Quantitative Imaging Technique Using the Layer-Stripping Algorithm

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Abstract. We present the layer-stripping algorithm for the solution of the hyperbolic coefficient inverse problem (CIP). Our numerical examples show quantitative reconstruction of small tumor-like inclusions in two-dimensions.

Introduction

In this work we present the layer-stripping algorithm applied for explicit reconstruction of the coefficient in the hyperbolic equation using data resulted from a single measurement. Our algorithm is based on the approximate globally convergent method of [4]. This method was verified on computationally simulated and on experimental data in [5, 6, 8] and references therein.

For the numerical discretization of an approximate globally convergent method we use the finite element method (FEM) [7]. The goal of our numerical simulations is to obtain quantitative images of small inclusions representing cancerous tumors. Thus, in our simulations we are interested in the accurate reconstruction of the location and the contrast of tumor-like inclusions. Examples of CIPs with applications in medicine are inverse problems of magnetic resonance elastography (MRE) which are studied recently in [2, 9] and references therein. The main feature of this medical imaging technique is that it allows perform measurements internally. In our numerical examples we use internal measurements and show very accurate and quantitative reconstruction of tumor-like inclusions which can be even of the very small sizes (for reconstruction of point-size inclusions see [7]). In the future work we plan to extend the layer-stripping algorithm of this paper to the case of CIPs with boundary measurements. Similarly with [1] an adaptive finite element method can be also considered as a topic for a future research.

Statements of Forward and Inverse Problems

We consider the Cauchy problem for the hyperbolic equation

$$a(x)u_{tt} = \Delta u \text{ in } \mathbb{R}^3 \times (0, \infty), \quad u(x, 0) = 0, \ u_t(x, 0) = \delta(x - x_0), \tag{1}$$

where δ is the Dirac delta function. In the case of the acoustic wave equation, $c(x) = 1/\sqrt{a(x)}$ is the sound speed. In the electromagnetic wave propagation with a field polarization in a non-magnetic medium, the function $a(x) = \varepsilon(x)$, where $\varepsilon(x)$ is the spatially distributed dielectric permittivity function. When

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the equation (1) is applied in the medical imaging with acoustic waves, the sound speed is defined as $c(x) = \sqrt{(\lambda(x) + 2\mu(x))/\rho(x)}$, where $\rho(x)$ is the density and $\lambda(x), \mu(x)$ are the Lamé constants of linear elasticity [3].

Let $\Omega \subset \mathbb{R}^3$ be a convex bounded domain with the boundary $\partial\Omega \in C^2$. We assume that the function $a(x)$ of equation (1) is such that

$$a(x) \in [1, M], \quad a(x) = 1 \text{ for } x \in \mathbb{R}^3 \diagdown \Omega, \tag{2}$$

where $M = const. > 1$ is *a priori* known constant.

Coefficient Inverse Problem (CIP). *Suppose that the coefficient $a(x)$ satisfies (2). Assume that the function $a(x)$ is unknown in the domain Ω. Determine the function $a(x)$ for $x \in \Omega$, by knowing the function $g(x, t)$ for a single source point $x_0 \notin \overline{\Omega}$*

$$u(x, t) = g(x, t) \quad \forall (x, t) \in \partial\Omega \times (0, \infty). \tag{3}$$

The Transformation Procedure for the Hyperbolic Case

In this section we present the main steps in the derivation of the approximate globally convergent method of [4]. First, we take the Laplace transform of the functions u in the hyperbolic equation (1):

$$w(x, s) = \int_0^\infty u(x, t) \mathrm{e}^{-st} dt \quad \text{for } s > \underline{s} = const. > 0, \tag{4}$$

where $\underline{s}$ is a certain sufficiently large number, which we choose in experiments. The parameter s is called *pseudo frequency*. It follows from (1) and (4) that the function w is the solution of the following problem

$$\Delta w - s^2 a(x) w = -\delta(x - x_0), \ x \in \mathbb{R}^3, \tag{5}$$

$$\lim_{|x| \to \infty} w(x, s) = 0, \tag{6}$$

where the limit in (6) is proven in [4]. In Theorem 2.7.2 of [4] was shown that $w(x, s) > 0$. Hence, we can consider functions $v(x, s)$ defined as

$$v(x, s) = \frac{\ln w(x, s)}{s^2}. \tag{7}$$

Assuming that the asymptotic behavior in Lemma 2.3 of [4] holds, substituting $w = \mathrm{e}^v$ in (5) and noting that the source point $x_0 \notin \overline{\Omega}$, we obtain

$$\Delta v + s^2 (\nabla v)^2 = a(x), \quad x \in \Omega. \tag{8}$$

Denote

$$q(x, s) = \partial_s v(x, s). \tag{9}$$

Using the asymptotic behavior in Lemma 2.3 of [4] and (9) we get

$$v(x, s) = -\int_s^\infty q(x, \tau) d\tau,$$

which can be rewritten as

$$v(x,s) = -\int_{s}^{\overline{s}} q(x,\tau)\,d\tau + V(x,\overline{s}), \tag{10}$$

where $V(x,\overline{s})$ is the unknown "tail" function. In [5, 6] we describe how this function can be approximated in computations. Differentiating this equation with respect to s and using (9) and (10), we obtain the following nonlinear integro-differential equation

$$\Delta q - 2s^2 \nabla q \int_{s}^{\overline{s}} \nabla q(x,\tau)\,d\tau + 2s\left[\int_{s}^{\overline{s}} \nabla q(x,\tau)\,d\tau\right]^2 + 2s^2 \nabla q \nabla V - 4s \nabla V \int_{s}^{\overline{s}} \nabla q(x,\tau)\,d\tau + 2s(\nabla V)^2 = 0, \quad x \in \Omega. \tag{11}$$

Assume now that we can solve (11) and find approximations for functions q and V in Ω together with their derivatives $D_x^{\alpha} q, D_x^{\alpha} V, |\alpha| \leq 2$. Then the the function $a(x)$ can be found explicitly via (8).

The Layer Stripping Algorithm

In this section we present the reduced version of the layer stripping algorithm for the solution of the integro-differential equation (11). We refer to [7] for the full details of it. To do that we make partition of the pseudo frequency interval $[\underline{s}, \overline{s}]$ into N sub-intervals $\overline{s} = s_0 > s_1 > \cdots > s_N = \underline{s}$ such that

$$\underline{s} = s_N < s_{N-1} < ... < s_1 < s_0 = \overline{s}, s_{i-1} - s_i = h,$$

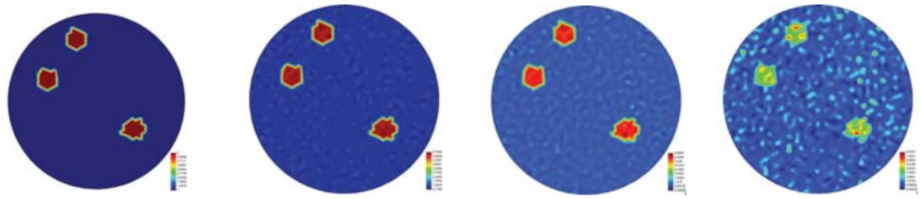
where h is the step size of every interval and $q(x,s) = q_n(x)$ for $s \in (s_{n+1}, s_n], n = 0, ..., N$. To solve (11) on every pseudo-frequency interval $(s_{n+1}, s_n]$, we use the following algorithm:

- Initialization: set $q_0 \equiv 0$ and compute the first tail function V_0 as described in [4, 5].
- For $n = 1, 2, \ldots, N$
  1. Set $q_{n,0} = q_{n-1}$, $V_{n,1} = V_{n-1}$
  2. For $i = 1, 2, \ldots, m_n$
     - Find $q_{n,i}$ by solving (11) on the interval $(s_{n+1}, s_n]$ with $V_n := V_{n,i}$.
     - Compute $v_{n,i} = -hq_{n,i} - h\sum_{j=0}^{n-1} q_j + V_{n,i}$.
     - Compute $a_{n,i}$ via FEM discretization of (8) with $a := a_{n,i}$ and $v := v_{n,i}$. Then solve the forward problem (1) with the new computed coefficient $a := a_{n,i}$, compute $w := w_{n,i}$ and update the tail $V_{n,i+1}$.
  3. Set $q_n = q_{n,m_n}$, $a_n = a_{n,m_n}$, $V_n = V_{n,m_{n+1}}$ and go to the next frequency interval $(s_{n+1}, s_n]$ if $n < N$. If $n = N$, then stop.

The stopping criteria for iterations m_n and n and step 3 in the above algorithm is derived computationally in [8]. The global convergence theorem was proven in [4, 5].

Numerical experiments in 2D

In this section we present the reconstruction of the function $a(x)$ at different values of pseudo-frequency s for the the case when the measured function $g(x,t)$ is known inside the domain of interest. Measuring of the field internally is allowed in some cases of medical imaging: for example, in medical resonance elastic imaging



a) exact function $a(x)$ b) $s = 19$ c) $s = 10$ d) $s = 5$

FIGURE 1. *a) The exact location of tumors. b), c), d) The reconstructed wave speed function $a(x)$ at different values of pseudo frequency s. On b) maximal reconstructed values of this function are 5.15 in tumor-like targets and $a(x) = 1$ outside of imaged targets what corresponds to the background medium. The image is highly accurate: compare with exact image on a) where maximal values of the exact function are 5. Again, on d) we observe that the image is deteriorated at pseudo frequency $s = 5$.*

[2]. For the full details of numerical implementation of the layer-stripping algorithm using FEM as well as for the more numerical results we refer to [7]. Here we present the reconstruction of the function $a(x)$ of Figure 1-a). On Figures 1-b), c) we observe almost perfect reconstruction when pseudo frequency s is taken as $s = 10$ and $s = 19$. Our numerical tests show that on the interval of pseudo frequencies $s = [8; 19]$ we get reconstruction similar to the exact one obtained on Figure 1-b). However, for pseudo-frequencies on the interval $s = [1; 7]$ we obtain reconstructed function $a(x)$ similar to the obtained on Figure 1-d). We observe that the image of Figure 1-d) is deteriorated for this value of pseudo-frequency.

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