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IMPROVED SURFACE RECONSTRUCTION FROM CONVENTIONAL GEOMETRY DATA FOR GENERAL SHELL FINITE ELEMENTS

MIKA MALINEN

CSC - IT Center for Science Ltd.
P.O. Box 405, FI-02101 Espoo, Finland
e-mail: `mika.malinen@csc.fi`, web page: `http://www.csc.fi/`

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Summary. We consider an alternate surface reconstruction strategy based on geometry data which is conventionally employed in the finite element modelling of elastic shells. In comparison with the usual strategy, a more accurate approximation of the shell mid-surface is obtained, so that differentiation of the chart giving the approximate mid-surface can solely be used to estimate the fundamental forms of the surface, which are needed so as to express the 2-D equations of shell theory. In the case of the lowest-order surface data, this is in contrast with the conventional approximation which does not make it possible to obtain curvature information naturally via the straightforward differentiation of the chart.

1 INTRODUCTION

Classical shell theory seeks simplifications to solving 3-D elasticity equations over a thin curved body by employing unknowns that depend on only two curvilinear coordinates associated with the shell mid-surface. Nevertheless, conventional finite element approximations of elastic shells applied in engineering are seldom formulated by use of a classical shell model, since traditional discrete geometry data does not generally offer the precise description of a smooth mapping (chart) which enables to express the placement of a shell in its reference configuration in terms of normal coordinates. Therefore, an approximate representation of the shell body must then be created in the first place.

The approximation of shell geometry is conventionally based on given information about the position and the director vector (the unit normal to the exact mid-surface) at the nodes of finite elements. The usual engineering representation of the shell body is then an outcome of simultaneous equal-order interpolation of the mid-surface coordinates and the director field. Importantly, this approach has been utilized to develop common structural analysis methods known as (isoparametric) general shell elements or, alternatively, as degenerated solid shell elements.

Although the Lagrange interpolation is usually employed to obtain the approximate mid-surface, the standard initial data actually contains enough information for reconstructing a significantly better approximation via requiring additionally that the normal to the approximate mid-surface agrees with the director specified at the nodes. Nevertheless, it seems that the possibility of utilizing such improved surface reconstruction has not been widely noted in engineering literature on shells, although we mention a paper[1] by Destuynder *et al.* where related ideas have been applied (it should also be noted that, from the same lowest-order data, higher-order approximations other than what we consider could be derived by using the macroelement strategies such as Powell-Sabin splines or reduced Hsieh-Clough-Tocher triangles). In addition to describing some technical aspects related to the improved surface reconstruction, we shall briefly discuss its utility, with special reference to the development of computational methods for elastic shells.

2 IMPROVED SURFACE RECONSTRUCTION

To describe the key ideas, assume that the lowest-order surface data is supplied so that we can construct transformations $\tilde{\mathbf{f}} : \hat{K} \to \mathbf{E}^3$ that elementwise yield a surface approximating the exact mid-surface as

$$\tilde{\mathbf{f}}(\hat{\mathbf{x}}) = \sum_{k=1}^{4} \hat{\lambda}_k(\hat{\mathbf{x}})\mathbf{p}_k, \tag{1}$$

where $\mathbf{p}_k$ give the coordinates of nodes with respect to a single frame of reference associated with an orthonormal basis and $\hat{\lambda}_k$ are the bilinear Lagrange interpolation basis functions defined over a reference element $\hat{K} = [-1,1] \times [-1,1]$.

Instead of performing the bilinear transformation to obtain a surface patch with straight edges, we may first apply the Hermite interpolation to obtain polynomial space curves of higher degree which shall be used to represent the boundaries of an alternate surface patch. This is possible since the shell director data basically specifies the set of tangent vectors to the exact mid-surface at a node $\mathbf{p}_k$ as

$$\mathcal{V}_k^T = \{\ \mathbf{p} - \mathbf{p}_k \in \mathbb{R}^3 \mid (\mathbf{p} - \mathbf{p}_k) \cdot \boldsymbol{d}_k = 0, \text{ with } \mathbf{p} \in \mathbf{E}^3\ \}, \tag{2}$$

where $\boldsymbol{d}_k$ is the shell director given at the node.

Technically we may generate polynomial space curves of degree 3 as follows. Given a pair of connected nodes $\mathbf{p}_i$ and $\mathbf{p}_j$ associated with the position vectors $\boldsymbol{r}_i$ and $\boldsymbol{r}_j$, respectively, we first create a local coordinate frame by associating its origin with the position vector $\boldsymbol{r}_o = 1/2(\boldsymbol{r}_i + \boldsymbol{r}_j)$ and by creating its orthonormal basis $\{\boldsymbol{k}_1, \boldsymbol{k}_2, \boldsymbol{k}_3\}$ in the following way. The third basis vector $\boldsymbol{k}_3$ is first required to point into the same direction as the vector $\boldsymbol{d}_i + \boldsymbol{d}_j$. We then define $\boldsymbol{v}_{ij} = \boldsymbol{r}_j - \boldsymbol{r}_i$ and perform the Gram-Schmidt procedure to obtain a new orthogonal vector, which we select to be the first basis vector $\boldsymbol{k}_1$. We finally use the requirement of right-handedness to select $\boldsymbol{k}_2$.

The positions of the two nodes are now expressed with respect to the local frame to obtain their coordinates $z^r = z_k^r$, with $k \in \{i, j\}$ and $z_k^2 = 0$ by construction. Setting $E = [z_i^1, z_j^1] \subset \mathbf{E}^1$, we then seek a polynomial curve $E \ni z^1 \mapsto (z^1, 0, \varphi_h(z^1))$ with

$$\varphi_h(z^1) = \sum_{k=1}^{2}[(\hat{\psi}_k \circ f_E^{-1})(z^1)\varphi_k + h_E/2 \cdot (\hat{\psi}_{2+k} \circ f_E^{-1})(z^1)\vartheta_k], \tag{3}$$

where $h_E = z_j^1 - z_i^1 > 0$ is a length parameter and f_E is an affine mapping between the points of $\hat{E} = [-1, 1]$ and E, with $f_E(-1) = z_i^1$ and $f_E(1) = z_j^1$. In addition, $\hat{\psi}_k : \hat{E} \to \mathbb{R}$ give the Hermite interpolation basis functions for the space of third-order polynomials defined over $\hat{E}$, with the degrees of freedom φ_k and ϑ_k chosen in the standard manner so that we have

$$\varphi_k = \varphi_h(z_k^1) \quad \text{and} \quad \vartheta_k = D\varphi_h(z_k^1)[\boldsymbol{k}_1]. \tag{4}$$

To select these scalar coefficients, we now reparametrize the curve to obtain the mapping $\mathbf{c}_{ij} : \hat{E} \to \mathbf{E}^3$ defined by $\hat{\sigma} \mapsto (f_E(\hat{\sigma}), 0, \varphi_h(f_E(\hat{\sigma})))$. Finally, the conditions that the curve passes through the nodes $\mathbf{p}_k$ and that the tangent vectors $\dot{\boldsymbol{c}}_{ij}(\hat{\sigma})$ to the curve evaluated at the nodes belong to the sets of tangent vectors $\mathcal{V}_k^T$ lead to the unique selection

$$\varphi_k = z_k^3 \quad \text{and} \quad \vartheta_k = -\frac{\boldsymbol{d}_k \cdot \boldsymbol{k}_1}{\boldsymbol{d}_k \cdot \boldsymbol{k}_3}, \tag{5}$$

where $\boldsymbol{d}_k \cdot \boldsymbol{k}_3 \neq 0$ can be assumed to hold without a practical restriction. We note that, as the tacit here is that the shell mid-surface is smooth, the boundary curves should give optimally accurate $O(h_E^4)$ descriptions of the mid-surface location in L_p (to derive such an estimate, one may consider the distance between the polynomial curve and the curve which is formed as an intersection of the exact mid-surface with the plane $z^2 = 0$).

After repeating the curve reconstruction for all connected nodes, the finite element blending technique (transfinite interpolation[2]) may be applied to obtain a surface patch whose boundaries agree with the cubic space curves obtained before. To proceed, we now redefine the mapping $\mathbf{c}_{ij} : \hat{E} \to \mathbf{E}^3$ such that $\mathbf{c}_{ij}(\hat{\sigma})$ gives the coordinates of the boundary curve points with respect to the global frame of reference. By utilizing the finite element blending, the improved approximation of the mid-surface is written as

$$\mathbf{f}(\hat{x}^1, \hat{x}^2) = \tilde{\mathbf{f}}(\hat{x}^1, \hat{x}^2) + \boldsymbol{f}_{12}(\hat{x}^2, \hat{x}^1) + \boldsymbol{f}_{23}(\hat{x}^1, \hat{x}^2) + \boldsymbol{f}_{43}(\hat{x}^2, \hat{x}^1) + \boldsymbol{f}_{14}(\hat{x}^1, \hat{x}^2) \tag{6}$$

where the functions $\boldsymbol{f}_{ij} : [-1, 1] \times [-1, 1] \to \mathbb{R}^3$ are associated with the curved edges and are of the form

$$\boldsymbol{f}_{ij}(r, s) = h_{ij}(r)[\mathbf{c}_{ij}(s) - 1/2(1 - s)\mathbf{p}_i - 1/2(1 + s)\mathbf{p}_j], \tag{7}$$

with $h_{ij}(r) = 1$ for the value of r corresponding to the edge curve $\mathbf{c}_{ij}(\hat{E})$ and $h_{ij}(r) = 0$ for the value of r corresponding to the opposite edge curve which does not intersect with $\mathbf{c}_{ij}(\hat{E})$.

The simplest way to satisfy the conditions for the blending functions h_{ij} is to employ linear interpolation functions. This leads to the mapping $\mathbf{f} : \hat{K} \rightarrow \mathbf{E}^3$ where $\mathbf{f}(\hat{\mathbf{x}}) \in [S_3(\hat{K})]^3$, with $S_3(\hat{K})$ the serendipity finite element space of degree 3. In view of approximation theory, the order of error in the serendipity approximation may generally depend on the shape of the surface patch. To avoid a possible degradation of convergence order, one may seek an alternate mapping $\mathbf{f}$ in terms of an augmented finite element space $V(\hat{K}) = S_3(\hat{K}) \oplus B(\hat{K})$ where $B(\hat{K})$ is spanned by bubble basis functions that vanish on the boundary $\partial\hat{K}$. The part $B(\hat{K})$ should ideally be chosen such that the condition $Q_r(\hat{K}) \subseteq V(\hat{K})$ holds for $r = 3$, with $Q_r(\hat{K})$ the space of all polynomials which are of at most degree r in each variable $\hat{x}^i$, $i = 1, 2$ (note that $Q_r(\hat{K}) \subseteq S_3(\hat{K})$ holds only for $r = 1$). The case $r = 3$ corresponding to use of bicubic polynomials can generally be expected to give fourth-order accurate approximation in L_p. To handle this case, we may generate two additional curves so as to connect diagonally two nodes which nevertheless are not connected via a real edge of the element (alternatively, consider $\hat{K}$ to be a macroelement for two alternate third-order subtriangulations of $\hat{K}$). Now, each of the two additional curves can be evaluated at two points to obtain $O(h_E^4)$ accurate surface location at four distinct interior points in order to find the coefficients for the bubble functions.

3 CONCLUDING REMARKS

If we let $\mathbf{f}_S : \hat{K} \rightarrow S \subset \mathbf{E}^3$ to denote the mapping between the points of the reference element and points that belong to the surface patch corresponding to the improved surface reconstruction, we can thus create a new surface mesh $\mathcal{T}_h$ contained in the physical point space such that its elements are the images $S = \mathbf{f}_S(\hat{K})$. By construction this surface mesh is continuous and, also, its tangent plane at each node $\mathbf{p}_k$ is uniquely defined.

A potentially useful aspect of the improved surface reconstruction is that it brings us closer to concepts of classic shell theory, as elementwise systems of normal coordinates can be defined and the differentiation of the mapping giving the approximate mid-surface can solely be used to obtain ingredients for expressing 2-D shell equations. For example, elementwise estimates of the curvatures of the mid-surface can be computed naturally. Indeed, the strategy based on the straightforward differentiation of the chart does not work in general when the lowest-order Lagrange interpolation of the mid-surface coordinates is used, so a surface reconstruction of higher degree than usual is indeed a necessity for making such approach to work.

REFERENCES

[1] Destuynder, P., Moguen, Y. & Salaün, M. Adaptive mesh refinements for thin shells whose middle surface is not exactly known. *Computer Methods in Applied Mechanics and Engineering* **197**, 4789–4811 (2008).

[2] Šolín, P., Segeth, K. & Doležel, I. *Higher-Order Finite Element Methods* (Chapman & Hall/CRC, 2004).