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A GENERALIZED FINITE ELEMENT METHOD FOR LINEAR THERMOELASTICITY

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Summary. We present and analyze a generalized finite element method for the quasistatic linear thermoelasticity problem. The numerical method proposed is based on the localized orthogonal decomposition technique first presented and analyzed in [5].

1 INTRODUCTION

In many applications the expansion and contraction of a material exposed to temperature changes are of great importance. This phenomenon is modelled as a system consisting of a hyperbolic elasticity equation, describing the displacement, coupled to a parabolic heat equation, see e.g. [1]. If the inertia effects are negligible, the hyperbolic term in the elasticity equation can be removed. This leads to an elliptic-parabolic system, often referred to as *quasistatic*, which we study here. This system is essentially of parabolic type. The classical finite element method for the thermoelastic system is analyzed in [2], where convergence rates of optimal order are derived for problems with solution in H^2 or higher.

When the elastic medium of interest is strongly heterogeneous, like in composite materials, the coefficients are highly varying and oscillating. Commonly, such coefficients are said to have *multiscale* features. For these problems classical polynomial finite elements fail to approximate the solution well unless the mesh width resolves the data variations. To overcome this difficulty, several numerical methods have been proposed, see for instance [3,4,5].

In this work we present a generalized finite element method based on the local orthogonal decomposition technique introduced in [5]. This method builds on ideas from the variational multiscale method [3,4], where the solution space is split into a coarse and a fine part. The coarse finite element space is modified such that the basis functions contain information from the heterogeneous material data and have support on small patches. We prove convergence of optimal order that does not depend on the derivatives of the rapidly varying coefficients.

2 PROBLEM FORMULATION

Let $\Omega \subseteq \mathbb{R}^d$, $d = 2, 3$, be a polygonal/polyhedral domain describing the reference configuration of an elastic body. For a given time $T > 0$ we let $u : [0, T] \times \Omega \to \mathbb{R}^d$ denote the displacement field and $\theta : [0, T] \times \Omega \to \mathbb{R}$ the temperature.

We assume small displacements and define the strain tensor as $\varepsilon(u) = \frac{1}{2}(\nabla u + \nabla u^{\intercal})$. Assuming further that the material is isotropic, Hooke's law gives the (total) stress tensor,

$$\bar{\sigma} = 2\mu\varepsilon(u) + \lambda(\nabla \cdot u)I - \alpha\theta I,$$

where I is the d-dimensional identity matrix, α is the thermal expansion coefficient, and μ and λ are the so called Lamé coefficients. The materials of interest are strongly heterogeneous which implies that α, μ, and λ are rapidly varying in space.

The linear quasistatic thermoelastic problem takes the form

$$-\nabla \cdot (2\mu\varepsilon(u) + \lambda\nabla \cdot uI - \alpha\theta I) = f, \quad \text{in } (0, T] \times \Omega, \tag{1}$$

$$\dot{\theta} - \nabla \cdot \kappa\nabla\theta + \alpha\nabla \cdot \dot{u} = g, \quad \text{in } (0, T] \times \Omega, \tag{2}$$

where κ is the heat conductivity parameter, which is assumed to be rapidly varying in space. We consider mixed boundary conditions for both equations.

We make the following assumptions on the data

1. $\kappa \in L_\infty(\Omega, \mathbb{R}^{d\times d})$, symmetric,

$$0 < \kappa_1 := ess\,inf_{x\in\Omega} \inf_{v\in\mathbb{R}^d\setminus\{0\}} \frac{\kappa(x)v \cdot v}{v \cdot v}, \quad \infty > \kappa_2 := ess\,sup_{x\in\Omega} \sup_{v\in\mathbb{R}^d\setminus\{0\}} \frac{\kappa(x)v \cdot v}{v \cdot v},$$

2. $\mu, \lambda, \alpha \in L_\infty(\Omega, \mathbb{R})$, and

$$0 < \mu_1 := ess\,inf_{x\in\Omega}\mu(x) \leq ess\,sup_{x\in\Omega}\mu(x) =: \mu_2 < \infty.$$

   Similarly, the constants $\lambda_1, \lambda_2, \alpha_1$, and α_2 are used to denote the corresponding upper and lower bounds for λ and α.

3. $f, \dot{f} \in L_\infty(L_2), \ddot{f} \in L_\infty(H^{-1})$, $g \in L_\infty(L_2)$, $\dot{g} \in L_\infty(H^{-1})$, and $\theta_0 \in V^2$.

We now turn to the weak formulation and define the following subspaces of H^1

$$V^1 := \{v \in (H^1(\Omega))^d : v = 0 \text{ on } \Gamma_D^u\}, \quad V^2 := \{v \in H^1(\Omega) : v = 0 \text{ on } \Gamma_D^\theta\}.$$

We arrive at the following weak formulation[2]. Find $u(t, \cdot) \in V^1$ and $\theta(t, \cdot) \in V^2$, such that,

$$(\sigma(u) : \varepsilon(v_1)) - (\alpha\theta, \nabla \cdot v_1) = (f, v_1), \quad \forall v_1 \in V^1, \tag{3}$$

$$(\dot{\theta}, v_2) + (\kappa\nabla\theta, \nabla v_2) + (\alpha\nabla \cdot \dot{u}, v_2) = (g, v_2), \quad \forall v_2 \in V^2, \tag{4}$$

and the initial value $\theta(0, \cdot) = \theta_0$ is satisfied. Here we use σ to denote the effective stress tensor $\sigma(u) := 2\mu\varepsilon(u) + \lambda(\nabla \cdot u)I$.

3 NUMERICAL APPROXIMATION

We will use the classical finite element approximation as a reference when designing the multiscale approach.

3.1 Classical finite element approximation

We consider finite element spaces of piecewise linear continuous functions V_h^1 and V_h^2 that resolves the data variations well. For the discretization in time we consider, for simplicity, a uniform time step τ such that $t_n = n\tau$ for $n \in \{0, 1, ..., N\}$ and $N\tau = T$. The classical finite element method with a backward Euler scheme in time reads; for $n \in \{1, ..., N\}$ find $u_h^n \in V_h^1$ and $\theta_h^n \in V_h^2$, such that

$$(\sigma(u_h^n) : \varepsilon(v_1)) - (\alpha\theta_h^n, \nabla \cdot v_1) = (f^n, v_1), \quad \forall v_1 \in V_h^1, \tag{5}$$

$$(\bar{\partial}_t\theta_h^n, v_2) + (\kappa\nabla\theta_h^n, \nabla v_2) + (\alpha\nabla \cdot \bar{\partial}_t u_h^n, v_2) = (g^n, v_2), \quad \forall v_2 \in V_h^2, \tag{6}$$

where $\bar{\partial}_t\theta_h^n := (\theta_h^n - \theta_h^{n-1})/\tau$ and similarly for $\bar{\partial}_t u_h^n$. Given initial data u_h^0 and θ_h^0 the system (5)-(6) is well posed and leads to an optimal convergence order[2].

3.2 Generalized finite element

We define V_H^1 and V_H^2 analogously to V_h^1 and V_h^2, but with a larger mesh size $H > h$. Furthermore, we use the notation $\mathcal{N} = \mathcal{N}^1 \times \mathcal{N}^2$ to denote the free nodes in $V_H^1 \times V_H^2$.

We introduce an interpolation operator $I_H = (I_H^1, I_H^2) : V_h^1 \times V_h^2 \to V_H^1 \times V_H^2$. Note that I_H^1 is vector-valued. We now define the kernels of I_H^1 and I_H^2

$$V_\mathrm{f}^1 := \{v \in V_h^1 : I_H^1 v = 0\}, \quad V_\mathrm{f}^2 := \{v \in V_h^2 : I_H^2 v = 0\}$$

The kernels are fine scale spaces in the sense that they contain all features that are not captured by the (coarse) finite element spaces V_H^1 and V_H^2. Now, we introduce a Ritz projection onto the fine scale spaces. For this we use the bilinear forms associated with the diffusion in (3)-(4). The projection of interest is thus $R_\mathrm{f} : V_h^1 \times V_h^2 \to V_\mathrm{f}^1 \times V_\mathrm{f}^2$, such that for all $(v_1, v_2) \in V_h^1 \times V_h^2$, $R_\mathrm{f}(v_1, v_2) = (R_\mathrm{f}^1 v_1, R_\mathrm{f}^2 v_2)$ fulfills

$$(\sigma(v_1 - R_\mathrm{f}^1 v_1) : \varepsilon(w_1)) = 0, \quad \forall w_1 \in V_\mathrm{f}^1, \tag{7}$$

$$(\kappa\nabla(v_2 - R_\mathrm{f}^2 v_2), \nabla w_2) = 0, \quad \forall w_2 \in V_\mathrm{f}^2. \tag{8}$$

Note that this is an uncoupled system and R_f^1 and R_f^2 are classical Ritz projections. We define the multiscale spaces

$$V_\mathrm{ms}^1 := \{v - R_\mathrm{f}^1 v : v \in V_H^1\}, \quad V_\mathrm{ms}^2 := \{v - R_\mathrm{f}^2 v : v \in V_H^2\}. \tag{9}$$

We can now formulate the multiscale method; for $n \in \{1, ..., N\}$ find $\tilde{u}_\mathrm{ms}^n = u_\mathrm{ms}^n + u_\mathrm{f}^n$, with $u_\mathrm{ms}^n \in V_\mathrm{ms}^1$, $u_\mathrm{f}^n \in V_\mathrm{f}^1$, and $\theta_\mathrm{ms}^n \in V_\mathrm{ms}^2$, such that

$$(\sigma(\tilde{u}_\mathrm{ms}^n) : \varepsilon(v_1)) - (\alpha\theta_\mathrm{ms}^n, \nabla \cdot v_1) = (f^n, v_1), \quad \forall v_1 \in V_\mathrm{ms}^1, \tag{10}$$

$$(\bar{\partial}_t\theta_\mathrm{ms}^n, v_2) + (\kappa\nabla\theta_\mathrm{ms}^n, \nabla v_2) + (\alpha\nabla \cdot \bar{\partial}_t\tilde{u}_\mathrm{ms}^n, v_2) = (g^n, v_2), \quad \forall v_2 \in V_\mathrm{ms}^2, \tag{11}$$

$$(\sigma(u_\mathrm{f}^n) : \varepsilon(w_1)) - (\alpha\theta_\mathrm{ms}^n, \nabla \cdot w_1) = 0, \quad \forall w_1 \in V_\mathrm{f}^1. \tag{12}$$

where $\theta_{\mathrm{ms}}^0 = R_{\mathrm{ms}}^2\theta_h^0$. Furthermore, we define $\tilde{u}_{\mathrm{ms}}^0 := u_{\mathrm{ms}}^0 + u_{\mathrm{f}}^0$, where $u_{\mathrm{f}}^0 \in V_{\mathrm{f}}^1$ is defined by (12) for $n = 0$ and $u_{\mathrm{ms}}^0 \in V_{\mathrm{ms}}^1$, such that

$$(\sigma(\tilde{u}_{\mathrm{ms}}^0) : \varepsilon(v_1)) - (\alpha\theta_{\mathrm{ms}}^0, \nabla \cdot v_1) = (f^0, v_1), \quad \forall v_1 \in V_{\mathrm{ms}}^1. \tag{13}$$

The term u_{f}^n accounts for variations in the coupling coefficient α which is not affecting the multiscale spaces.

The multiscale basis functions can be proven to decay exponentially. This allows for localizetion of the computation of the multiscale basis functions to vertex patches of k layers of coarse elements with $k \approx \log(H^{-1})$. We let the localized approximations be denoted by $\{\tilde{u}_{\mathrm{ms},k}^n\}_{n=1}^N$ and $\{\theta_{\mathrm{ms},k}^n\}_{n=1}^N$.

4 ERROR ANALYSIS

The main theoretical result is an a priori error bound presented in the following theorem.

Theorem 4.1. *Let* $\{u_h^n\}_{n=1}^N$ *and* $\{\theta_h^n\}_{n=1}^N$ *be the solution to* (5)-(6) *and* $\{\tilde{u}_{\mathrm{ms},k}^n\}_{n=1}^N$ *and* $\{\theta_{\mathrm{ms},k}^n\}_{n=1}^N$ *the solution to the localized version of* (10)-(12). *For* $n \in \{1, ..., N\}$ *we have*

$$\begin{aligned}\|u_h^n - \tilde{u}_{\mathrm{ms},k}^n\|_{H^1} + \|\theta_h^n - \theta_{\mathrm{ms},k}^n\|_{H^1} \leq C(H + k^{d/2}\xi^k)\big(&\|g\|_{L_\infty(L_2)} + \|\dot{g}\|_{L_\infty(H^{-1})} \\ &+ \|f\|_{L_\infty(L_2)} + \|\dot{f}\|_{L_\infty(L_2)} + \|\ddot{f}\|_{L_\infty(H^{-1})} \\ &+ t_n^{-1/2}\|\theta_h^0\|_{H^1}\big).\end{aligned}$$

We will also provide numerical examples which confirms our theoretical findings.

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