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# BUCKLING OF THIN RING-STIFFENED CYLINDRICAL SHELLS UNDER UNIFORM PRESSURE

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**Summary.** Buckling of the thin elastic ring-stiffened cylindrical shells under the uniform external pressure is analyzed. To find the critical pressure eigenvalue problems for linear differential equations are solved with the help of asymptotic methods. Simple approximate formulas for the critical pressure are derived.

## 1 INTRODUCTION

Ring-stiffened cylindrical shells are widely used in industrial applications. The important characteristic for these shells is the critical pressure. To find it one should solve an eigenvalue problem for linear differential equations. Equations of thin shells contain the dimensionless shell thickness as a small parameter. Therefore its solutions can be found with the help of asymptotic methods.

In Refs.[2],[1] and in almost all publications about ring-stiffened shell buckling the rings are modeled as circular beams. However, wide rings should be considered as annular plates. Buckling of an annular plate attached to the edge of a cylindrical shell was studied in Ref.[3].

In both cases the solution of the initial eigenvalue problem is represented as a sum of slowly varying functions and integrals of edge effect. The solution of the approximate eigenvalue problem for the shell stiffened by identical circular beams can be obtained by means of homogenization. In the first approximation for the shell joined with annular plates we get the eigenvalue problem for the annular plate. The solution of the last problem for the narrow plate is found with the perturbation method.

The aim of this essay is to present briefly the results obtained in the buckling theory of ring-stiffened shell by means of asymptotic methods.

## 2 PROBLEM STATEMENT

After separating variables the dimensionless equations describing buckling of the cylindrical shell stiffened by $n_r$ identical rings (see Figure 1) may be written as

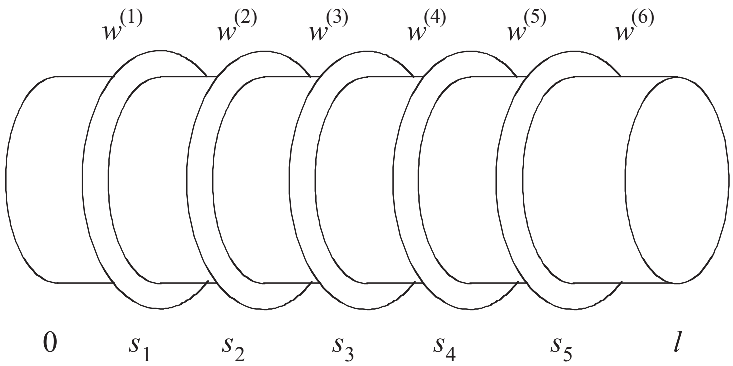


Figure 1: Page layout

$$\mu^4 \Delta\Delta w^{(j)} - \sigma \frac{d^2\Phi^{(j)}}{ds^2} - \lambda m^2 w^{(j)} = 0, \qquad \Delta\Delta\Phi^{(j)} + \frac{d^2 w^{(j)}}{ds^2} = 0, \quad j = 1, 2, \ldots n, \quad n = n_r + 1, \tag{1}$$

where

$$\Delta w = \frac{d^2 w}{ds^2} - m^2 w, \quad \mu^4 = \frac{h^2}{12}, \quad \sigma = 1 - \nu^2, \quad \lambda = \frac{\sigma p}{Eh},$$

$m$ is the circumferential wave number, $w^{(j)}$ is the normal deflection, $\Phi^{(j)}$ is the force function, $\mu > 0$ is a small parameter, $h$ is the dimensionless shell thickness, $\nu$ is Poisson's ratio, $p$ is the external pressure, and $E$ is Young's modulus. The solutions of equations (1) satisfy eight boundary conditions on the shell edges $s = 0$ and $s = l$, where $l$ is a non-dimension shell length, and $8n_r$ continuity conditions on the parallels $s = s_j$, $j = 1, 2, \ldots n_r$.

## 3 BEAM MODEL OF A RING

Consider buckling of a cylindrical shell stiffened by identical circular beams under the uniform external pressure $p$. Let us seek some unknown functions in equation (1) as

$$f^{(j)}(s) = f_0^{(j)}(s) + \sum_{k=1}^{4} D_k^{(j)} F_k^{(j)}(s), \quad j = 1, 2, \ldots n. \tag{2}$$

Here $f_0^{(j)}$ are slowly varying functions, $D_k^{(j)}$ are the arbitrary constants. The edge effect functions $F_k^{(j)}$, are localized in small neighborhoods of the parallels $s_j$, $j = 0, 1, \ldots n$.

To a first approximation we get the following equation

$$\frac{d^4 w_0^{(j)}}{ds^4} - \alpha^4 w_0^{(j)} = 0, \quad \sigma\alpha^4 = m^6\lambda - \mu^4 m^8, \quad j = 1, 2, \dots n. \tag{3}$$

The solutions of equations (3) satisfy four boundary conditions on shell edges and $4n_r$ continuity conditions on the parallels $s = s_j$.

It results from the second formula (3) that the approximate critical value of the buckling stress parameter is

$$\lambda_c = \min_m \lambda_1(m), \quad \lambda_1(m) = \sigma\alpha_1^4(m)/m^6 + \mu^4 m^2, \tag{4}$$

where $\alpha_1(m)$ is the minimal positive eigenvalue of the eigenvalue problem for equations (3).

Assume that rings are uniformly arranged, i. e. $s_j = jl/n$. If the number of rings $n_r$ is large and the stiffness of every ring is small, one can use the homogenization procedure for the evaluation of the eigenvalue $\alpha_1$ and get the following approximate formula

$$\lambda_c = 4\mu^3\beta(1+\eta)^{3/4}/(3\sigma)^{3/4}, \tag{5}$$

where $\beta = \pi/l$, $\eta = \sigma n I(\mu^4 hl)^{-1}$, $I$ is the dimensionless moment of inertia of the ring cross-section. For a ring with the rectangular cross-section $I = ab^3/12$, where $a$ and $b$ are the thickness and the width of the ring.

Consider buckling of the shell stiffened by rings with the rectangular cross-section of the width $b_j = f(j)b$. Using instead homogenization procedure Rayleigh-Ritz method with the Rayleigh function $\sin(\pi s/l)$ we obtain

$$\lambda_c = \frac{4\mu^3\beta}{(3\sigma)^{3/4}}\left(1 + \frac{2T(n)\eta}{n}\right)^{3/4}, \quad T_n = \sum_{j=1}^{n_r} f^3(j)\sin^2\left(\frac{\pi j}{n}\right) \tag{6}$$

One can use formulas (4)–(6) if the width of the rings is sufficiently small. Wide rings should be considered as annular plates.

## 4 PLATE MODEL OF A RING

To analyze buckling of a cylindrical shell stiffened by identical annular plates under the external pressure $p$ we seek the approximate solution of shell equations in the form (2). In the first approximation we get the eigenvalue problem for the following non-dimensional plate equation

$$\frac{d^4 w_p}{dr^4} + \frac{2}{r}\frac{d^3 w_p}{dr^3} - \frac{2m^2+1+\beta t_1}{r^2}\frac{d^2 w_p}{dr^2} + \frac{2m^2+1-\beta t_2}{r^3}\frac{dw_p}{dr} + \frac{m^2(m^2-4+\beta t_2)}{r^4} w_p = 0. \tag{7}$$

Here $r \in [1, r_1]$ is the radial coordinate, $m$ is the circumferential wave number, $w_p$ is the normal deflection,

$$\beta = \frac{12T_{1p}(1)}{a^2}, \quad t_k = \frac{r^2 T_{kp}(r)}{T_{kp}(1)}, \quad k = 1, 2,$$

$T_{kp}(r)$ are pre-buckling stresses

$$T_{1p} = \frac{\sigma p \gamma (r_1^2 - r^2)}{EhA_p r^2}, \quad T_{2p} = -\frac{\sigma p \gamma (r_1^2 + r^2)}{EhA_p r^2},$$

$A = \gamma(\gamma r_1^2 + \delta) + (3\sigma)^{1/4} a \gamma (r_1^2 - 1) h^{-3/2}/2$, $a$ is the plate thickness.

One can get an approximate analytical solution of equation (7) assuming that the plate is narrow, i.e. $\varepsilon = r_1 - 1 \ll 1$. Usually this assumption is valid for ring-stiffened shells.

Replacing variable $r = 1 + \varepsilon x$ in equation (7) and neglecting small terms we obtain the following approximate equation

$$\frac{d^4 w}{dx^4} - 2\varepsilon^2 m^2 \frac{d^2 w}{dx^2} + \varepsilon^4 m^4 (1 - \beta_0^2) w = 0, \quad \beta_0 = \sqrt{\beta}/(m\sqrt{\varepsilon}),$$

which has an analytical solution. Substituting this solution in boundary conditions we finally obtain the following approximate formula for the dimensionless critical pressure

$$\lambda_b = \frac{a^2 A \varepsilon}{12\gamma(r_1^2 - 1)} \min_m [m^2 \beta_0^2], \tag{8}$$

where $\beta_0$ is the least positive root of the equation

$$F \sinh\gamma \sin\alpha + G \cosh\gamma \cos\alpha + H = 0,$$

$$F = \beta_0^2(1 - 2\nu) - \delta^2, \quad G = (\beta_0^2 + \delta^2)\sqrt{\beta_0^2 - 1}, \quad H = (\beta_0^2 + \delta^2]\sqrt{\beta_0^2 - 1},$$
$$\alpha = \varepsilon m \sqrt{\beta_0 - 1}, \quad \gamma = \varepsilon m \sqrt{\beta_0 + 1}.$$

Formula (8) is valid for the shell stiffened by wide rings.

## 5 CONCLUSIONS

The approximate asymptotic formulas for the critical pressure can be used for an optimal design of thin ring-stiffened cylindrical shells.

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