\ldots\mu_k}$ are not antisymmetric, we have the relation

$$A_{\mu_1\ldots\mu_k} dx^{\mu_1} \wedge \ldots \wedge dx^{\mu_k} = A_{[\mu_1\ldots\mu_k]} dx^{\mu_1} \wedge \ldots \wedge dx^{\mu_k} \ . \tag{C.5}$$

As mentioned earlier one of the great advantages of differential forms is that they allow for differentiation and integration. We are now ready to define a derivative operator called the *exterior derivative.*

Definition C.1.2. The exterior derivative is an operator $\mathrm{d} : \Lambda^k \to \Lambda^{k+1}$ defined as an antisymmetric normalized partial derivative,

$$(\mathrm{d}A)_{\mu_1\ldots\mu_{k+1}} = (k+1)\partial_{[\mu_1} A_{\mu_2\ldots\mu_{k+1}]} \ . \tag{C.6}$$

An important consequence of this definition is that $\mathrm{d}^2 A = 0$ for any k-form A, (often simply denoted as $\mathrm{d}^2 = 0$). This follows from the antisymmetry property of the exterior derivative and the fact that partial derivatives commute. The linearity property of the partial derivative is inherited by the exterior derivative. Note that the exterior derivative of a zero-form is just the familiar differential of a scalar function. Another property of the exterior derivative that follows from the definition is that it satisfies the *Leibniz product rule*,

$$\mathrm{d}(A \wedge B) = \mathrm{d}A \wedge B + (-1)^k A \wedge \mathrm{d}B, \quad A \in \Lambda^k, \quad B \in \Lambda^l \ . \tag{C.7}$$

A k-form A is said to be *closed* if $\mathrm{d}A = 0$ and *exact* if $A = \mathrm{d}B$ for some $(k-1)$-form B. All exact forms are closed since $\mathrm{d}^2 = 0$ but the converse is not true in general. In Minkowski space all closed forms are exact except for zero-forms which cannot be exact since there are no k-forms for negative integers k. When we have a k-form expressed in terms of the basis vectors dx^μ, as in (C.4), there is a more useful representation of the exterior derivative than the one given in the definition above. In terms of the basis vectors for the cotangent space and tangent space the exterior derivative can be represented as $\mathrm{d} = dx^\mu \partial_\mu \wedge [\ ]$, where any k-form can be inserted in to the square brackets. As a demonstration of this, consider a general k-form A of the form (C.4). The exterior derivative of A can then be expressed as

$$\mathrm{d}A = dx^\mu \partial_\mu \wedge A = dx^{\mu_1} \partial_{\mu_1} \wedge \frac{1}{k!} A_{\mu_2\ldots\mu_{k+1}} dx^{\mu_2} \wedge \ldots \wedge dx^{\mu_{k+1}} = \frac{1}{k!} \partial_{\mu_1} A_{\mu_2\ldots\mu_{k+1}} dx^{\mu_1} \wedge \ldots \wedge dx^{\mu_{k+1}} \ .$$

We see that the above representation of the exterior derivative is indeed equivalent to our definition (C.6).

In the context of gauge theory it is useful to define the graded commutator for Lie algebra-valued forms:

Definition C.1.3. For Lie algebra-valued p- and q-forms ω and η, we define the *graded commutator* $[\cdot,\cdot]$ according to

$$[\omega,\eta] = \omega \wedge \eta - (-1)^{pq} \eta \wedge \omega \ . \tag{C.8}$$

There is another important operator on forms to be defined, namely the *Hodge dual operator* (or *Hodge star operator*).

Definition C.1.4. Let A be a k-form on an n-dimensional manifold M. The Hodge dual operator $* : \Lambda^k(M) \to \Lambda^{n-k}(M)$ is defined in terms of the Levi-Civita tensor ϵ as

$$(*A)_{\mu_1 \ldots \mu_{n-k}} = \frac{1}{k!} \epsilon^{\nu_1 \ldots \nu_k}{}_{\mu_1 \ldots \mu_{n-k}} A_{\nu_1 \ldots \nu_k} \ . \tag{C.9}$$

Since the Levi-Civita tensor is dependent on the metric of the manifold, so is the Hodge dual operator, (see Appendix A.2.1 for a treatment of the Levi-Civita tensor). It is worth noting that the dimensions of $\Lambda^k(M)$ and $\Lambda^{n-k}(M)$ are equal since the number of ways of picking k of n basis vectors equals the number of ways of picking $n-k$ of n basis vectors, (see the discussion after (C.4)). If we want to apply the Hodge dual operator to a k-form A expressed in terms of the coefficients $A_{\mu_1 \ldots \mu_k}$ and basis vectors dx^μ, as in (C.4), it may be more convenient to use the following formula, (as opposed to the definition (C.9)):

$$* \left(A_{\mu_1 \ldots \mu_k} dx^{\mu_1} \wedge \ldots \wedge dx^{\mu_k} \right) = \frac{1}{(n-k)!} A_{\mu_1 \ldots \mu_k} \epsilon^{\mu_1 \ldots \mu_k}{}_{\mu_{k+1} \ldots \mu_n} dx^{\mu_{k+1}} \wedge \ldots \wedge dx^{\mu_n} \ . \tag{C.10}$$

It can be shown that the Hodge dual satisfies the following relation

$$A \wedge *B = B \wedge *A \ . \tag{C.11}$$

As a specific example we can consider differential forms on the three-dimensional Euclidean space $\mathbb{R}^3$. The Hodge dual of two one-forms A and B is

$$*(A \wedge B)_k = \epsilon^{ij}{}_k A_i B_j \ .$$

Since there is no distinction between contravariant and covariant vectors in Euclidean space, one-forms are just vectors. The operation above on A and B is therefore just the conventional cross product.

Earlier we mentioned that differential forms allow for differentiation and integration to be defined operations on a general differential manifold. So far we have only been concerned with differentiation by the exterior derivative operator. As we will not need a theory of integration on a general differential manifold, we will not treat this subject to any greater extent. However, we will at least mention the relation between differential forms and the volume element $d^n x$ through the Levi-Civita tensor density, and state the *general Stokes theorem*.

Recall how in ordinary calculus on the Euclidean space $\mathbb{R}^n$, the volume element $d^n x$ transforms as

$$d^n x' = \left| \frac{\partial x'}{\partial x} \right| d^n x \ ,$$

under a change of coordinates $x \to x'$. Here $\left| \frac{\partial x'}{\partial x} \right|$ is the Jacobian of the coordinate transformation. From (A.6) we see that the volume element transforms exactly as a tensor density of weight one. On a n-dimensional manifold, however, the integrand is really a n-form. Thus we need to construct a n-form from the tensor density $d^n x$. Since a n-form is an antisymmetric tensor, an invariant object, we should try to construct an object with these qualities from the volume element. This can be done by multiplying the volume element with an antisymmetric tensor density of weight -1. In Appendix A.2.1 we derived such an object, the Levi-Civita symbol $\varepsilon^{\mu_1 \cdots \mu_n}$. A n-form is related to the volume element by

$$dx^{\mu_1} \wedge \ldots \wedge dx^{\mu_n} = \varepsilon^{\mu_1 \cdots \mu_n} d^n x \ . \tag{C.12}$$

This provides a way of converting an integral over a general differential manifold to an integral over Euclidean space, where the usual results of ordinary calculus applies.

Finally, we state the general Stokes theorem:

$$\int_M \mathrm{d}\omega = \int_{\partial M} \omega \ , \tag{C.13}$$

where ω is a differential form defined on a differential manifold M with boundary ∂M. All the familiar integration theorems of vector calculus, such as Greens theorem, Gauss theorem and Stokes (not so general) theorem are actually special cases of C.13.

C.2 Maxwell's equations in differential forms

As another illustration of differential forms we can consider the covariant form of Maxwell's equations. As we have seen in 2.1 Maxwell's equations can be stated as two tensorial equations, containing the antisymmetric electromagnetic field strength $F_{\mu\nu}$ and the four-current J^μ. These tensorial equations are

$$\partial_\nu F^{\mu\nu} = J^\mu \ , \tag{C.14}$$

and

$$\partial_{[\mu} F_{\nu\sigma]} = 0 \ . \tag{C.15}$$

In the formalism of differential forms the electromagnetic field strength is a two-form $F = \frac{1}{2} F_{\mu\nu} dx^\mu \wedge dx^\nu$. From the definition of the exterior derivative above we see that equation (C.15) is nothing but the statement that F is closed, that is, $\mathrm{d}F = 0$. Since Maxwell's theory of electromagnetism is formulated on Minkowski space, it follows from the discussion above that F must also be exact, which means that it can be written as $F = \mathrm{d}A$ for some one-form A. This one-form A is nothing but the dual of the familiar four-vector potential A_μ. We might have started from the vector potential one-form A. Then equation (C.15) would have followed as an identity of the formalism. Equation (C.14) can in turn be expressed as an equation between 3-forms by the use of the Hodge dual operator as $\mathrm{d}{*F} = {*J}$. Here $J = J_\mu dx^\mu$ is the current one-form. Let us prove this equation. First we begin by calculating $*F$ using (C.10) above. The result is

$$*F = * \left(\frac{1}{2} F_{\mu\nu} dx^\mu \wedge dx^\nu \right) = \frac{1}{(4-2)!} \frac{1}{2} F_{\rho\sigma} \epsilon^{\rho\sigma}{}_{\mu\nu} dx^\mu \wedge dx^\nu = \frac{1}{4} F^{\rho\sigma} \epsilon_{\rho\sigma\mu\nu} dx^\mu \wedge dx^\nu \ .$$

Applying the exterior derivative to this expression, and change some of the dummy indices, yields

$$\mathrm{d} * F = dx^\mu \partial_\mu \wedge \left(\frac{1}{4} F^{\alpha\beta} \epsilon_{\alpha\beta\nu\rho} dx^\nu \wedge dx^\rho \right) = \frac{1}{4} \partial_\mu F^{\alpha\beta} \epsilon_{\alpha\beta\nu\rho} dx^\mu \wedge dx^\nu \wedge dx^\rho \ .$$

Next we calculate the Hodge dual of the one-form J. Using (C.10) once again we find

$$*J = *(J_\mu dx^\mu) = \frac{1}{(4-1)!} J_\sigma \epsilon^\sigma{}_{\mu\nu\rho} dx^\mu \wedge dx^\nu \wedge dx^\rho = \frac{1}{6} J^\sigma \epsilon_{\sigma\mu\nu\rho} dx^\mu \wedge dx^\nu \wedge dx^\rho \ .$$

It is not immediately obvious from the expressions above for $\mathrm{d}{*F}$ and $*J$ that the equation $\mathrm{d}{*F} = {*J}$ is equivalent to $\partial_\mu F^{\nu\mu} = J^\nu$. To see that this is indeed the case, we apply the Hodge dual operator to both sides of the equation $\mathrm{d}{*F} = {*J}$. Starting with the left hand side, we find

$$\begin{aligned}
*\mathrm{d} * F &= * \left(\frac{1}{4} \partial_\mu F^{\alpha\beta} \epsilon_{\alpha\beta\nu\rho} dx^\mu \wedge dx^\nu \wedge dx^\rho \right) \\
&= \frac{1}{(4-3)!} \frac{1}{4} \partial_\mu F^{\alpha\beta} \epsilon_{\alpha\beta\nu\rho} \epsilon^{\mu\nu\rho}{}_\sigma dx^\sigma \\
&= \frac{1}{4} \partial_\mu F^{\alpha\beta} \epsilon_{\alpha\beta\nu\rho} \epsilon^{\mu\nu\rho\kappa} g_{\kappa\sigma} dx^\sigma \\
&= \frac{1}{4} \partial_\mu F^{\alpha\beta} \epsilon_{\alpha\beta\nu\rho} \epsilon^{\mu\kappa\nu\rho} g_{\kappa\sigma} dx^\sigma \\
&= -\frac{1}{2} \partial_\mu F^{\alpha\beta} \delta^{\mu\kappa}_{\alpha\beta} g_{\kappa\sigma} dx^\sigma \\
&= -\frac{1}{2} \partial_\mu (F^{\mu\kappa} - F^{\kappa\mu}) g_{\kappa\sigma} dx^\sigma \\
&= \partial_\mu F^{\kappa\mu} g_{\kappa\sigma} dx^\sigma \\
&= J^\kappa g_{\kappa\sigma} dx^\sigma \\
&= J_\sigma dx^\sigma = J \ .
\end{aligned} \tag{C.16}$$

Here we have used properties of the Levi-Civita tensor (see Appendix A.2.1) as well as the covariant inhomogeneous Maxwell equation (C.14)). Now we compute the Hodge dual of the right hand side of the equation $\mathrm{d}{*F} = {*J}$. The steps are

$$
\begin{aligned}
**J &= *\left(\frac{1}{6}J^{\sigma}\epsilon_{\sigma\mu\nu\rho}dx^{\mu}\wedge dx^{\nu}\wedge dx^{\rho}\right) \\
&= \frac{1}{(4-3)!}\frac{1}{6}J^{\kappa}\epsilon_{\kappa\mu\nu\rho}\epsilon^{\mu\nu\rho}{}_{\sigma}dx^{\sigma} \\
&= \frac{1}{6}J^{\kappa}\epsilon_{\kappa\mu\nu\rho}\epsilon^{\mu\nu\rho}{}_{\sigma}dx^{\sigma} \\
&= \frac{1}{6}J^{\kappa}\epsilon_{\kappa\mu\nu\rho}\epsilon^{\mu\nu\rho\tau}g_{\tau\sigma}dx^{\sigma} \\
&= -\frac{1}{6}J^{\kappa}\epsilon_{\kappa\mu\nu\rho}\epsilon^{\tau\mu\nu\rho}g_{\tau\sigma}dx^{\sigma} \\
&= J^{\kappa}\delta^{\tau}{}_{\kappa}g_{\tau\sigma}dx^{\sigma} \\
&= J_{\sigma}dx^{\sigma} = J\ .
\end{aligned}
\tag{C.17}
$$

Properties of the Levi-Civita tensor were once again used in this calculation, (see Appendix A.2.1). We have now proved by (C.16) and (C.17) that $*(\mathrm{d}{*F}) = *(*J)$, and since the Hodge dual operator is bijective it follows that $\mathrm{d}{*F} = {*J}$.

As a summary, Maxwell's equations can be formulated in terms of differential forms by

$$\mathrm{d}F = 0\ , \tag{C.18}$$

and

$$\mathrm{d} * F = *J\ . \tag{C.19}$$

Note that in vacuum, where $J = 0$, these equations become invariant under the "duality transformations" $F \to *F$, $*F \to -F$. There is much more to be said about this and for more information we refer to [16]. Finally, the gauge invariance of Maxwell's theory is incorporated in the formalism by the fact that $\mathrm{d}^2 = 0$. This implies that the equations will be invariant under the transformation $A \to A + \mathrm{d}f$ for any zero-form (scalar) f, since the equations only involve exterior derivatives of the field strength $F = \mathrm{d}A$ and its Hodge dual.

Appendix D

Lagrangians in Field Theory

In this thesis *Lagrangians*, or to be more specific, *Lagrangian fields* are frequently used to derive equations of motion of systems involving scalar fields, vector fields or more general tensor fields. The term Lagrangian usually refer to discrete systems, where the Lagrangian is a function of generalized coordinates (often denoted q^i), whereas *Lagrangian field density* is the correct term to use in the continuous case. However, for the sake of simplicity and due to the fact that we have considered only systems with infinite degrees of freedom, i.e. continuous systems, we simply refer to Lagrangian field densities as Lagrangians. Nevertheless, it is important to state the less obvious differences between the discrete Lagrangian and the Lagrangian field density. Lagrangian mechanics, or analytical mechanics, is a reformulation of the classical Newtonian mechanics where one expresses the physics through so called action integrals. The action integral, commonly denoted $\mathcal{S}$, is defined as

$$\mathcal{S} = \int L \, \mathrm{d}t \ , \tag{D.1}$$

where $L = T - V$ is the scalar Lagrangian depending on the kinetic energy (T) and the potential energy (V). By demanding the principle of least action and thus requiring that the Lagrangian is a stationary point of the action the system's development can be determined. In other words, the action is invariant under an infinitesimal change in the Lagrangian, i.e. $\delta\mathcal{S} = 0$. Using this one can derive the *Euler-Lagrange equation*

$$\frac{\partial L}{\partial q^i} - \frac{d}{dt}\frac{\partial L}{\partial \dot{q}^i} = 0 \ , \tag{D.2}$$

where we introduced generalized coordinates q^i and velocities $\dot{q}^i$. This Euler-Lagrange equation, however, is the equation of motion for a discrete system with finite degrees of freedom. To establish the behavior of a continuous system one has to define the Lagrangian field density. The definition of this is easily seen from the action integral:

$$\mathcal{S} = \int \Big(\int \mathcal{L} \, \mathrm{d}^n x \Big) \, \mathrm{d}t = \int \mathcal{L} \, \mathrm{d}^n x \, \mathrm{d}t \ , \tag{D.3}$$

where $\mathcal{L}$ is the Lagrangian field density. To retrieve the scalar Lagrangian L one therefore has to integrate the field density over the whole space. The Lagrangian field density is a very powerful tool when it comes to developing gauge theories. If $\mathcal{L}$ (or more precisely the action) is invariant under a symmetry transformation it follows that the equations of motion (the corresponding Euler-Lagrange equations in the continuous case) are invariant as well.

As an example of a field Lagrangian we will explicitly construct the *Klein-Gordon Lagrangian*, often denoted $\mathcal{L}_{KG}$. In order to do this we will first derive the Klein-Gordon equation and then guess the correct Lagrangian. The Klein-Gordon equation was actually first written down by Schrödinger before he settled on his most famous equation, the Schrödinger equation. The equation was then rediscovered in 1926 by Klein[1] and Gordon[2]in an attempt to create a relativistic theory of the electron. Nowadays we know that this endeavour was doomed to fail since the Klein-Gordon equation describes particles of spin 0, not of one half. To derive the Klein-Gordon equation we start from Einstein's famous energy identity, with $c = 1$

$$E^2 = m^2 + p^2 \ ,$$

where m is mass and p is momentum. We then perform the standard substitutions of quantum mechanics (with $\hbar = 1$)

$$p \mapsto -i\nabla \qquad\qquad E \mapsto i\frac{\partial}{\partial t} \ .$$

[1] Oskar Klein (1894-1977) was a Swedish theoretical physicist most famous for *Kaluza-Klein theory*, a unified theory of electromagnetism and gravity.

[2] Walter Gordon (1893-1939), German physicist.

Since a derivative does not make much sense unless it has a function to act on we also introduce a wave function, ϕ. Rearranging some terms we reach the equation

$$\Box\phi - m^2\phi = 0 \ ,$$

where we have introduced the *d'Alembert operator* defined as $\Box \equiv \partial^\mu\partial_\mu = -\frac{\partial^2}{\partial t^2} + \nabla^2$. To derive the Lagrangian we practice the fine art of trial and error, guessing our way to the correct answer:

$$\mathcal{L}_{KG} = -\frac{1}{2}\partial^\mu\phi\partial_\mu\phi - \frac{1}{2}m^2\phi^2 \ .$$

That this indeed does give us the Klein-Gordon equation is easily seen by applying the Euler-Lagrange equation. Explicitly

$$\frac{\partial\mathcal{L}_{KG}}{\partial\phi} - \partial_\mu\frac{\partial\mathcal{L}_{KG}}{\partial(\partial_\mu\phi)} = m^2\phi - \partial_\mu\partial^\mu\phi = 0 \ .$$

The reason for our particular sign convention can be understood by writing out the whole derivative term explicitly

$$-\frac{1}{2}\partial^\mu\phi\partial_\mu\phi = \frac{1}{2}(+\frac{\partial^2}{\partial t^2} - \nabla^2)\phi \ .$$

Since it is natural to associate $\frac{1}{2}\frac{\partial^2}{\partial t^2}\phi$ with kinetic energy we choose this term positive since we define the Lagrangian (from analytical mechanics) as $L = T - V$.

Appendix E

Group Theory

In this appendix we give an introduction to group theory as it is applied in physics. We will be covering the basic definition of a group, and provide some additional definitions. After this we give a brief introduction to representation theory, and then discuss a special class of groups called *Lie groups.* A Lie group is a group that is also a differential manifold, which is of fundamental importance in physics. To provide a more thorough understanding and some physical context, we also provide a cursory introduction to *group manifolds* and *spin representations.* The chapter is then concluded with explicit calculations of Lie algebras for use throughout the thesis.

Definition E.0.1. A group is defined as a set of unique elements ($\mathbf{G}$) which together with an operation ($*$) fulfills the following:

Closure : *if* a *and* $b \in \mathbf{G}$*, then* $a * b \in \mathbf{G}$*.*

Associativity : $\forall a, b$ *and* $c \in \mathbf{G}$*,* $(a * b) * c = a * (b * c)$*.*

Unit element : *there exists an object* $e \in \mathbf{G}$ *such that* $a * e = e * a = a$*.*

Inverse element : *for each* $a \in \mathbf{G}$*, there is an element* $b \in \mathbf{G}$ *such that* $a * b = b * a = e$*.*

The most common example of an easy to understand group is the set of all integers combined with the operation of addition. The sum of two integers is always an integer, and we already know that addition is associative. The unit element of the group is 0, since adding it to any integer will yield that integer as a result. Finally, the inverse element of any integer is just the same integer with opposite sign.

In physics we use the group structure to express certain symmetries of a system, such as invariance under a rotation, or Lorentz invariance. It can be shown that, with an appropriate operator, these symmetries can be expressed as groups and represented as matrices. Group theory can then be used to find the equations of motions for a system purely by using the symmetries that the theory possesses.

To make use of group theory there are some tools we need to understand and use, so we continue with a few more definitions:

Definition E.0.2. A group is called **abelian** if the elements commute with regards to the group operation ($*$), that is:

$$\forall a,b \in \mathbf{G} \ : \quad a * b = b * a \ .$$

This is important because the difference between what mathematical tools are applicable to abelian and non-abelian groups is very big. A prime example of a non-abelian group is the set of all invertible $n \times n$ matrices combined with the matrix multiplication operator. The difference is also readily observable in sections 2.2 and 3.

Definition E.0.3. If $\mathbf{G}$ and $\mathbf{H}$ are two groups with the same group operator ($*$), $\mathbf{H}$ is said to be a subgroup of $\mathbf{G}$ if

$$a \in \mathbf{H} \ \Rightarrow \ a \in \mathbf{G}$$

There are always two trivial subgroups; the unit element and the group itself.

Definition E.0.4. A map $\psi : \mathbf{G} \mapsto \mathbf{H}$ is a **homomorphism** if

$$g_i * g_j = g_k \ ,$$

and

$$\psi(g_i) * \psi(g_j) = \psi(g_k) \ .$$

where $g_i, g_j, g_k \in \mathbf{G}$, $\psi(g_i) * \psi(g_j), \psi(g_k) \in \mathbf{H}$. If ψ is also bijective it is an isomorphism, which we denote with $\mathbf{G} \cong \mathbf{H}$.

E.1 Representation theory

The most simple, but perhaps most instructive example of a continuous group is the group of all rotations in two dimensions. While we can write down a general element of the group of all rotations in two dimensions as $g(\theta)$, where θ is a continuous parameter, we would like to construct a more explicit representation. To do this we associate each element of the group with a matrix, taking the matrix multiplication as the group operator. If we fix a coordinate system we may express $g(\theta)$ with the rotations matrix $R(\theta)$ defined as

$$R(\theta) = \begin{bmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{bmatrix} .$$

The idea of associating group elements with matrices is formally known as *representation theory.* In mathematics group theory and representation theory can certainly be considered different subjects but in physics a distinction is seldom made.

Now that we have introduced the idea of representing group elements with matrices we are ready to continue. While it was easy to write down the rotation matrix in two dimensions we would certainly like to be able to rotate things in more than one plane. Since our intuition as humans is for the most part only good in two to three dimensions we would like to find a general way of constructing rotation matrices in arbitrary dimension. Thus, we must ask ourselves what property defines a rotation. The common answer is that a rotation is a transformation that preserves the length of vectors. Thus we want to find all matrices that leaves the quantity $x^T x$ invariant. We consider an arbitrary transformation with a matrix M

$$x^T x \mapsto (Mx)^T (Mx) = x^T M^T M x \ .$$

Now, if this quantity is to be equal to $x^T x$ we must have $M^T M = 1$. We call matrices that satisfies this condition *orthogonal.* But we are not done, if we take the determinant of the defining equation of a orthogonal matrix we find that $\det(M) = \pm 1$. Matrices which switch parity are orthogonal matrices with determinant -1. To exclude these we also demand that a rotation matrix M satisfies $\det(M) = 1$. These two demands are usually summarized by saying that a rotation is an element of the group $SO(n)$ where S stands for *special*, meaning that we have a determinant of one. O is for *orthogonal* and n is the dimension of the matrix and, obviously, of the space.

E.2 Lie Groups

In physics we are mostly interested in continuous groups. A simple example apart from rotations are the group of real numbers. Of special interest are the continuous groups known as *Lie groups* (named after the brilliant Norwegian mathematician Sophus Lie (1842-1899)), as they are differential manifolds meaning they describe some continuous geometry. This allows us to perform certain operations on the group manifolds, namely integration and differentiation. Lie groups are defined as follows:

Definition E.2.1. A Lie group ($\mathbf{G}$) is a finite-dimensional differential manifold with an associated smooth multiplication map:

$$(g_j, g_i) \in \mathbf{G} \times \mathbf{G} \to g_i g_j \in \mathbf{G}$$

and a smooth inverse map

$$g \in \mathbf{G} \to g^{-1} \in \mathbf{G}$$

that satisfy the group axioms in E.0.1.

When we construct the *covariant derivative* in section 2.1 there is a necessity for a theory of the first spatial derivative of the transformations corresponding to the Lie groups. We refer to these terms as being *Lie algebra-valued.* The theory regarding these objects is that of *Lie algebra.* The Lie algebra can be seen as a minimal representation of a group using group elements infinitely close to the identity element as *generators* that can span the entire group.

Definition E.2.2. A Lie algebra $\mathbf{g}$ is a vector space which has a bilinear mapping $[.,.] : \mathbf{g} \times \mathbf{g} \mapsto \mathbf{g}$ such that:

For all $X,Y \in \mathbf{g}$

$$[X,Y] = -[Y,X] \ .$$

The Jacobi identity,

$$[X,[Y,Z]] + [Y,[Z,X]] + [Z,[X,Y]] = 0 \ .$$

The objects X, Y, Z of this definition are usually denoted T_i where i is some index, and are called *generators*. The brackets([.,.]) are generally referred to as *Lie brackets*.

Now, how can we construct our original rotation matrix from these generators? To proceed we will have to use a idea, courtesy of Sophus Lie. Lie proposed that instead of performing the whole transformation at once, we can split it into many small transformations. Let $U(\phi)$ be a rotation transformation and, as discussed before, also a matrix. For a small transformation we may expand our transformation around the identity as

$$U(\delta\phi) = \mathbb{1} + \delta\phi T \ .$$

Now we can express a full transformation $U(\phi)$ by first dividing up the parameter in N pieces, $\phi = N\delta\phi$. To compensate we of course have to perform the transformation N times. We can now take the limit as N tends towards infinity of N infinitesimal transformations performed in succession:

$$U(\phi) = \lim_{N\to\infty} (\mathbf{1} + \frac{\phi T}{N})^N .$$

The infinity-limit on N defines the exponential, so we have

$$U(\phi) = \mathrm{e}^{\phi T} \ .$$

We see that to construct a full transformation we only need to find the generator and then exponentiate it. It is important that the exponential of a matrix should be interpreted as an infinite series according to

$$\mathrm{e}^X = \sum_{i=0}^{\infty} \frac{X^n}{n!} \ .$$

Let us return to our rotation matrix and find it again using the method described above. First we need to find the generator T. By expanding the defining equation for $SO(2)$ infinitesimal we find

$$(\mathbb{1} + i\delta\phi T)^T(\mathbb{1} + i\delta\phi T) = 1 \implies T^T + T = 0 \ ,$$

if the equation is to hold to first order. Now, there are not many 2×2 matrices which satisfy this condition. Truth is, down to a scale factor, c, there is only one:

$$T = c \cdot \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix} .$$

Now, using the exponential equation we find

$$e^{\phi cT} = \sum \frac{(\phi cT)^n}{n!} = \begin{bmatrix} \sum_{n=1} \frac{(-c)^{(n+1)}(\phi)^{2n}}{(2n)!} & \sum \frac{(-c)^{n+1}(\phi)^{2n-1}}{(2n-1)!} \\ -\sum \frac{(-c)^{n+1}(\phi)^{2n-1}}{(2n-1)!} & \sum \frac{(-c)^{n+1}(\phi)^{2n}}{2n!} \end{bmatrix}_{c=1} = \begin{bmatrix} \cos\phi & \sin\phi \\ -\sin\phi & \cos\phi \end{bmatrix} ,$$

where we in the last step see that we need to set $c = 1$ to get the rotation matrix out of our endeavour. We have managed to retrieve the classical rotation matrix, and now you may wonder for which groups are these manipulations actually allowed. It turns out that this procedure works for any group whose elements we may write as $U(\phi_1,\phi_2,..\phi_n)$ given that the parameters are continuous and that there exists $U(\phi_1,\phi_2,..\phi_n) = \mathbf{1}$ for some set of ϕ. This restriction is the same as saying that the parameters define a differentiable manifold, as a manifold consisting of a set of orthogonal, continuous parameters is trivially differentiable.

Now that we have confirmed that Lie's claim holds we can do this the other way around! Remembering that $U(\phi) = \mathrm{e}^{T\phi}$, differentiating both sides with respect to ϕ and then letting $\phi \to 0$ we get:

$$\frac{\mathrm{d}}{\mathrm{d}\phi}U(\phi) = TU(\phi) \ , \tag{E.1}$$

$$\left[\frac{\mathrm{d}}{\mathrm{d}\phi}U(\phi)\right]_{\phi=0} = T \cdot U(0) \ , \tag{E.2}$$

since U is a transformation around the unit element $U(0)$ is just the identity, giving us an expression for the generator T in terms of the parametrized representation matrix.

Now we have only discussed groups with one parameter, so we turn our attention to groups with multiple parameters. For groups with multiple generators we can further develop our understanding of them by attempting to combine them. We begin with expanding the group operator according to

$$U_a(\delta\phi) = \mathrm{e}^{(T_a \ \delta\phi_a)} = 1 + T_a\delta\phi_a + \frac{1}{2}T_a^2(\delta\phi_a)^2 \ .$$

Using this notation we can combine two generators T_a and T_b:

$$U_b^{-1}U_a^{-1}U_bU_a = 1 + \delta\phi_a\delta\phi_b[T_a,T_b] + \ldots \ .$$

We can choose to ignore higher order terms because we have assumed $\delta\phi_a, \delta\phi_b$ to be small. Since the group is closed $[T_a,T_b]$ must either be a group generator or zero. We see from this that

$$[T_a, T_b] = f_{ab}{}^cT_c \ , \tag{E.3}$$

must hold. T_c denotes some arbitrary group generator for the same group as T_a, T_b. The constants, $f_{ab}{}^c$ are called the *structure constants*. It can be shown that $f_{ab}{}^c$ is antisymmetric in its three indices, although some care needs to be taken when exchanging raised and lowered indices. The structure constant provides a representation independent characterization of a Lie group. This is a good time to look again at the definition of a Lie algebra. The commutator is the bilinear operation in its definition and it fulfills the Jacobi identity.

Finally, we note that the sum of two elements in the Lie algebra are in the Lie algebra, since the product of two elements in the Lie group is in the Lie group, and multiplication corresponds to addition in the exponent.

E.2.1 Killing forms

So far we have introduced the Lie bracket of a Lie algebra so that given two elements in the algebra we may construct a third one. However, we may ask for another operator that given two group elements returns a scalar. What we are seeking is thus a generalisation of the scalar product from linear algebra. Before we define this operator we first introduce a new notation for the Lie bracket with

$$\mathrm{ad}_x(y) = [x,y] \ .$$

The operator $\mathrm{ad}_x(y)$ is called the *adjoint action*. The reason we introduce this notation is because the use of Lie brackets may become cumbersome. For example consider the expression

$$[x[x,[x,[x,y]]]] = \mathrm{ad}_x \circ \mathrm{ad}_x \circ \mathrm{ad}_x \circ \mathrm{ad}_x(y) = \mathrm{ad}_x^4(y) \ .$$

It is clear that the right hand side is a more appropriate notation. We are now ready to give a definition:

Definition E.2.3. Given a finite-dimensional Lie algebra g its **Killing form** is the symmetric bilinear form given by the formula

$$\kappa(x,y) = \mathrm{tr}[\mathrm{ad}_x \circ \mathrm{ad}_y] \ .$$

Remember that in our notation this reads $ad_x \circ ad_x = [x[y,\cdot]]$. The Killing form is therefore a matrix, given a basis. Of course we shouldn't be surprised, there is after all a trace in the definition. Now, let us consider the special case of the adjoint action of a generator. To emphasize we write $x = T^a$. Its action on another generator is by definition

$$\mathrm{ad}_{T^a}(T^b) = [T^a,T^b] = f^{ab}{}_cT^c \ ,$$

where f is the structure constant. Since all elements of a Lie algebra can be written as a linear combination of the generators we can use these as a basis. Construct a vector, x, where the a:th entry indicates the multiplicity of T^a. Then we may write down a matrix for the adjoint action as

$$(\mathrm{ad}_{T^a})_{bc} = f^{ab}{}_c \ .$$

Just applying the same idea we find

$$\mathrm{ad}_{T^a} \circ \mathrm{ad}_{T^b} = (\mathrm{ad}_{T^a})_{cd}(\mathrm{ad}_{T^b})_{de} = f^{ac}{}_d f^{bd}{}_e \ ,$$

and we thus reach

$$\mathrm{tr}[\mathrm{ad}_{T^a}\mathrm{ad}_{T^b}] = \mathrm{tr}[f^{ac}{}_d f^{bd}{}_e] = f^{ac}{}_d f^{bd}{}_c \ ,$$

where we have summation over c and e.

E.3 An introduction to group manifolds

The properties of the group manifold has some implications for the physics that come from that group. Because of this it is necessary to be at least somewhat familiar with group manifolds, so we introduce them here. In general, the group manifold is an isomorphism class of manifolds, rather than some particular manifold. One way of finding a manifold that is part of this isomorphism class for a matrix representation of a group is by finding the parameter space of the matrix representation. To build intuition we will go over a few examples and discuss them afterwards.

The purpose of this section is for the reader to gain some understanding for the group manifold, a general manifold and the *direct product*. Examples of direct products relevant throughout this thesis are $SL(2) \times SL(2)$ and $SL(3) \times SL(3)$, which are the basis of the theory of Chern-Simons gravity that we present.

The special unitary group of order 2 can be written on the following parametric form

$$\begin{bmatrix} a+ib & -c+id \\ c+id & a-ib \end{bmatrix} \ ,$$

with the restriction $a^2+b^2+c^2+d^2 = 1$ coming from the restriction on the determinant. It is easily seen that the group manifold of $SU(2)$ in this representation is a hypersphere. The hypersphere is isomorphic to the three dimensional ball by isomorphisms of the form

$$\begin{cases} x = a \ , \\ y = b \ , \\ z = c \ , \\ r^2 = 1 - d^2 \ . \end{cases}$$

We see this because all of a,b,c,d are already limited by values between 0 and 1, and we've transferred the equation to one of the form $x^2 + y^2 + z^2 = r^2$ where the radius can be between 0 and 1.

The group $SO(1{,}1)$ is just the group of all Lorentz tranformations in one direction. The transformation matrix can be written as

$$\begin{bmatrix} p & s \\ s & p \end{bmatrix}$$

where the orthogonality condition is taken care of by the Minkowski metric. The determinant restriction gives us the parameter restriction $p^2 - s^2 = 1$ which defines the unit hyperbola. The hyperbola is an obvious example of a *non-compact* Lie group, as the hyperbola continues off into infinity meaning there is no well-defined final point. It is also not fully connected, because the allowed values for p are $p \leq -1, p \geq 1$, leaving a hole in the middle.

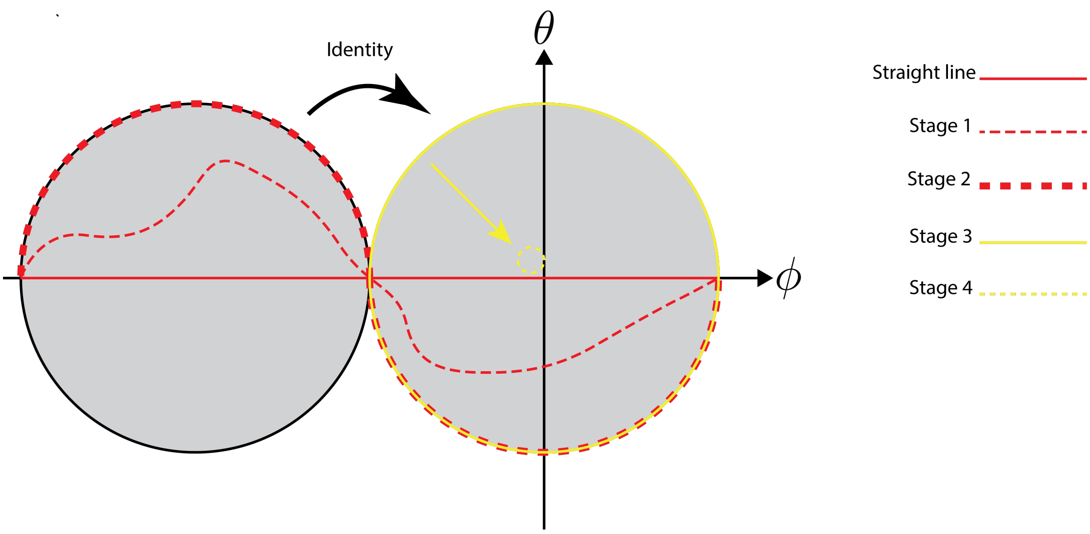
As we have seen in E.5.1 rotations in three dimensions can be represented as a multiplication of three rotations around different axes by some angle. Each of these matrices takes as a parameter an angle, that can take any real value. What is interesting is that this space is degenerate since two points 2π apart in the parameter space correspond to the exact same transformation. This means that all points in the $\mathbb{R}^3$ space are *identified* with a set of points contained within a sphere of radius π, so the group manifold of $SO(3)$ is actually a sphere with radius π.

We can choose this sphere to be centered on the origin, so that the range of each of the parameters is given by $\theta^2 + \phi^2 + \psi^2 \leq \pi^2$. At this point we have *almost* defined the group manifold of $SO(3)$. To complete the description we have to note that a rotation by π and $-\pi$ around some axis is equivalent, so the points on the boundary of the ball are identified with their antipodal points.

In conclusion, the group manifold of $SO(3)$ is a ball, of radius π in the parameter space of the matrix representation. The antipodal points of the surface of the ball are connected by an identity, that is, we identify them as the same point. An important consequence of this is that $SO(3)$ is not *simply connected*.

On a simply connected manifold any closed curve can be shrunk to a point by a continuous transformation. If we connect two antipodal points of the ball with some curve, that curve is closed, but cannot be continuously shrunk to a point since its' end points have to remain antipodal or the closed curve will open up. It turns out that if we let a curve run through two adjacent spheres in the parameter space this curve will in fact be shrinkable to a point. To show this we will provide a simple illustration, restricting us to the $\theta - \phi$ axis cross-section of the sphere for graphical simplicity.

Beginning with a straight line from $\phi = -3\pi$ to π, we can continuously deform the line, keeping the points $\phi = -3\pi, -\pi, \pi$ stationary. We can do this until the lines are on the boundary of the sphere as in stage 2 in the image. We then use the identity to transfer the line at the boundary of the extra sphere to the primary sphere. We now have a curve that can be trivially closed by continuous deformation, as is illustrated below



Comparing the manifolds of $SO(3)$ and $SU(2)$ we see that they are very similar, and they also have the same Lie algebra. This connection is no accident, if two groups have sufficiently similar manifolds their Lie algebra is the same. This also shows us a clear example of a group property not given by the Lie algebra, namely *simple connectedness*.

With this introduction to group manifolds we are ready for another definition:

Definition E.3.1. The *direct product*, $G \times H$ between the groups G, H with operators $*, \cdot$ is defined as follows

Elements in the new group are defined according to the Cartesian product, as $(g, h) : g \in G, h \in H$ on these elements we define the group operator $(\star)$ according to

$$(g,h) \star (g',h') = (g * g', h \cdot h') \ .$$

Some obvious consequences of this definition is that the direct product always has the two constituent groups as subgroups, corresponding to elements of the type g,1 and $(1,h)$. In addition, it is clear that the Lie algebra of the composite group is $\mathbf{g} \oplus \mathbf{h}$ because the subgroups approach the identity independent of each other. The group manifold of the composite group is the Cartesian product of the constituent group manifolds. In the examples in fig. E.1 the isomorphism symbol is used instead of equality because we are only interested in the isomorphism class of a manifold.

Furthermore, the definition of the direct product makes it straightforward to construct a matrix representation of the composite group. Picking two matrix representations of G and H according to T_G, T_H it is easy to see that the block diagonal matrix is in accordance with the definition:

$$\begin{bmatrix} T_G & \mathbf{0} \\ \mathbf{0} & T_H \end{bmatrix} \ .$$

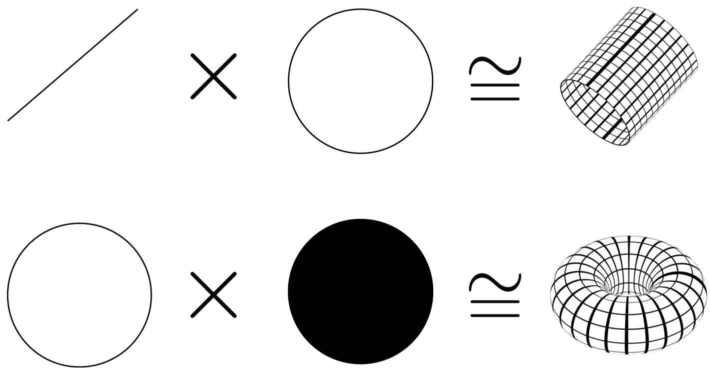


Figure E.1: Two different direct products between manifolds. The direct product of a line and a circle is shown to be a cylinder. The direct product of a circle and a disk embedded in two separate spaces should normally be embedded in 4-space, but it is isomorphic to the solid torus in three dimensions.

E.4 Spin representation

In this thesis we discuss Yang-Mills theory, which is formulated in terms of a group transformation acting on an abstract vector field. In section 2.2 the field was a scalar, and in 3.1 it was unspecified. Apart from the scalar field, classifications include tensor- and spinor fields. In this section we provide a basic discussion on classifying fundamental fields by their *spin j*. A similar classification can also be applied to the symmetry group itself, independent of whether it acts on any fundamental field. In this case, the group is a spin j representation of the special orthogonal group $SO(n,k)$. For the gauge group itself to be classified in this manner it must be decomposable into subgroups that are isomorphic to $SO(n,k)$.

The importance of this classification lies in quantum physics, where the *spin-statistics theorem* restricts the statistics of a quantized field theory. In particular, quantized spinor fields must obey Fermi-Dirac statistics and quantized scalar and tensor fields must obey Bose-Einstein statistics. That is, a quantized spinor field obeys the Pauli exclusion principle while scalar and tensor fields do not.

The definitions of scalars, spinors and tensors all specify how they transform under a rotation. For some arbitrary field, we can find out how it transforms under a rotation by relating the generators of its gauge group to the generators of the rotation group. For any gauge group, if we can rescale its generators in a way such that it has the same commutation relations as the special orthogonal group (E.5.1), it means that the *group manifolds* of the gauge groups are locally isomorphic. A local isomorphism of the group manifolds is a local equivalence between the transforms that they perform. If this isomorphism exists, we can characterize a fundamental field by performing a rotation infinitesimally.

If we denote this angle of rotation as $\mathbf{\Omega}$, we can define the *spin j* of the fundamental field as the largest j such that

$$\exp\left(\frac{2\pi}{j}T\right) = \mathbb{I} \tag{E.4}$$

where T is the generators of the gauge group in question, rescaled to fulfill the commutation relations of $SO(k,n)$.

As an example of this, we use the isomorphism between the gauge groups $SU(2)$ and $SO(3)$. The rotation matrix is given, denoting its generators as T_n and the angle of rotation as Ω, by

$$U_{\mathbf{\Omega}} = \exp\left(\Omega_x T_1 + \Omega_y T_2 + \Omega_z T_3\right).$$

The Pauli matrices E.5.3, with slight modifications fulfill the same commutation relations, so locally a rotation can be expressed as

$$\begin{aligned} U_{\mathbf{\Omega}} &= \exp\left(\frac{i\Omega_x\sigma_1}{2} + \frac{i\Omega_y\sigma_2}{2} + \frac{i\Omega_z\sigma_3}{2}\right) = \exp\left(\frac{i}{2}\begin{bmatrix} \Omega_z & \Omega_x - i\Omega_y \\ \Omega_x + \Omega_y & -\Omega_z \end{bmatrix}\right) \\ &= \begin{bmatrix} \cos\left(\frac{|\Omega|}{2}\right) + i\Omega_z \frac{1}{|\Omega|}\sin\left(\frac{|\Omega|}{2}\right) & i(\Omega_x - i\Omega_y)\frac{1}{\Omega}\sin\left(\frac{|\Omega|}{2}\right) \\ i(\Omega_x + i\Omega_y)\frac{1}{\Omega}\sin\left(\frac{|\Omega|}{2}\right) & \cos\left(\frac{|\Omega|}{2}\right) - i\Omega_z \frac{1}{|\Omega|}\sin\left(\frac{|\Omega|}{2}\right) \end{bmatrix}. \end{aligned}$$

We see that under a rotation Ω by a total of 2π radians the result is

$$U_{\mathbf{\Omega}} = \begin{bmatrix} -1 & 0 \\ 0 & -1 \end{bmatrix},$$

and to return to the identity, a rotation by 4π radians is required. From (E.4) we see that the fundamental field coupled to the gauge group $SU(2)$ has spin 1/2. A fundamental field with a half-integer spin is called *spinor*, and $SU(2)$ is referred to as a *spinor representation* of $SO(3)$. In general, this method cannot be applied to a completely general gauge group because isomorphies like $SU(2) \cong SO(3)$ do not nessecarily exist. In a similar fashion it can be shown that $sl(2) \cong so(1{,}2)$ is a spinor representation of $SO(1{,}2)$. For the rotation group of arbitrary order and signature, $SO(k{,}n)$, it's spinor representation is called $Spin(k{,}n)$. In section 6 and onwards we investigate theories that we refer to as spin-2 and spin-3 gravity, respectively. The spin in this refers to the properties of the metric tensor $g_{\mu\nu}$ and the metric-like higher-spin field $\psi_{(\mu\nu\rho)}$. The metric tensor is a symmetric tensor of rank two. A rotation is given locally by $\exp(\Omega T)$. Acting infinitesimally on $g_{\mu\nu}$, we have that

$$\begin{aligned}\delta g_{\alpha\beta} &= \delta\Omega(T^{\mu}_{\alpha}\delta^{\nu}_{\beta} + T^{\nu}_{\beta}\delta^{\mu}_{\alpha})g_{\mu\nu} \\ &= 2\delta\Omega T^{\mu}_{\alpha} g_{\mu\beta}\end{aligned}$$

where symmetry was used to exchange α,β and μ, ν in the second term in the parenthesis. Thus, the rotation generators acting on $g_{\mu\nu}$ act as $\exp(2\Omega T)$, which just a rotation by an angle of 2Ω. We see that the spin of the metric tensor must be two. In the same way we have for $\psi_{(\mu\nu\rho)}$ that

$$\begin{aligned}\delta\psi_{\alpha\beta\sigma} &= \delta\Omega(T^{\mu}_{\alpha}\delta^{\nu\rho}_{\beta\sigma} + (T^{\nu}_{\beta}\delta^{\rho\mu}_{\sigma\alpha} + (T^{\rho}_{\sigma}\delta^{\mu\nu}_{\alpha\beta})\psi_{\mu\nu\rho} \\ &= 3\delta\Omega T^{\mu}_{\alpha}\psi_{\mu\beta\sigma},\end{aligned}$$

and we see that $\psi_{(\mu\nu\rho)}$ is a spin-3 field.

E.5 Special Lie groups

In this section we will present some of the Lie groups that will be important throughout our work. We will present their generators, Lie algebra and dimensions. We also note some important Lie algebra isomorphisms that are used throughout the thesis.

In general we will refer to groups by their abbreviation, followed by brackets containing the dimension of the group. For example, $SO(5)$ would be the special orthogonal group in five dimensions. If we have two indices inside the brackets as in $SO(m{,}n)$ this means that the group describes a geometry with m time-like dimensions, and n space-like dimensions. The difference between the timelike and space-like dimensions are what metric they have, which will make the representation of $SO(1{,}2)$ different from $SO(3)$ despite them having the same dimension. The generators for $SO(1{,}2)$ are calculated in E.5.1. When referring to the Lie algebra of a Lie group we denote the Lie algebra with the abbreviation of the group in *lowercase* letters. This is standard convention used in order to avoid confusion when referring to Lie groups and Lie algebras simultaneously.

E.5.1 Special orthogonal group, $SO(n)$

The $SO(n)$ group is the special orthogonal group in n dimensions. *Special* denotes a determinant of 1, and *orthogonal* refers to matrix orthogonality.

$SO(2)$

One of the most familiar groups in $SO(n)$ is $SO(2)$, which is the group of rotations in $\mathbb{R}^2$, represented by the single operation

$$U(\phi) = \begin{bmatrix} \cos(\phi) & \sin(\phi) \\ -\sin(\phi) & \cos(\phi) \end{bmatrix} . \tag{E.5}$$

To obtain the Lie algebra we apply equation (E.1) to get

$$T = -[\begin{bmatrix} -\sin(\phi) & -\cos(\phi) \\ \cos(\phi) & -\sin(\phi) \end{bmatrix}_{\phi=0} = \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix} .$$

Since $SO(2)$ only contains one generator, it has a trivial Lie algebra.

$SO(3)$

To apply the same method to $SO(3)$ we first need its matrix representation. We can split $SO(3)$ up into a rotation in each orthogonal 2d plane in 3d space, that is, a rotation in the xy-plane, a rotation in the yz-plane and a rotation in the xz-plane.

$$U_0(\phi) = \begin{bmatrix} 1 & 0 & 0 \\ 0 & \cos(\phi) & \sin(\phi) \\ 0 & -\sin(\phi) & \cos(\phi) \end{bmatrix}, U_1(\phi) = \begin{bmatrix} \cos(\phi) & 0 & -\sin(\phi) \\ 0 & 1 & 0 \\ \sin(\phi) & 0 & \cos(\phi) \end{bmatrix}, U_2(\phi) = \begin{bmatrix} \cos(\phi) & \sin(\phi) & 0 \\ -\sin(\phi) & \cos(\phi) & 0 \\ 0 & 0 & 1 \end{bmatrix} \tag{E.6}$$

Which yield the following generators (once again by applying (E.1)):

$$T_0 = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & -1 & 0 \end{bmatrix}, T_1 = \begin{bmatrix} 0 & 0 & -1 \\ 0 & 0 & 0 \\ 1 & 0 & 0 \end{bmatrix}, T_2 = \begin{bmatrix} 0 & 1 & 0 \\ -1 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} . \tag{E.7}$$

Finally, we find the structure factor by commuting all possible combinations of T_0, T_1, T_2. We will only show two commutators here, and then write down a general expression for the structure factor.

$$[T_0, T_1] = T_0T_1 - T_1T_0 = \begin{bmatrix} 0 & 1 & 0 \\ -1 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} = T_2$$

$$[T_0, T_2] = T_0T_2 - T_2T_0 = \begin{bmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ -1 & 0 & 0 \end{bmatrix} = -T_1 .$$

These two commutators show the beginnings of a symmetry for the structure factor, and the general formula becomes

$$[T_i, T_j] = \epsilon_{ijk}T_k . \tag{E.8}$$

We see that the structure constants $f_{ijk} = \epsilon_{ijk}$ as per the notation in eq (E.3).

$SO(1,2)$

$SO(1,2)$ is the special orthogonal group with associated metric

$$\begin{bmatrix} -1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} .$$

Inutitively, we want to have one length-preserving parametric transformation for every orthogonal plane we can find in $\mathbb{R}^n$. If a plane consists of one space-like coordinate and one time-like coordinate then the length preserving transformation is the Lorentz transformation instead of the rotation that we have seen before.

The regular Lorentz transformation with an inertial system S and a boosted system S', S' travelling with velocity v with respect to S, is defined as follows (in x variable and t variable):

$$x' = \gamma(x - vt) \ , \tag{E.9}$$

$$t' = \gamma(t - vx/c^2) \ , \tag{E.10}$$

where γ is the Lorentz factor, $\gamma = \frac{1}{\sqrt{1-v^2/c^2}}$.

If we introduce the hyperbolic parameter θ, also called *rapidity*, defined as $\mathrm{e}^\theta = \gamma(1 + v/c)$ we may rewrite the Lorentz transform [12]. We compute $\sinh\theta$ and $\cosh\theta$ for reasons which will become apparent soon. We do moreover, from now and on, set $c = 1$ for simplicity:

$$\sinh\theta = \frac{\mathrm{e}^\theta - \mathrm{e}^{-\theta}}{2} = \frac{\gamma(1+v) - 1/(\gamma(1+v))}{2} = \frac{\gamma^2(1+v)^2 - 1}{2\gamma(1+v)} = \frac{1 + v - (1-v)}{2\gamma(1-v^2)} = v\gamma \ ,$$

and

$$\cosh\theta = \frac{\mathrm{e}^\theta + \mathrm{e}^{-\theta}}{2} = \frac{\gamma(1+v) + 1/(\gamma(1+v))}{2} = \frac{\gamma^2(1+v)^2 + 1}{2\gamma(1+v)} = \frac{1 + v + (1-v)}{2\gamma(1-v^2)} = \gamma \ .$$

Now we can express the Lorentz transform on hyperbolic form, yielding:

$$x' = x\cosh(\theta) - t\sinh(\theta) \ , \tag{E.11}$$

$$t' = t\cosh(\theta) - x\sinh(\theta) \ . \tag{E.12}$$

Equation (E.11) is very conveniently dependent on a parameter θ so that we can apply equation (E.1). The transformation matrices for $SO(1{,}2)$ are then simply:

$$U_0(\theta) = \begin{bmatrix} 1 & 0 & 0 \\ 0 & \cos(\phi) & -\sin(\phi) \\ 0 & \sin(\phi) & \cos(\phi) \end{bmatrix}, U_1(\theta) = \begin{bmatrix} \cosh(\theta) & -\sinh(\theta) & 0 \\ -\sinh(\theta) & \cosh(\theta) & 0 \\ 0 & 0 & 1 \end{bmatrix}, U_2(\theta) = \begin{bmatrix} \cosh(\theta) & 0 & -\sinh(\theta) \\ 0 & 1 & 0 \\ -\sinh(\theta) & 0 & \cosh(\theta) \end{bmatrix}. \tag{E.13}$$

Inserting into equation (E.1) we get the generators

$$T_0 = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & -1 \\ 0 & 1 & 0 \end{bmatrix}, T_1 = \begin{bmatrix} 0 & -1 & 0 \\ -1 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}, T_2 = \begin{bmatrix} 0 & 0 & -1 \\ 0 & 0 & 0 \\ -1 & 0 & 0 \end{bmatrix}. \tag{E.14}$$

We find the structure factor to be ${f_{ij}}^k = -\epsilon_{ij}^k$. To lower the k we need to note that for $k = 0$ we get an extra minus sign because the manifold of the group has a Minkowski metric. This we see is the only difference between the Lie algebra of $SO(3)$ and $SO(1{,}2)$.

E.5.2 Special linear group, $SL(n)$

$SL(n)$ is the group of $n \times n$ matrices with determinant 1. We will explicitly calculate the Lie algebra for $n = 2{,}3$. To find the Lie algebrae of $SL(2)$ and $SL(3)$ we will use LU-factorization, meaning we split the the arbitrary square matrix into a lower triangular matrix and an upper triangular matrix. This operation in reality depends on none of the diagonal elements of the matrix in question being 0, so it is only valid around the identity. Also, the decomposition actually adds n degrees of freedom, so we must add another restriction to obtain a unique decomposition. A valid such restriction is the requirement that the lower triangular matrix must be a unit triangular matrix, that is all of its diagonal elements are 1.

SL(2)

We write down $SL(2)$ as the product of a lower- and upper triangular matrix right away:

$$g = \begin{bmatrix} 1 & 0 \\ a & 1 \end{bmatrix} \begin{bmatrix} r & b \\ 0 & r^{-1} \end{bmatrix} ,$$

where the rs come from the determinant condition. As usual, using (E.1) we obtain the generators:

$$S_0 = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix} \quad , \quad S_1 = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix} \quad , \quad S_2 = \begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix} \quad .$$

To obtain a simple form of the commutation relations we rescale the generators as

$$S_0 = \frac{1}{2} \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix} \quad , \quad S_1 = \frac{1}{\sqrt{2}} \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix} \quad , \quad S_2 = \frac{1}{\sqrt{2}} \begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix} \quad .$$

making the commutation relations

$$[S_0,S_1] = S_1 \quad , \quad [S_2,S_0] = S_2 \quad , \quad [S_1,S_2] = S_0 \quad .$$

We can recombine the generators of $SL(2)$ so that its isomorphism to $SO(1{,}2)$ is readily apparent. We define a new set of generators as:

$$\begin{aligned} T_0 = S_0 &= \frac{1}{2} \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix} , \\ T_1 = \frac{S_1 + S_2}{\sqrt{2}} &= \frac{1}{2} \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} , \\ T_2 = \frac{S_1 - S_2}{\sqrt{2}} &= \frac{1}{2} \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix} . \end{aligned}$$

We evaluate the commutators of rising order and then write down the general expression:

$$\begin{aligned} [T_0,T_1] &= \frac{1}{\sqrt{2}}[S_0,S_1 + S_2] \\ &= \frac{S_1 - S_2}{\sqrt{2}} = T_2 , \\ [T_2,T_0] &= \frac{1}{\sqrt{2}}[S_1 - S_2,S_0] \\ &= -\frac{S_2 + S_3}{\sqrt{2}} = -T_1 , \\ [T_1,T_2] &= \frac{1}{2}[S_1 + S_2,S_1 - S_2] , \\ &= \frac{1}{2}\left([S_2,S_1] - [S_1,S_2]\right) = \frac{1}{2}(-S_0 - S_0) = -T_0 . \end{aligned}$$

Going off of these commutators we can write down the commutation relations as

$$[T_a,T_b] = \epsilon_{ab}{}^{c}T_c ,$$

where, as usual, the c can be lowered by the Minkowski metric, however in this case the minus sign is attached to the three. For the Minkowski metric to follow the convention of the rest of this appendix we switch the names of T_0 and T_2 so that our final generators are:

$$T_0 = \frac{1}{2} \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix} , \quad T_1 = \frac{1}{2} \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} , \quad T_2 = \frac{1}{2} \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix} .$$

and the structure constants are still $\epsilon_{ab}{}^{c}$.

$SL(3)$

We will find the Lie algebra of $SL(3)$, and then show how to find a diagonal embedding of $SL(2)$ in the larger $SL(3)$ gauge group. For ease of finding, we have split off the other possible recombinations of the $SL(3)$ Lie algebra to their own Appendix, E.5.2. We do not try to show how to find these particular recombinations, but rather just present their matrices and commutation relations. These recombinations specify a *principal embedding*, where the commutation relations between the $sl(2)$ subalgebra and the five remaining generators are non-trivial.

We write down the LU-decomposition of a three-dimensional matrix with a determinant equal to one:

$$g = \begin{bmatrix} 1 & 0 & 0 \\ a & 1 & 0 \\ b & c & 1 \end{bmatrix} , \begin{bmatrix} k^{-1} & d & e \\ 0 & kr & f \\ 0 & 0 & r^{-1} \end{bmatrix} .$$

The generators can easily be read off, as the off-diagonal parameters are entirely free:

$$T_k = \begin{bmatrix} 1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 0 \end{bmatrix} , \quad T_r = \begin{bmatrix} 0 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 1 \end{bmatrix} ,$$
$$T_a = \begin{bmatrix} 0 & 0 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} , \quad T_b = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 1 & 0 & 0 \end{bmatrix} ,$$
$$T_c = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix} , \quad T_d = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} ,$$
$$T_e = \begin{bmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} , \quad T_f = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix} ,$$

where we number the matrices by the usual reading order, and the subscript indicates from which parameter the generator was obtained.

Performing a recombination according to:

$$T_0 = -\frac{1}{2} T_k = \frac{1}{2} \begin{bmatrix} -i & 0 & 0 \\ 0 & i & 0 \\ 0 & 0 & 0 \end{bmatrix} ,$$
$$T_1 = \frac{T_d + T_a}{2} = \frac{1}{2} \begin{bmatrix} 0 & -i & 0 \\ -i & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} ,$$
$$T_2 = \frac{T_d - T_a}{2} = \frac{1}{2} \begin{bmatrix} 0 & -i & 0 \\ i & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} ,$$

we see that these three generators T_1, T_2, T_3 obviously satisfy the commutation relations in E.5.2. Finding the matrices fulfilling the $sl(2)$ commutation relations by replicating the $sl(2)$ algebra in 2 our of the three indices of $sl(3)$ is called the *diagonal* embedding. The remaining five generators $W_{1\ldots5}$ are just $T_k - T_r$ together with T_b, T_c, T_e, T_f. The most important property of this embedding is that the higher spin generators W_a have trivial commutation relations with the $SL(2)$ generators T_b. Listing a preliminary set of higher spin generators in a diagonal embedding, we have

$$W_1 = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -1 \end{bmatrix} , \quad W_2 \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 1 & 0 & 0 \end{bmatrix} , \quad W_3 = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix} ,$$
$$W_4 = \begin{bmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} , \quad W_5 \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix} .$$

Since we do not use this embedding in our thesis, we do not present any commutation relations. It is likely that the higher spin generators W should be recombined to obtain simple commutation relations before they are employed in an actual gauge theory. The purpose of showing these generators is to illustrate the difference between a diagonal and a principal embedding, such as the recombinations of $SL(3)$ that we will state now.

Conventions for Lie algebras in spin-3

Here we state the explicit matrix form of the different generators used in higher spin calculations. We follow the conventions of [32]. We also give the Lie algebra and commutators. The standard $sl(3)$ algebra is generated by T_a and T_{ab} where T_a forms a $sl(2)$ subalgebra.

$$\begin{aligned}
[T_a,T_b] &= \epsilon_{ab}{}^c T_c \ , \\
[T_a,T_{bc}] &= 2\epsilon^d{}_{a(b}T_{c)d} \ , \\
[T_{ab},T_{cd}] &= -2\left(\eta_{a(c}\epsilon_{d)b}{}^e + \eta_{b(c}\epsilon_{d)a}{}^e\right) T_e \ .
\end{aligned}$$

The invariant bilinear form denoted by "tr" gives

$$\begin{aligned}
\mathrm{tr}[T_a T_{ab}] &= 2\eta_{ab} \ , \\
\mathrm{tr}[T_a T_{bc}] &= 0 \ , \\
\mathrm{tr}[T_{ab} T_{cd}] &= -\frac{4}{3}\eta_{ab}\eta_{cd} + 2(\eta_{ac}\eta_{bd} - \eta_{ad}\eta_{bc}) \ .
\end{aligned}$$

It is useful to define new generators, L_i and W_j, by linearly combining T_a and T_{ab} according to

$$\begin{aligned}
T_0 &= \frac{1}{2}(L_1 + L_{-1}) \ , & T_1 &= \frac{1}{2}(L_1 - L_{-1}) \ , & T_2 &= L_0 \ , \\
T_{00} &= \frac{1}{4}(W_2 + W_{-2} + 2W_0) \ , & T_{01} &= \frac{1}{4}(W_2 - W_{-2}) \ , & T_{02} &= \frac{1}{2}(W_1 + W_{-1}) \ , \\
T_{11} &= \frac{1}{4}(W_2 - W_{-2} - 2W_0) \ , & T_{12} &= \frac{1}{2}(W_1 - W_{-1}) \ , & T_{22} &= W_0 \ .
\end{aligned}$$

The generators L and W obey the following commutation relations

$$\begin{aligned}
[L_i,L_j] &= (i-j)L_{i+j} \ , \\
[L_i,W_j] &= (2i-j)Wi+j \ , \\
[W_i,W_j] &= -\frac{1}{3}(i-j)(2i^2+2j^2-ij-8)L_{i+j} \ .
\end{aligned}$$

and all the non-zero components of the invariant bilinear form are

$$\begin{aligned}
&\mathrm{tr}[L_0L_0] = 2, & &\mathrm{tr}[L_1L_{-1}] = -4 \ , & & \\
&\mathrm{tr}[W_0W_0] = \frac{8}{3}, & &\mathrm{tr}[W_1W_{-1}] = -4 \ , & &\mathrm{tr}[W_2W_{-2}] = 16 \ .
\end{aligned}$$

The explicit matrix representation of the generators L_i and W_j is

$$\begin{aligned}
L_0 &= \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & -1 \end{bmatrix} \ , &
L_1 &= \begin{bmatrix} 0 & 0 & 0 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix} \ , &
L_{-1} &= \begin{bmatrix} 0 & -2 & 0 \\ 0 & 0 & -2 \\ 0 & 0 & 0 \end{bmatrix} \ , \\
W_0 &= \frac{2}{3}\begin{bmatrix} 1 & 0 & 0 \\ 0 & -2 & 0 \\ 0 & 0 & 1 \end{bmatrix} \ , &
W_1 &= \frac{2}{3}\begin{bmatrix} 0 & 0 & 0 \\ 1 & 0 & 0 \\ 0 & -1 & 0 \end{bmatrix} \ , &
W_2 &= 2\begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 1 & 0 & 0 \end{bmatrix} \ , \\
W_{-1} &= \begin{bmatrix} 0 & -2 & 0 \\ 1 & 0 & 2 \\ 0 & -1 & 0 \end{bmatrix} \ , &
W_{-2} &= 2\begin{bmatrix} 0 & 0 & 4 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} \ .
\end{aligned}$$

E.5.3 Special unitary group, $SU(n)$

The $SU(n)$ group is the group of unitary matrices with determinant 1. To find the generators for some arbitrary matrix A that fulfills $A^\dagger A = 1$ we will begin by observing that such a matrix can be written as its *Cholesky decomposition*:

$$A = U^\dagger U \ , \tag{E.15}$$

where U is an upper diagonal matrix. This makes enforcing the determinant equal to one condition a lot easier then for an arbitrary general $n \times n$ matrix A. The diagonal of U will have $n-1$ degrees of freedom since we have an equation of n real variables equal to a constant. Off the diagonal we have two degrees of freedom per index, for a total of $2 \cdot \frac{1}{2}((n-1)^2 + n - 1)$. The sum of the degrees of freedom is then $n^2 - 1$.

SU(2)

$SU(2)$ is a convenient example to start with due to its simplicity and importance in particle physics. The symmetries of the group play a crucial role when describing electroweak interaction. $SU(2)$ is represented by matrices of rank 2 and determinant 1. These matrices are also unitary. From the fact that $SU(2)$ is *special*, i.e. its group elements have determinant 1, it follows that the trace of the generators of the group are zero. This can be shown by using the exponential relationship between the generators (T_i), free parameters θ^i and the group elements U:

$$U(\theta^i) = \mathrm{e}^{\theta^i T_i} \ , \tag{E.16}$$

and the following identity[1]

$$\det(\mathrm{e}^R) = \mathrm{e}^{\mathrm{tr}\,[R]} \ , \tag{E.17}$$

where R is a matrix.

If we substitute $R = \theta^i T_i$ we find

$$\det(\mathrm{e}^{\theta^i T_i}) = \det(U) = 1 = \mathrm{e}^{\mathrm{tr}[\theta^i T_i]}. \Rightarrow \mathrm{tr}[T_i] = 0 \ , \tag{E.18}$$

where we used the linearity property of the trace. We moreover note that the generators T_i are *anti-hermitian*. This follows directly from expanding the exponential in equation (E.16) to first order and compute the product $U^\dagger U$:

$$U^\dagger U \simeq (1 + \theta^i T_i^\dagger)(1 + \theta^i T_i) = 1 + \theta^i (T_i + T_i^\dagger) + ... = 1 \Rightarrow T_i = -T_i^\dagger \ . \tag{E.19}$$

Thus we conclude that these generators of $SU(2)$ are both traceless *and* anti-hermitian.[2] In $SU(2)$ the generators are therefore given of the form

$$T_i = \begin{bmatrix} \alpha & \beta \\ \beta^* & -\alpha \end{bmatrix} \ , \tag{E.20}$$

where $\alpha \in \mathbb{R}$ and $\beta \in \mathbb{C}$. By setting $\alpha = 0$ we find two different kind of generators:

$$T_1 = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} \ , \tag{E.21}$$

and

$$T_2 = \begin{bmatrix} 0 & -i \\ i & 0 \end{bmatrix} \ . \tag{E.22}$$

The third and last generator of $SU(2)$ we get by setting $\beta = 0$:

$$T_3 = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix} \ . \tag{E.23}$$

[1]This identity can be shown by triangulizing R, i.e let $R = PTP^{-1}$, where T is upper-triangular with eigenvalues $\lambda_1 ... \lambda_n$ on the diagonal. Then it follows that e^T is a diagonal matrix with $\mathrm{e}^{\lambda_1} ... \mathrm{e}^{\lambda_n}$ on the diagonal. Since the determinant is the product of eigenvalues it directly follows that $\det(\mathrm{e}^T) = \mathrm{e}^{\mathrm{tr}\,T}$. Finally we observe that R and T have the same eigenvalues, thus we must have $\mathrm{tr}\,R = \mathrm{tr}\,T$. Moreover $P\mathrm{e}^T P^{-1} = \mathrm{e}^R$ (from $PT^kP^{-1} = R^k$ for all k) and therefore $\det(\mathrm{e}^R) = \det(\mathrm{e}^T) = \mathrm{e}^{\mathrm{tr}\,T} = \mathrm{e}^{\mathrm{tr}\,R}$ and we have proven our identity.

[2]Particle physicists often tend to choose a different representation of $SU(2)$, where the matrices are hermitian.

Note that the derived generators are not unique, however they form the smallest non-trivial representation of the Lie algebra, also called the *fundamental* representation which is unique for $SU(2)$. The generators T_1, T_2 and T_3 are called *Pauli matrices* and are commonly denoted σ_1, σ_2 and σ_3. From now on we stick to conventional notation and simply let $T_1 = \sigma_1$, $T_2 = \sigma_2$ and $T_3 = \sigma_3$. To derive the Lie algebra to $SU(2)$ we simply compute the commutators $[\sigma_1,\sigma_2], [\sigma_2,\sigma_3]$ and $[\sigma_1,\sigma_3]$ (the rest follow in the same manner):

$$[\sigma_1,\sigma_2] = \begin{bmatrix} i & 0 \\ 0 & -i \end{bmatrix} - \begin{bmatrix} -i & 0 \\ 0 & i \end{bmatrix} = \begin{bmatrix} 2i & 0 \\ 0 & -2i \end{bmatrix} = 2i\sigma_3 \ ,$$

$$[\sigma_2,\sigma_3] = \begin{bmatrix} 0 & i \\ i & 0 \end{bmatrix} - \begin{bmatrix} 0 & -i \\ -i & 0 \end{bmatrix} = \begin{bmatrix} 0 & 2i \\ 2i & 0 \end{bmatrix} = 2i\sigma_1 \ ,$$

$$[\sigma_1,\sigma_3] = \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix} - \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix} = \begin{bmatrix} 0 & -2 \\ 2 & 0 \end{bmatrix} = 2i\sigma_2 \ .$$

We can thus summarize the Lie algebra of $SU(2)$ according to[3]

$$[T_a,T_b] = [\sigma_a,\sigma_b] = 2i\epsilon_{abc}\sigma_c \ , \tag{E.24}$$

where ϵ_{abc} is the completely antisymmetric Levi-Civita symbol and the structure constant(s). From the Lie algebra of $SO(3)$ and the commutator relation in equation (E.8) we deduce that the structure factors of $SO(3)$ and $SU(2)$ only differ in sign why we may conclude that $SU(2)$ is isomorphic to $SO(3)$.

$SU(3)$

We may generalize the discussion in the last section by considering the group $SU(3)$. The matrix U can be written as:

$$U = \begin{bmatrix} a & d+i\rho & e+i\epsilon \\ 0 & b & f+it \\ 0 & 0 & c \end{bmatrix} \ . \tag{E.25}$$

We then use the restriction $abc = 1$ to rewrite (E.25) as $a = cr, b = r^{-1}, c = c^{-2}$. To get a matrix to plug into equation (E.1) we just need to evaluate $U^\dagger U$. We will do this and evaluate the partial derivatives in each of the parameters around the unit matrix ($r = c = 1, e = \epsilon = d = \rho = f = t = 0$).

$$\begin{aligned} U^\dagger U = A &= \begin{bmatrix} cr & 0 & 0 \\ d-i\rho & r^{-1} & 0 \\ e-i\epsilon & f-it & c^{-1} \end{bmatrix} \begin{bmatrix} cr & d+i\rho & e+i\epsilon \\ 0 & r^{-1} & f+it \\ 0 & 0 & c^{-1} \end{bmatrix} \\ &= \begin{bmatrix} c^2r^2 & cr(d+i\rho) & cr(e+i\epsilon) \\ (d-i\rho)cr & d^2+\rho^2+r^{-2} & (d-i\rho)(e+i\epsilon)+r^{-1}(f+it) \\ (e-i\epsilon)cr & (e-i\epsilon)(d+i\rho)+(f-it)r^{-1} & e^2+\epsilon^2+f^2+t^2+c^{-2} \end{bmatrix} \ . \end{aligned}$$

We continue by evaluating the partial derivatives around the identity matrices, defining $\mathbf{V} = \{r,c,e,\epsilon,d,\rho,f,t\}$ with $A(V_0)$ defining the identity matrix.

$$T_1 = -i\left[\frac{\partial A}{\partial r}\right]_{\mathbf{V}=V_0} = -i\begin{bmatrix} 2 & 0 & 0 \\ 0 & -2 & 0 \\ 0 & 0 & 0 \end{bmatrix} \ , \qquad T_2 = -i\left[\frac{\partial A}{\partial c}\right]_{\mathbf{V}=V_0} = -i\begin{bmatrix} 2 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & -2 \end{bmatrix} \ ,$$

$$T_3 = -i\left[\frac{\partial A}{\partial d}\right]_{\mathbf{V}=V_0} = -i\begin{bmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} \ , \qquad T_4 = -i\left[\frac{\partial A}{\partial \rho}\right]_{\mathbf{V}=V_0} = -i\begin{bmatrix} 0 & i & 0 \\ -i & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} \ ,$$

$$T_5 = -i\left[\frac{\partial A}{\partial e}\right]_{\mathbf{V}=V_0} = -i\begin{bmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ 1 & 0 & 0 \end{bmatrix} \ , \qquad T_6 = -i\left[\frac{\partial A}{\partial \epsilon}\right]_{\mathbf{V}=V_0} = -i\begin{bmatrix} 0 & 0 & i \\ 0 & 0 & 0 \\ -i & 0 & 0 \end{bmatrix} \ ,$$

$$T_7 = -i\left[\frac{\partial A}{\partial f}\right]_{\mathbf{V}=V_0} = -i\begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{bmatrix} \ , \qquad T_8 = -i\left[\frac{\partial A}{\partial t}\right]_{\mathbf{V}=V_0} = -i\begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & i \\ 0 & -i & 0 \end{bmatrix} \ .$$

[3]One can get rid of the factor 2 in the commutator by normalizing the generators T_i such that $T_i = \frac{\sigma_i}{2}$. This is conventionally done to yield a neater expression.

It is very common to rescale this Lie algebra to another form so that the algebra is more similar to that of $SU(2)$. Replacing S_1, S_2 with $T_1 = \frac{1}{2}S_1$ and $T_2 = \frac{2S_2 - S_1}{2\sqrt{3}}$ one gets the *Gell-Mann matrices.* This particular set of generators (which clearly spans all of $SU(3)$) were chosen by the American physicist Murray Gell-Mann (1929-) because they naturally extend the Pauli matrices from $SU(2)$ to $SU(3)$, which formed the basis for his model of quarks [45].

$SO(2,2)$

The Lie group $SO(2,2)$ consisting of all 4×4 matrices with determinant 1 and that forfill the orthogonality relation $X^\dagger \eta X = \eta$ with respect to the Lorentzian metric $\eta = \mathrm{diag}(-1,1,1,1)$ is of great importance in fundamental physics as it is the isometry group of AdS_3, the three-dimensional Anti de-Sitter space. In this thesis we make use of the fact that the Lie algebra of $SO(2,2)$ is isomorphic to that of $SO(1,2) \times SO(1,2)$ when relating the Chern-Simons gauge action to Einstein-Hilbert action. From the orthogonality relation for an element X in $SO(2,2)$ we may derive a condition for the generators of $SO(2,2)$ by expanding the exponential definition of the group element:

$$X = \exp(\theta^i A_i) \Rightarrow X \simeq 1 + \theta^i A_i \ . \tag{E.26}$$

For simplicity we consider only real elements X and generators A_i. Plugging the expansion into the orthogonality relation we find

$$X^T \eta X = (1 + \theta^i A_i^T)\eta(1 + \theta^i A_i) = \eta \Rightarrow A_i^T \eta + \eta A_i = 0 \ . \tag{E.27}$$

Moreover, since $SO(2,2)$ is a special group its generators are traceless. We have therefore two conditions which have to be satisfied by the generators. We do now perform a block decomposition of our generators A_i according to

$$A_i = \begin{bmatrix} S & T \\ U & V \end{bmatrix} ,$$

where S,T,U and V are real 2×2 matrices. Substituting the decomposition into our derived equation (I.3) we find

$$\begin{bmatrix} S^T & U^T \\ T^T & V^T \end{bmatrix} \begin{bmatrix} -\mathbb{1} & 0 \\ 0 & \mathbb{1} \end{bmatrix} + \begin{bmatrix} -\mathbb{1} & 0 \\ 0 & \mathbb{1} \end{bmatrix} \begin{bmatrix} S & T \\ U & V \end{bmatrix} = \begin{bmatrix} -S^T - S & U^T - T \\ -T^T + U & V^T + V \end{bmatrix} = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix} .$$

From this relation we may rewrite our generators as follows

$$A_i = \begin{bmatrix} S & T \\ T^T & V \end{bmatrix} ,$$

and we find that S,T, U and V must obey the following relations

$$\begin{cases} U = T^T \\ V = -V^T \\ S = -S^T \end{cases} .$$

From these conditions combined with the condition $\mathrm{tr}\,[A_i] = 0$ it is possible to find six different generators in a fundamental representation. This should be no surprise since there are six orthogonal planes in 4 dimensions to Lorentz boost and rotate in. We list these generators:

$$A_1 = \begin{bmatrix} 0 & 1 & 0 & 0 \\ -1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}, A_2 = \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & -1 & 0 \end{bmatrix}, A_3 = \begin{bmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

$$A_4 = \begin{bmatrix} 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 \end{bmatrix}, A_5 = \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}, A_6 = \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \end{bmatrix} .$$

It is a quite tedious task to compute the commutators $[A_i,A_j]$ in order to derive the Lie algebra of $SO(2,2)$ so this is done numerically. However, at this point we are settled for showing the isomorphism between $SO(2,2)$ and $SO(1,2) \times SO(1,2)$, since this result is of importance in our thesis.

What does it even mean that the algebra of $SO(2,2)$ is isomorphic to that of $SO(1,2) \times SO(1,2)$? In proper English we have verified the statement if we can show that $SO(2,2)$ can be separated into two parts, each part having its own subalgebra, in this case that of $SO(1,2)$. To prove this it will be useful to consider linear combinations of the derived generators A_i, $i = 1...6$. Consider the cleverly chosen linear combinations[4]

$$\begin{cases} A_{11} = \frac{1}{2}(A_1 + A_2),\ A_{21} = \frac{1}{2}(A_1 - A_2) \\ A_{12} = \frac{1}{2}(X + Y),\ A_{22} = \frac{1}{2}(X - Y) \\ A_{13} = \frac{1}{2}(P + Q),\ A_{23} = \frac{1}{2}(P - Q) \end{cases} ,$$

where X, Y, P and Q are yet unknown 4×4 matrices and the factor $\frac{1}{2}$ a normalization factor. We begin with the condition $[A_{1i}, A_{2j}] = 0$ and construct our unknowns A_{12}, A_{22}, A_{13} and A_{23} from it. Starting off by evaluating $[A_{11}, A_{22}]$ we obtain:

$$[A_{11}, A_{22}] = \frac{1}{4}([A_1, X] + [A_1, -Y] + [A_2, X] + [A_2, -Y]) \ .$$

We now make the "arbitrary" choice $X = A_3$ leading to

$$[A_{11}, A_{22}] = \frac{1}{4}([A_1, A_3] + [A_1, -Y] + [A_2, A_3] + [A_2, -Y]) = \frac{1}{4}(-A_4 + [A_1, -Y] - A_5 + [A_2, -Y]) \ ,$$

where we used $[A_1, A_3] = -A_4$ and $[A_2, A_3] = -A_5$. In order to achieve $[A_{11}, A_{22}] = 0$ we must now choose $Y = -A_6$, due to the fact that $[A_1, A_6] = +A_5$ and $[A_2, A_6] = +A_4$. Thus we have $X = A_3$ and $Y = -A_6$ and deduce $A_{12} = \frac{1}{2}(A_3 - A_6)$ and $A_{22} = \frac{1}{2}(A_3 + A_6)$. We use the same procedure to solve for P and Q. Consider

$$[A_{13}, A_{22}] = \frac{1}{4}([P, A_3] + [P, A_6] + [Q, A_3] + [Q, A_6]) \ ,$$

and let $P = A_4$, leading to

$$\frac{1}{4}([A_4, A_3] + [A_4, A_6] + [Q, A_3] + [Q, A_6]) = \frac{1}{4}(-A_1 + A_2 + [Q, A_3] + [Q, A_6]) \ ,$$

and to get $[A_{13}, A_{22}] = 0$ we therefore have to choose $Q = A_5$:

$$\frac{1}{4}(-A_1 + A_2 + [A_5, A_3] + [A_5, A_6]) = \frac{1}{4}(-A_1 + A_2 + A_1 - A_2) = 0 \ .$$

Thus we have found $A_{13} = \frac{1}{2}(A_4 + A_5)$ and $A_{23} = \frac{1}{2}(A_4 - A_5)$. Well, is all work done now? No, we obviously need to check that all commutators $[A_{1i}, A_{2j}]$ vanish and not just a few of them. This is done numerically and amazingly the relation holds for our seemingly non-arbitrary choices of $A_{11}, A_{12}, A_{13}, A_{21}, A_{22}$ and A_{23}! Thus we have shown that A_{1i} and A_{2j} can be separated and form their own subalgebra. By forming the commutators $[A_{1i}, A_{1j}]$ and $[A_{2i}, A_{2j}]$ we find

$$[A_{1i}, A_{1j}] = -\epsilon_{ij}{}^{k} A_{1k} \ ,$$
$$[A_{2i}, A_{2j}] = -\epsilon_{ij}{}^{k} A_{2k} \ ,$$

and this is precisely the Lie algebra of $SO(1,2)$ (see section E.5.1). Thus A_{1i} and A_{2j} each constitute a $SO(1,2)$ algebra.

[4] the choices of combinations are inspired by Thyssen and Ceulemans *Shattered Symmetry: Group Theory From the Eightfold Way to the Periodic Table*, page 307 [46].

Appendix F

Anti-de Sitter and Minkowski Spacetime

F.1 The Poincaré group

When we study Chern-Simons theory of gravity we will need to investigate the symmetries of both Anti-de Sitter space and Minkowski space. This is done by studying the group of transformations that leaves the metric invariant. We call this group the *isometry group* of the space. Since the Minkowski metric is rather familiar to us it is not to hard to guess which transformations should leave the metric invariant, however since this is certainly not the case for Anti-de Sitter space we need a general method to generate the transformations. Remember that a general transformation of the metric can be written as

$$g(x)_{\mu\nu} \mapsto g'_{\nu\mu}(x') = g(x)_{\rho\sigma}\frac{\partial x^\rho}{\partial x'^\mu}\frac{\partial x^\rho}{\partial x'^\nu} \ ,$$

In order for this transformation to preserve the metric we require that $g'_{\mu\nu}(x) = g_{\mu\nu}(x)$ and we rewrite the first equation as

$$g_{\nu\mu}(x') = g(x)_{\rho\sigma}\frac{\partial x^\rho}{\partial x'^\mu}\frac{\partial x^\sigma}{\partial x'^\nu} \ . \tag{F.1}$$

We will not attempt to solve this equation, instead we will study the specific case of an infinitesimal transformation. We thus write $x'^\rho = x^\rho + \epsilon\xi^\rho$. With this expansion we get $g_{\nu\mu}(x^\rho + \epsilon\xi^\rho) = g_{\nu\mu}(x^\rho) + \epsilon\xi^\rho\partial_\rho g_{\nu\mu}(x^\rho)$ and $\frac{\partial x^\rho}{\partial x'^\mu} = \delta^\rho_\mu - \epsilon\partial_\mu\xi^\rho$. With these transformation equation (F.1) becomes

$$g_{\nu\mu} + \epsilon\xi^\rho\partial_\rho g_{\nu\mu} = g_{\rho\sigma}(\delta^\rho_\mu - \epsilon\partial_\mu\xi^\rho)(\delta^\sigma_\nu - \epsilon\partial_\nu\xi^\sigma) \ .$$

If this equation is to hold to first order we must have

$$\xi^\rho\partial_\rho g_{\nu\mu} + g_{\rho\nu}\partial_\mu\xi^\rho + g_{\mu\sigma}\partial_\nu\xi^\sigma = 0 \ . \tag{F.2}$$

Let us consider a scalar field $\phi(x)$ and study its transformation under $x'^\rho = x^\rho + \epsilon\xi^\rho$ where the ξ satisfies equation (F.2). We have

$$\psi(x'^\rho) \approx \psi(x^\rho) + \epsilon\xi^\mu\partial_\mu\psi(x^\rho) = (1 + \epsilon\xi^\mu\partial_\mu)\psi(x^\rho) = U(\epsilon)\psi(x^\rho) \ .$$

The point is that we can pretend that this transformation is actually an infinitesimal transformation due to an operator $U(\epsilon)$. Since we know that this particular transformation leaves the metric invariant, so must the operator. To construct the full operator we use the exact same trick we did with the Lie group. That is, we divide the operator's argument into N pieces and take the limit. In this way we find

$$U(\varphi) = \lim_{N\to\infty} U^N\left(\frac{\varphi}{N}\right) = \lim_{N\to\infty}(1 + \frac{\varphi\xi^\mu\partial_\mu}{N})^N = e^{\varphi\xi^\mu\partial_\mu} \ .$$

We now see clearly the use of ξ^μ. By constructing $\xi^\mu\partial_\mu$, known as a *Killing vector* (field), we have actually found a generator to a group of transformations that leaves the metric invariant. We say that the Killing vectors generate the isometry group.
Let us now turn our attention to the specific case of the Minkowski metric and thus set $g_{\mu\nu} = \eta_{\mu\nu}$. Since this metric is constant we must have $\xi^\rho\partial_\rho\eta_{\nu\mu} \equiv 0$. We can thus reduce equation (F.2) to

$$\eta_{\rho\nu}\partial_\mu\xi^\rho + \eta_{\rho\mu}\partial_\nu\xi^\rho = 0 \ . \tag{F.3}$$

We first notice that this is a symmetric tensor equation in the free μ and ν indices. Since there exist 10 independent elements we expect to find ten equations for the ξ. If they are all linearly independent we will find a total of ten Killing vectors. The first four solutions are very easy. Since the equation above involves derivatives we can simply set ξ^μ equal to a constant in one of the indices. Since there is no need to be fancy we simply choose the constant to equal 1. Thus we have $\xi^\mu_{(1)} = (1{,}0{,}0{,}0)$, $\xi^\mu_{(2)} = (0{,}1{,}0{,}0)$ and so on. To find the remaining six ξ we have to be a bit more crafty. Let us make the specific choice of $\nu = 0$ and $\mu = 1$ in equation (F.3). Then we have two equations of interest, namely

$$\begin{aligned} -\partial_1\xi^0 + \partial_0\xi^1 &= 0 \ , \\ \partial_1\xi^1 &= 0 \ . \end{aligned}$$

Acting with ∂_1 on the first equation and using the second we find

$$\partial_0^2\xi = 0 \ .$$

Since the choice of $\nu = 0$ and $\mu = 1$ was arbitrary we will get the same identity for all choices of ν and μ. The conclusion is thus that ξ^μ have to be linear in all variables except the one corresponding to the position. With this restriction and of course demanding that ξ solves the Killing equation we quickly find all ten solutions as

$$\begin{aligned} &\xi^\mu_{(1)} = (1{,}0{,}0{,}0) \ , && \xi^\mu_{(2)} = (0{,}1{,}0{,}0) \ , && \xi^\mu_{(3)} = (0{,}0{,}1{,}0) \ , && \xi^\mu_{(4)} = (0{,}0{,}0{,}1) \ , \\ &\xi^\mu_{(5)} = (x{,}t{,}0{,}0) \ , && \xi^\mu_{(6)} = (y{,}0{,}t{,}0) \ , && \xi^\mu_{(7)} = (z{,}0{,}0{,}t) \ , && \\ &\xi^\mu_{(8)} = (0,-y{,}x{,}0) \ , && \xi^\mu_{(9)} = (0{,}0,-z{,}y) \ , && \xi^\mu_{(10)} = (0{,}z{,}0,-x) \ . && \end{aligned}$$

With these identities we can now form 10 Killing vectors and find the generators of 10 different transformations. But since the linear combination of one or more Killing vectors is still a Killing vector we will group the generators as indicated by the rows in the listing of the ξ. The first combination we will form is a combination of the first four and defined as

$$P_\mu \equiv \partial_\mu \ .$$

We call P^μ the generator of translations. The second row gives the generator of Lorentz boosts defined as

$$K_i \equiv x^i\partial_0 + x_0\partial_i \ .$$

Finally we use the last row to define the generator of rotations as

$$J_{ij} \equiv x^j\partial_k - x_k\partial^j \ .$$

It simplifies things if we combine the rotations and the boosts in a single entity defined as

$$M^{\mu\nu} = x^\mu\partial^\nu - x^\nu\partial^\mu$$

We have thus found the generators of the Minkowski isometry group. We refer to this group, consisting of Lorentz transformations, rotations and translations, as the *Poincaré group*.

Lie algebra of the Poincaré group

We now turn our attention to the problem of determining the Lie algebra of the Poincaré group. We begin with the commutator $[M^{\mu\nu}, M^{\rho\sigma}]$. First observe that

$$[x^\mu\partial^\nu, x^\rho\partial^\sigma] = x^\mu(\partial^\nu x^\rho)\partial^\sigma - x^\rho(\partial^\sigma x^\mu)\partial^\nu = \eta^{\nu\rho}x^\mu\partial^\sigma - \eta^{\sigma\mu}x^\rho\partial^\nu \ ,$$

then we have that

$$\begin{aligned} [M^{\mu\nu}, M^{\rho\sigma}] &= [x^\mu\partial^\nu, x^\rho\partial^\sigma] - [x^\nu\partial^\mu, x^\rho\partial^\sigma] + [x^\mu\partial^\nu, x^\sigma\partial^\rho] - [x^\nu\partial^\mu, x^\sigma\partial^\rho] \\ &= \eta^{\nu\rho}M^{\mu\sigma} - \eta^{\mu\rho}M^{\nu\sigma} - \eta^{\nu\sigma}M^{\mu\rho} + \eta^{\mu\sigma}M^{\nu\rho} \ . \end{aligned} \tag{F.4}$$

It is a bit easier to calculate the commutator between the Lorentz transformation and the translation generator

$$\begin{aligned} [M^{\mu\nu}, P^\rho] &= [x^\mu\partial^\nu - x^\nu\partial^\mu, \partial^\rho] = -(\partial^\rho x^\mu)\partial^\nu + \partial^\rho(x^\nu)\partial^\mu \\ &= -\eta^{\rho\mu}\partial^\nu + \eta^{\rho\nu}\partial^\mu = -\eta^{\rho\mu}P^\nu + \eta^{\rho\nu}P^\mu \ . \end{aligned} \tag{F.5}$$

and finally

$$[P^\mu, P^\nu] = [\partial^\mu, \partial^\nu] = 0 \ ,$$

since derivatives commute. To summarize we have

$$[M^{\mu\nu}, M^{\rho\sigma}] = \eta^{\nu\rho}M^{\mu\sigma} - \eta^{\mu\rho}M^{\nu\sigma} - \eta^{\nu\sigma}M^{\mu\rho} + \eta^{\mu\sigma}M^{\nu\rho} \ , \tag{F.6}$$

$$[M^{\mu\nu}, P^\rho] = -\eta^{\rho\mu}P^\nu + \eta^{\rho\nu}P^\mu \ , \tag{F.7}$$

$$[P^\mu, P^\nu] = 0 \ . \tag{F.8}$$

F.1.1 The special case of 2+1 dimensions

When we study Chern-Simons theory we will do so in 2+1 dimensions. This will further simplify our commutators. Remember that the indices in $M^{\nu\mu}$ represent the axis in the plane in which we rotate or boost. In three dimensions we have a very special relation, the number of components in a vector is precisely equal to the number of planes we can rotate in! Thus, we expect that we can write the generator $M^{\nu\mu}$ as a vector M^a. We define M^a as

$$M^a = \frac{1}{2}\epsilon^a{}_{\mu\nu}M^{\mu\nu} \ . \tag{F.9}$$

It is possible to express $M^{\mu\nu}$ as a function of M^a. To do so we contract both sides of (F.9) with $\epsilon_a{}^{\rho\sigma}$ reaching

$$\frac{1}{2}\epsilon_a{}^{\rho\sigma}\epsilon^a{}_{\mu\nu}M^{\mu\nu} = -\delta^{\rho\sigma}_{\mu\nu}M^{\mu\nu} = -M^{\sigma\rho} \ ,$$

where we in the last step used the anti-symmetry of $M^{\sigma\rho}$. Thus we may write

$$M^{\mu\nu} = -\epsilon^{\mu\nu}{}_a M^a \ .$$

Using this we may now start to simplify our commutators. Starting with the commutator of $M^{\mu\nu}$, (F.6), we first see that

$$[M^{\mu\nu},M^{\rho\sigma}] = \epsilon^{\mu\nu}{}_a\epsilon^{\rho\sigma}{}_b[M^a,M^b] \ . \tag{F.10}$$

The epsilon symbols may be eliminated by contraction with new epsilon symbols, explicitly

$$\epsilon_{\mu\nu}{}^c\epsilon_{\rho\sigma}{}^d\epsilon^{\mu\nu}{}_a\epsilon^{\rho\sigma}{}_b[M^a,M^b] = 4\delta^c_a\delta^d_b[M^a,M^b] = 4[M^c,M^d] \ . \tag{F.11}$$

We now also want to contract the right hand side of (F.6) with $\epsilon_{\mu\nu}{}^c\epsilon_{\rho\sigma}{}^d$. The first term, $\eta^{\nu\rho}M^{\mu\sigma}$, becomes

$$\begin{aligned}\epsilon_{\mu\nu}{}^c\epsilon_{\rho\sigma}{}^d(\eta^{\nu\rho}M^{\mu\sigma}) = \epsilon^{\mu\rho c}\epsilon_{\rho\sigma e}\eta^{de}(M_\mu{}^\sigma) = 2\delta^{\mu c}_{\sigma e}\eta^{de}M_\mu{}^\sigma &= -\eta^{de}M_e{}^c \\ &= M^{cd} = \epsilon^{cd}{}_aM^a \ .\end{aligned} \tag{F.12}$$

This calculation may be used to easily calculate the rest of terms by rearranging and renaming indices until we reach the identity above. For example the second term in the right hand side of (F.6)

$$\begin{aligned}\epsilon_{\mu\nu}{}^c\epsilon_{\rho\sigma}{}^d(-\eta^{\mu\rho}M^{\nu\sigma}) = \epsilon_{\nu\mu}{}^c\epsilon_{\rho\sigma}{}^d(\eta^{\mu\rho}M^{\nu\sigma}) = \{\nu\to\mu,\mu\to\nu\} = \\ \epsilon_{\mu\nu}{}^c\epsilon_{\rho\sigma}{}^d(\eta^{\nu\rho}M^{\mu\sigma}) = \epsilon^{cd}{}_aM^a \ ,\end{aligned}$$

where we in the last step used (F.12). We summarize the result for all terms as

$$\begin{aligned}&\epsilon_{\mu\nu}{}^c\epsilon_{\rho\sigma}{}^d\Big(\eta^{\nu\rho}M^{\mu\sigma} - \eta^{\mu\rho}M^{\nu\sigma} - \eta^{\nu\sigma}M^{\mu\rho} + \eta^{\mu\sigma}M^{\nu\rho}\Big) \\ &= \epsilon^{cd}{}_aM^a + \epsilon^{cd}{}_aM^a + \epsilon^{cd}{}_aM^a + \epsilon^{cd}{}_aM^a = 4\epsilon^{cd}{}_aM^a \ .\end{aligned} \tag{F.13}$$

Finally we are ready to state the commutator of $M^{\mu\nu}$. Using that the right hand side is given by (F.13) and the left hand side by (F.11) we see that

$$[M^a,M^b] = \epsilon^{ab}{}_cM^c \ . \tag{F.14}$$

Moving on, we want to compute the commutator $[M^a,P^b]$. To do so we start out with (F.7) and use our ability to rewrite the $M^{\mu\nu}$ as M^a. We also contract with an epsilon symbol just as before.

$$\epsilon_{\mu\nu}{}^a[M^{\mu\nu},P^b] = -\epsilon_{\mu\nu}{}^a\epsilon^{\mu\nu}{}_c[M^c,P^b] = 2\eta^{ad}\delta^c_d[M_c,P^b] = -2[M^a,P^b] \ . \tag{F.15}$$

Contracting the right hand side of (F.7) gives

$$\epsilon_{\mu\nu}{}^a\Big(-\eta^{b\mu}P^\nu + \eta^{b\nu}P^\mu\Big) = 2\epsilon^{ab}{}_\mu P^\mu \ , \tag{F.16}$$

and combining (F.15) with (F.16) gives us

$$[M^a,P^b] = \epsilon^{ab}{}_cP^c \ .$$

The commutator $[P^a,P^b] = 0$ is of course unchanged. We end this section by summarizing the results, demonstrating the Lie algebra of the Poincaré group:

$$\begin{aligned}[M^a,M^b] &= \epsilon^{ab}{}_cM^c \ , \\ [M^a,P^b] &= \epsilon^{ab}{}_cP^c \ , \\ [P^a,P^b] &= 0 \ .\end{aligned}$$

F.1.2 Chern-Simons and Einstein-Hilbert equivalence for Minkowski

In section 6.1 we showed that we can translate the Chern-Simons action in 2+1 dimensions in AdS_3 space into an Einstein-Hilbert action. We now wish to show the same equivalence but in Minkowski space. As done previously we have to choose a gauge group to express the Chern-Simons action in. The isometry group of Minkowski space in 2+1 dimensions is $SO(1,2)$ or the *Poincaré group* consisting of two sets of generators P_a and M_a which form the Lie algebra:

$$[M^a,M^b] = \epsilon^{ab}{}_c M^c \ ,$$
$$[M^a,P^b] = \epsilon^{ab}{}_c P^c \ ,$$
$$[P^a,P^b] = 0 \ .$$

Furthermore we construct the generators in such a way that they obey the following trace relations

$$\mathrm{tr}\,[P_a M_b] = \eta_{ab} \ , \tag{F.17}$$
$$\mathrm{tr}\,[P_a P_b] = \mathrm{tr}\,[M_a M_b] = 0 \ . \tag{F.18}$$

We have now specified the gauge group of the gauge connection, A, and wish to express the gauge connection in terms of frame fields and spin connections in order to relate the Chern-Simons action to Einstein-Hilbert action. If we associate the frame fields with the translation generators and spin connection with the Lorentz boosts generators we may define the gauge connection as

$$A = e^a P_a + \omega^a M_a \ ,$$

and now we have all we need in order to express Chern-Simons action in 2+1 dimensions in the language of vielbeins and spin connections. Consider the first term in the CS-action as stated in equation (6.1):

$$\mathrm{tr}\,[A \wedge \mathrm{d}A] = \mathrm{tr}\,[(e^a P_a + \omega^a M_a) \wedge (\mathrm{d}e^b P_b + d\omega^b M_b)] \ .$$

Using the trace relations $\mathrm{tr}\,[M_a M_b] = \mathrm{tr}\,[P_a P_b] = 0$ we can rewrite the expression:

$$\mathrm{tr}\,[A \wedge \mathrm{d}A] = e^a \wedge \mathrm{d}\omega^b\,\mathrm{tr}\,[P_a M_b] + \omega^a \wedge \mathrm{d}e^b\,\mathrm{tr}\,[M_a P_b] = e^a \wedge \mathrm{d}\omega_a + \omega^a \wedge \mathrm{d}e_a \ ,$$

where we used the trace relations and the cyclicity property of the trace on the last term. By performing a partial integration and dropping the boundary term on the second term the expression reduces to

$$e^a \wedge \mathrm{d}\omega_a + \omega^a \wedge \mathrm{d}e_a = 2e^a \wedge \mathrm{d}\omega_a \ .$$

The second term in the Chern-Simons action requires a little more work:

$$\mathrm{tr}\,[A \wedge A \wedge A] = \mathrm{tr}\,[(e^a P_a + \omega^a M_a) \wedge (e^b P_b + \omega^b M_b) \wedge (e^c P_c + \omega^c M_c)] \ .$$

However, expanding this expression may seem horrifying, but it is not so bad if we make use of the commutator relations between our generators and thus can state the following relations

$$e^a P_a \wedge e^b P_b = \frac{1}{2} e^a \wedge e^b [P_a,P_b] = 0 \ , \tag{F.19}$$
$$\omega^a M_a \wedge \omega^b M_b = \frac{1}{2}\omega^a \wedge \omega^b [M_a,M_b] = \frac{1}{2}\epsilon^c{}_{ab}\omega^a \wedge \omega^b M_c \ . \tag{F.20}$$

Thus only three terms will survive in our former complicated expression:

$$\mathrm{tr}\,[A \wedge A \wedge A] = e^a \wedge \omega^b \wedge \omega^c\,\mathrm{tr}\,[P_a M_b M_c] + \omega^a \wedge e^b \wedge \omega^c\,\mathrm{tr}\,[M_a P_b M_c] + \omega^a \wedge \omega^b \wedge e^c\,\mathrm{tr}\,[M_a M_b P_c] \ .$$

We consider each term individually. The first term gives us

$$e^a \wedge \omega^b \wedge \omega^c\,\mathrm{tr}\,[P_a M_b M_c] = \frac{1}{2} e^a \wedge \omega^b \wedge \omega^c\,\mathrm{tr}\,[P_a [M_b,M_c]] = \frac{1}{2}\epsilon_{bc}{}^d e^a \wedge \omega^b \wedge \omega^c\,\mathrm{tr}\,[P_a M_d] = \frac{1}{2}\epsilon_{abc} e^a \wedge \omega^b \wedge \omega^c \ ,$$

where we used the trace relation $\mathrm{tr}\,[P_a M_d] = \eta_{ad}$. The second term can be rewritten by using the commutator relation $M_a P_b = P_b M_a + \epsilon_{ab}{}^d P_d$:

$$\begin{aligned}
\omega^a \wedge e^b \wedge \omega^c\,\mathrm{tr}\,[M_a P_b M_c] &= \omega^a \wedge e^b \wedge \omega^c(\mathrm{tr}\,[P_b M_a M_c] + \epsilon_{ab}{}^d\,\mathrm{tr}\,[P_d M_c]) \\
&= \omega^a \wedge e^b \wedge \omega^c(\frac{1}{2}\,\mathrm{tr}\,[P_b[M_a,M_c]] + \epsilon_{abc}) \\
&= \omega^a \wedge e^b \wedge \omega^c(\frac{1}{2}\epsilon_{ac}{}^d\,\mathrm{tr}\,[P_b M_d] + \epsilon_{abc}) \\
&= \frac{1}{2}\epsilon_{abc} e^a \wedge \omega^b \wedge \omega^c \ .
\end{aligned}$$

Now only the last term in $\mathrm{tr}\,[A \wedge A \wedge A]$ remains:

$$\begin{aligned}\omega^a \wedge \omega^b \wedge e^c\,\mathrm{tr}\,[M_a M_b P_c] &= \frac{1}{2}\omega^a \wedge \omega^b \wedge e^c\,\mathrm{tr}\,[[M_a,M_b]P_c] \\ &= \frac{1}{2}\epsilon_{ab}{}^{d}\omega^a \wedge \omega^b \wedge e^c\,\mathrm{tr}\,[M_d P_c] \\ &= \frac{1}{2}\epsilon_{abc}e^a \wedge \omega^b \wedge \omega^c \ ,\end{aligned}$$

where we once again used $\mathrm{tr}\,[M_d P_c] = \eta_{dc}$. Thus all three terms contribute with $\frac{1}{2}\epsilon_{abc}e^a \wedge \omega^b \wedge \omega^c$ and we conclude

$$\mathrm{tr}\,[A \wedge A \wedge A] = \frac{3}{2}\epsilon_{abc}e^a \wedge \omega^b \wedge \omega^c \ .$$

Substituting this term and $\mathrm{tr}\,[A \wedge \mathrm{d}A]$ into the original Chern-Simons action, equation (6.1) we find

$$\mathcal{S}_{CS}[e,\omega] = \frac{k}{2\pi}\int_M e^a \wedge \mathrm{d}\omega_a + \frac{\epsilon_{abc}}{2}e^a \wedge \omega^b \wedge \omega^c = \frac{k}{2\pi}\int_M e^a \wedge R_a \ ,$$

where we have extracted a factor of 2 in order to compare this action to Einstein-Hilbert action. We note, upon comparison with EH-action, that our Chern-Simons action in Minkowski space is equivalent to Einstein-Hilbert action if we set $k = -\frac{1}{4G}$ and the cosmological constant to zero corresponding to flat space.

F.2 Anti-de Sitter space

We now turn our attention to flat three-dimensional Anti-de Sitter space, or AdS_3 for short. One can realize AdS_d, where d is the dimension of the spacetime, by embedding a hyperboloide in $d{+}1$ dimensions. In the case $d = 3$ we may therefore express the hyperboloide as

$$-U^2 - V^2 + X^2 + Y^2 = -l^2 \ , \tag{F.21}$$

where $\mu = 0,1,2$ and 3 are indices in the Minkowski space $Minkowski_2 \times Minkowski_2$ with the metric

$$g_{\mu\nu}dx^\mu dx^\nu = -dU^2 - dV^2 + dX^2 + dY^2 = ds^2 \ . \tag{F.22}$$

By introducing parameters

$$\begin{cases} U = l\cosh\rho\cos t \ , \\ V = l\cosh\rho\sin t \ , \\ X = l\sinh\rho\cos\phi \ , \\ Y = l\sinh\rho\sin\phi \ , \end{cases} \tag{F.23}$$

we manage to solve equation (F.21). We now calculate the metric by computing the derivatives dx_μ:

$$\begin{cases} dU = l\sinh\rho\cos t d\rho - l\cosh\rho\sin t dt \ , \\ dV = l\sinh\rho\sin t d\rho + l\cosh\rho\cos t dt \ , \\ dX = l\cosh\rho\cos\phi d\rho - l\sinh\rho\sin\phi d\phi \ , \\ dY = l\cosh\rho\sin\phi d\rho + l\sinh\rho\cos\phi d\phi \ , \end{cases} \tag{F.24}$$

which inserted in equation (F.22) yields the metric ds^2:

$$ds^2 = -l^2\cosh^2\rho dt^2 + l^2 d\rho^2 + l^2\sinh^2\rho d\phi^2 \ , \tag{F.25}$$

and from equation (F.25) we conclude that $g_{\rho\rho} = l^2, g_{\phi\phi} = l^2\sinh^2\rho$ and $g_{tt} = -l^2\cosh^2\rho$. Now we are able to construct the metric tensor:

$$g_{\mu\nu} = \begin{bmatrix} -l^2\cosh^2\rho & 0 & 0 \\ 0 & l^2 & 0 \\ 0 & 0 & l^2\sinh^2\rho \end{bmatrix} \ . \tag{F.26}$$

F.2.1 AdS_3 and Einstein's field equations

Having derived the metric of AdS_3 we ask the natural question: does it solve Einstein's equations? For the sake of the reader we once again state Einstein's field equations expressed in Cartan formalism:

$$\begin{aligned} \mathrm{d}e^a &= -\epsilon^a{}_{bc} e^b \wedge \omega^c \ , \\ R_a &= \frac{\Lambda}{2} \epsilon_{abc} e^a \wedge e^b \ . \end{aligned} \tag{F.27}$$

Using the hyperbolic coordinates introduced in the previous section we may read the correct vielbeins from equation (F.25) as

$$e^0 = l \cosh \rho dt \ , \qquad e^1 = l d\rho \ , \qquad e^2 = l \sinh \rho d\phi \ .$$

To find the spin connection we use the equation

$$\mathrm{d}e^a = -\epsilon^a{}_{bc} e^b \wedge \omega^c \ ,$$

which gives

$$\begin{aligned} \mathrm{d}e^0 = l \sinh(\rho) d\rho \wedge dt = -e^1 \wedge \omega^2 + e^2 \wedge \omega^1 \implies \omega^2_t = -\sinh(\rho) \ , \\ \mathrm{d}e^2 = l \cosh(\rho) d\rho \wedge d\phi = e^0 \wedge \omega^1 - e^1 \wedge \omega^0 \implies \omega^0_\phi = -\cosh(\rho) \ . \end{aligned}$$

It is straightforward to see that all other spin-connections vanish. Summarizing we have

$$\omega_0 = \cosh(\rho) d\phi \ , \qquad \omega_1 = 0 \ , \qquad \omega_2 = -\sinh(\rho) dt \ .$$

Finally we can write down R_a.

$$\begin{aligned} R_0 &= \mathrm{d}\omega_0 + \frac{1}{2} \epsilon_{0bc} \omega^b \wedge \omega^c = \sinh(\rho) d\rho \wedge d\phi \ , \\ R_1 &= \sinh(\rho) \cosh(\rho) d\phi \wedge dt \ , \\ R_2 &= \cosh(\rho) dt \wedge d\rho \ . \end{aligned}$$

Turning our attention to (F.27) we compute the right hand side and find

$$\begin{aligned} \frac{\Lambda}{2} \epsilon_{0bc} e^b \wedge e^c &= -\Lambda l^2 \sinh(\rho) d\rho \wedge d\phi \ , \\ \frac{\Lambda}{2} \epsilon_{1bc} e^b \wedge e^c &= -\Lambda l^2 \sinh(\rho) \cosh(\rho) d\phi \wedge dt \ , \\ \frac{\Lambda}{2} \epsilon_{2bc} e^b \wedge e^c &= -\Lambda l^2 \cosh(\rho) dt \wedge d\rho \ . \end{aligned}$$

Upon comparison with R_a we see that if we set $\Lambda = -\frac{1}{l^2}$, AdS_3 will indeed be a solution to the Einstein equations.

F.2.2 Lie algebra of AdS_3

For reasons that will become apparent we will use from now on use coordinates x,y,τ and t such that

$$-\tau^2 - t^2 + x^2 + y^2 = -l^2 \ , \tag{F.28}$$

$$ds^2 = -d\tau^2 - dt^2 + dx^2 + dy^2 \ . \tag{F.29}$$

Thus these coordinates parametrize AdS_3.

To study AdS_3 we want to find its isometry group. That is the group of all transformations that preserves the metric and equation (F.28). But this is precisely the isometry group of the embedding space if we do not allow translations. So the isometry group of AdS^3 must be $SO(2{,}2)$. Another example of this kind of reasoning is that the isometry group of S^d is $SO(d+1)$. To find the generators we thus face the Killing equation

$$g_{\rho\nu} \partial_\mu \xi^\rho + g_{\mu\sigma} \partial_\nu \xi^\sigma = 0 \ .$$

Notice that while $\xi = \text{const}$ is a solution this transformation will not preserve (F.28). We can find the remaining six Killing generators by the same procedure as for the Poincaré group. The result is:

$$\xi_{(1)} = (0,x,t,0) \ , \qquad \xi_{(2)} = (0,y,0,t) \ , \qquad \xi_{(3)} = (0,0,-y,x) \ ,$$
$$\xi_{(4)} = (x,0,\tau,0) \ , \qquad \xi_{(5)} = (y,0,0,\tau) \ , \qquad \xi_{(6)} = (-t,\tau,0,0) \ .$$

With these we can construct M^a and P^a through linearly combining the Killing vectors. We define them to be

$$\begin{aligned} M^a &= (y\partial^x - x\partial^y, y\partial^t - t\partial^y, t\partial^x - x\partial^t) \ , \\ P^a &= (\tau\partial^t - t\partial^\tau, \tau\partial^x - x\partial^\tau, \tau\partial^y - y\partial^\tau) \ . \end{aligned}$$

Notice that the first vector is precisely the generator of Lorentz transformations in three dimensions! (This is of course why we defined it this way). It is tempting to associate the P^a with translations, but we have some more work to do before we can make any sort of claim.

With these generators we obviously want to write down the corresponding Lie algebra. The first commutator involving only M^a is easy since we can use our result from Minkowski spacetime. We must therefore have

$$[M^a,M^b] = \epsilon^{ab}{}_c M^c \ .$$

The remaining commutators can be computed by hand, or by using matrices. We will be content with summarizing the result

$$\begin{aligned} [M^a,M^b] &= \epsilon^{ab}{}_c M^c \ , \\ [M^a,P^b] &= \epsilon^{ab}{}_c P^c \ , \\ [P^a,P^b] &= \epsilon^{ab}{}_c M^c \ . \end{aligned}$$

If we define

$$\begin{aligned} J_+^a &\equiv \frac{1}{2}\Big(M^a + P^a\Big) \ , \\ J_-^a &\equiv \frac{1}{2}\Big(M^a - P^a\Big) \ , \end{aligned}$$

we can compute

$$\begin{aligned} [J_+^a,J_-^b] = \frac{1}{4}[M^a + P^a,M^b - P^b] &= \frac{1}{4}\Big([M^a,M^b] - [M^a,P^b] + [P^a,M^b] - [P^a,P^b]\Big) \\ &= \frac{1}{4}\Big(\epsilon^{ab}{}_c M^c + \epsilon^{ab}{}_c P^c - \epsilon^{ab}{}_c P^c - \epsilon^{ab}{}_c M^c\Big) = 0 \ , \end{aligned} \tag{F.30}$$

and

$$\begin{aligned} [J_\pm^a,J_\pm^b] = \frac{1}{4}\Big([M^a,M^b] \pm [M^a,P^b] \pm [P^a,M^b] + [P^a,P^b]\Big) = \\ \frac{1}{2}\epsilon^{ab}{}_c\Big(M^c \pm P^c\Big) = \epsilon^{ab}{}_c J_\pm^c \ , \end{aligned}$$

Thus we can write down a new Lie algebra of $SO(2,2)$ as

$$[J_a^+,J_b^+] = \epsilon_{ab}{}^c J_c^+ \ , \tag{F.31}$$
$$[J_a^-,J_b^-] = \epsilon_{ab}{}^c J_c^- \ , \tag{F.32}$$
$$[J_a^+,J_b^-] = 0 \ . \tag{F.33}$$

The surprise here is that we can clearly see that the algebra of the two generators is isomorphic to $so(2,1)$, we have thus found that

$$SO(2,2) = SO(2,1) \times SO(2,1) \ .$$

F.2.3 From AdS_3 to Minkowski

Let us now discuss something rather remarkable, the contraction of the AdS_3 Lie algebra to the Poincaré algebra. First, let us single out the τ-dimension. This dimension is indeed rather confusing, what do we mean with another time-like dimension? Let us try to get rid of it! The procedure is surprisingly trivial. First we define $P^a_\tau = \lim_{\tau\to\infty} \frac{P^a}{\tau}$. The commutator involving only M^a is of course unchanged but so is the commutator involving both M^a and P^a_τ

$$[M^a,P^a_\tau] = \lim_{\tau\to\infty} \frac{1}{\tau}[M^a,P^b_\tau] = \lim_{\tau\to\infty} \frac{1}{\tau}[M^a,P^b] = \lim_{\tau\to\infty} \epsilon^{ab}{}_c \frac{1}{\tau} P^c = \epsilon^{ab}{}_c P^c_\tau \ .$$

The only difference from the algebra of AdS_3 is

$$[P^a_\tau,P^b_\tau] = \lim_{\tau\to\infty} \frac{1}{\tau^2}[P^a,P^b] = \lim_{\tau\to\infty} \frac{1}{\tau^2}\epsilon^{ab}{}_c P^c = 0 \ ,$$

so we may state our new algebra as

$$\begin{aligned}
[M^a,M^b] &= \epsilon^{ab}{}_c M^c \ , \\
[M^a,P^b_\tau] &= \epsilon^{ab}{}_c P^c_\tau \ , \\
[P^a,P^b] &= 0 \ ,
\end{aligned}$$

which is precisely the algebra of the Poincaré group! This method of taking a limit in a Lie algebra to obtain a new one is called a *Inönu-Wigner contraction*. But we can go even further, let us observe that using the explicit form of P^a we find that

$$P^a_\tau = \lim_{\tau\to\infty} \frac{1}{\tau}\Big(\tau\partial^t - t\partial^\tau, \tau\partial^x - x\partial^\tau, \tau\partial^y - y\partial^\tau\Big) = \partial^a \ .$$

So if we contract P^a we actually get the generator of translations in Minkowski spacetime. This is why we spoke of P^a as a momentum generator earlier. We started out in AdS_3 and ended up in Minkowski, what happened to the cosmological constant? The defining equation of AdS_3 is

$$-\tau^2 - t^2 + x^2 + y^2 = -l^2 \ .$$

If we now let $\tau \to \infty$ we still have to satisfy this equation. However, we do not want to touch t,x nor y so we are only left with the alternative that $\lim_{\tau\to\infty} \frac{l}{\tau} = 1$ and thus we see that $l \to \infty$ at the same rate as τ. This explains why we end up in Minkowski space since the cosmological constant, $\Lambda = \frac{1}{l^2}$, vanishes as we let τ and therefore also l goes to infinity.

Appendix G

General Relativity in Affine Connection

In this appendix we derive the torsion tensor, the Riemann curvature tensor (as well as the related Ricci tensor and Ricci scalar) and Einstein's field equations, in the *affine* formulation of general relativity. Some other useful results and properties of the various quantities are derived in the process. Most of the results and the discussion follows the chapter 3 of [16].

The difference between two connections $\tilde{\Gamma}^{\rho}{}_{\mu\nu}$ and $\Gamma^{\rho}{}_{\mu\nu}$, transforming as (4.12), is a tensor. In particular, this applies to both the affine connection and the spin connection.

Let $T^{\rho}{}_{\mu\nu} = \tilde{\Gamma}^{\rho}{}_{\mu\nu} - \Gamma^{\rho}{}_{\mu\nu}$. Under a general coordinate transformation, $x^{\mu} \to x'^{\mu}$, the object $T^{\rho}{}_{\mu\nu}$ transforms as

$$
\begin{aligned}
T^{\rho}{}_{\mu\nu} \longrightarrow T'^{\rho}_{\mu\nu} &= \tilde{\Gamma}'^{\rho}_{\mu\nu} - \Gamma'^{\rho}_{\mu\nu} \\
&= \frac{\partial x^{\sigma}}{\partial x'^{\mu}}\frac{\partial x^{\lambda}}{\partial x'^{\nu}}\frac{\partial x'^{\rho}}{\partial x^{\tau}}\tilde{\Gamma}^{\tau}_{\sigma\lambda} - \frac{\partial x^{\sigma}}{\partial x'^{\mu}}\frac{\partial x^{\lambda}}{\partial x'^{\nu}}\frac{\partial^2 x'^{\rho}}{\partial x^{\sigma}\partial x^{\lambda}} - \frac{\partial x^{\sigma}}{\partial x'^{\mu}}\frac{\partial x^{\lambda}}{\partial x'^{\nu}}\frac{\partial x'^{\rho}}{\partial x^{\tau}}\Gamma^{\tau}_{\sigma\lambda} \\
&\quad + \frac{\partial x^{\sigma}}{\partial x'^{\mu}}\frac{\partial x^{\lambda}}{\partial x'^{\nu}}\frac{\partial^2 x'^{\rho}}{\partial x^{\sigma}\partial x^{\lambda}} \\
&= \frac{\partial x^{\sigma}}{\partial x'^{\mu}}\frac{\partial x^{\lambda}}{\partial x'^{\nu}}\frac{\partial x'^{\rho}}{\partial x^{\tau}}(\tilde{\Gamma}^{\tau}_{\sigma\lambda} - \Gamma^{\tau}_{\sigma\lambda}) \\
&= \frac{\partial x^{\sigma}}{\partial x'^{\mu}}\frac{\partial x^{\lambda}}{\partial x'^{\nu}}\frac{\partial x'^{\rho}}{\partial x^{\tau}}T^{\tau}_{\sigma\lambda} ,
\end{aligned}
$$

which shows that $T^{\rho}{}_{\mu\nu}$ indeed transforms as a tensor, proving our earlier assertion. As a consequence of this result the variation of a connection is a tensor.

There is a particularly interesting tensor that can be formed from any connection in this manner. From the Christoffel connection $\Gamma^{\rho}{}_{\mu\nu}$ we can form a new connection by permuting its lower indices. Taking the difference between our original connection and the one with permuted lower indices produces the *torsion tensor*:

$$T^{\rho}_{\mu\nu} = \Gamma^{\rho}_{\mu\nu} - \Gamma^{\rho}_{\nu\mu} = 2\Gamma^{\rho}_{[\mu\nu]} \ . \tag{G.1}$$

To establish a unique connection on a manifold with a metric $g_{\mu\nu}$ we impose two additional conditions. First, we require the connection to be *torsion-free*, a condition which is realized by demanding the connection to be symmetric in its lower indices. Secondly, we require the connection to be metric compatible, meaning that the covariant derivative of the metric with respect to that connection vanishes everywhere. Using these two requirements we can prove our assertion that the connection now is uniquely determined from the metric by finding a unique expression of the connection in terms of the metric. To find such an expression we start by writing out the covariant derivative of the metric,,

$$D_{\rho}g_{\mu\nu} = \partial_{\rho}g_{\mu\nu} - \Gamma^{\tau}{}_{\rho\mu}g_{\tau\nu} - \Gamma^{\tau}{}_{\rho\nu}g_{\mu\tau} = 0 \ .$$

Using this formula together with the symmetric property of the lower indices of the Christoffel symbol as well as the metric tensor, we find that

$$D_{\rho}g_{\mu\nu} - D_{\mu}g_{\nu\rho} - D_{\nu}g_{\rho\mu} = \partial_{\rho}g_{\mu\nu} - \partial_{\mu}g_{\nu\rho} - \partial_{\nu}g_{\rho\mu} + 2\Gamma^{\tau}{}_{\mu\nu}g_{\tau\rho} = 0 \ .$$

After rearranging terms and multiplying by $\frac{1}{2}g^{\sigma\rho}$ we find the following expression of the Christoffel symbol in terms of the metric tensor:

$$\Gamma^{\sigma}{}_{\mu\nu} = \frac{1}{2}g^{\sigma\rho}\left(\partial_{\mu}g_{\nu\rho} + \partial_{\nu}g_{\rho\mu} - \partial_{\rho}g_{\mu\nu}\right) \ . \tag{G.2}$$

There remains yet to prove that a connection of this form transforms as (4.12). This is indeed the case, but for the sake of brevity we omit to perform this calculation.

It would be useful to have a local description of the curvature of our space. For this purpose we introduce the Riemann curvature tensor. Perhaps the most straightforward way to derive this object is to evaluate the commutator of the covariant derivative. Using the earlier imposed condition that the Christoffel symbol be symmetric in its lower indices we find that

$$[D_\mu,D_\nu]V^\rho = (\partial_\mu \Gamma^\rho{}_{\nu\sigma} - \partial_\nu \Gamma^\rho{}_{\mu\sigma} + \Gamma^\rho{}_{\mu\tau}\Gamma^\tau{}_{\nu\sigma} - \Gamma^\rho{}_{\nu\tau}\Gamma^\tau{}_{\mu\sigma})V^\sigma \ . \tag{G.3}$$

We identify the expression inside the bracket of the right-hand side as the Riemann curvature tensor,

$$R^\rho{}_{\sigma\mu\nu} = \partial_\mu \Gamma^\rho{}_{\nu\sigma} - \partial_\nu \Gamma^\rho{}_{\mu\sigma} + \Gamma^\rho{}_{\mu\tau}\Gamma^\tau{}_{\nu\sigma} - \Gamma^\rho{}_{\nu\tau}\Gamma^\tau{}_{\mu\sigma} \ . \tag{G.4}$$

Since we have expressed the Riemann curvature tensor in terms of non-tensorial elements it is by no means obvious that it actually is a tensor. By making a general coordinate transformation it is possible to show that it indeed transforms as a tensor. However, we will omit from presenting the details of this calculation.

From the formula (G.4) the anti-symmetry of $R^\rho{}_{\sigma\mu\nu}$ in its last two indices is apparent, as we would expect from the way we derived it from the commutator of the covariant derivative,

$$R^\rho{}_{\sigma\mu\nu} = -R^\rho{}_{\sigma\nu\mu} \ . \tag{G.5}$$

A number of other useful symmetries of the Riemann curvature tensor can be derived. We will settle for stating some of these symmetry relations and refer the reader to [16] for a derivation of a few of them. The Riemann curvature tensor is anti-symmetric in its first two indices:

$$R_{\rho\sigma\mu\nu} = -R_{\sigma\rho\mu\nu} \ , \tag{G.6}$$

and it is symmetric under an interchange of the first pair of indices with the second pair of indices:

$$R_{\rho\sigma\mu\nu} = R_{\mu\nu\rho\sigma} \ . \tag{G.7}$$

The anti-symmetric part of the last three indices vanishes:

$$R_{\rho[\sigma\mu\nu]} = 0 \ , \tag{G.8}$$

and finally the Riemann curvature tensor satisfies the Bianchi identity:

$$D_{[\mu}R_{\rho\sigma]\mu\nu} = 0 \ . \tag{G.9}$$

Not all of these symmetry relations are independent but they are all useful in their own way. It should also be pointed out that it is of fundamental importance that the connection be torsion-free in order for these symmetry relations to hold.

By contracting indices of the Riemann curvature tensor we can produce other useful tensors. The Ricci tensor is defined in the following way:

$$R_{\mu\nu} = R^\rho{}_{\mu\rho\nu} \ . \tag{G.10}$$

Note that without involving the metric tensor in the process there are only three possible contractions of $R^\rho{}_{\sigma\mu\nu}$. Because the Riemann curvature tensor is anti-symmetric in its first two indices (see (G.6)), the contraction of its first two indices vanishes,

$$R^\rho{}_{\rho\mu\nu} = g^{\rho\sigma}R_{\sigma\rho\mu\nu} = -g^{\sigma\rho}R_{\rho\sigma\mu\nu} = -R^\rho{}_{\rho\mu\nu} \quad \Rightarrow \quad R^\rho{}_{\rho\mu\nu} = 0 \ .$$

Since the Riemann tensor is also anti-symmetric in its last two indices by (G.5) it follows that contracting the first and last index will result in the Ricci tensor, up to a minus sign. Some authors choose to define the Ricci tensor in this way so one has to be careful about sign conventions.

As a consequence of the symmetry (G.7) of the Riemann curvature tensor, the Ricci tensor is completely symmetric,

$$R_{\mu\nu} = R^\rho{}_{\mu\rho\nu} = g^{\rho\sigma}R_{\sigma\mu\rho\nu} = g^{\sigma\rho}R_{\rho\nu\sigma\mu} = R^\sigma{}_{\nu\sigma\mu} = R_{\nu\mu} \ .$$

By contracting the Ricci tensor we form the Ricci scalar R. This time, however, we do need to use the metric tensor to perform the contraction,

$$R = R^\mu{}_\mu = g^{\mu\nu}R_{\mu\nu} \ . \tag{G.11}$$

In other words, the Ricci scalar is the trace of the Ricci tensor.

Compared with the Riemann curvature tensor the Ricci tensor contains somewhat less information about the curvature of the space, and the Ricci scalar contains even less information than the Ricci tensor. Essentially, the Ricci tensor and the Ricci scalar contains information about traces of the Riemann curvature tensor. However, Einstein's field equations, the governing physical equations of general relativity describing how the curvature of space is affected by the presence of matter and energy, is formulated solely in terms of the Ricci tensor and the Ricci scalar as far as the curvature describing objects goes.

Before presenting Einstein's field equations we first introduce the Einstein tensor $G_{\mu\nu}$. It is defined in terms of the Ricci tensor, Ricci scalar and the metric tensor in the following way:

$$G_{\mu\nu} = R_{\mu\nu} - \frac{1}{2} g_{\mu\nu} R \; . \tag{G.12}$$

Clearly, it is a symmetric tensor. Furthermore, as a consequence of the Bianchi identity, this tensor is divergence-free:

$$D^{\mu} G_{\mu\nu} = 0 \; . \tag{G.13}$$

Finally, we present the Einstein field equations:

$$G_{\mu\nu} + \Lambda g_{\mu\nu} = \frac{8\pi G}{c^4} T_{\mu\nu} \; , \tag{G.14}$$

where Λ is the cosmological constant, G is Newton's gravitational constant, c is the speed of light in vacuum and $T_{\mu\nu}$ is the stress-energy tensor. The stress-energy tensor describes the gravitating sources which curve spacetime. For a treatment of the cosmological constant and the stress-energy tensor, see chapter (4.4).

Appendix H

Stress-Energy Tensor for a Conformally Coupled Scalar Field in 2+1 Dimensions

The main purpose of this appendix is to derive the stress-energy tensor for the scalar field

$$\mathcal{L}_{\mathrm{S}} = -\frac{1}{2}g^{\mu\nu}(\partial_\mu\phi\partial_\nu\phi) - \frac{1}{2}m^2\phi^2 - \frac{1}{2}\xi R\phi^2 - \xi'\phi^p \ , \tag{H.1}$$

under the assumption that we have a conformal coupling. The result is

$$T_{\mathrm{S}}^{\mu\nu} = \xi G^{\mu\nu}\phi^2 + \xi(g^{\mu\nu}\Box(\phi^2) - D^\mu D^\nu(\phi^2)) - g^{\mu\nu}\xi'\phi^6 - \frac{g^{\mu\nu}}{2}(\partial^\rho\phi\partial_\rho\phi + m^2\phi^2) + \partial^\mu\phi\partial^\nu\phi \ .$$

However, in order to reach this result we require several identities for which we present proofs. The derivation of the stress-energy tensor start in section H.5.

H.1 Variation of determinant of metric

Let g denote the determinant of the metric tensor $g_{\mu\nu}$. In other words, let

$$g = \det(g_{\mu\nu}) \ .$$

Now we make use of the very handy matrix identity

$$\ln\Big(\det(A)\Big) = \mathrm{tr}\,[\ln(A)] \ ,$$

where A is a matrix. This relation follows directly from taking the logarithm of equation E.17. If we set $A = g_{\mu\nu}$ and take the variation of the left hand side we find

$$\delta(\ln(\det(g_{\mu\nu}))) = \delta\ln(g) = \frac{\delta g}{g} \ ,$$

and doing the same on the right side yields

$$\delta\,\mathrm{tr}\,(\ln[g_{\mu\nu}]) = \mathrm{tr}\,(g^{\nu\sigma}\delta g_{\sigma\mu}) = g^{\mu\nu}\delta g_{\mu\nu} \ ,$$

where we in the first step substituted $(g^{-1})_{\sigma\nu}$ for $g^{\nu\sigma}$ and in the second step we renamed our indices. By now comparing the right hand side and the left hand side we find the variation of the determinant g in terms of the variation of the metric:

$$\delta g = g g^{\mu\nu}\delta g_{\mu\nu} \ . \tag{H.2}$$

By using this equation we can find an expression for $\delta(\sqrt{-g})$[1]:

$$\delta\sqrt{-g} = -\frac{\delta g}{2\sqrt{-g}} = \frac{-g g^{\mu\nu}\delta g_{\mu\nu}}{2\sqrt{-g}} = \frac{1}{2}\sqrt{-g}g^{\mu\nu}\delta g_{\mu\nu} \ . \tag{H.3}$$

[1] We could of course be more general and compute $\delta f(g)$ where f is a function. However, in order to compute the stress tensor of a Klein Gordon Lagrangian coupled to a scalar field we only need $f(g) = \sqrt{-g}$.

H.2 Variation of Ricci scalar

Before we get started with this quite cumbersome derivation we remind ourselves that the Ricci scalar R is formed by contracting the Ricci tensor $R_{\mu\nu}$ and the metric $g_{\mu\nu}$ according to

$$R = R_{\mu\nu} g^{\mu\nu} \ .$$

If we now take the variation of this object we get stuck with taking the variation of the Ricci tensor instead:

$$\delta R = \delta g^{\mu\nu} R_{\mu\nu} + g^{\mu\nu} \delta R_{\mu\nu} \ ,$$

However, recall that the Ricci tensor is a contracted Riemann tensor. From equation (G.4) we found that the Riemann tensor can be written in terms of the Christoffel symbol:

$$R^{\rho}{}_{\sigma\mu\nu} = \partial_\mu \Gamma^{\rho}{}_{\nu\sigma} - \partial_\nu \Gamma^{\rho}{}_{\mu\sigma} + \Gamma^{\rho}{}_{\mu\tau}\Gamma^{\tau}{}_{\nu\sigma} - \Gamma^{\rho}{}_{\nu\tau}\Gamma^{\tau}{}_{\mu\sigma} \ ,$$

and the Christoffel symbol is related to the metric as seen in appendix G restated here for convenience:

$$\Gamma^{\sigma}{}_{\mu\nu} = \frac{1}{2} g^{\sigma\rho} \left(\partial_\mu g_{\nu\rho} + \partial_\nu g_{\rho\mu} - \partial_\rho g_{\mu\nu} \right) \ .$$

Thus, when taking the variation of the Riemann tensor above and performing a contraction we will be able to rewrite the Ricci scalar in terms of variation of the metric alone. Eventually we have to tackle some obstacles on the way. Let us compute the variation of the Riemann tensor:

$$\delta R^{\rho}{}_{\sigma\mu\nu} = \partial_\mu \delta\Gamma^{\rho}{}_{\nu\sigma} - \partial_\nu \delta\Gamma^{\rho}{}_{\mu\sigma} + \delta\Gamma^{\rho}{}_{\mu\tau}\Gamma^{\tau}{}_{\nu\sigma} + \Gamma^{\rho}{}_{\mu\tau}\delta\Gamma^{\tau}{}_{\nu\sigma} - \delta\Gamma^{\rho}{}_{\nu\tau}\Gamma^{\tau}{}_{\mu\sigma} - \Gamma^{\rho}{}_{\nu\tau}\delta\Gamma^{\tau}{}_{\mu\sigma} \ .$$

It may seem we didn't get anywhere since we still do not know how to take the variation of a Christoffel symbol. However, before we continue we try to rewrite the expression. While the Christoffel symbol is not a tensor the variation of it is as proven in Appendix G. Thus, we are allowed to compute its covariant derivative D:

$$D_\tau(\delta\Gamma^{\sigma}{}_{\mu\nu}) = \partial_\tau \delta\Gamma^{\sigma}{}_{\mu\nu} + \Gamma^{\sigma}{}_{\tau\lambda}\delta\Gamma^{\lambda}{}_{\mu\sigma} - \Gamma^{\lambda}{}_{\tau\mu}\delta\Gamma^{\sigma}{}_{\lambda\nu} - \Gamma^{\lambda}{}_{\tau\nu}\delta\Gamma^{\sigma}{}_{\mu\lambda} \ .$$

where we used the definition of the covariant derivative.

Using this fact we may simplify the variation of the Riemann tensor by noting that it is a difference between two covariant derivatives D_μ and D_ν:

$$\delta R^{\rho}{}_{\sigma\mu\nu} = D_\mu(\delta\Gamma^{\rho}{}_{\nu\sigma}) - D_\nu(\delta\Gamma^{\rho}{}_{\mu\sigma}) \ .$$

and the second term in the variation of the Ricci scalar can be rewritten as

$$g^{\mu\nu}\delta R_{\mu\nu} = g^{\mu\nu}\delta R^{\rho}{}_{\mu\rho\nu} = g^{\mu\nu}\left[D_\rho(\delta\Gamma^{\rho}{}_{\nu\mu}) - D_\nu(\delta\Gamma^{\rho}{}_{\rho\mu}) \right] = D_\gamma(g^{\mu\nu}\delta\Gamma^{\gamma}{}_{\nu\mu} - g^{\mu\gamma}\delta\Gamma^{\rho}{}_{\rho\mu}) \ ,$$

where we in the last step contracted indices to be able to express the variation with only one covariant derivative. Now we are ready for taking the variation of the Christoffel symbol. From equation (G.2) it follows

$$\begin{aligned}
\delta\Gamma^{\sigma}{}_{\mu\nu} &= \frac{1}{2}\delta g^{\sigma\rho}\left(\partial_\mu g_{\nu\rho} + \partial_\nu g_{\rho\mu} - \partial_\rho g_{\mu\nu} \right) + \frac{1}{2} g^{\sigma\rho}\left(\partial_\mu \delta g_{\nu\rho} + \partial_\nu \delta g_{\rho\mu} - \partial_\rho \delta g_{\mu\nu} \right) \\
&= \delta g^{\sigma\rho} g_{\rho\gamma}\Gamma^{\gamma}{}_{\mu\nu} + \frac{1}{2} g^{\sigma\rho}\left(\partial_\mu \delta g_{\nu\rho} + \partial_\nu \delta g_{\rho\mu} - \partial_\rho \delta g_{\mu\nu} \right) \\
&= \frac{1}{2} g^{\sigma\rho}(D_\mu \delta g_{\nu\rho} + D_\nu \delta g_{\rho\mu} - D_\rho \delta g_{\mu\nu}) \\
&= -\frac{1}{2}(g_{\nu\alpha} D_\mu \delta g^{a\sigma} + g_{\mu\beta} D_\nu \delta g^{\sigma\beta} - g_{\mu\alpha} g_{\nu\beta} D^\sigma \delta g^{\alpha\beta}) \ ,
\end{aligned} \tag{H.4}$$

where we in the last step used the fact that $\delta g_{\mu\nu} = -g_{\mu\alpha} g_{\nu\beta} g^{\alpha\beta}$. Now let's consider the terms $g^{\mu\nu}\delta\Gamma^{\gamma}{}_{\nu\mu}$ and $g^{\mu\gamma}\delta\Gamma^{\rho}{}_{\rho\mu}$:

$$g^{\mu\nu}\delta\Gamma^{\gamma}{}_{\nu\mu} = -\frac{1}{2}(D_\alpha \delta g^{\alpha\gamma} + D_\beta \delta g^{\gamma\beta} - g_{\alpha\beta} D^\gamma \delta g^{\alpha\beta}) = -D_\alpha \delta g^{\alpha\gamma} + \frac{1}{2} g_{\alpha\beta} D^\gamma \delta g^{\alpha\beta} \ , \tag{H.5}$$

$$g^{\mu\gamma}\delta\Gamma^{\rho}{}_{\rho\mu} = -\frac{1}{2}(D_\rho \delta g^{\gamma\rho} + g_{\alpha\beta} D^\gamma \delta g^{\alpha\beta} - \delta g^{\gamma\alpha} D_\alpha) = -\frac{1}{2} g_{\alpha\beta} D^\gamma \delta g^{\alpha\beta} \ . \tag{H.6}$$

We are almost done, we just need to subtract the terms and take the covariant derivative in order to get the second term in the variation of the Ricci scalar. We have

$$g^{\mu\nu}\delta\Gamma^{\gamma}{}_{\nu\mu} - g^{\mu\gamma}\delta\Gamma^{\rho}{}_{\rho\mu} = g_{\alpha\beta}D^{\gamma}\delta g^{\alpha\beta} - D_{\alpha}\delta g^{\alpha\gamma} \ .$$

Finally we take the covariant derivative D_γ:

$$g^{\mu\nu}\delta R_{\mu\nu} = D_\gamma(g^{\mu\nu}\delta\Gamma^{\gamma}{}_{\nu\mu} - g^{\mu\gamma}\delta\Gamma^{\rho}{}_{\rho\mu}) = D^2 g_{\alpha\beta}\delta g^{\alpha\beta} - D_\gamma D_\alpha \delta g^{\alpha\beta} \ , \tag{H.7}$$

and at last we have found our identity for δR (renaming the indices $\alpha \to \mu$, $\beta \to \nu$):

$$\delta R = \delta g^{\mu\nu} R_{\mu\nu} + D^2 g_{\mu\nu}\delta g^{\mu\nu} - D_\mu D_\nu \delta g^{\mu\nu} \ . \tag{H.8}$$

H.3 d'Alembert operator on a scalar field in curved spacetime

In this section we prove that

$$D^\mu D_\mu \phi = \frac{1}{\sqrt{-g}}\partial_\mu\left(g^{\mu\nu}\sqrt{-g}\partial_\nu\phi\right) \ , \tag{H.9}$$

for a scalar field ϕ. Let us first consider the right hand side. We have to be a bit careful here because we can not, in general, raise or lower indices on partial derivatives. This is because $\partial_\mu V^\nu$ is not a tensor! However, since $\partial_\mu\phi = D_\mu\phi$ if ϕ is a scalar field we may write $g^{\mu\nu}\partial_\nu\phi = D^\mu\phi$. Using this and acting with the second derivative gives

$$\frac{1}{\sqrt{-g}}\partial_\mu\left(\sqrt{-g}D^\mu\phi\right) = \partial_\mu D^\mu\phi + \frac{(\partial_\mu g)}{2g}D^\mu\phi \ .$$

We can compute the derivative of the metric just as we did the variation of it. Since this has already been done we simple state the slightly altered version of (H.2)

$$\partial_\mu g = g g^{\sigma\rho}\partial_\mu(g_{\sigma\rho}) \ ,$$

and we can conclude that

$$\frac{1}{\sqrt{-g}}\partial_\mu\left(\sqrt{-g}D^\mu\phi\right) = \partial_\mu D^\mu\phi + \frac{g^{\sigma\rho}\partial_\mu(g_{\sigma\rho})}{2}D^\mu\phi \ .$$

We now move on to the left hand side of (H.9)

$$D_\mu D^\mu\phi = \partial_\mu D^\mu\phi + \Gamma^{\mu}{}_{\mu\rho}D^\rho\phi \ .$$

The last term may be simplified further

$$\Gamma^{\mu}{}_{\mu\rho} = \frac{1}{2}g^{\mu\nu}\left(\partial_\mu g_{\nu\rho} + \partial_\rho g_{\nu\mu} - \partial_\nu g_{\mu\rho}\right) = \frac{1}{2}g^{\mu\nu}\partial_\rho g_{\nu\mu} \ ,$$

since the metric is symmetric. Thus the left hand side becomes

$$D_\mu D^\mu\phi = \partial_\mu D^\mu\phi + \frac{g^{\mu\nu}\partial_\rho(g_{\nu\mu})}{2}D^\rho\phi \ ,$$

which is equal to the right hand side. With this we have proven equation (H.9).

H.4 Weyl transformation of the Ricci scalar

In this section we will prove that the Ricci scalar, R, transforms according to

$$R \mapsto \tilde{R} = \Omega^{-2}(R - 4\Box\ln(\Omega) - 2(\partial^a\ln\Omega)(\partial_a\ln\Omega) \ ,$$

under the conformal (Weyl) transformation $g_{\mu\nu} \mapsto \tilde{g}_{ab} = \Omega^2(x)g_{ab}$ in $2+1$ dimensions. We will do this the long way, that is we will first construct the transformed Christoffel symbol, use it to find our Riemann tensor and finally contract it twice to get the Ricci scalar. The transformation of the Christoffel symbol is straight forward to find using its definition and the identity $\tilde{g}^{ab} = \Omega^2(x)g_{ab}$. Explicitly

$$\tilde{\Gamma}^\mu_{\nu\rho} = \tilde{g}^{\mu\lambda}\frac{1}{2}(\partial_\nu\tilde{g}_{\lambda\rho} + \partial_\rho\tilde{g}_{\lambda\nu} - \partial_\lambda\tilde{g}_{\nu\rho}) = \Omega^{-2}g^{\mu\lambda}\frac{1}{2}\Big(\partial_\nu(\Omega^2 g_{\lambda\rho}) + (\partial_\rho\Omega^2 g_{\lambda\nu}) - \partial_\lambda(\Omega^2 g_{\nu\rho})\Big) \tag{H.10}$$

$$= \Gamma^\mu_{\nu\rho} + \Big(\delta^\mu_\rho D_\nu \ln\Omega + \delta^\mu_\nu D_\rho \ln\Omega - g_{\nu\rho}D^\mu \ln\Omega\Big) = \Gamma^\mu_{\nu\rho} + f(\Omega)^\mu_{\rho\nu} \ , \tag{H.11}$$

where we in the third step used that $g^{\mu\nu}g_{\nu\rho} = \delta^\mu_\rho$ and the fact that since Ω is a scalar we may write $\partial_\mu \ln\Omega = D_\mu \ln\Omega$. D_μ is of course the standard covariant derivative. We can also see that $f^\mu_{\rho\nu}$ is symmetric in its lower indices. Next we consider the Riemann tensor which can be written as $R^\mu{}_{\nu\rho\lambda} = \partial_\rho\Gamma^\mu{}_{\nu\lambda} - \partial_\lambda\Gamma^\mu{}_{\nu\rho} + \Gamma^\mu{}_{\sigma\rho}\Gamma^\sigma{}_{\nu\lambda} - \Gamma^\mu{}_{\sigma\lambda}\Gamma^\sigma{}_{\nu\rho}$. Plugging in the expression for $\tilde{\Gamma}$ derived above we find that

$$\begin{aligned}\tilde{R}^\mu{}_{\nu\rho\lambda} &= R^\mu{}_{\nu\rho\lambda} + 2\partial_{[\rho|}f^\mu{}_{\nu|\lambda]} + 2f^\mu{}_{\sigma[\rho|}f^\sigma{}_{\nu|\lambda]} - 2f^\mu{}_{\sigma[\lambda|}\Gamma^\sigma{}_{\nu|\rho]} + 2\Gamma^\mu{}_{\sigma[\rho|}f^\sigma{}_{\nu|\lambda]} \\ &= 2D_{[\rho|}f^\mu{}_{\nu|\lambda]} + 2f^\mu{}_{\sigma[\rho|}f^\sigma{}_{\nu|\lambda]} \ .\end{aligned} \tag{H.12}$$

While this expression is nice and short we will have use for a more explicit. We start by examining the expression $D_{[\rho|}f^\mu{}_{\nu|\lambda]}$.

$$D_{[\rho|}f^\mu{}_{\nu|\lambda]} = \delta^\mu_\nu D_{[\rho}D_{\lambda]}\ln\Omega + D_{[\rho}\delta^\mu_{\lambda]}D_\nu \ln\Omega - g_{\nu[\lambda}D_{\rho]}D^\mu \ln\Omega \ .$$

While it is not true that covariant derivatives commute in general we may use the fact that ours act on a scalar function to see that

$$D_\rho D_\lambda \ln\Omega = \partial_\rho\partial_\lambda \ln\Omega - \Gamma^\sigma{}_{\rho\lambda}\partial_\sigma \ln\Omega = \partial_\lambda\partial_\rho \ln\Omega - \Gamma^\sigma{}_{\lambda\rho}\partial_\sigma \ln\Omega = D_\lambda D_\rho \ln\Omega \ ,$$

from which we may conclude that $\delta^\mu_\nu D_{[\rho}D_{\lambda]}\ln\Omega = 0$ and that

$$D_{[\rho|}f^\mu{}_{\nu|\lambda]} = D_{[\rho}\delta^\mu_{\lambda]}D_\nu \ln\Omega - g_{\nu[\lambda}D_{\rho]}D^\mu \ln\Omega \ .$$

We move on to the second term in (H.12).

$$f^\mu{}_{\sigma\rho}f^\sigma{}_{\nu\lambda} = \Big(\delta^\mu_\sigma D_\rho \ln\Omega + \delta^\mu_\rho D_\sigma \ln\Omega - g_{\rho\sigma}D^\mu \ln\Omega\Big)\Big(\delta^\sigma_\nu D_\lambda \ln\Omega + \delta^\sigma_\lambda D_\nu \ln\Omega - g_{\lambda\nu}D^\sigma \ln\Omega\Big) \ .$$

Now, antisymmetrize the whole equation in order to achieve the second term in the transformed Riemann tensor. We moreover expand the parenthesis:

$$\begin{aligned}f^\mu{}_{\sigma[\rho|}f^\sigma{}_{\nu|\lambda]} &= \delta^\mu_\nu(D_{[\rho|}\ln\Omega)(D_{|\lambda]}\ln\Omega) + \delta^\mu_{[\lambda|}(D_\nu \ln\Omega)(D_{|\rho]}\ln\Omega) - g_{\nu[\lambda|}(D^\mu \ln\Omega)(D_{|\rho]}\ln\Omega) \\ &+\delta^\mu_{[\rho|}(D_\nu \ln\Omega)(D_{|\lambda]}\ln\Omega) + \delta^\mu_{[\rho|}(D_{\lambda]}\ln\Omega)(D_\nu \ln\Omega) - \delta^\mu_{[\rho|}g_{\nu|\lambda]}(D_\sigma \ln\Omega)(D^\sigma \ln\Omega) \\ &-g_{\nu[\rho|}(D^\mu \ln\Omega)(D_{|\lambda]}\ln\Omega) - g_{[\rho\lambda]}(D_\nu \ln\Omega)(D^\mu \ln\Omega) + g_{\nu[\lambda|}(D_{|\rho]}\ln\Omega)(D^\mu \ln\Omega) \ .\end{aligned}$$

and after some cancellations we find

$$f^\mu{}_{\sigma[\rho}f^\sigma{}_{|\nu|\lambda]} = \delta^\mu_{[\rho|}(D_{|\lambda]}\ln\Omega)(D_\nu \ln\Omega) - \delta^\mu_{[\rho|}g_{\nu|\lambda]}(D_\sigma \ln\Omega)(D^\sigma \ln\Omega) - g_{\nu[\rho|}(D_{|\lambda]}\ln\Omega)(D^\mu \ln\Omega) \ .$$

Summarizing we have found

$$\begin{aligned}\tilde{R}^\mu{}_{\nu\rho\lambda} = R^\mu{}_{\nu\rho\lambda} + 2\Big(&\delta^\mu_{[\rho}(D_{\lambda]}\ln\Omega)(D_\nu \ln\Omega) - \delta^\mu_{[\rho|}g_{\lambda]\nu}(D_\sigma \ln\Omega)(D^\sigma \ln\Omega) \\ &- g_{\nu[\rho|}(D_{\lambda]}\ln\Omega)(D^\mu \ln\Omega) + D_{[\rho}\delta^\mu_{\lambda]}D_\nu \ln\Omega - g_{\nu[\lambda}D_{\rho]}D^\mu \ln\Omega\Big) \ .\end{aligned}$$

We may now procceed and calculate the Ricci tensor, $R_{\nu\lambda}$. By definition we have that $R_{\nu\lambda} = R^\mu{}_{\nu\mu\lambda}$. We remember that, since we work in $2+1$ dimensions, we have $\delta^\mu_\mu = 3$, the rest of the calculation is straight-forward and gives

$$\begin{aligned}\tilde{R}_{\nu\lambda} &= R_{\nu\lambda} + (3-1)(D_\lambda \ln\Omega)(D_\nu \ln\Omega) - (3-1)g_{\lambda\nu}(D_\sigma \ln\Omega)(D^\sigma \ln\Omega) - (D_\lambda \ln\Omega)(D_\nu \ln\Omega) \\ &+ g_{\lambda\nu}(D_\sigma \ln\Omega)(D^\sigma \ln\Omega) + (1-3)D_\lambda D_\nu \ln\Omega - g_{\lambda\nu}(D_\sigma D_\sigma \ln\Omega) + D_\lambda D_\nu \ln\Omega \\ &= R_{\nu\lambda} + (D_\lambda \ln\Omega)(D_\nu \ln\Omega) - g_{\lambda\nu}(D_\sigma \ln\Omega)(D^\sigma \ln\Omega) - g_{\lambda\nu}(D_\sigma D^\sigma \ln\Omega) - D_\lambda D_\nu \ln\Omega \ .\end{aligned}$$

We have finally reached the last step. By using that $\tilde{R} = \tilde{g}^{\nu\lambda}\tilde{R}_{\nu\lambda}$ we can write down the transformed Ricci scalar as

$$\begin{aligned}\tilde{R} &= \Omega^{-2}\Big(R + (1-3)(D_\sigma \ln\Omega)(D^\sigma \ln\sigma) - (1+3)(D_\sigma D^\sigma \ln\Omega)\Big) \\ &= \Omega^{-2}\Big(R - 2(D_\sigma \ln\Omega)(D^\sigma \ln\sigma) - 4\Box \ln\Omega\Big) \ .\end{aligned}$$

which is precisely what we set out to prove.

H.5 The stress-energy tensor

Consider the Lagrangian for a real scalar field ϕ:

$$\mathcal{L} = -\frac{1}{2}g^{\mu\nu}(\partial_\mu\phi\partial_\nu\phi) - \frac{1}{2}m^2\phi^2 - \frac{1}{2}\xi R\phi^2 - \xi'\phi^p \ , \tag{H.13}$$

where R is the Ricci scalar and ξ, ξ' and p are constants. By rewriting the first term $g^{\mu\nu}(\partial_\mu\phi\partial_\nu\phi) = \partial_\mu\phi\partial^\mu\phi = (\partial\phi)^2$ we note that the first two terms in the Lagrangian is precisely the Klein-Gordon Lagrangian, $\mathcal{L}_{KG}$, which is derived in Appendix D. The third and fourth terms are coupling terms. We may now write down the corresponding action

$$\mathcal{S}_{scalarfield} = \int \sqrt{-g}d^3x(-\frac{1}{2}g^{\mu\nu}(\partial_\mu\phi\partial_\nu\phi) - \frac{1}{2}m^2\phi^2 - \frac{1}{2}\xi R\phi^2 - \xi'\phi^p) \ ,$$

where g denotes the determinant of the metric $g^{\mu\nu}$. Are there any constraints on ξ and ξ' and p? These constants are often picked so that the scalar field is conformally coupled to gravity. This means that our matter action is unchanged, up to a boundary term, under the transformations

$$g_{\mu\nu} \mapsto \Omega^2(x)g_{\mu\nu}, \qquad\qquad \phi \mapsto \Omega^{-1/2}(x)\phi \ ,$$

where $\Omega(x)$ is a scalar function and we set $m = 0$ [47]. The transformation considered above is a rescaling of the metric, thus invariance under this transformation means that we will have scale-invariance. We can now understand why we set $m = 0$, this is since m induces a natural length-scale and thus spoil conformal invariance. The transformation of the scalar field can be understood through a dimensional argument since a scalar field in $2+1$ dimensions has the dimension of $[L]^{-1/2}$. We can also use the above transformations to see that $\sqrt{-g} \mapsto \Omega^3(x)\sqrt{-g}$ and $g^{\mu\nu} \mapsto \Omega^{-2}(x)g^{\mu\nu}$. The first identity follows simply because the determinant will make the transformation cubic and the second because the inverse relationship must hold. The transformation of the Ricci scalar was worked out in section H.4. The result is $R \mapsto \Omega^{-2}\Big(R - 4\Box\ln\Omega - 2\partial^\mu(\ln\Omega)\partial_\mu(\ln\Omega)\Big)$. Using these transformation identities we can compute the transformation of the action as

$$\begin{aligned}\mathcal{S}_{Scalarfield} \mapsto \int d^3x\sqrt{-g}\Big(&-\frac{\Omega}{2}g^{\mu\nu}\partial_\mu(\Omega^{-1/2}\phi)\partial_\nu(\Omega^{-1/2}\phi)\\ &-\frac{1}{2}\xi\Big(R - 4\Box\ln\Omega - 2\partial^\mu(\ln\Omega)\partial_\mu(\ln\Omega\Big)\phi^2 - \Omega^{-p/2+3}\xi'\phi^p\Big) \ .\end{aligned} \tag{H.14}$$

We now use the result from section H.3 as well as partial integration on the term with the d'Alembert operator resulting in

$$\begin{aligned}2\xi\int d^3x\sqrt{-g}\Box\ln\Omega\phi^2 &= 2\xi\int d^3x\partial_\mu(\sqrt{-g}\partial^\mu\ln\Omega)\phi^2\\ &= -4\xi\int d^3x\sqrt{-g}\partial^\mu(\ln\Omega)\partial_\mu(\phi)\phi + \text{Boundary term} \ .\end{aligned}$$

If we drop the boundary term and set $p = 6$ we can write (H.14) as

$$\mathcal{S}_{\text{scalarfield}} \mapsto \mathcal{S}_{\text{scalarfield}} + (\xi - \frac{1}{8})\int d^3x\sqrt{-g}\Big(\frac{1}{\Omega^2}\partial^\mu(\Omega)\partial_\mu(\Omega)\phi^2 - \frac{4}{\Omega}(\partial_\mu)(\Omega\partial^\mu\phi)\phi\Big) \ ,$$

and we can clearly see that our action will indeed be invariant under a conformal transformation if we set $\xi = \frac{1}{8}$. Because of this we usually refer to this specific value as a *conformal coupling*. Before we continue and calculate the stress-energy tensor we derive the equation of motion for the scalar field ϕ. This is as usual given by the Euler-Lagrange equation

$$\partial_\nu\frac{\partial\Big(\sqrt{-g}\mathcal{L}\Big)}{\partial\Big(\partial_\nu\phi\Big)} - \frac{\partial\Big(\sqrt{-g}\mathcal{L}\Big)}{\partial\phi} = 0 \ .$$

The first term can be rewritten as

$$\partial_\nu\frac{\partial\Big(\sqrt{-g}\mathcal{L}\Big)}{\partial\Big(\partial_\nu\phi\Big)} = \partial_\nu\frac{\partial\Big(-\sqrt{-g}\frac{1}{2}\partial^\mu\phi\partial_\mu\phi\Big)}{\partial\Big(\partial_\nu\phi\Big)} = -\partial_\nu\Big(\sqrt{-g}g^{\nu\mu}\partial_\mu\phi\Big) = -\sqrt{-g}\Box\phi \ ,$$

where we in the last step used the identity $\frac{1}{\sqrt{-g}}\partial_\nu\Big(\sqrt{-g}g^{\nu\mu}\partial_\mu\phi\Big) = g_{\nu\mu}D^\nu D^\mu\phi = \Box\phi$ shown in section H.3. The second term in the Euler-Lagrange equations becomes

$$\frac{\partial\Big(\sqrt{-g}\mathcal{L}\Big)}{\partial\phi} = -\sqrt{-g}\Big(m^2\phi + \xi R\phi + 6\xi'\phi^5\Big) \ ,$$

and so the Euler-Lagrange equations give the equation of motion

$$\Big(\Box - m^2 - \xi R\Big)\phi - 6\xi'\phi^5 = 0 \ , \tag{H.15}$$

We now consider the stress-energy tensor. For simplicity we divide the total action $S_{\text{scalarfield}}$ into three parts such that

$$\mathcal{S}_{\text{scalarfield}} = \mathcal{S}_{KG} + \mathcal{S}_1 + \mathcal{S}_2 \ ,$$

where $S\mathcal{S}_{\mathcal{KG}}$ is the Klein-Gordon action (generated by $\mathcal{L}_{\mathcal{KG}}$), $\mathcal{S}_1 = \int -\frac{1}{2}\xi R\phi^2\sqrt{-g}d^3x$ and $\mathcal{S}_2 = -\int \xi'\phi^6\sqrt{-g}d^3x$. Using the action $\mathcal{S}_{\text{scalarfield}}$ we wish to derive the corresponding stress-energy tensor $T^{\mu\nu}$. For this purpose we need to find the variation of the action with respect to the metric $g^{\mu\nu}$ and then use the fact that

$$T^{\mu\nu} = \frac{2}{\sqrt{-g}}\frac{\delta S_{\text{scalarfield}}}{\delta g_{\mu\nu}} \ .$$

For the variation of $\mathcal{S}_{\text{scalarfield}}$ we make great use of the following identities

$$\delta\sqrt{-g} = \frac{1}{2}\sqrt{-g}\delta g_{\mu\nu}g^{\mu\nu} \ ,$$
$$\delta g^{\rho\sigma} = -g^{\rho\mu}\delta g_{\mu\nu}g^{\nu\sigma} \ ,$$

moreover, when varying $\mathcal{S}_1$ the following result is needed:

$$\delta R = R_{\mu\nu}\delta g^{\mu\nu} - D_\mu D_\nu\delta g^{\mu\nu} + g_{\mu\nu}D^2\delta g^{\mu\nu} \ ,$$

where D represents a covariant derivative. These results are by no means obvious, and rather tedious to prove, why refer to the earlier sections of this appendix. Now we are settled for taking the variation of our action. Starting off with the Klein-Gordon action:

$$\begin{aligned}\delta\mathcal{S}_{KG} = \delta(\sqrt{-g}(-\frac{1}{2}g^{ab}(\partial_a\phi\partial_b\phi) - \frac{1}{2}m^2\phi^2)) &= \delta(\sqrt{-g})\mathcal{L}_{KG} + \sqrt{-g}(-\frac{1}{2}\delta g^{ab}(\partial_a\phi\partial_b\phi)) \ , \\ &= \frac{1}{2}\sqrt{-g}\delta g_{\mu\nu}g^{\mu\nu}\mathcal{L}_{KG} + \sqrt{-g}(\frac{1}{2}g^{a\mu}\delta g_{\mu\nu}g^{\nu b}(\partial_a\phi\partial_b\phi) \ , \\ &= \frac{1}{2}\sqrt{-g}\delta g_{\mu\nu}(g^{\mu\nu}\mathcal{L}_{KG} + \partial^\mu\phi\partial^\nu\phi) \ ,\end{aligned}$$

and thus we can find the first part of our stress-energy tensor by cancelling the factor in front of the parenthesis:

$$T^{\mu\nu}_{KG} = g^{\mu\nu}\mathcal{L}_{KG} + \partial^\mu\phi\partial^\nu\phi = -\frac{g^{\mu\nu}}{2}(\partial^\rho\phi\partial_\rho\phi + m^2\phi^2) + \partial^\mu\phi\partial^\nu\phi \ .$$

It remains to vary $\mathcal{S}_1$ and $\mathcal{S}_2$. Before attacking $\mathcal{S}_1$ we consider the much simpler action, $\mathcal{S}_2$. $\mathcal{S}_2$ is only affected by the variation of $\sqrt{-g}$ and thus we may write

$$\delta S_2 = -\delta\sqrt{-g}\xi'\phi^6 = -\frac{1}{2}\sqrt{-g}\delta g_{\mu\nu}g^{\mu\nu}\xi'\phi^6 \ ,$$

and we conclude that

$$T_2^{\mu\nu} = \frac{2}{\sqrt{-g}}\frac{\delta S_2}{\delta g_{\mu\nu}} = -g^{\mu\nu}\xi'\phi^6 \ .$$

Finally, we are ready to attack the variation of $\mathcal{S}_1$. By definition we have

$$\delta\mathcal{S}_1 = -\frac{1}{2}\delta(\sqrt{-g}\xi R\phi^2) = -\frac{1}{2}(\delta\sqrt{-g}\xi R\phi^2 + \sqrt{-g}\xi\delta R\phi^2) \ .$$

The first term is easy: $-\frac{1}{2}\delta\sqrt{-g}\xi R\phi^2 = -\frac{1}{4}\sqrt{-g}\delta g_{\mu\nu}g^{\mu\nu}\xi R\phi^2$ (from our beloved identity above). The second term is more tricky:

$$\begin{aligned}-\frac{1}{2}\sqrt{-g}\xi\delta R\phi^2 &= -\frac{1}{2}\sqrt{-g}\xi(R_{ab}\delta g^{ab} - D_aD_b\delta g^{ab} + g_{ab}D^2\delta g^{ab})\phi^2 \ , \\ &= -\frac{1}{2}\sqrt{-g}\xi(-R_{ab}g^{a\mu}g^{\nu b}\delta g_{\mu\nu} + D_aD_bg^{a\mu}g^{\nu b}\delta g_{\mu\nu} - g_{ab}D^2g^{a\mu}g^{\nu b}\delta g_{\mu\nu})\phi^2 \ .\end{aligned}$$

$$= -\frac{1}{2}\sqrt{-g}\xi(-R^{\mu\nu}\delta g_{\mu\nu} + D^\mu D^\nu \delta g_{\mu\nu} - g^{\mu\nu} D^2 \delta g_{\mu\nu})\phi^2 \ .$$

The first term in this expression contains the Ricci tensor and combined with the term $-\frac{1}{4}\sqrt{-g}\delta g_{\mu\nu} g^{\mu\nu}\xi R\phi^2$ containing the Ricci scalar we may form the Einstein tensor $G^{\mu\nu}$:

$$-\frac{1}{2}\sqrt{-g}\xi\delta g_{\mu\nu}(-R^{\mu\nu} + \frac{1}{2}g^{\mu\nu}R)\phi^2 = \frac{1}{2}\sqrt{-g}\xi\delta g_{\mu\nu}G^{\mu\nu}\phi^2 \ .$$

Thus $\delta\mathcal{S}_1$ can be rewritten as

$$\delta\mathcal{S}_1 = \frac{1}{2}\sqrt{-g}\xi\delta g_{\mu\nu}G^{\mu\nu}\phi^2 + \frac{1}{2}(g^{\mu\nu}D^2\delta g_{\mu\nu} - D^\mu D^\nu \delta g_{\mu\nu})\phi^2\sqrt{-g}\xi \ .$$

Now we remind ourselves that the variation of the action δS does really sit under an integral sign why we are allowed to perform integration by parts. Thus, we can rewrite the terms involving the covariant derivatives. Let us attack the integral

$$\begin{aligned}\frac{1}{2}\int d^3x\sqrt{-g}\xi\phi^2(g^{\mu\nu}D^2\delta g_{\mu\nu} - \nabla^\mu D^\nu \delta g_{\mu\nu}) &= \frac{1}{2}\int d^3x\sqrt{-g}\xi(g^{\mu\nu}D^2(\phi^2)\delta g_{\mu\nu} - D^\mu D^\nu(\phi^2)\delta g_{\mu\nu}) \\ &= \frac{1}{2}\int d^3x\sqrt{-g}\xi(g^{\mu\nu}D^2(\phi^2) - D^\mu D^\nu(\phi^2))\delta g_{\mu\nu} \ .\end{aligned} \tag{H.16}$$

We have thus rewritten $\delta\mathcal{S}_1$ according to

$$\delta\mathcal{S}_1 = \frac{1}{2}\sqrt{-g}\xi\delta g_{\mu\nu}G^{\mu\nu}\phi^2 + \frac{1}{2}\sqrt{-g}\xi(g^{\mu\nu}D^2(\phi^2) - D^\mu D^\nu(\phi^2))\delta g_{\mu\nu} \ ,$$

leading to the final contribution of the stress tensor

$$T_1^{\mu\nu} = \frac{2}{\sqrt{-g}}\frac{\delta\mathcal{S}_1}{\delta g_{\mu\nu}} = \xi G^{\mu\nu}\phi^2 + \xi(g^{\mu\nu}D^2(\phi^2) - D^\mu D^\nu(\phi^2)) \ .$$

Now, at last, we are ready to present the total stress tensor $T^{\mu\nu}$:

$$T^{\mu\nu} = T_1^{\mu\nu} + T_2^{\mu\nu} + T_{KG}^{\mu\nu} = \xi G^{\mu\nu}\phi^2 + \xi(g^{\mu\nu}\Box(\phi^2) - D^\mu D^\nu(\phi^2)) - g^{\mu\nu}\xi'\phi^6 - \frac{g^{\mu\nu}}{2}(\partial^\rho\phi\partial_\rho\phi + m^2\phi^2) + \partial^\mu\phi\partial^\nu\phi \ ,$$

where we substituted D^2 for the *D' Alembert operator* $\Box$. Precisely this result can also be found in Zelnikov and Frolovs' *Introduction to Black Hole Physics*, written in a covariant way (lowered indices), omitting the ξ'-term [27].

Appendix I

More on Morris-Thorne Solutions from Conformally Coupled Scalar Fields

Here we provide the rest of the components of the energy momentum-tensor for a conformally coupled scalar field in the Morris-Thorne case and a more detailed discussion on the differential equations which appear when relating the stress-tensor of the conformally coupled scalar field to the Einstein tensor obtained from the Morris and Thorne-solution. Remember that we consider the stress-tensor of a conformally coupled scalar field:

$$T_{\mu\nu} = \frac{1}{8}G_{\mu\nu}\phi^2 + \frac{1}{8}(g_{\mu\nu}\Box\phi^2 - D_\mu D_\nu\phi^2) - g_{\mu\nu}\xi'\phi^6 - \frac{g_{\mu\nu}}{2}(\partial^\rho\phi\partial_\rho\phi) + \partial_\mu\phi\partial_\nu\phi \ ,$$

where $G_{\mu\nu}$ is the Einstein tensor. We first consider

$$\Box\phi^2 = g^{\mu\nu}D_\mu D_\nu\phi^2 = g^{\mu\nu}D_\mu(2\phi D_\nu\phi) = 2g^{\mu\nu}(D_\mu\phi D_\nu\phi) + 2g^{\mu\nu}(\phi D_\mu D_\nu\phi) = 2g^{\mu\nu}(\partial_\mu\phi\partial_\nu\phi) + 2\phi\Box\phi \ ,$$

where we have used the fact that a covariant derivative acting on a scalar is just the partial derivative. Now using the e.o.m for our scalar field, (5.22), and exploiting the fact that it only depends on r we see that

$$\Box\phi^2 = 2g^{rr}(\phi')^2 + 2\phi\Big(\frac{1}{8}R\phi + 6\xi'\phi^5\Big) = 2(1-\frac{b}{r})(\phi')^2 + \frac{1}{4}R\phi^2 + 12\xi'\phi^6$$

Using the fact that $D^tD^t\phi^2 = 0$ since ϕ only depends on r we may now write down the time-component of our stress-energy tensor:

$$T_{tt} = \frac{1}{8}G_{tt}\phi^2 - \frac{e^{2\Phi}}{16}R\phi^2 + \frac{e^{2\Phi}}{4}(1-\frac{b}{r})\phi'^2 - \frac{e^{2\Phi}}{2}\xi'\phi^6 \ .$$

When computing the rr-component we note that

$$\begin{aligned} g_{\mu\nu}\Box\phi^2 - D_\mu D_\nu\phi^2 &= g_{\mu\nu}g^{\rho\sigma}D_\sigma D_\rho\phi^2 - D_\mu D_\nu\phi^2 \\ &= g_{\mu r}g^{r\sigma}D_\sigma\partial_r\phi^2 - D_\mu\partial_r\phi^2 = g_{rr}g^{rr}D_r\partial_r\phi^2 - D_r\partial_r\phi^2 = 0 \ . \end{aligned}$$

Thus all we have for the rr-component is

$$\begin{aligned} T_{rr} &= \frac{1}{8}G_{rr}\phi^2 - g_{rr}\xi'\phi^6 - \frac{g_{rr}}{2}g^{rr}(\phi')^2 + (\phi')^2 \\ &= \frac{1}{8}G_{rr}\phi^2 - \frac{1}{(1-\frac{b}{r})}\xi'\phi^6 + \frac{(\phi')^2}{2} \ . \end{aligned}$$

Lastly, for the $\varphi\varphi$-component we again use the identity $\Box\phi^2 = 2g^{\mu\nu}\partial_\mu\phi\partial_\nu + 2\phi\Box\phi$ (since $D_\varphi D_\varphi\phi^2 = 0$) and we find

$$T_{\varphi\varphi} = \frac{1}{8}G_{\varphi\varphi}\phi^2 + \frac{r^2}{16}R\phi^2 - \frac{r^2}{4}(1-\frac{b}{r})\phi'^2 + \frac{r^2}{2}\xi'\phi^6 \ .$$

T_{tt} and $T_{\varphi\varphi}$ can also be written in a way which allows us to relate the components:

$$T_{tt} = \frac{1}{8}G_{tt}\phi^2 + g_{tt}C \ ,$$
$$T_{\varphi\varphi} = \frac{1}{8}G_{\varphi\varphi}\phi^2 + g_{\varphi\varphi}C \ ,$$

where we introduced the term $C = \frac{1}{16}R\phi^2 - \frac{1}{4}(1-\frac{b}{r})\phi'^2 + \frac{1}{2}\xi'\phi^6$. Now, reminding ourselves that $T_{\mu\nu} = \frac{1}{\kappa}G_{\mu\nu}$ in the absence of a cosmological constant, where $\kappa = \frac{8\pi G}{c^4}$ we have

$$\frac{G_{tt}}{g_{tt}}(\frac{1}{\kappa} - \frac{1}{8}\phi^2) = C = \frac{G_{\varphi\varphi}}{g_{\varphi\varphi}}(\frac{1}{\kappa} - \frac{1}{8}\phi^2) \Rightarrow \frac{G_{\varphi\varphi}}{g_{\varphi\varphi}} = \frac{G_{tt}}{g_{tt}} \ . \tag{I.1}$$

Given the explicit expression for G_{00} and G_{22}, in an orthonormal basis as $G_{00} = \frac{b'r-b}{r^3}$ and $G_{22} = \frac{1}{1-\frac{b}{r}}(\Phi'' + (\Phi')^2) + \frac{b-b'r}{2r^2}\Phi'$ and the fact that $G_{00} = g^{tt}G_{tt}$, $G_{22} = g^{\varphi\varphi}G_{\varphi\varphi}$ with $g^{tt} = \mathrm{e}^{-2\Phi}$ and $g^{\varphi\varphi} = \frac{1}{r^2}$ we must have

$$\frac{G_{\varphi\varphi}}{g_{\varphi\varphi}} = \frac{G_{tt}}{g_{tt}} \implies G_{00} = G_{22} \ , \tag{I.2}$$

and thus get the differential equation

$$\frac{b'r-b}{r^3} = \frac{1}{1-\frac{b}{r}}(\Phi'' + (\Phi')^2) + \frac{b-b'r}{2r^2}\Phi' \ . \tag{I.3}$$

This is a very complicated differential equation and we will not solve it in general. We will instead consider the equation in a specific case. If we let the redshift function $\Phi(r) = 0$ (negectible tidal forces) the equation reduces to a simple Euler equation[1]

$$\frac{b'r-b}{r^3} = 0 \implies b(r) = Kr \ , \tag{I.4}$$

with K being a constant. This implies that the metric component, $g_{rr} = \frac{1}{1-K}$, is constant, not very interesting! If we want non-trivial solutions we should look for $\Phi(r) \neq 0$. We could for example specify to the case of a Ellis wormhole, i.e with $b(r) = \frac{b_0^2}{r}$. Then we have the following equation

$$-\frac{2b_0^2}{r^4} = \frac{1}{1-\frac{b_0^2}{r^2}}(\Phi'' + (\Phi')^2) + \frac{b_0^2}{r^3}\Phi' \ , \tag{I.5}$$

which is exactly soluble, but with a very tedious solution. Let us attack the problem from a different point of view, namely from the equation of motion for our conformally coupled scalar field ϕ:

$$\Box\phi = \frac{1}{8}R\phi + 6\xi'\phi^5 \ . \tag{I.6}$$

$\Box\phi$ can be rewritten

$$\begin{aligned}
\Box\phi &= g^{\mu\nu}D_\nu D_\mu\phi = g^{\mu\nu}D_\nu\partial_\mu\phi \\
&= g^{\mu\nu}(\partial_\mu\partial_\nu\phi - \partial_\sigma\phi\Gamma^\sigma_{\mu\nu}) = g^{rr}\phi'' - g^{rr}\phi'\Gamma^r_{rr} - g^{\varphi\varphi}\phi'\Gamma^r_{\varphi\varphi} \\
&= g^{rr}\phi'' - g^{rr}\phi'\frac{g^{rr}}{2}\partial_r(g_{rr}) - g^{\varphi\varphi}\phi'\frac{1}{2}g^{rr}(\partial_\varphi g_{\varphi r} + \partial_\varphi g_{r\varphi} - \partial_r g_{\varphi\varphi}) \\
&= g^{rr}\phi'' - (g^{rr})^2\phi'\frac{1}{2}\partial_r(g_{rr}) + g^{\varphi\varphi}\phi'\frac{1}{2}g^{rr}\partial_r(g_{\varphi\varphi}) \\
&= (1-\frac{b}{r})\phi'' + \phi'\left[\frac{b}{2r^2} - \frac{b'}{2r} + \frac{1}{r}(1-\frac{b}{r})\right] \ ,
\end{aligned}$$

resulting in another complicated differential equation for ϕ and $b(r)$:

$$(1-\frac{b}{r})\phi'' + \phi'\left[\frac{b}{2r^2} - \frac{b'}{2r} + \frac{1}{r}(1-\frac{b}{r})\right] = \frac{1}{8}R\phi + 6\xi'\phi^5 \ . \tag{I.7}$$

One could, once again, consider the case $b(r) = \frac{b_0^2}{r}$, i.e. Ellis wormhole. However, this also leads to a very difficult differential equation, this time for ϕ, which we will not bother to solve here. It seems we could not get any new information from either our equation of motion, (I.6), or the stress tensor for the conformally coupled scalar field, (I.1). We may notice something interesting if we combine the

[1] We could of course also consider the more general case $\Phi(r) = \mathit{const}$ and get the same result. This solution will be encountered later on.

equations (I.6) and (I.1) and omit the ϕ^5/ϕ^6-term in (I.6) and (I.1) respectively. With these restrictions the equations reduce to

$$\begin{cases} T_{\mu\nu} = \frac{1}{8}G_{\mu\nu}\phi^2 + \frac{1}{8}(g_{\mu\nu}\Box\phi^2 - D_\nu D_\mu\phi^2) - g_{\mu\nu}\frac{1}{2}(\partial^\rho\phi\partial_\rho\phi) + \partial_\nu\phi\partial_\mu\phi \ , \\ (\Box - \frac{1}{8}R)\phi = 0 \ . \end{cases}$$

If we take the trace of the reduced stress-tensor we find:

$$\begin{aligned} g^{\mu\nu}T_{\mu\nu} &= \frac{1}{8}g^{\mu\nu}G_{\mu\nu}\phi^2 + \frac{1}{8}(3\Box\phi^2 - D^\mu D_\mu\phi^2) - \frac{3}{2}(\partial^\rho\phi\partial_\rho\phi) + \partial^\mu\phi\partial_\mu\phi \\ &= -\frac{R}{16}\phi^2 + \frac{1}{4}(D^\mu D_\mu\phi^2) - \frac{1}{2}\partial^\mu\phi\partial_\mu\phi \\ &= -\frac{R}{16}\phi^2 + \frac{1}{4}g^{\mu\nu}D_\nu(2\phi D_\mu\phi) - \frac{1}{2}\partial^\mu\phi\partial_\mu\phi \\ &= -\frac{R}{16}\phi^2 + \frac{1}{4}g^{\mu\nu}(2D_\nu\phi D_\mu\phi + 2\phi D_\mu D_\nu\phi) - \frac{1}{2}\partial^\mu\phi\partial_\mu\phi \\ &= -\frac{R}{16}\phi^2 + \frac{1}{2}\phi D^\mu D_\mu\phi \\ &= \frac{\phi}{2}(D^\mu D_\mu - \frac{R}{8})\phi = \frac{\phi}{2}(\Box - \frac{R}{8})\phi = 0 \ , \end{aligned}$$

where we in the last step used our equation of motion. Thus we have shown that the stress-energy tensor is traceless in the case of (massless) conformal coupling in 2+1 dimensions (the result can of course be generalized to higher dimensions). Moreover, since we also have $T_{\mu\nu} = \frac{1}{\kappa}G_{\mu\nu}$ and

$$g^{\mu\nu}G_{\mu\nu} = g^{\mu\nu}R_{\mu\nu} - \frac{1}{2}g^{\mu\nu}g_{\mu\nu}R = R - \frac{3}{2}R = -\frac{R}{2} \ , \tag{I.8}$$

we must have

$$g^{\mu\nu}T_{\mu\nu} = \frac{1}{\kappa}g^{\mu\nu}G_{\mu\nu} = -\frac{1}{\kappa}\frac{R}{2} = 0 \implies R = 0 \ . \tag{I.9}$$

This means that we have a Ricci scalar which is zero in the case of conformal coupling with massless scalar field! We may also note that, since

$$R = -R_{00} + R_{11} + R_{22} = 2({R^0}_{101} + {R^2}_{121} + {R^0}_{202}) \ , \tag{I.10}$$

and $G_{00} = {R^2}_{121}$, $G_{22} = {R^0}_{101}$, found in section 5.4, and $G_{00} = G_{22}$ from equation (I.2) we must have

$$R = 2({R^0}_{101} + {R^2}_{121} + {R^0}_{202}) = 2(G_{00} - G_{22} + {R^0}_{202}) = 2{R^0}_{202} = 0 \ , \tag{I.11}$$

leading to

$${R^0}_{202} = \frac{1 - b/r}{r}\Phi' = 0 \ , \tag{I.12}$$

which implies $\Phi(r) = const$ and/or $b(r) = r$. The case $\Phi(r) = const$ results in $b(r) = Kr$ from equation (I.3), which we earlier discarded as an uninteresting solution. The case $b(r) = r$ corresponds to a divergent differential equation (I.3). Thus we have no interesting solutions! This means that we can not find a (traversable) wormhole solution to a conformally coupled (massless) scalar field in 2+1 dimensions![2]

[2]In four (3+1) dimensions the situation is radically different, see e.g. [28]

Appendix J

Miscellaneous Calculations in Higher Spin Gravity

In this appendix we carry out some of the calculations that are too tedious for the main text of chapter 7. We do this to prevent the obfuscation of the main points of the chapter by hiding it behind a big wall of math.

J.1 Chern-Simons equations of motion for $SL(3) \times SL(3)$

The Chern-Simons equations of motion are given by:

$$F = \mathrm{d}A + A \wedge A = 0 \ .$$

where the connections are given by

$$A = \mathrm{d}x^\mu \left(\omega_\mu^a + \frac{e_\mu^a}{l}\right) T_a + \mathrm{d}x^\mu \left({\omega_\mu}^{bc} + \frac{{e_\mu}^{bc}}{l}\right) T_{bc} \equiv E^a T_a + E^{bc} T_{bc} \ , \tag{J.1}$$

$$\bar{A} = \mathrm{d}x^\mu \left(\omega_\mu^a - \frac{e_\mu^a}{l}\right) \bar{T}_a + \mathrm{d}x^\mu \left({\omega_\mu}^{bc} - \frac{{e_\mu}^{bc}}{l}\right) \bar{T}_{bc} \equiv \bar{E}^a T_a + \bar{E}^{bc} \bar{T}_{bc} \ . \tag{J.2}$$

Using the connections find the equations of motion. We begin by writing down $F(A)$ for the A-connection in an appropriate form, we can then easily see what terms change sign for $\bar{A}$. We then add/subtract the equations of motion for A and $\bar{A}$ to obtain four independent equations of motion. Calculations are performed in Appendix J.1.

$$\begin{aligned} F(A) &= \mathrm{d}\left(E^a T_a + E^{bc} T_{bc}\right) + \left(E^a T_a + E^{bc} T_{bc}\right) \wedge \left(E^d T_d + E^{ef} T_{ef}\right) \\ &= \mathrm{d}\left(E^a T_a + E^{bc} T_{bc}\right) + \frac{1}{2}[T_a,T_d] E^a \wedge E^d + \frac{1}{2}[T_a,T_{ef}] E^a \wedge E^{ef} + \frac{1}{2}[T_{bc},T_d] E^{bc} \wedge E^d + \frac{1}{2}[T_{bc},T_{ef}] E^{bc} \wedge E^{ef} \\ &= \mathrm{d}\left(E^a T_a + E^{bc} T_{bc}\right) + \frac{1}{2}{\epsilon_{ad}}^g T_g E^a \wedge E^d + {\epsilon^g}_{a(e} T_{f)g} E^a \wedge E^{ef} \\ &\qquad + {\epsilon^g}_{a(e} T_{f)g} E^d \wedge E^{bc} - \left(\eta_{b(e} {\epsilon_{f)c}}^g + \eta_{c(e} {\epsilon_{f)b}}^g\right) T_g E^{bc} \wedge E^{ef} \ , \end{aligned}$$

where we may rewrite the last term $\left(\eta_{b(e} {\epsilon_{f)c}}^g + \eta_{c(e} {\epsilon_{f)b}}^g\right) T_g E^{bc} \wedge E^{ef} = 2{\epsilon^g}_{fc} T_g E^{bc} \wedge E^f_{\ b}$.

The index a is name choice for the free index of all terms. The terms attached to single-index Lie algebra coffecients and double-index coefficients are linearly independent, so this equation can be split into two:

$$\begin{aligned} F_1(A) &= \Big(\mathrm{d}E^a + \frac{1}{2}{\epsilon^a}_{bc} E^b \wedge E^c - 2{\epsilon^a}_{fc} E^{bc} \wedge E^f_{\ b}\Big) T_a = 0 \ , \\ F_2(A) &= \mathrm{d}E^{bc} T_{bc} + 2{\epsilon^g}_{a(e} T_{f)g} E^a \wedge E^{ef} = 0 \ . \end{aligned}$$

We expand the E-terms and drop the radius of curvature, the l:s, from the vielbeins for brevity, reintroducing them later via dimensional analysis:

$$F_1(A) = \mathrm{d}(w^a + e^a) + \frac{1}{2}{\epsilon^a}_{bc}\left(w^b \wedge w^c + 2e^b \wedge w^c + e^b \wedge e^c\right) \tag{J.3}$$

$$- 2{\epsilon^a}_{fc}\left(w^{bc} \wedge w^f_{\ b} + 2e^{bc} \wedge w^f_{\ b} + e^{bc} \wedge e^f_{\ b}\right) \ , \tag{J.4}$$

$$F_2(A) = \mathrm{d}(w^{bc} + e^{bc}) T_{bc} + 2{\epsilon^g}_{a(e} T_{f)g}\left(w^a \wedge e^{ef} + w^a \wedge w^{ef} + e^a \wedge w^{ef} + e^a \wedge e^{ef}\right) \ , \tag{J.5}$$

where all of the generators can be dropped in the first equation because their indices are termwise free. $F(\bar{A})$ is just $F(A)$ with a minus sign in front of every term containing only one e-term. Using this we can write down two equations of motion as the difference and sum of $F_1(A)$ and $F_1(\bar{A})$:

$$\begin{aligned}
\frac{F_1(A)-F_1(\bar{A})}{2} &= \mathrm{d}e^a + \epsilon^a{}_{bc}\left(e^b\wedge w^c\right) - 4\epsilon^a{}_{fc}e^{bc}\wedge w^f{}_b = 0 \ , \\
\frac{F_1(A)+F_1(\bar{A})}{2} &= \mathrm{d}w^a + \frac{1}{2}\epsilon^a_{bc}\left(w^b\wedge w^c + \frac{e^b\wedge e^c}{l^2}\right) - 2\epsilon^a{}_{fc}\left(w^{bc}\wedge w^f{}_b + \frac{e^{bc}\wedge e^f{}_b}{l^2}\right) = 0 \ .
\end{aligned}$$

To give the same treatment to $F_2(A)$ we first want to be able to drop the generators. To do this we need to move the symmetrization brackets from the generator to the e, w.

$$\begin{aligned}
F_2(A) &= \mathrm{d}(w^{bc}+e^{bc})T_{bc} + 2\epsilon^g{}_{a(e}T_{f)g}\left(w^a\wedge e^{ef} + w^a\wedge w^{ef} + e^a\wedge w^{ef} + e^a\wedge e^{ef}\right) \\
&\Rightarrow \mathrm{d}(w^{bc}+e^{bc})T_{bc} + 2\epsilon^{ga(e|}T_{fg}\left(w_a\wedge e_e{}^{|f)} + w_a\wedge w_e{}^{|f)} + e_a\wedge w_e{}^{|f)} + e_a\wedge e_e{}^{|f)}\right) \\
&\Rightarrow \mathrm{d}(w^{bc}+e^{bc}) + 2\epsilon^{ga(e|}\left(w_a\wedge e_e{}^{|f)} + w_a\wedge w_e{}^{|f)} + e_a\wedge w_e{}^{|f)} + e_a\wedge e_e{}^{|f)}\right) = 0 \ .
\end{aligned}$$

With this, we can construct two more equations of motion for $F_2(A)$:

$$\begin{aligned}
\frac{F_2(A)-F_2(\bar{A})}{2} &= \mathrm{d}e^{ab} + 2\epsilon^{ga(a|}w_a\wedge e_e{}^{|b)} + 2\epsilon^{ga(e|}e_a\wedge w_e{}^{|f)} = 0 \ , \\
\frac{F_2(A)+F_2(\bar{A})}{2} &= \mathrm{d}w^{ef} + 2\epsilon^{ag(e|}e_a\wedge e_g{}^{|f)} + 2\epsilon^{ga(e|}w_a\wedge w_e{}^{|f)} = 0 \ .
\end{aligned}$$

At this point we can write down all four equations of motion in one place, according to:

$$\begin{aligned}
&\mathrm{d}e^a + \epsilon^a{}_{bc}e^b\wedge w^c - 4\epsilon^a{}_{fc}e^{bc}\wedge w^f{}_b = 0 \ , \\
&\mathrm{d}w^a + \frac{1}{2}\epsilon^a_{bc}\left(w^b\wedge w^c + \frac{e^b\wedge e^c}{l^2}\right) - 2\epsilon^a{}_{fc}\left(w^{bc}\wedge w^f{}_b + \frac{e^{bc}\wedge e^f{}_b}{l^2}\right) = 0 \ , \\
&\mathrm{d}e^{ab} + 2\epsilon^{ga(a|}w_a\wedge e_e{}^{|b)} + 2\epsilon^{ga(e|}e_a\wedge w_e{}^{|f)} = 0 \ , \\
&\mathrm{d}w^{ef} + 2\epsilon^{ag(e|}e_a\wedge e_g{}^{|f)} + 2\epsilon^{ga(e|}w_a\wedge w_e{}^{|f)} = 0 \ .
\end{aligned} \tag{J.6}$$

J.2 Turning Chern-Simons action into a modified Einstein-Hilbert action in spin-3

In section 7.1 we postulated that a spin 3 Chern-Simons action can be expressed as the regular Einstein-Hilbert action combined with some extra terms due to spin-3. At first we considered a Chern Simons action of the form

$$\mathcal{S}_{CS} = \frac{k}{2\pi}\int_M \mathrm{tr}\left[e\wedge R\right] + \frac{1}{3l^2}\mathrm{tr}\left[e\wedge e\wedge e\right] \ , \tag{J.7}$$

where $e = e^aT_a + e^{bc}T_{bc}$, T_a and T_{bc} denote generators of the gauge group SL(3,R) obeying the Lie algebra

$$[T_a,T_b] = \epsilon_{ab}^{\ \ c}T_c \ , \tag{J.8}$$

$$[T_a,T_{bc}] = 2\epsilon_{a(b}^{\ \ \ d}T_{c)d} \ , \tag{J.9}$$

$$[T_{ab},T_{cd}] = -2(\eta_{a(c}\epsilon_{d)b}^{\ \ \ e} + \eta_{b(c}\epsilon_{d)a}^{\ \ \ e})T_e \ . \tag{J.10}$$

(J.11)

Consider the first term in (J.7): $\mathrm{tr}\left[e\wedge R\right]$. We remind ourselves that we may write

$$R = \mathrm{d}\omega + \omega\wedge\omega \ , \tag{J.12}$$

where ω can be expanded in terms of the generators T_a and T_{ab} according to $\omega = \omega^a T_a + \omega^{bc} T_{bc}$. Thus we have

$$\begin{aligned}
\mathrm{d}\omega + \omega \wedge \omega &= \mathrm{d}\omega^a T_a + \mathrm{d}\omega^{ab} T_{ab} + \omega^a T_a \wedge \omega^b T_b + \omega^{ab} T_{ab} \wedge \omega^{cd} T_{cd} + 2\omega^a T_a \wedge \omega^{bc} T_{bc} \\
&= \mathrm{d}\omega^a T_a + \mathrm{d}\omega^{ab} T_{ab} + \frac{1}{2}\omega^a \wedge \omega^b [T_a,T_b] + \frac{1}{2}\omega^{ab} \wedge \omega^{cd}[T_{ab},T_{cd}] + \omega^a \wedge \omega^{bc}[T_a,T_{bc}] \\
&= \mathrm{d}\omega^a T_a + \mathrm{d}\omega^{ab} T_{ab} + \frac{1}{2}\omega^a \wedge \omega^b \epsilon_{ab}^{\ \ c} T_c - \omega^{ab} \wedge \omega^{cd}(\eta_{a(c}\epsilon_{d)b}^{\ \ e} + \eta_{b(c}\epsilon_{d)a}^{\ \ e}) T_e + \omega^a \wedge \omega^{bc}(\epsilon_{ab}^{\ \ d} T_{cd} + \epsilon_{ac}^{\ \ d} T_{bd}) \\
&= (\mathrm{d}\omega^a + \frac{1}{2}\epsilon_{bc}^{\ \ a}\omega^b \wedge \omega^c - 2\omega^{bc} \wedge \omega^{de}\eta_{b(d}\epsilon_{e)c}^{\ \ a}) T_a + (\mathrm{d}\omega^{ab} + \omega^c \wedge \omega^{da}\epsilon_{cd}^{\ \ b} + \omega^k \wedge \omega^{al}\epsilon_{kl}^{\ \ b}) T_{ab} \\
&= (R^a - 2\omega^{bc} \wedge \omega^{de}\eta_{b(d}\epsilon_{e)c}^{\ \ a}) T_a + (\mathrm{d}\omega^{ab} + \omega^c \wedge \omega^{da}\epsilon_{cd}^{\ \ b} + \omega^k \wedge \omega^{al}\epsilon_{kl}^{\ \ b}) T_{ab} \ ,
\end{aligned}$$

where we made use of the fact $e^a T_a \wedge e^b T_b = \frac{1}{2} e^a \wedge e^b [T_a,T_b]$ and $R^a = \mathrm{d}\omega^a + \frac{1}{2}\epsilon_{bc}^{\ \ a}\omega^b \wedge \omega^c$. Now we are almost ready for taking the trace $\mathrm{tr}\,(e \wedge R)$, but we will need some trace relationships first:

$$\mathrm{tr}\,[T_a T_b] = 2\eta_{ab} \ , \tag{J.13}$$

$$\mathrm{tr}\,[T_a T_{bc}] = 0 \ , \tag{J.14}$$

$$\mathrm{tr}\,[T_{ab} T_{cd}] = -\frac{4}{3}\eta_{ab}\eta_{cd} + 2(\eta_{ac}\eta_{bd} + \eta_{ad}\eta_{bc}) \ . \tag{J.15}$$

Taking the wedge product $e \wedge R$ and invoking these relations lead to

$$\begin{aligned}
\mathrm{tr}\,[e \wedge R] &= 2e^a \wedge (R_a - 2\eta_{b(d}\epsilon_{e)ca}\omega^{bc} \wedge \omega de) + e^{ab} \wedge (\mathrm{d}\omega^{cd} + \omega^h \wedge \omega^{ic}\epsilon_{hi}^{\ \ d} + \omega^k \wedge \omega^{cl}\epsilon_{kl}^{\ \ d})\,\mathrm{tr}\,[T_{ab}T_{cd}] \\
&= 2e^a \wedge (R_a - 2\eta_{b(d}\epsilon_{e)ca}\omega^{bc} \wedge \omega^{de}) + 4e^{ab} \wedge \mathrm{d}\omega_{ab} + e^{ab} \wedge \left[\omega^h \wedge \omega_a^{\ i}\epsilon_{bhi} + \omega^h \wedge \omega_b^{\ i}\epsilon_{ahi} + \omega^k \wedge \omega_a^{\ l}\epsilon_{bkl} + \omega^k \wedge \omega_b^{\ l}\epsilon_{ahl}\right] \\
&= 2e^a \wedge (R_a - 2\eta_{b(d}\epsilon_{e)ca}\omega^{bc} \wedge \omega_{de}) + 4e^{ab} \wedge \mathrm{d}\omega_{ab} + 4e^{ab} \wedge \left[\omega^h \wedge \omega_a^{\ i}\epsilon_{bhi} + \omega^h \wedge \omega_b^{\ i}\epsilon_{ahi}\right] .
\end{aligned}$$

The last term can be written in a more compact and sophisticated way with the use of symmetrization:

$$4e^{ab} \wedge \left[\omega^h \wedge \omega_a^{\ i}\epsilon_{bhi} + \omega^h \wedge \omega_b^{\ i}\epsilon_{ahi}\right] = 2e^{ab} \wedge \epsilon_{hi(a|}\omega^h \wedge \omega^{\ i}_{|b)} \ .$$

We have now rewritten the first term in (J.7) according to

$$\mathrm{tr}\,[e \wedge R] = 2e^a \wedge (R_a - 2\eta_{b(d}\epsilon_{e)ca}\omega^{bc}\omega^{de}) + 4e^{ab} \wedge \mathrm{d}\omega_{ab} + 2e^{ab}\epsilon_{hi(a|}\omega^h \wedge \omega^{\ i}_{|b)} \ .$$

However, the expression can be further simplified. The term $\eta_{b(d}\epsilon_{e)ca}\omega^{bc}\omega^{de}$ can be rewritten:

$$\begin{aligned}
\eta_{b(d}\epsilon_{e)ca}\omega^{bc} \wedge \omega^{de} &= \frac{1}{2}[\eta_{bd}\epsilon_{eca}\omega^{bc} \wedge \omega^{de} + \eta_{be}\epsilon_{dca}\omega^{bc} \wedge \omega^{de}] \\
&= \frac{1}{2}[\epsilon_{eca}\omega^{bc} \wedge \omega_b^{\ e} + \epsilon_{dca}\omega^{bc} \wedge \omega_b^{\ d}] = \omega^{bc} \wedge \omega_b^{\ e}\epsilon_{eca} \ ,
\end{aligned}$$

and we conclude

$$\mathrm{tr}\,[e \wedge R] = 2e^a \wedge R_a - 4e^a \wedge \omega^{bc} \wedge \omega_b^{\ e}\epsilon_{eca} + 4e^{ab} \wedge \mathrm{d}\omega_{ab} + 2e^{ab}\epsilon_{ih(a|}\omega^h \wedge \omega^{\ i}_{|b)} \ .$$

Moving on to the second term, $\mathrm{tr}\,(e \wedge e \wedge e)$. By expanding e in terms of the generators of $SL3$ we find

$$\begin{aligned}
e \wedge e \wedge e &= (e^a T_a + e^{de} T_{de}) \wedge (e^b T_b + e^{fg} T_{fg}) \wedge (e^c T_c + e^{hi} T_{hi}) \\
&= e^a T_a \wedge e^b T_b \wedge e^c T_c + e^a T_a \wedge e^{fg} T_{fg} \wedge e^{hi} T_{hi} + e^{de} T_{de} \wedge e^b T_b \wedge e^{hi} T_{hi} + e^{de} T_{de} \wedge e^{fg} T_{fg} \wedge e^c T_c + X \ ,
\end{aligned}$$

where "X" denotes terms which will vanish when taking the trace, i.e. $\mathrm{tr}\,[X] = 0$. Thus we conclude that we need to compute four different traces:

$$\mathrm{tr}\,[e \wedge e \wedge e] = [1] + [2] + [3] + [4] \ ,$$

where

$$\begin{aligned}
[1] &:= \mathrm{tr}\,[e^a T_a \wedge e^b T_b \wedge e^c T_c] \\
[2] &:= \mathrm{tr}\,[e^a T_a \wedge e^{fg} T_{fg} \wedge e^{hi} T_{hi}] \\
[3] &:= \mathrm{tr}\,[e^{de} T_{de} \wedge e^b T_b \wedge e^{hi} T_{hi}] \\
[4] &:= \mathrm{tr}\,[e^{de} T_{de} \wedge e^{fg} T_{fg} \wedge e^c T_c] \ .
\end{aligned}$$

Let's start with [1]. We recall that $e^aT_a \wedge e^bT_b = \frac{1}{2}e^a \wedge e^b[T_a,T_b]$ and find:

$$\mathrm{tr}\,[e^aT_a \wedge e^bT_b \wedge e^cT_c] = \frac{1}{2}e^a \wedge e^b \wedge e^c\epsilon_{ab}^{\ \ d}\,\mathrm{tr}\,[T_cT_d] = e^a \wedge e^b \wedge e^c\epsilon_{abc}\ ,$$

where we in the last step used $\mathrm{tr}\,[T_cT_d] = \mathrm{tr}\,[T_dT_c] = \eta_{dc}$. The second term ([2]) requires a bit more work:

$$\begin{aligned}\mathrm{tr}\,[e^aT_a \wedge e^{fg}T_{fg} \wedge e^{hi}T_{hi}] &= \mathrm{tr}\,[e^aT_a \wedge e^{bc}T_{bc} \wedge e^{de}T_{de}]\\ &= -\,\mathrm{tr}\,[e^a \wedge e^{bc} \wedge e^{de}(\eta_{b(d}\epsilon_{e)c}^{\ \ \ f} + \eta_{c(d}\epsilon_{e)b}^{\ \ \ f})T_aT_f]\end{aligned}$$

$$= -2e^a \wedge e^{bc} \wedge e^{de}(\eta_{b(d}\epsilon_{e)c}^{\ \ \ f} + \eta_{c(d}\epsilon_{e)b}^{\ \ \ f})\eta_{af} = -e^a \wedge e^{bc} \wedge e^{de}\Big[\eta_{bd}\epsilon_{aec} + \eta_{be}\epsilon_{adc} + \eta_{cd}\epsilon_{aeb} + \eta_{ce}\epsilon_{adb}\Big]\ ,$$

where we once again used the commutator (in this case $[T_{bc},T_{de}]$) and the trace relationship (J.13). The third term [3] follows from a completely analogous computation (same commutator and trace as [2]) and we find

$$\mathrm{tr}\,[e^{de}T_{de} \wedge e^bT_b \wedge e^{hi}T_{hi}] = ... = -2e^a \wedge e^{bc} \wedge e^{de}(\eta_{b(d}\epsilon_{e)c}^{\ \ \ f} + \eta_{c(d}\epsilon_{e)b}^{\ \ \ f})\eta_{af} = [2]\ .$$

The fourth term in $\mathrm{tr}\,[e \wedge e \wedge e]$ is the most cumbersome. We remind ourselves of the commutator relation $T_{bc}T_a = T_aT_{bc} - 2\epsilon_{a(b}^{\ \ \ f}S_{f)}$ and may then rewrite [4] as follows

$$\mathrm{tr}\,[e^{bc}T_{bc} \wedge e^aT_a \wedge e^{de}T_{de}] = e^{bc} \wedge e^a \wedge e^{de}\,\mathrm{tr}\,[T_aT_{bc}T_{de} - 2\epsilon_{a(b}^{\ \ \ f}T_{f)}]\ . \tag{J.16}$$

By swapping the wedge product $e^{bc} \wedge e^a$ and thus obtaining a minus sign we may rewrite the first part of the expression

$$\mathrm{tr}\,[e^{bc} \wedge e^a \wedge e^{de}(T_aT_{bc}T_{de})] = -\,\mathrm{tr}\,[e^aT_a \wedge e^{bc}T_{bc} \wedge e^{de}T_{de}] = -[2]\ .$$

The second term in (J.16) needs to get investigated as well:

$$\begin{aligned}-2\,\mathrm{tr}\,[e^{bc} \wedge e^a \wedge e^{de}\epsilon_{a(b}^{\ \ \ f}T_{f)}] &= 2\,\mathrm{tr}\,[e^a \wedge e^{bc} \wedge e^{de}\epsilon_f^{\ a(b}T_{c)f}T_{de}]\\ &= \mathrm{tr}\,[e^a \wedge e^{bc} \wedge e^{de}\epsilon_f^{\ ab}T_{cf}T_{de}] + \mathrm{tr}\,[e^a \wedge e^{bc} \wedge e^{de}\epsilon_f^{\ ac}T_{bf}T_{de}]\end{aligned}$$

$$= e^a \wedge e^{bc} \wedge e^{de}\Big[-\frac{4}{3}\epsilon_f^{\ ab}\eta_{cf}\eta_{de} + 2\epsilon_f^{\ ab}(\eta_{cd}\eta_{fe} + \eta_{ce}\eta_{fd}) - \frac{4}{3}\epsilon_f^{\ ac}\eta_{bf}\eta_{de} + 2\epsilon_f^{\ ac}(\eta_{bd}\eta fe + \eta_{be}\eta_{fd}\Big]$$

$$= 2e^a \wedge e^{bc} \wedge e^{de}\Big[\epsilon_{eab}\eta_{cd} + \epsilon_{dab}\eta_{ce} + \epsilon_{eac}\eta_{bd} + \epsilon_{dac}\eta_{be}\Big]\ .$$

Thus [4] can be rewritten according to

$$[4] = -[2] + 2e^a \wedge e^{bc} \wedge e^{de}\Big[\epsilon_{eab}\eta_{cd} + \epsilon_{dab}\eta_{ce} + \epsilon_{eac}\eta_{bd} + \epsilon_{dac}\eta_{be}\Big]\ .$$

Adding [2], [3] and [4] together we find

$$\begin{aligned}[2] + [3] + [4] &= e^a \wedge e^{bc} \wedge e^{de}\Big[2\epsilon_{eab}\eta_{cd} + 2\epsilon_{dab}\eta_{ce} + 2\epsilon_{eac}\eta_{bd} + 2\epsilon_{dac}\eta_{be} - \eta_{bd}\epsilon_{aec} - \eta_{be}\epsilon_{adc} - \eta_{cd}\epsilon_{aeb} - \eta_{ce}\epsilon_{adb}\Big]\\ &= -e^a \wedge e^{bc} \wedge e^{de}\Big[2\eta_{bd}\epsilon_{aec} + 2\eta_{cd}\epsilon_{aeb} - 4\epsilon_{eab}\eta_{cd} - 4\epsilon_{eac}\eta_{bd}\Big]\\ &= -e^a \wedge e^{bc} \wedge e^{de}\Big[6\eta_{bd}\epsilon_{aec} + 6_{cd}\epsilon_{aeb}\Big] = -12e^a \wedge e^{bc} \wedge e^{de}\epsilon_{aec}\eta_{bd} = -12e^a \wedge e^{bc} \wedge e_b^{\ e}\epsilon_{aec}\ .\end{aligned}$$

Now we add all terms together to find $\mathrm{tr}\,[e \wedge e \wedge e]$ at last

$$\mathrm{tr}\,[e \wedge e \wedge e] = e^a \wedge e^b \wedge e^c\epsilon_{abc} - 12e^a \wedge e^{bc} \wedge e_b^{\ e}\epsilon_{aec}\ .$$

Substituting this term and our expression for $\mathrm{tr}\,[e \wedge R]$ into the C-S action we find

$$\begin{aligned}\mathcal{S}_{CS}[e,\omega] = \frac{k}{\pi}\int_M e^a \wedge R_a &+ \frac{1}{6}e^a \wedge e^b \wedge e^c\epsilon_{abc} - 2e^a \wedge \omega^{bc} \wedge \omega_b^{\ e}\epsilon_{eca} + 2e^{ab} \wedge \mathrm{d}\omega_{ab}\\ &+ e^{ab}\epsilon_{ih(a|}\omega^h \wedge \omega_{|b)}^{\ \ i} - 2e^a \wedge e^{bc} \wedge e_b^{\ e}\epsilon_{aec}\ .\end{aligned}$$

J.3 Gauge transformations and coordinate transformations

In section 7.1 we stated that the variations:

$$\delta e = \frac{\delta A - \delta \overline{A}}{2} = \mathrm{d}\Lambda_- + [e,\Lambda_+] + [\omega,\Lambda_-] \ , \quad \delta\omega = \frac{\delta A + \delta\overline{A}}{2} = \mathrm{d}\Lambda_+ + [e,\Lambda_-] + [\omega,\Lambda_+] \ ,$$

can be seen to vanish if we pick $\Lambda_\pm$ correctly. Let us first simply consider $\Lambda_\pm = \tau_\pm^a T^a + \tau_\pm^{bc} T^{bc}$ where $\tau_\pm$ is a constant parameter and T^a, T^{bc} are the usual $sl(3)$ generators. To evaluate the commutators we use (7.1). The calculations resembles those from the previous section and we will be content with summarizing the results:

$$\begin{aligned}
\delta e^a_\mu &= \partial_\mu \tau^a_- + \epsilon^a{}_{bc} e^b_\mu \tau^c_+ - 4\epsilon^a{}_{bc} e^{cd}_\mu \tau_{+d}{}^b + \epsilon^a{}_{bc}\omega^b_\mu \tau^c_- - 4\epsilon^a{}_{bc}\omega^{cd}_\mu \tau_{-d}{}^b \ ,\\
\delta e^{ab}_\mu &= \partial_\mu \tau^{ab}_- + 2\epsilon^{cd(a|} e_c \tau_{+d}{}^{|b)} - 2\epsilon^{cd(a|}\tau_{+c} e_d{}^{|b)} + 2\epsilon^{cd(a|}\omega_c \tau_{-d}{}^{|b)} - 2\epsilon^{cd(a|}\tau_{-c}\omega_d{}^{|b)} \ ,\\
\delta \omega^a_\mu &= \partial_\mu \tau^a_+ + \epsilon^a{}_{bc} e^b_\mu \tau^c_- - 4\epsilon^a{}_{bc} e^{dc} \tau_{-d}{}^b + \epsilon^a{}_{bc}\omega^b_\mu \tau^c_+ - 4\epsilon^a{}_{bc}\omega^{db} \wedge \tau_{+d}{}^b \ ,\\
\delta \omega^{ab}_\mu &= \partial_\mu \tau^{ab}_+ + 2\epsilon^{cd(a|} e_c \tau_{-d}{}^{|b)} - 2\epsilon^{cd(a|}\tau_{-a} e_d{}^{|b)} + 2\epsilon^{cd(a|}\omega_c \tau_{+d}{}^{|b)} - 2\epsilon^{cd(a|}\tau_{+c}\omega_d{}^{|b)} \ .
\end{aligned}$$

We are now ready to see that the variation under a gauge transformation is precisely that of the variation under a coordinate transformation. We set $\tau^a_+ = \xi^\rho \omega^a_\rho$ and $\tau^a_- = \xi^\rho e^a_\rho$ then

$$\begin{aligned}
\delta e^a - \delta_\xi e^a &= 2\xi^\nu (\partial_{[\mu} e^a_{\nu]} + \epsilon^a{}_{bc} e^b_{[\mu}\omega^c_{\nu]} - 4\epsilon^a{}_{bc} e^{cd}_{[\mu}\omega^b_{\nu]d}) = 0 \ ,\\
\delta e^{ab} - \delta_\xi e^{ab} &= 2\xi^\nu (\partial_{[\mu} e^{ab}_{\nu]} + 2\epsilon^{cd(a|} e_{c[\mu}\omega_{\nu]d}{}^{|b)} + 2\epsilon^{cd(a|}\omega_{c[\mu} e_{\nu]d}{}^{|b)}) = 0 \ ,\\
\delta \omega^a - \delta_\xi \omega^a &= 2\xi^\nu (\partial_{[\mu}\omega^a_{\nu]} + \frac{\epsilon^a{}_{bc}}{2}(e^b_{[\mu} e^c_{\nu]} + \omega^b_{[\mu}\omega^c_{\nu]}) - 2\epsilon^a{}_{kc}(e^{bc}_{[\mu} e_{\nu]b}{}^k + \omega^{bc}_{[\mu}\omega_{\nu]b}{}^k) = 0 \ ,\\
\delta \omega^{ab} - \delta_\xi \omega^{ab} &= 2\xi^\nu (\partial_{[\mu}\omega^{ab}_{\nu]} + 2\epsilon^{cd(a|} e_{c[\mu} e_{\nu]d}{}^{|b)} + 2\epsilon^{cd(a|}\omega_{c[\mu}\omega_{\nu]d}{}^{|b)}) = 0 \ .
\end{aligned}$$

As indicated all the equations equal zero. This is because the second identity is precisely the equations of motion for the higher spin Chern-Simons action.

J.4 Calculations of black holes and wormholes in spin-3 gravity

This section is devoted to make explicit some of the more lengthy calculations concerning black holes and wormholes in spin-3 gravity theory.

J.4.1 Calculations of a spin-3 black hole

We consider the connections

$$A = b^{-1} a b + b^{-1}\mathrm{d}b \ , \qquad \bar{A} = b\bar{a}b^{-1} + b\mathrm{d}b^{-1} \ , \tag{J.17}$$

with $b = e^{\rho L_0}$ and

$$a = [l_D W_2 + \mathcal{W} W_{-2} - Q W_0] dx^+ + [l_P L_1 - \mathcal{L} L_{-1} + \Phi W_0] dx^- \ , \tag{J.18}$$

$$\bar{a} = [l_D W_{-2} + \mathcal{W} W_2 - Q W_0] dx^- - [l_P L_{-1} - \mathcal{L} L_1 - \Phi W_0] dx^+ \ . \tag{J.19}$$

Here we have used the notation $x^\pm = t \pm \phi$. We have chosen a coordinate system with coordinates (t,ρ,ϕ) as discussed in (8.1). The generators L_i and W_i as well as their Lie algebra and trace relations, is presented in Appendix E.5.2. l_D, l_P, $\mathcal{W}$, $\mathcal{L}$ and Φ are parameters specifying the charges of our black hole.

First of all we want to rewrite our connections in a more convenient form. The second terms in the expressions for A and $\bar{A}$ is straight forward to compute;

$$b^{-1}\mathrm{d}b = e^{-\rho L_0}\mathrm{d}e^{\rho L_0} = L_0 d\rho \ , \tag{J.20}$$

and similarly

$$b\mathrm{d}b^{-1} = -L_0 d\rho \ . \tag{J.21}$$

To perform an analogous evaluation of the first terms in (J.17), we need to calculate the commutation relations of b with the generators L_i and W_i. This is shown explicitly for the W_2 generator; from Appendix E.5.2 we find the commutation relation $[W_2, L_0] = 2W_2$ and

$$\begin{aligned}
b^{-1}W_2b &= b^{-1}W_2\sum_{n=0}^{\infty}\frac{\rho^nL_0^n}{n!} = b^{-1}\sum_{n=0}^{\infty}\frac{\rho^n(L_0W_2+[W_2,L_0])L_0^{n-1}}{n!} \\
&= b^{-1}\sum_{n=0}^{\infty}\frac{\rho^n(L_0+2I)W_2L_0^{n-1}}{n!} = ... = b^{-1}\sum_{n=0}^{\infty}\frac{\rho^n(L_0+2I)^n}{n!}W_2 \\
&= b^{-1}e^{\rho(L_0+2I)}W_2 = b^{-1}be^{2\rho}W_2 \\
&= e^{2\rho}W_2
\end{aligned}$$

By analogous calculations, using the commutation relations between the generators L_i and W_i stated in Appendix E.5.2, we find

$$\begin{aligned}
b^{-1}W_2b &= e^{2\rho}W_2\ , & bW_2b^{-1} &= e^{-2\rho}W_2 \\
b^{-1}W_{-2}b &= e^{-2\rho}W_{-2}\ , & bW_{-2}b^{-1} &= e^{2\rho}W_{-2} \\
b^{-1}W_0b &= W_0\ , & bW_0b^{-1} &= W_0 \\
b^{-1}L_1b &= e^{\rho}L_1\ , & bL_1b^{-1} &= e^{-\rho}L_1 \\
b^{-1}L_{-1}b &= e^{-\rho}L_{-1}\ , & bL_{-1}b^{-1} &= e^{\rho}L_{-1}
\end{aligned} \tag{J.22}$$

With these relations, (J.20) and (J.21), we can rewrite the connections (J.17) as

$$A = [e^{2\rho}l_DW_2 + e^{-2\rho}\mathcal{W}W_{-2} - QW_0]dx^+ + [e^{\rho}l_PL_1 - e^{-\rho}\mathcal{L}L_{-1} + \Phi W_0]dx^- + L_0d\rho\ , \tag{J.23}$$

$$\bar{A} = [e^{2\rho}l_DW_{-2} + e^{-2\rho}\mathcal{W}W_2 - QW_0]dx^- - [e^{\rho}l_PL_{-1} - e^{-\rho}\mathcal{L}L_1 - \Phi W_0]dx^+ - L_0d\rho\ . \tag{J.24}$$

The equations of motion for our connections in higher spin theories is $F = \mathrm{d}A + A\wedge A = 0$, and similarly for $\bar{A}$. We continue by working out what the equations of motion imply for the parameters. The exterior derivative of A is straightforward to evaluate;

$$\begin{aligned}
\mathrm{d}A &= (\partial_t dt + \partial_\rho d\rho + \partial_\phi d\phi)\wedge A \\
&= \partial_\rho d\rho \wedge \big([e^{2\rho}l_DW_2 + e^{-2\rho}\mathcal{W}W_{-2} - QW_0]dx^+ + [e^{\rho}l_PL_1 - e^{-\rho}\mathcal{L}L_{-1} + \Phi W_0]dx^- + L_0d\rho\big) \\
&= 2[e^{2\rho}l_DW_2 - e^{-2\rho}\mathcal{W}W_{-2}]d\rho\wedge dx^+ + [e^{\rho}l_PL_1 + e^{-\rho}\mathcal{L}L_{-1}]d\rho\wedge dx^-\ .
\end{aligned} \tag{J.25}$$

When calculating $A\wedge A$ it is convenient to introduce some shorthand notation in order to make the calculation more transparent. We write $A = A_+dx^+ + A_-dx^- + L_0d\rho$ for the connection A given by (J.23), that is

$$A_+ = e^{2\rho}l_DW_2 + e^{-2\rho}\mathcal{W}W_{-2} - QW_0\ , \tag{J.26}$$

and

$$A_- = e^{\rho}l_PL_1 - e^{-\rho}\mathcal{L}L_{-1} + \Phi W_0\ . \tag{J.27}$$

Now we can evaluate $A\wedge A$ in terms of A_+, A_- and L_0;

$$\begin{aligned}
A\wedge A &= (A_+dx^+ + A_-dx^- + L_0d\rho)\wedge(A_+dx^+ + A_-dx^- + L_0d\rho) \\
&= A_+A_-dx^+\wedge dx^- + A_+L_0dx^+\wedge d\rho + A_-A_+dx^-\wedge dx^+ \\
&\quad + A_-L_0dx^-\wedge d\rho + L_0A_+d\rho\wedge dx^+ + L_0A_-d\rho\wedge dx^- \\
&= [A_+,A_-]dx^+\wedge dx^- + [L_0,A_+]d\rho\wedge dx^+ + [L_0,A_-]d\rho\wedge dx^-
\end{aligned} \tag{J.28}$$

To evaluate the commutators we need to use the commutation relations of the generators L_i and W_i, see Appendix E.5.2. By the use of these commutation relation, (J.26) and (J.27) we find

$$\begin{aligned}
[A_+,A_-] &= [(e^{2\rho}l_DW_2 + e^{-2\rho}\mathcal{W}W_{-2} - QW_0), (e^{\rho}l_PL_1 - e^{-\rho}\mathcal{L}L_{-1} + \Phi W_0)] \\
&= e^{3\rho}l_Dl_P[W_2,L_1] - e^{\rho}l_D\mathcal{L}[W_2,L_{-1}] + e^{2\rho}l_D\Phi[W_2,W_0] + e^{-\rho}\mathcal{W}l_P[W_{-2},L_1] \\
&\quad - e^{-3\rho}\mathcal{W}\mathcal{L}[W_{-2},L_{-1}] + e^{-2\rho}\mathcal{W}\Phi[W_{-2},W_0] - e^{\rho}Ql_P[W_0,L_1] + e^{-\rho}Q\mathcal{L}[W_0,L_{-1}] \\
&= 2e^{\rho}(l_PQ - 2l_D\mathcal{L})W_1 + 2e^{-\rho}(\mathcal{L}Q - 2l_P\mathcal{W})W_{-1}\ .
\end{aligned}$$

In the same way we find that

$$[L_0, A_+] = -2(e^{2\rho} l_D W_2 - e^{-2\rho} \mathcal{W} W_{-2}) \ ,$$

and

$$[L_0, A_-] = -e^{\rho} l_P L_1 + e^{-\rho} \mathcal{L} L_{-1} \ .$$

Comparing the latter two commutation relations with (J.25) we see that (J.28) can be rewritten as $A \wedge A = [A_+, A_-] dx^+ \wedge dx^- - \mathrm{d}A$, so the equation of motion simply states that

$$\mathrm{d}A + A \wedge A = [A_+, A_-] dx^+ \wedge dx^- = 0 \ .$$

This directly implies that $[A_+, A_-] = 0$, that is

$$2e^{\rho}(l_P Q - 2l_D \mathcal{L}) W_1 + 2e^{-\rho}(\mathcal{L} Q - 2l_P \mathcal{W}) W_{-1} = 0 \ .$$

The coefficient of each generator must equal to zero in order for this equality to hold since the generators are linearly independent. Therefore we find the following relations between the parameters;

$$l_P Q = 2l_D \mathcal{L} \ , \qquad \mathcal{L} Q = 2l_P \mathcal{W} \ .$$

We restate these relations as

$$Q = \frac{2\mathcal{W} l_P}{\mathcal{L}} \ , \qquad \frac{\mathcal{L}^2}{l_P^2} = \frac{\mathcal{W}}{l_D} \ . \tag{J.29}$$

These are the restrictions of the parameters of (J.23) and (J.24) by the flatness condition on the connections. Note that we have not yet derived how the equation $\mathrm{d}\bar{A} + \bar{A} \wedge \bar{A} = 0$ constrain the connections. However, if one performs this calculation in an analogous manner one finds that this restricts the parameters exactly as the flatness condition on A did, i.e. by (J.29).

Having investigated the equations of motion for the connections A and $\bar{A}$, given by (J.23) and (J.24), respectivly, we now calculate the corresponding metric.

In the principal embedding of SL(2) into SL(3), the metric tensor is given by

$$g_{\mu\nu} = \frac{1}{2} \mathrm{tr}[e_{(\mu} e_{\nu)}] \ , \tag{J.30}$$

where the vielbein is defined as $e_\mu = e_\mu{}^a J_a + e_\mu{}^{ab} T_{ab}$. The vielbein can be expressed in terms of the connections A and $\bar{A}$ as

$$e = e_\mu dx^\mu = \frac{1}{2}(A - \bar{A}) \ , \tag{J.31}$$

(see (insert ref)). Using (J.23) and (J.24) for A and $\bar{A}$ we find

$$\begin{aligned} e = {} & \frac{1}{2}([e^{2\rho} l_D W_2 + e^{-2\rho} \mathcal{W} W_{-2} - Q W_0] dx^+ + [e^{\rho} l_P L_1 - e^{-\rho} \mathcal{L} L_{-1} + \Phi W_0] dx^- + L_0 d\rho \\ & - [e^{2\rho} l_D W_{-2} + e^{-2\rho} \mathcal{W} W_2 - Q W_0] dx^- + [e^{\rho} l_P L_{-1} - e^{-\rho} \mathcal{L} L_1 - \Phi W_0] dx^+ + L_0 d\rho) \\ = {} & \frac{1}{2}[(e^{2\rho} l_D - e^{-2\rho} \mathcal{W})(W_2 - W_{-2}) + (e^{\rho} l_P - e^{-\rho} \mathcal{L})(L_1 + L_{-1})] dt + L_0 d\rho \\ & + \frac{1}{2}[(e^{2\rho} l_D + e^{-2\rho} \mathcal{W})(W_2 + W_{-2}) - (e^{\rho} l_P + e^{-\rho} \mathcal{L})(L_1 - L_{-1}) - 2(Q + \Phi) W_0] d\phi \ . \end{aligned} \tag{J.32}$$

At last, to calculate the metric tensor, or rather the spacetime interval $ds^2 = g_{\mu\nu} dx^\mu dx^\nu$, we make use of some trace relations for the generators L_i and W_i, (see Appendix E.5.2). We restate these trace relations here for convenience. The only nonzero trace of a product of two generators L_i and W_i are

$$\begin{aligned} & \mathrm{tr}[L_0 L_0] = 2 \ , \qquad \mathrm{tr}[L_1 L_{-1}] = -4 \\ & \mathrm{tr}[W_0 W_0] = \frac{8}{3} \ , \qquad \mathrm{tr}[W_1 W_{-1}] = -4 \ , \qquad \mathrm{tr}[W_2 W_{-2}] = 16 \ . \end{aligned} \tag{J.33}$$

Using these trace relations and (J.32) we calculate the metric;

$$
\begin{aligned}
ds^2 = \frac{1}{2}\mathrm{tr}[e^2] &= \frac{1}{8}[-2(e^{2\rho}l_D - e^{-2\rho}\mathcal{W})^2\mathrm{tr}[W_2W_{-2}] + 2(e^{\rho}l_P - e^{-\rho}\mathcal{L})^2\mathrm{tr}[L_1L_{-1}]]dt^2 \\
&+ \frac{1}{8}[2((e^{2\rho}l_D + e^{-2\rho}\mathcal{W})^2\mathrm{tr}[W_2W_{-2}] - 2(e^{\rho}l_P + e^{-\rho}\mathcal{L})^2 + 4(Q+\Phi)^2\mathrm{tr}[W_0^2]]d\phi^2 \\
&+ \frac{1}{2}\mathrm{tr}[L_0^2]d\rho \\
&= -\left[4l_D^2\left(e^{2\rho} - e^{-2\rho}\frac{\mathcal{W}}{l_D}\right)^2 + l_P^2\left(e^{\rho} - e^{-\rho}\frac{\mathcal{L}}{l_P}\right)^2\right]dt^2 + d\rho^2 \\
&+ \left[4l_D^2\left(e^{2\rho} + e^{-2\rho}\frac{\mathcal{W}}{l_D}\right)^2 + l_P^2\left(e^{\rho} + e^{-\rho}\frac{\mathcal{L}}{l_P}\right)^2 + \frac{4}{3}(Q+\Phi)^2\right]d\phi^2 \ .
\end{aligned}
$$

Finally, by invoking the relations (J.29), we arrive at

$$
\begin{aligned}
ds^2 = &-\left[4l_D^2\left(e^{2\rho} - e^{-2\rho}\frac{\mathcal{L}^2}{l_P^2}\right)^2 + l_P^2\left(e^{\rho} - e^{-\rho}\frac{\mathcal{L}}{l_P}\right)^2\right]dt^2 + d\rho^2 \\
&+ \left[4l_D^2\left(e^{2\rho} + e^{-2\rho}\frac{\mathcal{L}^2}{l_P^2}\right)^2 + l_P^2\left(e^{\rho} + e^{-\rho}\frac{\mathcal{L}}{l_P}\right)^2 + \frac{4}{3}(Q+\Phi)^2\right]d\phi^2 \ . \qquad \text{(J.34)}
\end{aligned}
$$

We continue by imposing the trivial holonomy constraint to further restrict the parameters of our black hole solution. This is done by solving (8.8). We restate these equations here for convinience:

$$
\det(a_t) = 0 \ , \qquad \frac{1}{2}\beta^2\mathrm{tr}[a_t^2] = 1 \ , \qquad \text{(J.35)}
$$

where a_t is the part of a, given by (J.18), proportional to dt. Since $x^{\pm} = t \pm \phi$, we have

$$
a_t = l_D W_2 + \mathcal{W}W_{-2} - QW_0 + l_P L_1 - \mathcal{L}L_{-1} + \Phi W_0 \ ,
$$

To calculate the determinant of a_t and the trace of a_t^2 we need to calculate the matrix representations of a_t and a_t^2 by using the matrix representations of the generators L_i and W_i given in Appendix E.5.2. Actually, we only need the diagonal elements of the matrix a_t^2 to calculate its trace. We find

$$
a_t = \begin{pmatrix} \frac{2}{3}(\Phi - Q) & 2\mathcal{L} & 8\mathcal{W} \\ l_P & -\frac{4}{3}(\Phi - Q) & 2\mathcal{L} \\ 2l_D & l_P & \frac{2}{3}(\Phi - Q) \end{pmatrix} \ , \qquad \text{(J.36)}
$$

and

$$
a_t^2 = \begin{pmatrix} \frac{4}{9}(\Phi - Q)^2 + 2l_P\mathcal{L} + 16l_D\mathcal{W} & - & - \\ - & \frac{16}{9}(\Phi - Q)^2 + 4l_P\mathcal{L} & - \\ - & - & \frac{4}{9}(\Phi - Q)^2 + 2l_P\mathcal{L} + 16l_D\mathcal{W} \end{pmatrix} \ . \qquad \text{(J.37)}
$$

The determinant of a_t is

$$
\det(a_t) = \frac{16}{27}(Q - \Phi)^3 + \frac{8}{3}(l_P\mathcal{L} - 8l_D\mathcal{W})(Q - \Phi) + 16l_D\mathcal{L}^2 \ ,
$$

where we have used (J.29) to simplify the expression. The trace of a_t^2 is simply the sum of the diagonal elements of (J.37):

$$
\mathrm{tr}[a_t^2] = \frac{8}{3}(Q - \Phi)^2 + 8l_P\mathcal{L} + 32l_D\mathcal{W} \ .
$$

We can now restate the trivial holonomy constraint (J.35) as two algebraic equations:

$$
\begin{aligned}
&\frac{4}{27}(Q - \Phi)^3 + \frac{2}{3}(l_P\mathcal{L} - 8l_D\mathcal{W})(Q - \Phi) + 4l_D\mathcal{L}^2 = 0 \ , \\
&\frac{4}{3}\beta^2(Q - \Phi)^2 + 4\beta^2(l_P\mathcal{L} + 4l_D\mathcal{W}) \ . \qquad \text{(J.38)}
\end{aligned}
$$

Using (J.29) and solving the first of the above equations above for $Q - \Phi$ (preferably by the aid of a computer) we find

$$Q - \Phi = -6\frac{\mathcal{W} l_P}{\mathcal{L}} \ , \tag{J.39}$$

and hence (by (J.29))

$$\Phi = 8\frac{\mathcal{W} l_P}{\mathcal{L}} \ . \tag{J.40}$$

Substituting (J.39) in the second equation of (J.38), we can solve for β^2. After some algebraic manipulations, we arrive at

$$\beta^2 = \frac{\mathcal{L}^2}{4 l_P}(16\mathcal{W}^2 l_P + \mathcal{L}^3)^{-1} \ . \tag{J.41}$$

The trivial holonomy constraint is seen to restrict the parameters of the connection by (J.40) and specifying the temperature potential β by (J.41).

Next, we calculate the holonomy invariants around the spatial cycle given by

$$\Theta_{0,A} = 2\pi \det(a_\phi) \ , \qquad \Theta_{1,A} = 2\pi^2 \mathrm{tr}[a_\phi^2] \ , \tag{J.42}$$

and

$$\Theta_{0,\bar{A}} = 2\pi \det(\bar{a}_\phi) \ , \qquad \Theta_{1,\bar{A}} = 2\pi^2 \mathrm{tr}[\bar{a}_\phi^2] \ , \tag{J.43}$$

where a_ϕ and $\bar{a}_\phi$ are the parts of a and $\bar{a}$ proportional to $d\phi$, respectively. It follows from (J.18) and (J.19) that

$$a_\phi = l_D W_2 + \mathcal{W} W_{-2} - (Q+\Phi) W_0 - l_P L_1 + \mathcal{L} L_{-1} \ ,$$

and

$$\bar{a}_\phi = -l_D W_{-2} - \mathcal{W} W_2 + (Q+\Phi) W_0 - l_P L_{-1} + \mathcal{L} L_1 \ .$$

Using the explicit matrix representations of the generators L_i and W_i given in Appendix E.5.2, we evaluate the matrix representations of a_ϕ and $\bar{a}_\phi$:

$$a_\phi = \begin{pmatrix} -\frac{2}{3}(Q+\Phi) & -2\mathcal{L} & 8\mathcal{W} \\ -l_P & \frac{4}{3}(Q+\Phi) & -2\mathcal{L} \\ 2l_D & -l_P & -\frac{2}{3}(Q+\Phi) \end{pmatrix} \ ,$$

$$\bar{a}_\phi = \begin{pmatrix} \frac{2}{3}(Q+\Phi) & 2l_P & -8l_D \\ \mathcal{L} & -\frac{4}{3}(Q+\Phi) & 2l_P \\ -2\mathcal{W} & \mathcal{L} & \frac{2}{3}(Q+\Phi) \end{pmatrix} \ . \tag{J.44}$$

Calculating the determinant of these matrices and multiplying by 2π, we find

$$\Theta_{0,A} = -\Theta_{0,\bar{A}} = \frac{32\pi}{27}(Q+\Phi)^3 + \frac{16\pi}{3}(Q+\Phi)(l_P\mathcal{L} - 8 l_D \mathcal{W}) + 32\pi\mathcal{L}^2 l_P \ , \tag{J.45}$$

Again, for the other holonomy invariants, it suffices to calculate only the diagonal elements of a_ϕ^2 and $\bar{a}_\phi^2$ since we are only interested in their traces. Doing so, and then summing these respective diagonal elements and multiplying by $4\pi^2$, we find

$$\Theta_{1,A} = \Theta_{1,\bar{A}} = \frac{48\pi^2}{9}(Q+\Phi)^2 + 16\pi^2(l_P\mathcal{L} + 4 l_D\mathcal{W}) \ . \tag{J.46}$$

Finally, we calculate the periodicity of the Euclidean time that assures regularity at the horizon.

By performing a Wick rotation, $t \to i t_E$, where t_E is the Euclidean time, we express our black hole metric (J.34) in Euclidean geometry:

$$\begin{aligned} ds^2 = &\left[4 l_D^2 \left(e^{2\rho} - e^{-2\rho}\frac{\mathcal{L}^2}{l_P^2}\right)^2 + l_P^2\left(e^{\rho} - e^{-\rho}\frac{\mathcal{L}}{l_P}\right)^2\right] dt_E^2 + d\rho^2 \\ &+ \left[4 l_D^2 \left(e^{2\rho} + e^{-2\rho}\frac{\mathcal{L}^2}{l_P^2}\right)^2 + l_P^2\left(e^{\rho} + e^{-\rho}\frac{\mathcal{L}}{l_P}\right)^2 + \frac{4}{3}(Q+\Phi)^2\right] d\phi^2 \ . \end{aligned} \tag{J.47}$$

We consider a Taylor expansion to second order of the metric near the horizon and determine the periodicity of the Euclidean time that assures regularity at the horizon in this quadratic approximation. We introduce a new radial coordinate

$$r = \frac{e^{\rho} - e^{\rho_h}}{e^{\rho_h}} \ ,$$

which is zero at the horizon and in units of the radial distance of the horizon. It follows that

$$e^{\rho} = e^{\rho_h}(1+r) \ , \qquad d\rho = \frac{1}{1+r} dr \ ,$$

and substituting this in the metric (J.47) above, yields

$$\begin{aligned} ds^2 = &\left[4l_D^2 \left(e^{2\rho_h}(1+r)^2 - e^{-2\rho_h}(1+r)^{-2}\frac{\mathcal{L}^2}{l_P^2}\right)^2 + l_P^2 \left(e^{\rho_h}(1+r) - e^{-\rho_h}(1+r)^{-1}\frac{\mathcal{L}}{l_P}\right)^2\right] dt_E^2 + \frac{1}{(1+r)^2}dr^2 \\ &+ \left[4l_D^2 \left(e^{2\rho_h}(1+r)^2 + e^{-2\rho_h}(1+r)^{-2}\frac{\mathcal{L}^2}{l_P^2}\right)^2 + l_P^2 \left(e^{\rho_h}(1+r) + e^{-\rho_h}(1+r)^{-1}\frac{\mathcal{L}}{l_P}\right)^2 + \frac{4}{3}(Q+\Phi)^2\right] d\phi^2 \ . \end{aligned} \tag{J.48}$$

The horizon is located at $e^{\rho_h} = \sqrt{\frac{\mathcal{L}}{l_P}}$ (see (8.13)). Using this and expanding near $r = 0$, neglecting terms of cubic or higher order in r, yields

$$\begin{aligned} ds^2 &= \left[4l_D^2 \left(e^{2\rho_h}(1+2r) - e^{2\rho_h}(1-2r)\right)^2 + l_P^2 \left(e^{\rho_h}(1+r) - e^{\rho_h}(1-r)\right)^2\right] dt_E^2 + \frac{1}{(1+r)^2}dr^2 \\ &\quad + \left[4l_D^2 \left(e^{2\rho_h}(1+2r) + e^{2\rho_h}(1-2r)\right)^2 + l_P^2 \left(e^{\rho_h}(1+r) + e^{\rho_h}(1-r)\right)^2 + \frac{4}{3}(Q+\Phi)^2\right] d\phi^2 \\ &= \left[64 l_D^2 e^{4\rho_h} r^2 + 4 l_P^2 e^{2\rho_h} r^2\right] dt_E^2 + \frac{1}{(1+r)^2}dr^2 + \left[16 l_D^2 e^{4\rho_h} + 4 l_P^2 e^{2\rho_h} + \frac{4}{3}(Q+\Phi)^2\right] d\phi^2 \\ &= \left[64 l_D \mathcal{W} + 4 l_P \mathcal{L}\right] r^2 dt_E^2 + \frac{1}{(1+r)^2}dr^2 + \left[16 l_D \mathcal{W} + 4 l_P \mathcal{L} + \frac{4}{3}(Q+\Phi)^2\right] d\phi^2 \ , \end{aligned} \tag{J.49}$$

where we in the last step also used (J.29) to simplify the expression. To remove the conical singularity at the horizon we must require t_E to be periodic with period

$$\frac{2\pi}{\sqrt{64 l_D \mathcal{W} + 4 l_P \mathcal{L}}} \ .$$

This shows that the trivial holonomy constraint is consistent with a regular Euclidean horizon.

J.4.2 Calculations of a spin-3 black hole in a wormhole gauge

Our starting point here are the connections A and $\bar{A}$ of the form (J.17) with $b = e^{\rho L_0}$,

$$a = (L_1 - l_0 L_{-1} - w_0 W_{-2})dx^+ + (\mu W_2 + \alpha L_{-1} + \beta W_{-2} - \gamma W_0)dx^- \ ,$$

and

$$\bar{a} = -(L_{-1} - l_0 L_1 + w_0 W_2)dx^- + (\mu W_{-2} - \alpha L_1 + \beta W_2 - \gamma W_0)dx^+ \ .$$

Here l_0, w_0, μ, α, β and γ are parameters, L_i and W_i are the usual generators of $SL(3)$ and the coordinates are (t,ρ,ϕ) with $x^{\pm} = t \pm \phi$. Using (J.22) we can write

$$A = (e^{\rho}L_1 - e^{-\rho}l_0 L_{-1} - e^{-2\rho}w_0 W_{-2})dx^+ + (e^{2\rho}\mu W_2 + e^{-\rho}\alpha L_{-1} + e^{-2\rho}\beta W_{-2} - \gamma W_0)dx^- + L_0 d\rho \ , \tag{J.50}$$

and

$$\bar{A} = -(e^{\rho}L_{-1} - e^{-\rho}l_0 L_1 + e^{-2\rho}w_0 W_2)dx^- + (e^{2\rho}\mu W_{-2} - e^{-\rho}\alpha L_1 + e^{-2\rho}\beta W_2 - \gamma W_0)dx^+ - L_0 d\rho \ . \tag{J.51}$$

The connections should satisfy the Chern-Simons equations of motion $\mathrm{d}A + A \wedge A = 0$ (and the same equation for $\bar{A}$) in order to be a solution of our spin-3 gravity theory. We solve these equations for A explicitly. The equations for $\bar{A}$ is solved analogously and the resulting restrictions on the parameters are identical to those imposed by the equations of motion for A.

We start by calculating the exterior derivative of A:

$$\begin{aligned}
\mathrm{d}A &= (\partial_t dt + \partial_\rho d\rho + \partial_\phi d\phi) \wedge A \\
&= \partial_\rho d\rho \wedge [(e^\rho L_1 - e^{-\rho} l_0 L_{-1} - e^{-2\rho} w_0 W_{-2})dx^+ \\
&\quad + (e^{2\rho}\mu W_2 + e^{-\rho}\alpha L_{-1} + e^{-2\rho}\beta W_{-2} - \gamma W_0)dx^- + L_0 d\rho] \\
&= (e^\rho L_1 + e^{-\rho} l_0 L_{-1} + 2e^{-2\rho} w_0 W_{-2}) d\rho \wedge dx^+ \\
&\quad + (2e^{2\rho}\mu W_2 - e^{-\rho}\alpha L_{-1} - 2e^{-2\rho}\beta W_{-2}) d\rho \wedge dx^- \ . \qquad \text{(J.52)}
\end{aligned}$$

To evaluate the second term $A \wedge A$, we once again make use of the shorthand notation $A = A_+ dx^+ + A_- dx^- + L_0 d\rho$ so that we can write

$$A \wedge A = [A_+, A_-] dx^+ \wedge dx^- + [L_0, A_+] d\rho \wedge dx^+ + [L_0, A_-] d\rho \wedge dx^- \ ,$$

just as we did in (J.28). Making use of the commutation relations of the generators L_i and W_i stated in Appendix E.5.2, we find

$$[A_+, A_{-1}] = 2(\alpha - 8\mu w_0)L_0 + 2e^\rho(2\mu l_0 - \gamma)W_1 + 2e^{-\rho}(2\beta - l_0\gamma)W_{-1} \ , \qquad \text{(J.53)}$$

and

$$[L_0, A_+] d\rho \wedge dx^+ + [L_0, A_-] d\rho \wedge dx^- = -\mathrm{d}A \qquad \text{(J.54)}$$

with $\mathrm{d}A$ given by (J.52). Thus, the equations of motion reads

$$\mathrm{d}A + A \wedge A = [A_+, A_-] dx^+ \wedge dx^- = 0 \ ,$$

and for this equation to hold we must have $[A_+, A_-] = 0$. The coefficient of each generator of (J.53) must then equal to zero. We can formulate this condition as a system of three equations in three unknowns α, β and γ:

$$\begin{cases} \alpha - 8\mu w_0 = 0 \\ 2\mu l_0 - \gamma = 0 \\ 2\beta - l_0 \gamma \end{cases}$$

It is straightforward to solve this system to get

$$\begin{cases} \alpha = 8\mu w_0 \\ \beta = \mu l_0^2 \\ \gamma = 2\mu l_0 \end{cases} \qquad \text{(J.55)}$$

Substituting (J.55) into our connections (J.56) and (J.57) we get

$$\begin{aligned}
A &= (e^\rho L_1 - e^{-\rho} l_0 L_{-1} - e^{-2\rho} w_0 W_{-2}) dx^+ \\
&\quad + \mu(e^{2\rho} W_2 + 8w_0 e^{-\rho} L_{-1} + l_0^2 e^{-2\rho} W_{-2} - 2l_0 W_0) dx^- + L_0 d\rho \ , \qquad \text{(J.56)}
\end{aligned}$$

and

$$\begin{aligned}
\bar{A} &= -(e^\rho L_{-1} - e^{-\rho} l_0 L_1 + e^{-2\rho} w_0 W_2) dx^- \\
&\quad + \mu(e^{2\rho} W_{-2} - 8w_0 e^{-\rho} L_1 + l_0^2 e^{-2\rho} W_2 - 2l_0 W_0) dx^+ - L_0 d\rho \ . \qquad \text{(J.57)}
\end{aligned}$$

These connections are a proper solution of our spin-3 gravity theory, and we continue by calculating the corresponding metric.

The vielbein is given in terms of the connections by (J.31). Inserting our connections (J.56) and (J.57) in this formula yields

$$\begin{aligned}
e &= \frac{1}{2}(A - \bar{A}) \\
&= \frac{1}{2}[(e^\rho + (8\mu w_0 - l_0)e^{-\rho})(L_1 + L_{-1}) + (\mu e^{2\rho} + (w_0 - \mu l_0^2)e^{-2\rho})(W_2 - W_{-2})]dt + L_0 d\rho \\
&\quad + \frac{1}{2}[(e^\rho + (8\mu w_0 + l_0)e^{-\rho})(L_1 - L_{-1}) - (\mu e^{2\rho} + (w_0 + \mu l_0^2)e^{-2\rho})(W_2 + W_{-2}) + 4l_0 w_0]d\phi \ , \qquad \text{(J.58)}
\end{aligned}$$

and the spacetime interval is given in terms of this vielbein by $ds^2 = \frac{1}{2}\mathrm{tr}[e^2]$. Most of the terms of $\mathrm{tr}(e^2)$ vanishes since most products of two $sl(3)$ matrices L_i and W_i are traceless. The only five non-zero traces are given by (J.33), and we find that

$$\begin{aligned} ds^2 = &-\left([e^\rho + (8\mu w_0 - l_0)e^{-\rho}]^2 + 4[\mu e^{2\rho} + (w_0 - \mu l_0^2)e^{-2\rho}]^2\right) dt^2 + d\rho^2 \\ &+ \left([e^\rho + (8\mu w_0 + l_0)e^{-\rho}]^2 + 4[\mu e^{2\rho} + (w_0 + \mu l_0^2)e^{-2\rho}]^2 + \frac{16}{3} l_0^2\right) d\phi^2 \ . \end{aligned} \tag{J.59}$$

Appendix K

Hamiltonian and Energy in Chern-Simons Theory

In this section we give further evidence to why we can associate the constant M in the BTZ metric, equation (5.18), with the energy of a black hole. We will do so by investigating the Hamiltonian of a Chern-Simons Lagrangian. Our discussion will be based on a paper by M. Bañados and I. Reyes where a more detailed discussion can be found, see [48]. The Hamiltonian, $H(q,\dot{q})$ (q denoting a generalized coordinate), can be seen as the total energy of a system. It is related to the Lagrangian through a *Legendre transform*:

$$\mathcal{L} = p_i \dot{q}^i - H(q,\dot{q}),$$

where $p_i \equiv \frac{\partial \mathcal{L}}{\partial \dot{q}^i}$ is the *conjugate momentum* and q^i our fields or generalized coordinates. We have also used the notation $\dot{q} = \frac{d}{dt}q$. The action will then take the following form

$$\mathcal{S} = \int d^3x (p_i \dot{q}^i - H(p,q) + \lambda^a \phi_a). \tag{K.1}$$

The λ^a introduced here are Lagrange multipliers and ϕ_a are constraints. For the reader's convenience we restate the Chern-Simons action as

$$\mathcal{S}_{CS} = \frac{k}{4\pi} \int \mathrm{tr}[A \wedge \mathrm{d}A + \frac{2}{3} A \wedge A \wedge A] \ .$$

Let us now try to manipulate the Chern-Simons action so that we may simply read off the Hamiltonian. First we "split" the connection according to $A_\mu = (A_0, A_i)$ where $i \in (1,2)$. We start by manipulating the first term

$$\begin{aligned} A \wedge \mathrm{d}A &= (A_0 dt + A_i dx^i) \wedge \mathrm{d}(A_0 dt + A_j dx^j) \\ &= A_0 dt \wedge \mathrm{d}A_j dx^j + A_i dx^i \wedge \mathrm{d}A_0 dt + A_i dx^i \wedge \mathrm{d}A_j dx^j \\ &= A_0 dt \wedge (\mathrm{d}A_j dx^j - \mathrm{d}A_i dx^i) + \underbrace{\mathrm{d}(A_i dx^i \wedge A_0 dt)}_{0} + A_i dx^i \wedge \mathrm{d}A_j dx^j \\ &= \epsilon^{ij}(A_0(\partial_i A_j - \partial_j A_i) - A_i \dot{A}_j) d^3x \ , \end{aligned}$$

where we have discarded a surface term as indicated. Proceeding with the second term and making the trace explicit

$$\begin{aligned} \mathrm{tr}[\frac{2}{3} A \wedge A \wedge A] &= \frac{2}{3} \mathrm{tr}[A_0 A_i A_j - A_i A_0 A_j + A_i A_j A_0] dt \wedge dx^i \wedge dx^j && \text{(K.2)} \\ &= \frac{2}{3} \mathrm{tr}[2A_0 A_i A_j - A_0 A_j A_i] dt \wedge dx^i \wedge dx^j && \text{(K.3)} \\ &= 2\,\mathrm{tr}[A_0 A_i A_j] dt \wedge dx^i \wedge dx^j && \text{(K.4)} \\ &= 2\epsilon^{ij}\,\mathrm{tr}[A_0 A_i A_j] d^3x \ . && \text{(K.5)} \end{aligned}$$

Notice that the relative minus sign in the first step is due to the wedge product. We use the cyclicity of the trace in the second step and in the third step we once again make use of the anti-symmetry of the wedge product. If we now use the trick of writing $A_i A_j = \frac{1}{2}[A_i, A_j]$, remember that A is matrix-valued, we find

$$\begin{aligned} \mathrm{tr}[A \wedge \mathrm{d}A + A \wedge A \wedge A] &= \epsilon^{ij}\,\mathrm{tr}[(A_0(\partial_i A_j - \partial_j A_i + [A_i, A_j]) - A_i \dot{A}_j] d^3x && \text{(K.6)} \\ &= \epsilon^{ij}\,\mathrm{tr}[A_0 F_{ij} - A_i \dot{A}_j] d^3x && \text{(K.7)} \\ &= \frac{1}{2} \eta^{ab} \epsilon^{ij} (A_0^a F_{ij}^b - A_i \dot{A}_j) d^3x \ . && \text{(K.8)} \end{aligned}$$

In the last step we used, as introduced in the previous section that $\mathrm{tr}[A_i A_j] = \mathrm{tr}[A^a_i S^a A^b_j S^b] = \frac{1}{2}\eta^{ab} A^a_i A^b_j$. The Chern-Simons action thus takes the following form

$$\mathcal{S}_{CS} = \frac{k}{8\pi}\int d^3x \epsilon^{ij}\eta_{ab}[-A^a{}_i \dot{A}^b{}_j + A^a{}_0 F^b{}_{ij}] \ . \tag{K.9}$$

Comparing with (K.1) we may now identify $A^a_0 = \lambda^a$ as a Lagrange multiplier with a constraint $\phi^b = F^b_{ij} = 0$. We also notice that we have $H \equiv 0$, does this mean that we have zero energy? No, the answer is that our action is incomplete, we are missing a boundary term. Remember that the e.o.m for a Chern-Simons theory is $F = 0$. We must now check that our action truly is stationary when we satisfy our e.o.m. To derive our equations of motion we vary the action with respect to all connections (independently) and find

$$\begin{aligned}
\delta\mathcal{S}_{CS} &= \frac{k}{8\pi}\int d^3x \epsilon^{ij}\eta_{ab}(-\delta A^a_i \dot{A}^b_j - A^a_i \frac{d}{dt}\delta A^b_j + \delta A^a_0 F^b{}_{ij} + A^a_0 \delta F^b{}_{ij}) \\
&= \frac{k}{8\pi}\int d^3x \epsilon^{ij}\eta_{ab}(-\delta A^a_i \dot{A}^b_j + \dot{A}^a_i \delta A^b_j + \delta A^a_0 F^b{}_{ij} + A^a_0 \delta F^b{}_{ij} + \underbrace{\frac{d}{dt}(A_i \delta A_j))}_{0} \\
&= \frac{k}{8\pi}\int d^3x \epsilon^{ij}\eta_{ab}(2\dot{A}^a_i \delta A^b_j + \delta A^a_0 F^b{}_{ij} + A^a_0 \delta F^b{}_{ij}) \ .
\end{aligned} \tag{K.10}$$

We can modify the term $\epsilon^{ij} A^a_0 \delta F^b_{ij}$ into a more convenient form.

$$\epsilon^{ij} A^a_0 \delta F^b_{ij} = \epsilon^{ij} A^a_0 \Big(\partial_i \delta A_j - \partial_j \delta A_i\Big) = 2\epsilon^{ij} A^a_0 \partial_i \delta A_j = 2\epsilon^{ij}\Big(\partial_i (A^a_0 \delta A_j) - \delta A_j \partial_i (A_0)\Big) \ ,$$

and thus we can see that

$$\epsilon^{ij}(2\dot{A}^a_i \delta A^b_j + A^a_0 \delta F^b_{ij}) = 2\epsilon^{ij}\Big(\partial_i (A^a_0 \delta A_j) + F^b_{0i}\delta A_j\Big) \ .$$

Using this we may restate (K.10) as

$$\delta\mathcal{S}_{CS} = \int \mathrm{e.o.m} + \underbrace{\frac{k}{4\pi}\int d^3x \epsilon^{ij}\partial_i \,\mathrm{tr}[A_0 \delta A_j]}_{\delta E} \ .$$

This term spoils our theory since our equation of motions will not be stationary points of the action. Fortunately we can easily deal with this problem by redefining our original H according to $H' = H + E$. The action is thus written as

$$\mathcal{S}_{CS} = \frac{k}{8\pi}\int d^3x \epsilon^{ij}\,\mathrm{tr}[-A_i \dot{A}_j + A_0 F_{ij}] - E \ .$$

The reason for this negative sign can be seen to arise naturally from (K.1). With this modification our action is well defined. The name E is certainly no coincidence. Since the old Hamiltonian simply was a constraint $\epsilon^{ij}\,\mathrm{tr}[A_0 F_{ij}]$ we will interpret this new term as the energy. Before we proceed we also note that the expression for δE may be simplified if we use Stoke's theorem, then

$$\delta E = \frac{k}{4\pi}\int d^3x \epsilon^{ij}\partial_i \,\mathrm{tr}[A_0 \delta A_j] = \frac{k}{4\pi}\int dt \int_{\rho\to\infty} d\phi \,\mathrm{tr}[A_0 \delta A_\phi] \ ,$$

where ρ is a radial variable and ϕ is an angular variable.

Let us now turn to the specific case of the BTZ black hole. For the BTZ black hole we have the connections

$$\begin{aligned}
A &= (e^\rho L_1 - e^{-\rho}\mathcal{L}L_{-1})(dt + d\phi) + L_0 d\rho, \\
\overline{A} &= (-e^\rho L_{-1} + \mathcal{L}e^{-\rho}L_1)(dt - d\phi) - L_0 d\rho \ ,
\end{aligned}$$

as introduced in the chapter on higher spin 7. In order to evaluate our boundary term and the energy we would like to find a way of relating A_ϕ andA_0 when $r \to \infty$. It is easy to see that we have $A_\phi = A_0$ and

$\overline{A}_\phi = -\overline{A}_0$. We call this a *chiral symmetry*. Using this fact we may rewrite the variation of the energy term according to

$$\delta E = \frac{k}{4\pi}\int dt \int_{r\to\infty} d\phi \,\mathrm{tr}[A_0 \delta A_\phi] = \delta \frac{k}{4\pi}\int dt \int_{r\to\infty} d\phi \,\mathrm{tr}\left[\frac{A_\phi^2}{2}\right] ,$$

and we can see that

$$E[A] = \frac{k}{4\pi}\int dt \int_{r\to\infty} d\phi \,\mathrm{tr}\left[\frac{A_\phi^2}{2}\right] .$$

The trace can be evaluated using the explicit matrix representation for L_1 and L_{-1} found in appendix E.5.2. The result is

$$\begin{aligned}\frac{1}{2}\,\mathrm{tr}[A_\phi^2] &= \frac{1}{2}\,\mathrm{tr}[(e^\rho L_1 - \mathcal{L}e^{-\rho}L_{-1})^2] = 4\mathcal{L} ,\\ \frac{1}{2}\,\mathrm{tr}[\overline{A}_\phi^2] &= \frac{1}{2}\,\mathrm{tr}[(e^\rho L_{-1} - \mathcal{L}e^{-\rho}L_1)^2] = 4\mathcal{L} .\end{aligned}$$

There is an analogous expression for $\overline{A}$, the only difference is a sign. It is now time to evaluate the energy of our BTZ black hole. It is important to remember that our action, as described in the previous section, is the difference between two Chern-Simons actions. The trace can be evaluated using the explicit matrix representation for L_1 and L_{-1} found in appendix E.5.2. So our total energy is

$$E = E[A] - E[\overline{A}] = \frac{k}{4\pi}\int dt \int_{r\to\infty} d\phi \Big(\mathrm{tr}\left[\frac{A_\phi^2}{2}\right] + \mathrm{tr}\left[\frac{A_\phi^2}{2}\right]\Big) = \frac{2k}{\pi}\int dt\mathcal{L} \propto k\mathcal{L} ,$$

It is possible to evaluate the integral by going to Euclidean time, but we will be satisfied with the knowledge that it must be proportional to $k\mathcal{L}$. We can now see that the energy depends on $k\mathcal{L}$ and since k is proportional to $\frac{1}{G}$ it is natural to assume that $\mathcal{L} \propto GM$ where M is the mass of the BTZ black hole.

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