Robust Discrete-Time Gain-Scheduled Guaranteed Cost PSD Controller Design

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***Abstract*—The most widely used controllers in industry are still the proportional, integral, and derivative (PID) and discrete-time proportional, summation, and difference (PSD) controllers, thanks to their simplicity and performance characteristics. However, with these conventional fixed gain controllers we could have difficulties to handle nonlinear or time-variant characteristics. The introduction of linear parameter-varying (LPV) systems led to various gain-scheduled controller design techniques in both state-space and frequency domain during the last 30 years. In spite of all these, there is still a lack of general approaches for advanced guaranteed cost PID/PSD controller design approaches for LPV systems. In this paper a new advanced controller design approach for discrete-time gain-scheduled guaranteed cost PSD controller design with input saturation and anti-windup is presented for uncertain LPV systems. In addition, the controller design problem is formulated in such a way, which gives convex dependency regarding the scheduled parameters. It results in a less conservative controller design compared to approaches using quadratic stability or the multiconvexity lemma and it's relaxations. Finally, a numerical example shows the benefits of the proposed approach.**

I. Introduction

It is well known that proportional, integral, and derivative (PID) and discrete-time proportional, summation, and difference (PSD) controllers are extensively used in industry [1]. Furthermore, the robust PID/PSD controller design theory is well established for linear systems [2], but almost all real processes are more or less nonlinear. If the plant's operating region is small, one can use the robust control approaches to design a linear robust PID/PSD controller where the nonlinearities are treated as model uncertainties. However, for nonlinear processes, where the operating region is large, the above mentioned controller synthesis may be inapplicable or provide unreasonable conservative designs with poor performance. For this reason, the PID/PSD controller design for nonlinear systems is nowadays a very active and important field of research.

Gain-scheduling is one of the most commonly used controller design approaches for nonlinear systems and has a wide range of use in industrial applications. Particularly, the introduction of the notion of linear parameter-varying (LPV) systems has accelerated the development [3]. For a more comprehensive survey of the field, readers are also referred to survey papers [4], [5] and [6].

Lyapunov theory and small-gain theorem are the two main (not independent) research directions for testing and synthetizing performance and stability of LPV systems. Convexification in the scheduling parameter dependency of the closed-loop conditions allows to transform the controller design problem to convex optimization problem subject to some finite number linear/bilinear matrix inequalities (LMI/BMI). The approaches based on the small-gain theorem and/or integral quadratic constraints are mainly equivalent with the quadratic stability, and highly depend on the structure of the applied multipliers, therefore may be numerically expensive, respectively conservative [7], [8], [9]. Nonetheless, these approaches have their own benefits highlighted by a significant amount of publications. Along this line, multi-convexification technique can balance conservatism, e.g. as pointed out in [10] within affine quadratic stability (AQS) framework. Furthermore, different relaxation techniques have been deployed to reduce the conservativeness caused by the multi-convexity requirement [11], [12], [13], [14]. Multi-convexity has been differently solved, usually by restricting the closed-loop LPV structure, system or controller to avoid cross term effects of the scheduling parameters [15], [16], [17], [18].

While relatively huge amount of literature is dealing with control of LPV systems, only few papers are devoted to PID/PSD controller design. Furthermore, most of them are based on quadratic stability with an H_∞ norm bound [19], [20]. In order to overcome the gap a new approach is introduced for discrete-time gain-scheduled guaranteed cost PSD controller design for uncertain LPV systems with input saturation and anti-windup.

The mathematical notation of the paper is as follows. Given a symmetric matrix $P = P^T \in \mathbb{R}^{n\times n}$, the inequality $P > 0$ ($P \geq 0$) denotes the positive definiteness (semi definiteness) of the matrix. Matrices, if not explicitly stated, are assumed to have compatible dimensions. I denotes the identity matrix of corresponding dimensions. Notation for interval of numbers between a and b including endpoints a and b is $\langle a, b\rangle = \{x \in \mathbb{R} | a \leq x \leq b\}$. $A \circ B$ denotes the Hadamard (or Schur) product between matrices $A \in \mathbb{R}^{n\times m}$ and $B \in \mathbb{R}^{n\times m}$. $A \,\bar{\circ}\, b$ denotes product defined in *Definition 2* in *Appendix* between the matrix $A \in \mathbb{R}^{n\times m}$ and vector $b \in \mathbb{R}^n$.

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II. PRELIMINARIES AND PROBLEM FORMULATION

The following class of discrete-time linear parameter varying systems is considered throughout the paper:

$$\begin{aligned} x(k+1) &= A(\theta(k))x(k) + B(\theta(k))u(k), \\ y(k) &= Cx(k), \end{aligned} \tag{1}$$

where $x(k) \in \mathbb{R}^n$, $u(k) \in \mathbb{R}^m$ and $y(k) \in \mathbb{R}^l$ are the state, control input and the measured output vectors, respectively. The matrix functions $A(\theta(k)) \in \mathbb{R}^{n\times n}$ and $B(\theta(k)) \in \mathbb{R}^{n\times m}$ are assumed to depend on the scheduling variable $\theta(k) \in \langle \underline{\theta}, \overline{\theta} \rangle \in \Omega$ as (3) with $S(\theta(k)) = \{A(\theta(k)),\ B(\theta(k))\}$. In addition A_0, B_0, A_i, B_i, $i = 1, 2, \ldots, p$ and C are constant matrices with appropriate dimensions.

The scheduling variable used in this paper is extended and distributed to:

$$\theta(k) = [\alpha_1, \ldots, \alpha_{N_\alpha}, \beta_1, \ldots, \beta_{N_\beta}], \tag{2}$$

where, it is assumed that the scheduling parameters $\alpha_i(k)$ $i = 1, 2, \ldots, N_\alpha$ are constant or time-varying and can be measured or estimated and therefore used in the controller, and the scheduling parameters $\beta_j(k)$, $j = 1, 2, \ldots, N_\beta$ are constant or time-varying but unknown (uncertain) parameters.

$$\begin{aligned} S(\theta(k)) &= S_0 + \sum_{i=1}^{N_\alpha} S_i\alpha_i(k) + \sum_{j=1}^{N_\beta} S_{(N_\alpha+j)}\beta_j(k) \\ &= S_0 + \sum_{i=1}^{p} S_i\theta_i(k). \end{aligned} \tag{3}$$

Furthermore, it is assumed that the maximal rate of change of scheduled parameters $\Delta\theta_i(k) \leq \rho_{\theta_i}$ are known and predefined.

The output feedback gain-scheduled PSD control law is defined in this paper as:

$$\begin{aligned} u(k) = G_f(z) \circ \Bigg(& K_P(\theta(k))e_p(k) + E_{EFM}(k) \\ & + K_D(\theta(k))\frac{e_d(k) - e_d(k-1)}{T_s} \Bigg) \circ \lambda(k), \end{aligned} \tag{4}$$

where $E_{EFM}(k)$ denotes the discretized integral term using the Euler's forward method:

$$E_{EFM}(k) = T_s \sum_{i=0}^{k-1} \Big(\big(K_S(\theta(i))e(i)\big) \circ \sigma(i) \Big), \tag{5}$$

furthermore, $e_p(k) = y(k) - c_p \circ w(k)$ is the control error vector for the proportional part, $e(k) = y(k) - w(k)$ is the control error vector for the summation part, $e_d(k) = y(k) - c_d \circ w(k)$ is the control error vector for the difference part, $w(k) \in \mathbb{R}^l$ is the reference signal vector, c_p, $c_d \in \mathbb{R}^l$ are the set-point weighting vectors, $\lambda(k) \in \mathbb{R}^m$ is a vector of known time-varying parameters $\lambda_i \in \langle \underline{\lambda_i}, 1 \rangle \in \Phi$ (which serves to ensure the hard input constraints $|u| \leq u_{max}$), $\sigma(k) \in \mathbb{R}^m$ is a vector of switching parameters $\sigma_i \in \langle 0, 1 \rangle$ for anti-windup, T_s is the sample time, and matrices $K_P(\theta(k))$, $K_S(\theta(k))$, $K_D(\theta(k)) \in \mathbb{R}^{m\times l}$ are controller gain matrices in the form (3) with $S(\theta(k)) = \{K_P(\theta(k)),\ K_S(\theta(k)),\ K_D(\theta(k))\}$.

Note 1. Notice that the controller gain matrices (K_{P_i}, K_{S_i}, K_{D_i}) which are related to $\beta(k)$ are equal to zero. Furthermore, for centralized controller design the gain matrices ($K_P(\theta(k))$, $K_S(\theta(k))$, $K_D(\theta(k))$) are full matrices. For decentralized control the structure of these matrices can be predefined. In the case when $m = l$ a fully decentralized control can be obtained by structuring the gain matrices to diagonal form.

The saturation vector variable $\lambda(k) = [\lambda_1(k), \ldots, \lambda_m(k)]^T$, $\lambda_i \in \langle \underline{\lambda_i}, 1 \rangle$ is guaranteeing the hard input constraints $|u_i| \leq u_{i_{max}}$, $i = 1, \ldots, m$ if it is calculated as:

$$\lambda_i(k) = \begin{cases} 1 & \rightarrow \text{if } |u_{i_s}(k)| <= u_{i_{max}} \\ \frac{u_{i_{max}}}{|u_{i_s}(k)|} & \rightarrow \text{if } |u_{i_s}(k)| > u_{i_{max}} \end{cases}, \quad i = 1, \ldots, m \tag{6}$$

where

$$\begin{aligned} u_s(k) = G_f(z) \circ \Bigg(& K_P(\theta(k))e_p(k) + E_{EFM}(k) \\ & + K_D(\theta(k))\frac{e_d(k) - e_d(k-1)}{T_s} \Bigg). \end{aligned} \tag{7}$$

It is assumed that the maximal rate of change of these saturation parameters are known and predefined $\Delta\lambda_i \leq \rho_{\lambda_i}$. An upper bound can be calculated as $\rho_{\lambda_i} = 1 - \lambda_i$.

Note 2. Notice that the lower bound of this parameter $\underline{\lambda_i}$, is need to set by the designer before the controller design. If the system is stable, this parameter can be chosen as $\underline{\lambda_i} \geq 0$. For unstable systems this lower bound should be grater then zero $\underline{\lambda_i} > 0$. To obtain the less conservative controller design, we suggest to design and tune first a controller without the input saturation and then determine the lower bound on this parameter as $\underline{\lambda_i} = \frac{u_{i_{max}}}{u_{i_{s_{max}}}}$, then redesign the controller with the input saturation using the obtained $\underline{\lambda_i}$.

The vector of switching parameters $\sigma(k) = [\sigma_1(k), \ldots, \sigma_m(k)]^T$, $\sigma_i \in \langle 0, 1 \rangle$ for integral (sum) windup is calculated as follows:

$$\sigma_i(k) = \begin{cases} 1 \rightarrow \text{if } \lambda(k) = 1 \\ 0 \rightarrow \text{if } \lambda(k) < 1 \end{cases}, \quad i = 1, \ldots, m. \tag{8}$$

The filter $G_f(z) = [G_{f_1}(z), \ldots, G_{f_m}(z)]^T$ serves as a filter for the derivative part. In this paper a first order filter is used with the transfer function:

$$G_{f_i}(z) = \frac{b_i}{z - a_i}, \quad i = 1, \ldots, m, \tag{9}$$

where $a_i = \frac{1}{e^{(T_s/T_{f_i})}}$ and $b_i = 1 - a_i$ obtained from discretizing a first order filter using the zero-order hold discretization method with sampling time T_s, and with filter coefficient T_{f_i}.

III. ROBUST DISCRETE-TIME GS-PSD CONTROLLER DESIGN

This section first describes the closed-loop system for controller design then presents the stability and performance conditions for the obtained closed-loop system. Finally, as the main result, a theorem is given for the advanced robust discrete-time PSD controller design for uncertain LPV systems with input saturation and anti-windup, which guarantees the closed-loop stability and the guaranteed cost.

A. Closed-loop system for controller design

It is assumed that the reference signal $w(k)$ is bounded, and that within the reference trajectory the reference target is reachable within the input constraints $|u(k)| \leq u_{max}$. Based on the previous assumption the control law for $w(k) = 0$ can be rewritten as follows:

$$\begin{aligned} u(k) = G_f(z) \circ \Bigg(K_P(\theta(k))y(k) + E_{EFM}(k) \\ + K_D(\theta(k))\frac{y(k) - y(k-1)}{T_s}\Bigg) \circ \lambda(k), \end{aligned} \tag{10}$$

where in the term $E_{EFM}(k)$ (5), $e(i) = y(i)$.

One can formulate the PSD controller design problem in different ways. In the paper [21] the authors formulated the PSD controller design as a time-delay control problem, because of the $y(k-1)$ term in the derivative part. In the paper [22] and in our previous papers [23], [24] two new state variables were used $z_1(k) = \sum_{i=0}^{k-2} y(i)$ and $z_2(k) = \sum_{i=0}^{k-1} y(i)$ to describe the closed-loop system. However, these state variables in our case can't be used due to the switching parameter $\sigma(k)$ inside the summation term. Because of that a new state variables are introduced:

$$z_1(k) = T_s \sum_{i=0}^{k-1} \big(\sigma(i) \circ y(i)\big), \tag{11}$$

$$z_2(k) = y(k-1). \tag{12}$$

Substituting expressions (11) and (12) to the control law (10), one can obtain:

$$\begin{aligned} u(k) = G_f(z) \circ \Bigg(\Big(K_P(\theta(k)) + \frac{1}{T_s}K_D(\theta(k))\Big)y(k) \\ + K_S(\theta(k))z_1(k) - \frac{1}{T_s}K_D(\theta(k))z_2(k)\Bigg) \circ \lambda(k). \end{aligned} \tag{13}$$

The control algorithm (13) can be transformed to the following state space form:

$$\begin{aligned} x_c(k+1) = A_c x_c(k) + B_c(\theta(k))\tilde{y}(k), \\ u(k) = C_c(\lambda(k))x_c(k), \end{aligned} \tag{14}$$

where $\tilde{y}(k) = [y(k),\, z_1(k),\, z_2(k)]^T$, $x_c(k) = [x_P(k),\, x_S(k),\, x_D(k)]^T$ are the extended measured output and the controller state vectors, respectively. In addition,

$$B_c(\theta(k)) = \begin{bmatrix} K_P(\theta(k))\bar{\circ}b_f, & 0, & 0 \\ 0, & K_S(\theta(k))\bar{\circ}b_f, & 0 \\ \frac{K_D(\theta(k))}{T_s}\bar{\circ}b_f, & 0, & -\frac{K_D(\theta(k))}{T_s}\bar{\circ}b_f \end{bmatrix},$$

$$A_c = \begin{bmatrix} A_f, & 0, & 0 \\ 0, & A_f, & 0 \\ 0, & 0, & A_f \end{bmatrix}, A_f = \begin{bmatrix} a_1, \dots, & 0 \\ \vdots, & \ddots, & \vdots \\ 0, & \dots, & a_m \end{bmatrix}, b_f = \begin{bmatrix} b_1 \\ \vdots \\ b_m \end{bmatrix},$$

$$C_c(\lambda(k)) = \begin{bmatrix} I, I, I \end{bmatrix} \bar{\circ}\, \lambda(k),$$

furthermore, a_i, b_i, $i = 1, \dots, m$ are the filter coefficients from (9). Substituting the control law (14) to the system (1), the following closed-loop system is obtained:

$$\tilde{x}(k+1) = A_{cl}(\theta(k), \sigma(k), \lambda(k))\tilde{x}(k), \tag{15}$$

where $\tilde{x}(k) = [x(k),\, z_1(k),\, z_2(k),\, x_P(k),\, x_S(k),\, x_D(k)]^T$ and

$$A_{cl}(\theta(k), \sigma(k), \lambda(k)) = \begin{bmatrix} A_{cl_{11}}(\theta(k), \sigma(k)), & A_{cl_{12}}(\theta(k), \lambda(k)) \\ A_{cl_{21}}(\theta(k)), & A_{cl_{22}} \end{bmatrix},$$

$$A_{cl_{11}}(\theta(k), \sigma(k)) = \begin{bmatrix} A(\theta(k)), & 0, 0 \\ T_s C \,\bar{\circ}\, \sigma(k), & I, 0 \\ C, & 0, 0 \end{bmatrix}, \quad A_{cl_{22}} = \begin{bmatrix} A_f, & 0, & 0 \\ 0, & A_f, & 0 \\ 0, & 0, & A_f \end{bmatrix},$$

$$A_{cl_{12}}(\theta(k), \lambda(k)) = \begin{bmatrix} B(\theta(k))(I\bar{\circ}\lambda(k)), & B(\theta(k))(I\bar{\circ}\lambda(k)), & B(\theta(k))(I\bar{\circ}\lambda(k)) \\ 0, & 0, & 0 \\ 0, & 0, & 0 \end{bmatrix},$$

$$A_{cl_{21}}(\theta(k)) = \begin{bmatrix} (K_P(\theta(k))\,\bar{\circ}\, b_f)C, & 0, & 0 \\ 0, & K_S(\theta(k))\,\bar{\circ}\, b_f, & 0 \\ (K_D(\theta(k))\frac{1}{T_s}\,\bar{\circ}\, b_f)C, & 0, & -K_D(\theta(k))\frac{1}{T_s}\,\bar{\circ}\, b_f \end{bmatrix}.$$

B. Stability conditions

A reasonable compromise between the solvability and the conservativeness is to chose the candidate for Lyapunov function in the following form:

$$V(\theta(k), \lambda(k)) = \tilde{x}^T(k)P(\theta(k), \lambda(k))\tilde{x}(k), \tag{16}$$

where

$$P(\theta(k), \lambda(k)) = P_0 + \sum_{i=1}^{p} P_i\theta_i(k) + \sum_{j=1}^{m} P_{p+j}\lambda_j(k). \tag{17}$$

The first difference of the Lyapunov function (16) is given as follows:

$$\begin{aligned} \Delta V(.) = \tilde{x}^T(k+1)\, P(\theta(k+1), \lambda(k+1))\, \tilde{x}(k+1) \\ - \tilde{x}^T(k)\, P(\theta(k), \lambda(k))\, \tilde{x}(k), \end{aligned} \tag{18}$$

where, on substituting $\theta(k+1) = \theta(k) + \Delta\theta(k)$, $\lambda(k+1) = \lambda(k) + \Delta\lambda(k)$ to $P(\theta(k+1), \lambda(k+1))$, and assuming that $\Delta\theta_i(k) \leq \rho_{\theta_i}$, $i = 1, \dots, p$ and $\Delta\lambda_j \leq \rho_{\lambda_j}$, $j = 1, \dots, m$ one obtains:

$$P(\theta(k+1), \lambda(k+1)) \leq P_\rho(\theta(k), \lambda(k)), \tag{19}$$

$$\begin{aligned} P(\theta(k+1), \lambda(k+1)) = P_0 + \sum_{i=1}^{p} P_i\theta_i(k) + \sum_{i=1}^{p} P_i\Delta\theta_i(k) \\ + \sum_{j=1}^{m} P_{p+j}\lambda_j(k) + \sum_{j=1}^{m} P_{p+j}\Delta\lambda_j(k), \end{aligned}$$

$$P_\rho(\theta(k),\lambda(k)) = P_0 + \sum_{i=1}^{p} P_i\theta_i(k) + \sum_{j=1}^{m} P_{p+j}\lambda_j(k) + \sum_{i=1}^{p} P_i\rho_{\theta_i} + \sum_{j=1}^{m} P_{p+j}\rho_{\lambda_j}.$$

Based on *Definition 2.1* (Affine Quadratic Stability) from [25] and on previous derivation the following definition can be formulated:

Definition 1. The closed-loop system (15) for all $\theta(k) \in \Omega$ and $\lambda(k) \in \Phi$ for given ρ_{θ_i}, $i = 1,\dots,p$ and ρ_{λ_j}, $j = 1,\dots,m$ is affinely quadratically stable if $p+m+1$ symmetric matrices $P_0, P_1, \dots, P_{p+m}$ exist such that $P(\theta(k),\lambda(k))$ (17), $P_\rho(\theta(k),\lambda(k))$ (19) are positive definite and for the first difference of the Lyapunov function (18) along the trajectory of closed-loop system (15) it holds:

$$\Delta V(\theta(k),\lambda(k)) \leq 0. \tag{20}$$

C. Performance quality

To assess the performance quality in LQR fashion, the following parameter-varying quadratic cost function has been chosen:

$$J_d = \sum_{k=0}^{\infty} J(k), \qquad J(k) = \tilde{x}(k)^T Q(\theta(k))\tilde{x}(k) + u(k)^T R u(k), \tag{21}$$

where $Q(\theta(k)) = Q_0 + \sum_{i=1}^{p} Q_i\theta_i(k) \geq 0$, $R > 0$, Q_0, $Q_i \in \mathbb{R}^{(n+2l+3m)\times(n+2l+3m)}$, $R \in \mathbb{R}^{m\times m}$ are symmetric positive definite (semidefinite) and definite matrices, respectively.

D. Controller design

The following lemma, is needed for the main result:

Lemma 1. *Consider the closed-loop system* (15) *with a control algorithm* (14)*. Control algorithm* (14) *will be a stabilizing and guaranteed cost algorithm if there exist a positive scalar* ϵ *such that for the first difference of the positive definite Lyapunov function* (16) *the following condition holds:*

$$\max_{u}\{\Delta V(k) + J(k)\} \leq -\epsilon x(k)^T x(k), \;\; \epsilon \to \infty. \tag{22}$$

Proof. Assume that the first difference of the Lyapunov function is $\Delta V(k) = V(k+1) - V(k)$ and that the Lyapunov function (16) is positive definite. For $\epsilon \to 0$ the Bellman-Lyapunov inequality (22) can be rewritten to:

$$\Delta V(k) + J(k) \leq 0 \to \Delta V(k) \leq -J(k), \tag{23}$$

from this follows that if the Lyapunov function (16) is positive definite, then the first difference of the Lyapunov function will be negative definite, so the system will be stable. Furthermore, summing both side from 0 to ∞:

$$\sum_{i=0}^{\infty} J(i) = J_d \leq V(0) - V(\infty) \leq V(0), \tag{24}$$

one can obtain the upper bound on the cost function (21) (i.e. the guaranteed cost). □

The main result for advanced robust discrete-time guaranteed cost PSD controller design for uncertain LPV systems with input saturation and anti-windup is given in the next theorem:

Theorem 1. *The closed-loop system* (15) *for all* $\theta(k) \in \Omega$ *and* $\lambda(k) \in \Phi$*, for given maximal rates of change of scheduled parameters* ρ_{θ_i}*,* $i = 1,\dots,p$*, maximal rates of change of saturation parameters* ρ_{λ_j}*,* $j = 1,\dots,m$*, lower bounds on saturation parameters* $\underline{\lambda_i}$*,* $i = 1,\dots,m$*, and for given weighting matrices* R*,* Q_i*,* $i = 0,2,\dots,p$*, is affinely quadratically stable with hard input constraints* $|u(k)| \leq u_{max}$*, with anti-windup and guaranteed cost, if* $p+m+1$ *symmetric matrices* $P_0, P_1, \dots, P_{p+m}$*, and* $p+1$ *controller gain matrices* K_{P_i}*,* K_{S_i}*,* K_{D_i} $i = 0,1,\dots,p$ *exists such that* $P(\theta(k),\lambda(k))$ (17)*,* $P_\rho(\theta(k),\lambda(k))$ (19) *are positive definite, and the following inequalities hold:*

$$\begin{bmatrix} -\overline{P}_i + \overline{Q}_i + \overline{F}_i^T R\overline{F}_i, & \overline{A}_{cl_i}^T \\ \overline{A}_{cl_i}, & X_\rho \end{bmatrix} \leq 0$$
$$X_\rho = X_i^{-1}\left(\overline{P}_{\rho_i} - X_i\right)X_i^{-1} - X_i^{-1}, \tag{25}$$
$$i = 1,2,\dots,2^{m+p},$$

where in each iteration holds $X_i|_j = \overline{P}_{\rho_i}|_{j-1}$ *(*j *– actual iteration step).*

Proof. For the ease of notation we drop the dependency on time (k) during the proof. Substituting the control law (14) to the quadratic cost function (21) one can obtain:

$$J(.) = \tilde{x}^T\left(Q(\theta) + F^T(\lambda)RF(\lambda)\right)\tilde{x}, \tag{26}$$

where $F(\lambda(k)) = \left[0,0,0,I\,\bar{\circ}\,\lambda(k), I\,\bar{\circ}\,\lambda(k), I\,\bar{\circ}\,\lambda(k)\right]^T$.

Furthermore, substituting the system equation from (1) to the first difference of the Lyapunov function (18), one can obtain:

$$\Delta V(.) = \tilde{x}^T\Big(A_{cl}^T\big(\theta,\sigma,\lambda\big)P_\rho\big(\theta,\lambda\big)A_{cl}^T\big(\theta,\sigma,\lambda\big) - P\big(\theta,\lambda\big)\Big)\tilde{x}. \tag{27}$$

Now, substituting the first difference of the Lyapunov function (27) and the quadratic cost function (26) to the Bellman-Lyapunov inequality (22), after some manipulation one can obtain:

$$A_{cl}^T(\theta,\sigma,\lambda)P_\rho(\theta,\lambda)A_{cl}(\theta,\sigma,\lambda) - P(\theta,\lambda) + Q(\theta) + F^T(\lambda)RF(\lambda) \leq 0. \tag{28}$$

Using the Schur complement, we can rewrite the previous inequality (28) as follows:

$$M(\theta,\sigma,\lambda) = \begin{bmatrix} M_{11}(\theta,\lambda), & M_{21}^T(\theta,\sigma,\lambda) \\ M_{21}(\theta,\sigma,\lambda), & M_{22}(\theta,\lambda) \end{bmatrix} \leq 0, \tag{29}$$

where

$$M_{11}(\theta,\lambda) = -P(\theta,\lambda) + Q(\theta) + F(\lambda)^T RF(\lambda),$$
$$M_{22}(\theta,\lambda) = -P_\rho^{-1}(\theta,\lambda), M_{21}(\theta,\sigma,\lambda) = A_{cl}(\theta,\sigma,\lambda).$$

The inequality (29) is convex regarding the scheduling variable $\theta(k)$, because M_{11} and M_{21} are affine regarding to this parameter, and M_{22} is an iverse of an affine function of this parameter, and regarding [26] (*Section 2.3.2*) an inverse of an affine function remains convex.

Furthermore, the inequality (29) is convex regarding the saturation parameter $\lambda(k)$, too. It follows from that the term M_{21} is affine regarding this parameter, M_{22} is convex (the proof is same as for $\theta(k)$), and finally M_{11} is convex, because $P(\theta(k), \lambda(k))$ is affine regarding $\lambda(k)$ and the term $F(\lambda(k))^T RF(\lambda(k))$ is a quadratic function of this parameter, and the second derivative regarding $\lambda(k)$ is also positive definite due to the quadratic form and that the matrix R is positive definite $\rightarrow$ and the sum of two convex functions remains convex [26].

Finally, the inequality (29) is convex regarding the switching parameter $\sigma(k)$, since its appears only affinely in the term $A_{cl}(\theta(k), \sigma(k), \lambda(k))$.

Now that we know that the inequality (29) is convex regarding the scheduling variable $\theta(k)$, the saturation variable $\lambda(k)$, and the switching variable $\sigma(k)$, we can conclude that the inequality (29) will be negative definite for $\forall\theta(k) \in \Omega$, $\lambda(k) \in \Phi$, and $\sigma_k \in \Phi_\sigma$, if it takes negative values at the corners of $\theta(k)$, $\lambda(k)$, and $\sigma(k)$. In addition, the switching parameter $\sigma(k)$ is connected to the saturation parameter so the number of vertices can be reduced. That is, the inequality (29) splits to 2^{m+p} inequalities $\rightarrow$ (25). The overlines in the inequality (25) indicates the given item at the vertices of $\theta(k)$, $\lambda(k)$ and $\sigma(k)$.

Finally, the inversion of $-\overline{P}_{\rho_i}$ in the inequality (29) can be linearized as follows (to obtain LMI design procedure):

$$lin(-\overline{P}_{\rho_i}) \leq X_i^{-1} \left(\overline{P}_{\rho_i} - X_i\right) X_i^{-1} - X_i^{-1}, \tag{30}$$

where in each iteration holds $X_i|_j = \overline{P}_{\rho_i}|_{j-1}$ (j – actual iteration step). $\square$

Note 3. For the first iteration $X_i|_1$ is a freely chosen positive definite matrix, or it can be calculated from Lyapunov function obtained by a standard LQR design for the nominal system or in the given vertex.

Note 4. The proposed theorem can be used also for:

- quadratic stability with respect to scheduled parameters. For this case $\Delta\theta_i \rightarrow \infty$ and the matrices in (16) $P_i = 0$, $i = 1, 2, \ldots, p$.
- quadratic stability with respect to saturation parameters. For this case $\Delta\lambda_i \rightarrow \infty$ and the matrices in (16) $P_j = 0$, $j = p+1, p+2, \ldots, p+m$.
- quadratic stability with respect to both scheduled and saturation parameters. For this case $\Delta\theta_i \rightarrow \infty$ and $\Delta\lambda_i \rightarrow \infty$, and the matrices in (16) $P_i = 0$, $i = 1, 2, \ldots, p+m$.

IV. Example

In order to show the viability of the previous proposed method, the following simple nonlinear system has been chosen (inspired from [27]):

$$\begin{aligned} \dot{x} &= -x|x|\gamma + u, \\ y &= x, \qquad\qquad -0.5 \leq u \leq 0.5, \end{aligned} \tag{31}$$

where $\gamma \in \langle 0.9, 1.1\rangle$ is unknown (uncertain) parameter. The system (31) can be transformed into the following form:

$$\begin{aligned} \dot{x} &= -a(\theta)x + bu, \\ y &= cx, \qquad\qquad -0.5 \leq u \leq 0.5, \end{aligned} \tag{32}$$

where $a(\theta) = a_0 + a_1\theta_1 + a_2\theta_2$, $b = 1$, $c = 1$, and $\theta_2 = \beta \in \langle -1, 1\rangle$ is unknown (uncertain) variable, furthermore,

$$\theta_1 = \alpha = \frac{|y| - a_0}{a_1} \in \langle -1, 1\rangle.$$

The coefficients a_0 and a_1 were calculated so as to maintain the scheduling parameter θ_1 in the range $\langle -1, 1\rangle$:

$$a_0 = \tfrac{\min(|y|)+\max(|y|)}{2}; \quad a_1 = \tfrac{\min(|y|)-\max(|y|)}{2}.$$

From the model (31) follows that $\max(|y|) = 0.7071$ and $\min(|y|) = 0$ and it follows that $a_0 = 0.3535$ and $a_1 = -0.3535$. The parameter $a_2 = 0.1$ (computed from γ).

The obtained LPV system (32) was transformed to discrete-time with sample time $T_s = 0.1$ using the Euler's forward method [28] to obtain the model for controller design in the form (1).

Using *Theorem 1* with weighting matrices $Q = q_i I$, $q_0 = 1 \times 10^6$, $q_1 = q_2 = 0$, $R = rI$, $r = 1 \times 10^{-4}$, sampling time $T_s = 0.1\,s$, filter time constant $T_f = 1 \times 10^{-3}$, maximal values of rate of change of scheduled parameters $\rho_1 = 0.2815$, $\rho_2 = 0$, with $\lambda \in \langle 0.5,\ 1\rangle$ and $\rho_\lambda = 0.5$, we obtained a robust discrete-time gain-scheduled PSD controller with hard input constraints and anti-windup in the form (4), where

$$\begin{aligned} K_P(\theta(k)) &= -0.5041 - 0.0090\,\theta_1(k), \\ K_S(\theta(k)) &= -0.5875 - 0.0408\,\theta_1(k), \\ K_D(\theta(k)) &= -0.0210 - 3.1398 \times 10^{-4}\,\theta_1(k). \end{aligned} \tag{33}$$

Numerical solution has been carried out by SDPT3 4.0 [29] solver under MATLAB 2014b using YALMIP R20150918 [30]. The simulations were done via SIMULINK.

Simulation results for $\gamma = 1$ (Fig. 1) confirm that the *Theorem 1* holds and the closed-loop system is stable with hard input constraints and anti-windup. In the simulations $w(t)$, $y(t)$, $u(t)$, $\theta(t)$ and $\lambda(t)$ are the reference signal, measured output, controller output, scheduled parameter and the saturation parameter, respectively. The *red* color denotes the closed-loop system with the proposed algorithm with input constraints and anti-windup. The *green* color denotes the constrained closed-loop system without anti-windup. Finally, the *black* dotted lines denote the closed-loop system without input constraints and anti-windup.

V. Conclusion

A novel methodology is presented in the paper for robust discrete-time gain-scheduled guaranteed cost PSD controller design with hard input constraints and anti-windup for uncertain LPV systems. The proposed approach ensures the robust affine quadratic stability, guaranteed cost and hard input constraints for all scheduled parameters and their prescribed maximal rate of change. The controller design problem with

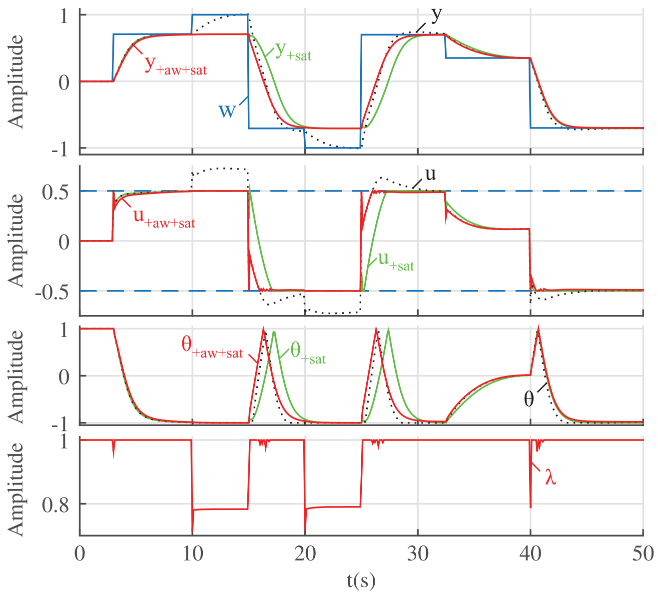

Fig. 1. Simulation results with the proposed approach with and without input constraints and anti-windup.

stability and performance conditions, is translated to an optimization problem subject to linear matrix inequality (LMI) constraints. This optimization problem is directly convex regarding the scheduled variable and the variable for hard input constraints. Therefore, the proposed controller design approach is less conservative compared with the approaches presented in the literatures or in our previous publications, where convexification had to be used. Numerical example shows the effectiveness of the introduced approach.

Appendix

Definition 2. For a matrix $A \in \mathbb{R}^{n\times m}$ and a vector $b \in \mathbb{R}^n$, $A \bar{\circ} b$ is a matrix, of the same dimension as A, with elements given by:

$$(A\bar{\circ}b)_{i,j} = (A)_{i,j}(b)_i \tag{34}$$

For example for matrix A and vector b:

$$A = \begin{bmatrix} A_{11}, A_{12} \\ A_{21}, A_{22} \end{bmatrix}, \; b = \begin{bmatrix} b_1 \\ b_2 \end{bmatrix}, \tag{35}$$

the expression $A\bar{\circ}b$ is equal to:

$$A\bar{\circ}b = \begin{bmatrix} A_{11}b_1, A_{12}b_1 \\ A_{21}b_2, A_{22}b_2 \end{bmatrix}. \tag{36}$$

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