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CLOSED-FORM FINITE ELEMENT SOLUTIONS FOR BEAMS AND PLATES

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Summary. We formulate a linear elastic beam problem by employing Saint Venant's principle so that end effects do not appear in the beam. Then the elasticity solution to the formulated interior problem is presented. By using mid-surface variables derived from the solution, a nodally-exact beam element is formulated by a force-based approach. A finite element calculation example is presented. In addition to the beam element, elasticity-based circular and rectangular plate elements may be developed on the basis of similar approaches founded on 2D stress functions or 3D displacement potentials.

1 INTRODUCTION

Elasticity solutions for beams are of fundamental interest in mechanical sciences. Two-dimensional (2D) interior elasticity solutions can be easily obtained, for example, for an end-loaded cantilever and a uniformly loaded simply-supported beam by employing the Airy stress function[1]. An interior solution excludes, by virtue of the Saint Venant's principle, the end effects that decay with distance from the ends of a beam. In the calculation of displacements, constraint conditions are applied at the beam supports to prevent it from moving as a rigid body.

We show how a general 2D interior solution for a linearly elastic isotropic beam can be discretized in order to obtain nodally-exact 1D rod and beam finite elements. The methodology applies to circular and rectangular plates as well.

2 PROBLEM FORMULATION AND SOLUTION

A 2D linearly elastic homogeneous isotropic plane beam under a uniform pressure p is shown in Fig. 1. The cross-sectional load resultants are calculated from

$$N(x) = t\int_{-h/2}^{h/2} \sigma_x dy, \quad M(x) = t\int_{-h/2}^{h/2} \sigma_x y dy, \quad Q(x) = t\int_{-h/2}^{h/2} \tau_{xy} dy. \tag{1}$$

The boundary conditions on the upper and lower surfaces of the beam are $\sigma_y(x, h/2) = -p$, $\sigma_y(x, -h/2) = 0$ and $\tau_{xy}(x, \pm h/2) = 0$. At the beam ends the tractions are specified only through the load resultants and, thus, the boundary conditions are imposed only in a *weak* sense[2]. This means that the exponentially decaying end effects of the isotropic plane beam are neglected by virtue of the Saint Venant's principle and only the interior solution of the beam is under consideration. Using the Airy stress function $\Psi(x, y)$, the stresses of the plane beam are obtained from the equations[2]

$$\sigma_x = \frac{\partial^2\Psi}{\partial y^2}, \ \sigma_y = \frac{\partial^2\Psi}{\partial x^2}, \ \tau_{xy} = -\frac{\partial^2\Psi}{\partial x\partial y}, \qquad \frac{\partial^4\Psi}{\partial x^4} + 2\frac{\partial^4\Psi}{\partial x^2\partial y^2} + \frac{\partial^4\Psi}{\partial y^4} = 0. \tag{2}$$

The solution to the interior plane beam problem that ensures compatibility is obtained by finding the solution of Eq. $(2)_4$ that satisfies the stress boundary conditions of the beam. By adapting a general procedure outlined by Barber[2], we find that the stress function for the interior problem of any plane beam under a uniform pressure p is

$$\Psi(x, y) = c_1 y^2 + c_2 y^3 + c_3 xy\left(1 - \frac{4y^2}{3h^2}\right) - \frac{q}{240I}\left[5h^3x^2 + 15h^2x^2y + 4y^3\left(y^2 - 5x^2\right)\right], \tag{3}$$

where $q = pt$ is the uniform load, $I = th^3/12$ is the second moment of the cross-sectional area and c_1, c_2 and c_3 are constant coefficients.

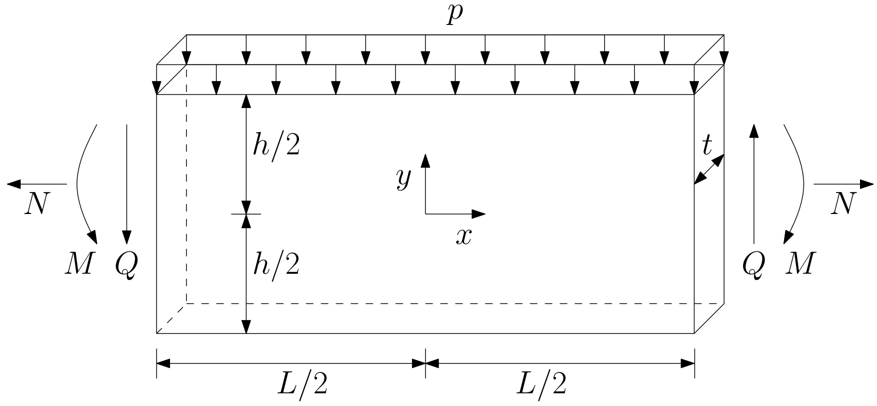

Figure 1: 2D linearly elastic, homogeneous, isotropic beam under uniform pressure p.

3 INTERIOR DISPLACEMENTS

Under plane stress conditions and through the strain-displacement relations, we obtain the 2D displacements[3]

$$
\begin{aligned}
U_x(x,y) &= \frac{1}{E}\left\{2c_1x + 6c_2xy - \frac{2c_3y}{3h^2}\left[3h^2(1+\nu) + 6x^2 - 2y^2(2+\nu)\right] - Cy + D_1\right\} \\
&\quad + \frac{qx}{24EI}\left[\nu h^3 + 3\nu h^2 y + 4x^2y - 4y^3(2+\nu)\right], \qquad (4) \\
U_y(x,y) &= \frac{1}{E}\left[-2c_1\nu y - 3c_2(x^2+\nu y^2) + \frac{4c_3(x^3+3\nu xy^2)}{3h^2} + Cx + D_2\right] \\
&\quad + \frac{q}{48EI}\left\{-2h^3y + 3h^2\left[x^2(2+\nu) - y^2\right] + 2\left[y^4(1+2\nu) - x^4 - 6\nu x^2y^2\right]\right\}, \qquad (5)
\end{aligned}
$$

where the integration constants C, D_1 and D_2 constitute the degrees of freedom of the beam as a rigid body. Constant C relates to a small clockwise rotation about the origin and D_1 and D_2 correspond to translations in the directions of x and y, respectively.

Next we obtain for the axial displacement and the transverse deflection of the mid-surface, and for the clockwise positive rotation of the cross-section on the mid-surface the expressions $u_x(x) = U_x(x,0)$, $u_y(x) = U_y(x,0)$, $\phi(x) = (\partial U_x/\partial y)(x,0)$, respectively.

4 EXACT ROD AND BEAM FINITE ELEMENTS

Let us consider a two-node finite element. For nodes $i = 1,2$, we have axial displacements $u_{x,i}$, transverse displacements $u_{y,i}$ and rotations of the cross-section ϕ_i. We write for nodes 1 and 2 six equations

$$
\begin{array}{lll}
u_{x,1} = u_x(-L/2) & u_{y,1} = u_y(-L/2) & \phi_1 = -\phi(-L/2) \\
u_{x,2} = u_x(L/2) & u_{y,2} = u_y(L/2) & \phi_2 = -\phi(L/2).
\end{array} \qquad (6)
$$

We can solve the six unknowns c_1, c_2, c_3, C, D_1 and D_2 from Eqs. (6). To obtain the finite element equations, we calculate the load resultants at nodes $i = 1,2$

$$
\begin{array}{lll}
N_1 = -N(-L/2) & Q_1 = -Q(-L/2) & M_1 = M(-L/2) \\
N_2 = N(L/2) & Q_2 = Q(L/2), & M_2 = -M(L/2).
\end{array} \qquad (7)
$$

The conventional presentation for the 1D rod and beam elements is obtained by writing Eqs. (7) in matrix form (below $\Phi = 3h^2(1+\nu)/L^2$; same as Timoshenko beam element for $\kappa = 2/3$)

$$
\frac{EA}{L}\begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix}\begin{Bmatrix} u_{x,1} \\ u_{x,2} \end{Bmatrix} = \begin{Bmatrix} N_1 \\ N_2 \end{Bmatrix} - \frac{q\nu h}{2}\begin{Bmatrix} 1 \\ -1 \end{Bmatrix}, \qquad (8)
$$

$$
\frac{EI}{(1+\Phi)L^3}\begin{bmatrix} 12 & 6L & -12 & 6L \\ 6L & (4+\Phi)L^2 & -6L & (2-\Phi)L^2 \\ -12 & -6L & 12 & -6L \\ 6L & (2-\Phi)L^2 & -6L & (4+\Phi)L^2 \end{bmatrix}\begin{Bmatrix} u_{y,1} \\ \phi_1 \\ u_{y,2} \\ \phi_2 \end{Bmatrix} = \begin{Bmatrix} Q_1 \\ M_1 \\ Q_2 \\ M_2 \end{Bmatrix} - \frac{q}{2}\begin{Bmatrix} L \\ \frac{10L^2-3h^2(2+5\nu)}{60} \\ L \\ -\frac{10L^2-3h^2(2+5\nu)}{60} \end{Bmatrix} \qquad (9)
$$

5 EXAMPLE – END-LOADED CANTILEVER

Let us consider a cantilever beam modeled with one beam element that is loaded with a point load P at node 1 and clamped at node 2. At node 2 we set $u_{y,2} = 0$ and $\phi_2 = 0$. The displacements at node 1 calculated from Eq. (9) are

$$u_{y,1} = \frac{PL^3}{3EI} + \frac{PLh^2}{8GI} \,, \qquad \phi_1 = -\frac{6u_{y,1}}{(4+\Phi)L} \,.$$

Substitution of the displacements into the equations for c_1, c_2, c_3, C, D_1 and D_2 that were obtained by the aid of Eqs. (7) and then substituting the result into Eqs. (4) and (5) leads to the exact 2D interior displacements

$$U_x(x,y) = \frac{Py}{24EI}\left[9L^2 - 12x(L+x) + 4y^2(2+\nu)\right]$$
$$U_y(x,y) = \frac{P}{48EI}\left[5L^3 - 18L^2x + 6h^2(1+\nu)(L-2x) + 12L(x^2+\nu y^2) + 8(x^3+3\nu xy^2)\right]$$

via which we obtain the interior stresses $\sigma_x = -Py(L+2x)/(2I)$, $\tau_{xy} = -P(b^2-y^2)/(2I)$.

6 CONCLUSIONS

A comparison between the current beam element and Euler–Bernoulli, Timoshenko and Reddy-Bickford theories in terms of analytical and finite element solutions can be done by looking at the work of Eisenberger[4]. In his Table 4, Eisenberger provides numerical results for the cantilever problem and the beam of the present paper corresponds to the “Elasticity”–column in that table. That is to say, the current beam element provides the solution which is usually considered to be the exact reference solution when approximate 1D beam theories and finite elements are benchmarked. The presented elasticity approach applies also to circular[5] and rectangular[6] plates.

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