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ON A POSTERIORI ESTIMATES FOR C^1 KIRCHHOFF ELEMENTS

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Summary. We study a posteriori estimates for C^1 continuous Kirchhoff elements in order to perform an a posteriori analysis for the Kirchhoff plate bending obstacle problem.

1 INTRODUCTION

The Kirchhoff plate bending problem is a classical fourth-order partial differential equation that describes the deflection of a thin elastic plate [1,2,3]. Combined with an inequality constraint for the deflection the related minimization problem leads to a fourth-order obstacle problem [4,5]. A priori and a posteriori analyses for various conforming and non-conforming finite element methods for the obstacle problem of a clamped Kirchhoff plate were recently studied by Brenner et al. [6,7,8].

We wish to derive a posteriori error estimates for the Kirchhoff plate bending obstacle problem with general boundary conditions. In order to successfully perform the analysis for the obstacle problem, one must fully understand the linear source problem. Finite element methods for the source problem have been studied for example by Beirao da Veiga et al. [9,10,11]. Our goal is to derive local upper and lower bounds for the error in a problem that combines clamped, free and simply-supported boundaries.

2 PROBLEM STATEMENT

Let $\Omega \subset \mathbb{R}^2$ be a convex polygon describing the midsurface of the plate with thickness d. The plate is clamped on $\Gamma_C \subset \partial\Omega$, simply supported on $\Gamma_S \subset \partial\Omega$ and free on $\Gamma_F \subset \partial\Omega$. The boundary satisfies $\partial\Omega = \overline{\Gamma_C} \cup \overline{\Gamma_S} \cup \overline{\Gamma_F}$.

Let $V = \{w \in H^2(\Omega) : w|_{\Gamma_C \cup \Gamma_S} = 0,\ \nabla w \cdot \boldsymbol{n}|_{\Gamma_C} = 0\}$. Define the bilinear form $a : V \times V \to \mathbb{R}$ and linear form $l : V \to \mathbb{R}$ as

$$a(w,v) = \int_\Omega \frac{d^3}{12}\mathbb{C}\,\boldsymbol{E}(\nabla w) : \boldsymbol{E}(\nabla v)\,\mathrm{d}x, \qquad l(w) = \int_\Omega f v\,\mathrm{d}x, \tag{1}$$

where

$$\boldsymbol{E}(\boldsymbol{v}) = \frac{1}{2}(\nabla \boldsymbol{v} + \nabla \boldsymbol{v}^T), \qquad \mathbb{C}\,\boldsymbol{A} = \frac{E}{1+\nu}\left(\boldsymbol{A} + \frac{\nu}{1-\nu}(\operatorname{tr}\boldsymbol{A})\boldsymbol{I}\right), \tag{2}$$

and f is the loading. The variational formulation reads: find $u \in V$ such that

$$a(u,v) = l(v) \quad \forall v \in V. \tag{3}$$

3 FINITE ELEMENT METHOD

We consider residual a posteriori estimates of conforming finite element approximations to (3). We apply C^1 continuous finite elements such as Argyris quintic triangle or Hsieh-Clough-Tocher macro element.

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