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Isoperimetric inequalities for Schatten norms of Riesz potentials ☆

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In this note we prove that the ball is a maximiser for integer order Schatten p-norms of the Riesz potential operators among all domains of a given measure in $\mathbb{R}^d$. In particular, the result is valid for the polyharmonic Newton potential operator, which is related to a nonlocal boundary value problem for the poly-Laplacian extending the one considered by M. Kac in the case of the Laplacian, so we obtain isoperimetric inequalities for its eigenvalues as well, namely, analogues of Rayleigh–Faber–Krahn and Hong–Krahn–Szegö inequalities.

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1. Introduction

1.1. Let $\Omega \subset \mathbb{R}^d$ be a set with finite nonzero Lebesgue measure. In $L^2(\Omega)$ we consider the Riesz potential operators

$$(\mathcal{R}_{\alpha,\Omega}f)(x) := \int_\Omega \varepsilon_{\alpha,d}(|x-y|)f(y)dy, \quad x \in \Omega, \quad f \in L^2(\Omega), \quad 0 < \alpha < d, \tag{1.1}$$

where

$$\varepsilon_{\alpha,d}(|x-y|) = c_{\alpha,d}|x-y|^{\alpha-d} \tag{1.2}$$

and $c_{\alpha,d}$ is a positive constant,

$$c_{\alpha,d} = 2^{\alpha-d}\pi^{-d/2}\frac{\Gamma(\alpha/2)}{\Gamma((d-\alpha)/2)}.$$

The operator $\mathcal{R}_{\alpha,\Omega}$ generalises the Riemann–Liouville ones to several variables and the Newton potential operators to fractional orders. Since $c_{\alpha,d}|x|^{\alpha-d}$ is the Fourier transform of the function $|\xi|^{-\alpha}$ in $\mathbb{R}^d$, it is the fundamental solution of $(-\Delta)^{\alpha/2}$, i.e. $(-\Delta_y)^{\alpha/2}\varepsilon_{\alpha,d}(x-y) = \delta_x$. In particular, for an even integer $\alpha = 2m$ with $0 < m < d/2$, the function

$$\varepsilon_{2m,d}(|x|) = c_{2m,d}|x|^{2m-d}, \tag{1.3}$$

is the fundamental solution to the polyharmonic equation of order $2m$ in $\mathbb{R}^d$.

So, the polyharmonic Newton potential

$$(\mathcal{L}^{-1}_{2m,\Omega}f)(x) := \int_\Omega \varepsilon_{2m,d}(|x-y|)f(y)dy,\ f \in L^2(\Omega), \tag{1.4}$$

is a particular case of the Riesz potential,

$$\mathcal{L}^{-1}_{2m,\Omega} = \mathcal{R}_{2m,\Omega}. \tag{1.5}$$

In the case $m = 1$, i.e., for the Laplacian, under the assumption of a sufficient regularity of the boundary of Ω (for example, piecewise C^1), it is known, see for example Mark Kac [13], that the equation

$$u(x) = (\mathcal{L}^{-1}_{2,\Omega}f)(x) = \int_\Omega \varepsilon_{2,d}(|x-y|)f(y)dy \tag{1.6}$$

is equivalent to the equation

$$-\Delta u(x) = f(x), \quad x \in \Omega, \tag{1.7}$$

with the following nonlocal boundary condition

$$-\frac{1}{2}u(x) + \int\limits_{\partial\Omega} \frac{\partial \varepsilon_{2,d}(|x-y|)}{\partial n_y} u(y) dS_y - \int\limits_{\partial\Omega} \varepsilon_{2,d}(|x-y|) \frac{\partial u(y)}{\partial n_y} dS_y = 0, \ x \in \partial\Omega, \tag{1.8}$$

where $\frac{\partial}{\partial n_y}$ denotes the outer normal derivative at the point $y \in \partial\Omega$. This approach was further expanded in Kac's book [14] with interesting applications, in particular, to the Weyl spectral asymptotics of the eigenvalue counting function for the Laplacian, see more in [15] and [27].

Considerations in the present note, as it concerns integer values $m \geq 1$, take care of a generalization of the boundary value problem (1.7)–(1.8). Moreover, for noninteger values of m (i.e., $\alpha \notin 2\mathbb{Z}$), the operator (1.1) acts as the interior term in the resolvent for boundary problems for the fractional power of the Laplacian, see, e.g. [28].

1.2. We are interested in questions of spectral geometry. The main reason why the results are useful, beyond the intrinsic interest of geometric extremum problems, is that they produce *a priori* bounds for spectral invariants of operators on arbitrary domains. For a good general review of isoperimetric inequalities for the Dirichlet, Neumann and other Laplacians we can refer to [1].

This note is motivated in part by a recent paper of Miyanishi and Suzuki [20], where it has been proved that the Schatten norm of the double layer potential is minimised on a circle among all C^2 domains with a given area in $\mathbb{R}^2$. In the case of $\Omega \subset \mathbb{R}^2$, a similar extremum problem for the logarithmic potential has been investigated by the second- and third named authors in [23]. There have been more recent developments of this case in [29].

The main results of this note consist in showing that (under certain restrictions for indices) the Schatten norms of the Riesz potentials $\mathcal{R}_{\alpha,\Omega}$ over sets of a given measure are maximised on balls. More precisely, we can summarise our results as follows:

- Let $0 < \alpha < d$ and let Ω^* be a ball in $\mathbb{R}^d$; we set $p_0 := d/\alpha$. Then for any integer p with $p_0 < p \leq \infty$ we have

$$\|\mathcal{R}_{\alpha,\Omega}\|_p \leq \|\mathcal{R}_{\alpha,\Omega^*}\|_p, \tag{1.9}$$

  for any domain Ω with $|\Omega| = |\Omega^*|$. Here $\|\cdot\|_p$ stands for the Schatten p-norm, $|\cdot|$ for the Lebesgue measure. The proof is based on the application of a suitably adapted Brascamp–Lieb–Luttinger inequality. Note that for $p = \infty$ this result gives a variant of the famous Rayleigh–Faber–Krahn inequality for the Riesz potentials (and hence also the Newton potential).
- We also establish the Hong–Krahn–Szegö inequality: the supremum of the second eigenvalue of $\mathcal{R}_{\alpha,\Omega}$ among bounded open sets with a given measure is approached by

the union of two identical balls with mutual distance going to infinity but it is never achieved.

1.3. There is a vast amount of papers dedicated to the above type of results for Dirichlet, Neumann and other Laplacians, see, for example, [5,12] and references therein. For instance, the questions are still open concerning boundary value problems for the bi-Laplacian (see [12, Chapter 11]). The main difficulty arises because the resolving operators of these boundary value problems are not positive for higher powers of the Laplacian. The same is the situation for the Schatten p-norm inequalities: the result for the Dirichlet Laplacian can be obtained from Luttinger's inequality [19] but very little is known for other boundary conditions (see [9]). The Hong–Krahn–Szegö inequality for the Robin Laplacian was proved recently [17] (see [6] for further discussions). So, in general, until now there were no examples of a boundary value problem for the poly-Laplacian ($m > 1$) for which all the above results had been proved. It seems that there are no isoperimetric results for the fractional order Riesz potentials either.

We believe that Kac's boundary value problem (1.10)–(1.11) serves as the first example of such boundary value problem, for which all the above results are true. This problem describes the nonlocal boundary conditions for the poly-Laplacian corresponding to the polyharmonic Newton potential operator.

In a bounded connected domain $\Omega \subset \mathbb{R}^d$ with a piecewise C^1 boundary $\partial\Omega$, as an analogue to (1.7) we consider the polyharmonic equation

$$(-\Delta_x)^m u(x) = f(x), \quad x \in \Omega, \quad m \in \mathbb{N}. \tag{1.10}$$

To relate the polyharmonic Newton potential (1.4) to the boundary value problem (1.10) in Ω, we can use the result of [16] asserting that for each function $f \in L^2(\Omega)$, the polyharmonic Newton potential (1.4) belongs to the class $H^{2m}(\Omega)$ and satisfies, for $i = 0, 1, \ldots, m-1$, the nonlocal boundary conditions

$$\begin{aligned}
&-\frac{1}{2}(-\Delta_x)^i u(x) + \\
&+\sum_{j=0}^{m-i-1} \int\limits_{\partial\Omega} \frac{\partial}{\partial n_y}(-\Delta_y)^{m-i-1-j} \varepsilon_{2(m-i),d}(|x-y|)(-\Delta_y)^j(-\Delta_y)^i u(y)dS_y \\
&-\sum_{j=0}^{m-i-1} \int\limits_{\partial\Omega} (-\Delta_y)^{m-i-1-j} \varepsilon_{2(m-i),d}(|x-y|)\frac{\partial}{\partial n_y}(-\Delta_y)^j(-\Delta_y)^i u(y)dS_y = 0, \\
&x \in \partial\Omega.
\end{aligned} \tag{1.11}$$

Conversely, if a function $u \in H^{2m}(\Omega)$ satisfies (1.10) and the boundary conditions (1.11) for $i = 0, 1, \ldots, m-1$, then it defines the polyharmonic Newton potential by the formulae (1.4).

Therefore, our analysis (of the special case of the Riesz potential) of the polyharmonic Newton potential (1.4) implies corresponding result for the boundary value problem (1.10)–(1.11). Note that the analogue of the problem (1.10)–(1.11) for the Kohn Laplacian and its powers on the Heisenberg group have been recently investigated in [24]. We note that there are certain interesting questions concerning such operators, lying beyond Schatten classes properties, see e.g. [10] for different regularised trace formulae.

1.4. In Section 2 we discuss main spectral properties of the Riesz potential operator and formulate the main results of this paper. Their proof will be given in Sections 3 and 4.

Some proofs of this paper can be easily extended to yield similar results for more general convolution type integral operators generalising the case of the Riesz transforms. We refer to [25] for the corresponding formulations.

Estimates for the first and second eigenvalues in this paper can be obtained also for Riesz and Riesz type transforms in the framework of spherical and hyperbolic geometries, see [26] for the corresponding results.

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2. Main results

2.1. The operator $\mathcal{R}_{\alpha,\Omega}$ and its properties

In a set $\Omega \subset \mathbb{R}^d$ with finite nonzero Lebesque measure we study the spectral problem of the Riesz potentials

$$\mathcal{R}_{\alpha,\Omega}u = \int_{\Omega} \varepsilon_\alpha(|x-y|)u(y)dy = \lambda u,\ u \in L^2(\Omega), \tag{2.1}$$

where

$$\varepsilon_{\alpha,d}(|x-y|) := c_{\alpha,d}|x-y|^{\alpha-d}$$

and $0 < \alpha < d$. (We may sometimes drop the subscripts α, d and Ω in the notation of the operator and the kernel, provided this does not cause a confusion.) Recall that in the case of the Newton potential operator it is the same as considering the spectrum of the operator corresponding to the boundary value problem (1.10)–(1.11), which we call $\mathcal{L} = \mathcal{L}_{2m,\Omega}$, under the assumption that $\Omega \subset \mathbb{R}^d$ is a bounded connected domain with piecewise C^1 continuous boundary $\partial\Omega$, that is,

$$(-\Delta_x)^m u(x) = \lambda^{-1}u(x),\ x \in \Omega,\ m \in \mathbb{N}, \tag{2.2}$$

with the nonlocal boundary conditions (1.11).

Recall that Ω has finite Lebesgue measure. The well-known Schur test shows immediately that $\mathcal{R}_{\alpha,\Omega}$ is bounded in $L^2(\Omega)$. Moreover, it will be shown soon that this operator is compact in $L^2(\Omega)$ as well and belongs to certain Schatten classes $\mathfrak{S}^p$. Since the Riesz kernel is symmetric, the operator $\mathcal{R}_{\alpha,\Omega}$ is self-adjoint.

Recall that the norm in Schatten class $\mathfrak{S}^p$ (the p-norm) of a compact operator T is defined as

$$\|T\|_p = \left(\sum_{j=1}^{\infty} s_j(T)^p \right)^{\frac{1}{p}}, \quad 1 \leq p < \infty, \tag{2.3}$$

for $s_1 \geq s_2 \geq ... > 0$ being the singular values of T. For $p = \infty$, it is usual to set

$$\|T\|_\infty := \|T\|,$$

i.e., the operator norm of T in $L^2(\Omega)$. For compact self-adjoint operators the singular values are equal to the moduli of (nonzero) eigenvalues, and the corresponding eigenfunctions form a complete orthogonal basis on L^2. Additionally, if the operator is nonnegative, the words 'moduli of' in the previous sentence may be deleted.

The eigenvalues of $\mathcal{R}_{\alpha,\Omega}$ may be enumerated in the descending order of their moduli,

$$|\lambda_1| \geq |\lambda_2| \geq ...$$

where λ_j is repeated in this series according to its multiplicity. We denote the corresponding eigenfunctions by $u_1, u_2, \ldots$, so that for each eigenvalue λ_j one and only one corresponding (normalised) eigenfunction u_j is fixed,

$$\mathcal{R}_{\alpha,\Omega} u_j = \lambda_j u_j, \quad j = 1, 2, ...$$

The following proposition asserts that the operator $\mathcal{R}_{\alpha,\Omega}$ is compact and evaluates the decay rate of its singular numbers.

Proposition 2.1. *Let $\Omega \subset \mathbb{R}^d$ be a measurable set with finite Lebesgue measure, $0 < \alpha < d$. Then*

(1) *The operator $\mathcal{R}_{\alpha,\Omega}$ is nonnegative; this means, in particular, that all eigenvalues are nonnegative,*

$$\lambda_j \equiv \lambda_j(\mathcal{R}_{\alpha,\Omega}) = |\lambda_j(\mathcal{R}_{\alpha,\Omega})| = s_j.$$

(2) *For the eigenvalues λ_j the following estimate holds:*

$$\lambda_j \leq C|\Omega|^{\vartheta} j^{-\vartheta},$$

where $\vartheta = \alpha/d$. (In particular, this implies the compactness of the operator.)

Proof. We start by recalling that

$$\varepsilon_{\alpha',d} * \varepsilon_{\alpha'',d}(|x-y|) \equiv \int_{\mathbb{R}^d} \varepsilon_{\alpha',d}(|x-z|)\varepsilon_{\alpha'',d}(|z-y|)dz = (2\pi)^d \varepsilon_{\alpha'+\alpha'',d}(|x-y|), \quad (2.4)$$

as soon as $0 < \alpha', \alpha'' < \alpha' + \alpha'' < d$. Since $|\xi|^{-(\alpha'+\alpha'')} = |\xi|^{-\alpha'}|\xi|^{-\alpha''}$, this well known relation follows, for example, from the fact, already mentioned, that $\varepsilon_{\alpha,d}$ is the Fourier transform of $|\xi|^{-\alpha}$, and from the relation between the Fourier transform of a product and the convolution of the Fourier transforms. We denote by χ_Ω the characteristic function of the set Ω. Consider the operator

$$\tilde{\mathcal{R}}_{\alpha,\Omega}: \quad f \in L^2(\mathbb{R}^d) \mapsto \chi_\Omega(x) \int_{\mathbb{R}^d} \varepsilon_{\alpha,d}(|x-y|)\chi_\Omega(x)(y)f(y)dy \in L^2(\mathbb{R}^d). \quad (2.5)$$

In the direct sum decomposition $L^2(\mathbb{R}^d) = L^2(\Omega) \oplus L^2(\mathbb{R}^d \setminus \Omega)$ the operator $\tilde{\mathcal{R}}_{\alpha,\Omega}$ is represented as $\mathcal{R}_{\alpha,\Omega} \oplus \mathbf{0}$, therefore the (nonzero) singular numbers of operators $\tilde{\mathcal{R}}_{\alpha,\Omega}$ and $\mathcal{R}_{\alpha,\Omega}$ coincide. Due to (2.4), the operator $\tilde{\mathcal{R}}_{\alpha,\Omega}$ can be represented as $(2\pi)^d T^*T$, where $T : L^2(\mathbb{R}^d) \to L^2(\mathbb{R}^d)$,

$$Tf(x) = \int \varepsilon_{\alpha/2,d}(|x-y|)\chi_\Omega(y)f(y)dy. \quad (2.6)$$

The above relations show that, in fact, the operator $\tilde{\mathcal{R}}_{\alpha,\Omega} = T^*T$ and, further on, the operator $\mathcal{R}_{\alpha,\Omega}$ are nonnegative; this proves the first statement in the Proposition. Further on, the eigenvalues of $\mathcal{R}_{\alpha,\Omega}$ equal the squares of the singular numbers of T. Now we can apply the Cwikel estimate, see [7], concerning the singular numbers estimates for integral operators with kernel of the form $h(x-y)g(y)$. In our case, $h = \varepsilon_{\alpha/2,d}$, $g = \chi_\Omega$, and thus the Cwikel estimate (1) in [7], with $p = 2d/\alpha$ gives

$$s_j(T) \le Cj^{-1/p}\|\chi_\Omega\|_{L^p} = Cj^{-\alpha/(2d)}|\Omega|^{\alpha/(2d)}, \quad (2.7)$$

with certain constant $C = C(\alpha, d)$ and, finally,

$$s_j(\mathcal{R}_{\alpha,\Omega}) \le Cj^{-\theta}|\Omega|^\theta, \quad \theta = \alpha/d. \quad (2.8)$$

The proof is complete. □

Remark 2.2. Actually, for the operator T, and therefore, for the operator $\mathcal{R}_{\alpha,\Omega}$, the estimate (2.8) is accompanied by the asymptotic formula, $s_j(T) \sim Cj^{-\alpha/(2d)}|\Omega|^{\alpha/(2d)}$, with an explicitly given constant C. For a bounded set Ω, this asymptotics is a particular case of general results of M. Birman–M. Solomyak's paper [2] concerning integral operators with weak polarity in the kernel. One can easily dispose of this boundedness condition, using the estimate (2.8) and the asymptotic approximation procedure, like this was done many times since early 70s, for example, in [3] and [21].

It follows from Proposition 2.1 that the operator $\mathcal{R}_{\alpha,\Omega}$ belongs to every Schatten class $\mathfrak{S}^p$ with $p > p_0 = \alpha/d$ and

$$\|\mathcal{R}_{\alpha,\Omega}\|_p = \left(\sum_{j=1}^{\infty} \lambda_j(\Omega)^p\right)^{\frac{1}{p}}, \ 1 \le p < \infty. \tag{2.9}$$

In the course of our further analysis, we will need to calculate the trace of certain trace class integral operators. For a positive trace class integral operator $\mathbf{K}$ with continuous kernel $K(x,y)$ on a nice set Ω, it is well known that $\mathrm{Tr}(\mathbf{K}) = \int_\Omega K(x,x)dx$. This result cannot be used in our more general setting, since our set Ω is not supposed to be nice and we cannot grant the continuity of the kernels in question. So we need some additional work.

For an operator $\mathbf{K}$ with kernel $K(x,y)$ an exact criterium for membership in $\mathfrak{S}^p$ in terms of the kernel exists only for $p = 2$, i.e., for Hilbert–Schmidt operators (but see also conditions for Schatten classes in terms of the regularity of the kernel in [8] and in Remark 2.5). Namely for $\mathbf{K}$ to belong to $\mathfrak{S}^2$, it is necessary and sufficient that $\iint_{\Omega\times\Omega} |K(x,y)|^2 dxdy < \infty$, moreover, $\|\mathbf{K}\|_2^2$ equals exactly the above integral and for the trace class operator $\mathbf{K}^*\mathbf{K}$ the same integral equals its trace. We are going to discuss some variations on these well known properties.

First, let us recall a sufficient condition for an integral operator to belong to the Schatten class $\mathfrak{S}^p$ for $p > 2$. This condition was found independently by several mathematicians; for us, it is convenient to refer to the paper [22]. The class $L^{p,q}$ is defined as consisting of functions $K(x,y)$, $x, y \in \Omega$ such that

$$\|K\|_{L^{p,q}} = \left((|K(x,y)|^p dx)^{q/p}\right)^{1/q} < \infty.$$

The main result in [22] asserts the following.

Theorem 2.3. *Let $p > 2$, $p' = p/(p-1)$ and let the kernel K belong to $L^2(\Omega\times\Omega)$. Suppose that K and the adjoint kernel $K^*(x,y) = \overline{K(y,x)}$ belong to $L^{p',p}$. Then the integral operator $\mathbf{K}$ with kernel $K(x,y)$ belongs to the Schatten class $\mathfrak{S}^p$. Moreover,*

$$\|\mathbf{K}\|_p \le (\|K\|_{L^{p',p}} \|K^*\|_{L^{p',p}})^{\frac{1}{2}}.$$

What we, actually, need are some consequences of this Theorem, established in [11]. First of all, it is shown there, quite elementarily, that the condition $K \in L^2(\Omega\times\Omega)$ is excessive and may be removed. However, what we need most is the following result (see Theorem 2.4 in [11] for the statement in most generality).

Theorem 2.4. *Let the kernel $K(x,y)$ satisfy the conditions of Theorem 2.3 for some $p > 2$. Then for the operator $\mathbf{K}^s$, which belongs to the trace class by this theorem for any integer $s > p$, the following formula holds*

$$\operatorname{Tr}(\mathbf{K}^s) = \int\limits_{\Omega^s} \left(\prod_{k=1}^{s} K(x_k, x_{k+1}) \right) dx_1 dx_2 \dots dx_s, \; x_{s+1} \equiv x_1. \tag{2.10}$$

We apply Theorem 2.4 to our kernel $K(x,y) = \varepsilon_{\alpha,d}(|x-y|)$, $x,y \in \Omega$. Since the measure of Ω is finite, the kernel $K(x,y)$ belongs to $L^{p',p}(\Omega \times \Omega)$ for any $p > \frac{d}{\alpha}$. Therefore, for the trace of the operator $\mathbf{K}^s$ formula (2.10) holds, and thus, for $s > p_0 = \frac{d}{\alpha}$ we have

$$\begin{aligned} \sum \lambda_j(\mathcal{R}_{\alpha,\Omega})^s &= \operatorname{Tr}(\mathcal{R}^s_{\alpha,\Omega}) \\ &= \int\limits_{\Omega^s} \left(\prod_{k=1}^{s} K(x_k, x_{k+1}) \right) dx_1 dx_2 \dots dx_s, \; x_{s+1} \equiv x_1. \end{aligned} \tag{2.11}$$

Remark 2.5. For the membership in the Schatten classes $\mathfrak{S}^p$ with $p < 2$ usually a certain regularity of the kernel is required. In [8] it was shown, among other things, that if the integral kernel K of an operator $\mathbf{K}f(x) = \int_\Omega K(x,y) f(y) dy$ satisfies

$$K \in H^\mu(\Omega \times \Omega)$$

for a manifold Ω of dimension d then

$$\mathbf{K} \in \mathfrak{S}^p(L^2(\Omega)) \text{ for } p > \frac{2d}{d+2\mu}.$$

In the case of the Riesz potential with $K(x,y) = \varepsilon_{\alpha,d}(|x-y|)$ it can be readily checked that it implies that $\mathcal{R}_{\alpha,\Omega} \in \mathfrak{S}^p(L^2(\Omega))$ for $p > \frac{d}{\alpha}$.

As it was already mentioned before, if the integral kernel $K^{(s)}$ of the operator $\mathbf{K}^s$ is not continuous, the formula $\operatorname{Tr}(\mathbf{K}^s) = \int_\Omega K^{(s)}(x,x)dx$ may fail but it can be replaced by the formula (2.10) (and hence also (2.11)). However, we can mention another expression for the trace: if $\widetilde{K^{(s)}}$ denotes the averaging of $K^{(s)}$ with respect to the martingale maximal function, we have

$$\operatorname{Tr}(\mathbf{K}^s) = \int\limits_{\Omega} \widetilde{K^{(s)}}(x,x) dx,$$

where we refer to [8, Section 4] for the description of $\widetilde{K^{(s)}}$, its properties, and further references.

2.2. Formulation of main results

We now formulate the main results of this note. Here $|\Omega|$ denotes the Lebesgue measure of Ω.

Theorem 2.6. *Let Ω^* be a ball in $\mathbb{R}^d$. Let $p_0 := \frac{d}{\alpha}$. Then for any integer p with $p_0 < p \leq \infty$, we have*

$$\|\mathcal{R}_{\alpha,\Omega}\|_p \leq \|\mathcal{R}_{\alpha,\Omega^*}\|_p, \tag{2.12}$$

for any domain Ω with $|\Omega| = |\Omega^*|$.

For $p = \infty$, the statement will follow from a variant of the Rayleigh–Faber–Krahn inequality for the operator $\mathcal{R}_{\alpha,\Omega}$. For all finite integers p, Theorem 2.6 follows from the Brascamp–Lieb–Luttinger inequality for symmetric rearrangements of Ω.

We also obtain the following Hong–Krahn–Szegö inequality:

Theorem 2.7. *The maximum of the second eigenvalue $\lambda_2(\Omega)$ of $\mathcal{R}_{\alpha,\Omega}$ among all sets $\Omega \subset \mathbb{R}^d$ with a given measure is approached by the union of two identical balls with mutual distance going to infinity.*

Similar result for the Dirichlet Laplacian is called the Hong–Krahn–Szegö inequality. See, for example, [5] and [17] for further references. We also refer to [6] which deals with the second eigenvalue of a nonlocal (and nonlinear) p-Laplacian operator. We note that in Theorem 2.7 we have $\lambda_2(\Omega) > 0$ since all the eigenvalues of $\mathcal{R}_{\alpha,\Omega}$ are nonnegative (see Proposition 2.1).

3. Proof of Theorem 2.6

Since the integral kernel of $\mathcal{R}_{\alpha,\Omega}$ is positive, the statement, sometimes called Jentsch's theorem, applies, see, e.g., [21].

Lemma 3.1. *The eigenvalue λ_1 of $\mathcal{R}_{\alpha,\Omega}$ with the largest modulus is positive and simple; the corresponding eigenfunction u_1 can be chosen to be positive, and any other eigenfunction u_j, $j > 1$, is sign changing in Ω.*

(Note that the positivity of λ_1 is already known, since the operator $\mathcal{R}_{\alpha,\Omega}$ is nonnegative; what is important now, it is the positivity of u_1.)

We will also use this Lemma in Section 4. Recall that we have already established in Proposition 2.1 that all $\lambda_j(\Omega)$, $i = 1, 2, \ldots$, are positive for any domain Ω.

Now we prove the following analogue of Rayleigh–Faber–Krahn theorem for the operator $\mathcal{R}_{\alpha,\Omega}$. We will use this fact further on, in Section 4. See [1] for a general discussion on this subject.

Lemma 3.2. *The ball Ω^* is a maximiser of the first eigenvalue of the operator $\mathcal{R}_{\alpha,\Omega}$ among all domains of a given volume, i.e.*

$$0 < \lambda_1(\Omega) \leq \lambda_1(\Omega^*)$$

for an arbitrary domain $\Omega \subset \mathbb{R}^d$ with $|\Omega| = |\Omega^*|$.

Remark 3.3. In other words Lemma 3.2 says that the operator norm of $\mathcal{R}_{\alpha,\Omega}$ is maximised on the ball among all Euclidean domains with a given volume.

Proof of Lemma 3.2. Recall that Ω is a bounded measurable set in $\mathbb{R}^d$. Its symmetric rearrangement Ω^* is the ball centred at 0 with the measure equal to the measure of Ω, i.e. $|\Omega^*| = |\Omega|$. Let u be a nonnegative measurable function in Ω, such that all its positive level sets have finite measure. With the definition of the symmetric-decreasing rearrangement of u we can use the layer-cake decomposition [18], which expresses a nonnegative function u in terms of its level sets as

$$u(x) = \int_0^\infty \chi_{\{u(x)>t\}} dt, \tag{3.1}$$

where χ is the characteristic function of the corresponding domain. The function

$$u^*(x) := \int_0^\infty \chi_{\{u(x)>t\}^*} dt \tag{3.2}$$

is called the (radially) symmetric-decreasing rearrangement of a nonnegative measurable function u.

Recalling the Riesz inequality [18] and the fact that $\varepsilon_\alpha(|x-y|)$ is a symmetric-decreasing function, we obtain

$$\int_\Omega \int_\Omega u_1(y)\varepsilon_\alpha(|y-x|)u_1(x)dydx \leq \int_{\Omega^*} \int_{\Omega^*} u_1^*(y)\varepsilon_\alpha(|y-x|)u_1^*(x)dydx. \tag{3.3}$$

In addition, for each nonnegative function $u \in L^2(\Omega)$ we have

$$\|u\|_{L^2(\Omega)} = \|u^*\|_{L^2(\Omega^*)}. \tag{3.4}$$

Therefore, from (3.3), (3.4) and the variational principle for $\lambda_1(\Omega^*)$, we get

$$\lambda_1(\Omega) = \frac{\int_\Omega \int_\Omega u_1(y)\varepsilon_\alpha(|y-x|)u_1(x)dydx}{\int_\Omega |u_1(x)|^2 dx} \leq$$
$$\frac{\int_{\Omega^*} \int_{\Omega^*} u_1^*(y)\varepsilon_\alpha(|y-x|)u_1^*(x)dydx}{\int_{\Omega^*} |u_1^*(x)|^2 dx} \leq$$
$$\sup_{v \in L^2(\Omega^*), v \neq 0} \frac{\int_{\Omega^*} \int_{\Omega^*} v(y)\varepsilon_\alpha(|y-x|)v(x)dydx}{\int_{\Omega^*} |v(x)|^2 dx} = \lambda_1(\Omega^*),$$

completing the proof. $\square$

Now we can finish the *Proof* of Theorem 2.6. For integer values of $p > p_0$ we have by Theorem 2.4:

$$\sum_{j=1}^{\infty} \lambda_j^p(\Omega) = \int_{\Omega} \dots \int_{\Omega} \varepsilon_\alpha(|y_1 - y_2|)\dots\varepsilon_\alpha(|y_p - y_1|)dy_1\dots dy_p, \;\; p > p_0, \;\; p \in \mathbb{N}. \tag{3.5}$$

It follows from the Brascamp–Lieb–Luttinger [4] inequality that

$$\begin{aligned} &\int_{\Omega^p} \varepsilon_\alpha(|y_1 - y_2|)\dots\varepsilon_\alpha(|y_p - y_1|)dy_1\dots dy_p \le \\ &\int_{\Omega^*} \dots \int_{\Omega^*} \varepsilon_\alpha(|y_1 - y_2|)\dots\varepsilon_\alpha(|y_p - y_1|)dy_1\dots dy_p, \end{aligned} \tag{3.6}$$

which proves

$$\sum_{j=1}^{\infty} \lambda_j^p(\Omega) \le \sum_{j=1}^{\infty} \lambda_j^p(\Omega^*), \;\; p \in \mathbb{N}, \;\; p > p_0, \tag{3.7}$$

for $\Omega \subset \mathbb{R}^d$ with $|\Omega| = |\Omega^*|$. Here we have used that the kernel ε_α is a symmetric-decreasing function in $\Omega^* \times \Omega^*$, i.e.

$$\varepsilon_\alpha^*(|x - y|) = \varepsilon_\alpha(|x - y|), \quad x, y \in \Omega^* \times \Omega^*.$$

The proof is complete.

4. Proof of Theorem 2.7

To prove Theorem 2.7 we will use the classical two-ball trick, Lemma 3.1 and Lemma 3.2.

Proof. Let us introduce the following sets:

$$\Omega^+ := \{x : u_2(x) > 0\}, \;\; \Omega^- := \{x : u_2(x) < 0\}.$$

Therefore,

$$\begin{aligned} u_2(x) > 0, \;\; \forall x \in \Omega^+ \subset \Omega, \;\; \Omega^+ \ne \{\emptyset\}, \\ u_2(x) < 0, \;\; \forall x \in \Omega^- \subset \Omega, \;\; \Omega^- \ne \{\emptyset\}, \end{aligned}$$

and it follows from Lemma 3.1 that the domains Ω^- and Ω^+ both have positive Lebesgue measure. Taking

$$u_2^+(x) := \begin{cases} u_2(x) & \text{in } \Omega^+, \\ 0 & \text{otherwise,} \end{cases} \tag{4.1}$$

and

$$u_2^-(x) := \begin{cases} u_2(x) & \text{in } \Omega^-, \\ 0 & \text{otherwise,} \end{cases}$$

we obtain

$$\lambda_2(\Omega)u_2(x) = \int\limits_{\Omega^+} \varepsilon_\alpha(|x-y|)u_2^+(y)dy + \int\limits_{\Omega^-} \varepsilon_\alpha(|x-y|)u_2^-(y)dy, \ x \in \Omega.$$

Multiplying by $u_2^+(x)$ and integrating over Ω^+ we get

$$\begin{aligned}\lambda_2(\Omega)\int\limits_{\Omega^+} |u_2^+(x)|^2dx &= \int\limits_{\Omega^+} u_2^+(x)\int\limits_{\Omega^+} \varepsilon_\alpha(|x-y|)u_2^+(y)dydx \\ &\quad + \int\limits_{\Omega^+} u_2^+(x)\int\limits_{\Omega^-} \varepsilon_\alpha(|x-y|)u_2^-(y)dydx, \ x \in \Omega.\end{aligned}$$

The second term on the right hand side is non-positive since the integrand is non-positive. Therefore,

$$\lambda_2(\Omega)\int\limits_{\Omega^+} |u_2^+(x)|^2dx \le \int\limits_{\Omega^+} u_2^+(x)\int\limits_{\Omega^+} \varepsilon_\alpha(|x-y|)u_2^+(y)dydx,$$

that is,

$$\frac{\int_{\Omega^+} u_2^+(x)\int_{\Omega^+} \varepsilon_\alpha(|x-y|)u_2^+(y)dydx}{\int_{\Omega^+} |u_2^+(x)|^2dx} \ge \lambda_2(\Omega).$$

By the variational principle,

$$\begin{aligned}\lambda_1(\Omega^+) &= \sup_{v\in L^2(\Omega^+), v\not\equiv 0} \frac{\int_{\Omega^+} v(x)\int_{\Omega^+} \varepsilon_\alpha(|x-y|)v(y)dydx}{\int_{\Omega^+} |v(x)|^2dx} \\ &\ge \frac{\int_{\Omega^+} u_2^+(x)\int_{\Omega^+} \varepsilon_\alpha(|x-y|)u_2^+(y)dydx}{\int_{\Omega^+} |u_2^+(x)|^2dx} \ge \lambda_2(\Omega).\end{aligned}$$

Similarly, we get

$$\lambda_1(\Omega^-) \ge \lambda_2(\Omega).$$

So we have

$$\lambda_1(\Omega^+) \geq \lambda_2(\Omega),\ \lambda_1(\Omega^-) \geq \lambda_2(\Omega). \tag{4.2}$$

We now introduce B^+ and B^-, the balls of the same volume as Ω^+ and Ω^-, respectively. Due to Lemma 3.2, we have

$$\lambda_1(B^+) \geq \lambda_1(\Omega^+),\ \lambda_1(B^-) \geq \lambda_1(\Omega^-). \tag{4.3}$$

Comparing (4.2) and (4.3), we obtain

$$\min\{\lambda_1(B^+), \lambda_1(B^-)\} \geq \lambda_2(\Omega). \tag{4.4}$$

Now let us consider the set $B^+ \cup B^-$, with the balls $B^\pm$ placed at distance l, i.e.

$$l = \operatorname{dist}(B^+, B^-).$$

Denote by $u_1^\circledast$ the first positive normalised eigenfunction of $\mathcal{R}_{\alpha,B^+\cup B^-}$ and take u_+ and u_- being the first normalised eigenfunctions of each single ball, i.e., of operators $\mathcal{R}_{\alpha,B^\pm}$. We introduce the function $v^\circledast \in L^2(B^+ \cup B^-)$, which equals u_+ in B^+ and γu_- in B^-. Since the functions u_+, u_-, $u^\circledast$ are positive, it is possible to find a real number γ so that $v^\circledast$ is orthogonal to $u_1^\circledast$. Observe that

$$\int\limits_{B^+\cup B^-} \int\limits_{B^+\cup B^-} v^\circledast(x)v^\circledast(y)\varepsilon_\alpha(|x-y|)dxdy = \sum_{i=1}^{4} \mathcal{I}_i, \tag{4.5}$$

where

$$\mathcal{I}_1 := \int\limits_{B^+}\int\limits_{B^+} u_+(x)u_+(y)\varepsilon_\alpha(|x-y|)dxdy,$$
$$\mathcal{I}_2 := \int\limits_{B^+}\int\limits_{B^-} u_+(x)u_-(y)\varepsilon_\alpha(|x-y|)dxdy,$$
$$\mathcal{I}_3 := \gamma\int\limits_{B^-}\int\limits_{B^+} u_-(x)u_+(y)\varepsilon_\alpha(|x-y|)dxdy,$$
$$\mathcal{I}_4 := \gamma^2\int\limits_{B^-}\int\limits_{B^-} u_-(x)u_-(y)\varepsilon_\alpha(|x-y|)dxdy.$$

By the variational principle,

$$\lambda_2(B^+ \cup B^-) = \sup_{v\in L^2(B^+\bigcup B^-),\, v\perp u_1,\, \|v\|=1} \int\limits_{B^+\cup B^-}\int\limits_{B^+\cup B^-} v(x)v(y)\varepsilon_\alpha(|x-y|)dxdy.$$

Since by construction $v^\circledast$ is orthogonal to u_1, we get

$$\lambda_2(B^+ \cup B^-) \geq \int\limits_{B^+\cup B^-} \int\limits_{B^+\cup B^-} v^{\circledast}(x)v^{\circledast}(y)\varepsilon_\alpha(|x-y|)dxdy = \sum_{i=1}^{4} \mathcal{I}_i.$$

On the other hand, since u_+ and u_- are the first normalised eigenfunctions (by Lemma 3.1 both are positive everywhere) of each single ball B^+ and B^-, we have

$$\lambda_1(B^\pm) = \int\limits_{B^\pm} \int\limits_{B^\pm} u_\pm(x)u_\pm(y)\varepsilon_\alpha(|x-y|)dxdy$$

Summarising the above facts, we obtain

$$\lambda_2(B^+ \cup B^-) \geq \frac{\int_{B^+}\int_{B^+} u_+(x)u_+(y)\varepsilon_\alpha(|x-y|)dxdy + \gamma^2 \int_{B^-}\int_{B^-} u_-(x)u_-(y)\varepsilon_\alpha(|x-y|)dxdy + \mathcal{I}_2 + \mathcal{I}_3}{\lambda_1(B^+)^{-1}\int_{B^+}\int_{B^+} u_+(x)u_+(y)\varepsilon_\alpha(|x-y|)dxdy + \gamma^2\lambda_1(B^-)^{-1}\int_{B^-}\int_{B^-} u_-(x)u_-(y)\varepsilon_\alpha(|x-y|)dxdy}. \tag{4.6}$$

Since the kernel $\varepsilon_\alpha(|x-y|)$ tends to zero as $x \in B^\pm$, $y \in B^\mp$ and $l \to \infty$, we observe that

$$\lim_{l\to\infty} \mathcal{I}_2 = \lim_{l\to\infty} \mathcal{I}_3 = 0,$$

thus

$$\lim_{l\to\infty} \lambda_2(B^+ \bigcup B^-) \geq \max\{\lambda_1(B^+), \lambda_1(B^-)\}, \tag{4.7}$$

where $l = \mathrm{dist}(B^+, B^-)$. The inequalities (4.4) and (4.7) imply that the optimal set for λ_2 does not exist. On the other hand, taking $\Omega \equiv B^+ \bigcup B^-$ with $l = \mathrm{dist}(B^+, B^-) \to \infty$, and B^+ and B^- being identical, from the inequalities (4.4) and (4.7) we obtain

$$\begin{aligned}\lim_{l\to\infty} \lambda_2(B^+ \bigcup B^-) &\geq \min\{\lambda_1(B^+), \lambda_1(B^-)\} = \lambda_1(B^+) \\ &= \lambda_1(B^-) \geq \lim_{l\to\infty} \lambda_2(B^+ \cup B^-),\end{aligned}$$

and this implies that the maximising sequence for λ_2 is given by a disjoint union of two identical balls with mutual distance going to ∞. $\square$

References

[1] R.D. Benguria, H. Linde, B. Loewe, Isoperimetric inequalities for eigenvalues of the Laplacian and the Schrödinger operator, Bull. Math. Sci. 2 (1) (2012) 1–56.

[2] M.Sh. Birman, M.Z. Solomyak, Asymptotics of the spectrum of weakly polar integral operators, Izv. Akad. Nauk SSSR Ser. Mat. 34 (5) (1970) 1142–1158 (Russian). English translation in: Math. USSR, Izv. 4 (1971) 1151–1168.

[3] M.Sh. Birman, M.Z. Solomyak, The principal term of the spectral asymptotics for “non-smooth” elliptic problems, Funktsional. Anal. i Prilozhen. 4 (4) (1970) 1–13 (Russian). English translation in: Funct. Anal. Appl. 4 (1971) 265–275.

[4] H.J. Brascamp, E.H. Lieb, J.M. Luttinger, A general rearrangement inequality for multiple integrals, J. Funct. Anal. 17 (1974) 227–237.
[5] L. Brasco, G. Franzina, On the Hong–Krahn–Szegö inequality for the p-Laplace operator, Manuscripta Math. 141 (3–4) (2013) 537–557.
[6] L. Brasco, E. Parini, The second eigenvalue of the fractional p-Laplacian, arXiv:1409.6284, 2014.
[7] M. Cwikel, Weak type estimates for singular values and the number of bound states of Schrödinger operators, Ann. of Math. (2) 106 (1) (1977) 93–100.
[8] J. Delgado, M. Ruzhansky, Schatten classes on compact manifolds: kernel conditions, J. Funct. Anal. 267 (3) (2014) 772–798.
[9] B. Dittmar, Sums of reciprocal eigenvalues of the Laplacian, Math. Nachr. 237 (2002) 45–61.
[10] M.R. Dostanić, Regularized trace of the operator of Riesz potential type on bounded domain in $\mathbb{C}$, Adv. Math. 272 (2015) 330–349.
[11] M. Goffeng, Analytic formulas for the topological degree of non-smooth mappings: the odd-dimensional case, Adv. Math. 231 (2012) 357–377.
[12] A. Henrot, Extremum Problems for Eigenvalues of Elliptic Operators, Frontiers in Mathematics, Birkhäuser Verlag, Basel, 2006.
[13] M. Kac, On some connections between probability theory and differential and integral equations, in: Proceedings of the Second Berkeley Symposium on Mathematical Statistics and Probability, 1950, University of California Press, Berkeley and Los Angeles, 1951, pp. 189–215.
[14] M. Kac, Integration in Function Spaces and Some of Its Applications, in: Lezioni Fermiane (Fermi Lectures), Accademia Nazionale dei Lincei, Pisa, 1980.
[15] T.Sh. Kal'menov, D. Suragan, On spectral problems for the volume potential, Dokl. Math. 80 (2) (2009) 646–649.
[16] T.Sh. Kal'menov, D. Suragan, Boundary conditions for the volume potential for the polyharmonic equation, Differ. Equ. 48 (4) (2012) 604–608, translation of Differ. Uravn. 48 (4) (2012) 595–599.
[17] J. Kennedy, An isoperimetric inequality for the second eigenvalue of the Laplacian with Robin boundary conditions, Proc. Amer. Math. Soc. 137 (2) (2009) 627–633.
[18] E.H. Lieb, M. Loss, Analysis, second edition, Graduate Studies in Mathematics, vol. 14, American Mathematical Society, Providence, RI, 2001.
[19] J.M. Luttinger, Generalized isoperimetric inequalities, Proc. Natl. Acad. Sci. USA 70 (1973) 1005–1006.
[20] Y. Miyanishi, T. Suzuki, Eigenvalues and eigenfunctions of double layer potentials, arXiv:1501.03627, 2015.
[21] M. Reed, B. Simon, Methods of Modern Mathematical Physics, Vol. IV: Analysis of Operators, Academic Press, 1977.
[22] B. Russo, On the Hausdorff–Young theorem for integral operators, Pacific J. Math. 68 (1) (1977) 241–253.
[23] M. Ruzhansky, D. Suragan, Isoperimetric inequalities for the logarithmic potential operator, J. Math. Anal. Appl. 434 (2) (2016) 1676–1689.
[24] M. Ruzhansky, D. Suragan, On Kac's principle of not feeling the boundary for the Kohn Laplacian on the Heisenberg group, Proc. Amer. Math. Soc. 144 (2) (2016) 709–721.
[25] M. Ruzhansky, D. Suragan, On Schatten norms of convolution type integral operators, Russian Math. Surveys 71 (1) (2016), http://dx.doi.org/10.1070/RM9665.
[26] M. Ruzhansky, D. Suragan, On first and second eigenvalues of Riesz transforms in spherical and hyperbolic geometries, Bull. Math. Sci. (2016), http://dx.doi.org/10.1007/s13373-016-0082-5.
[27] N. Saito, Data analysis and representation on a general domain using eigenfunctions of Laplacian, Appl. Comput. Harmon. Anal. 25 (1) (2008) 68–97.
[28] B.W. Schulze, Boundary Value Problems and Singular Pseudo-Differential Operators, John Wiley and Sons, Ltd., Chichester, 1998.
[29] S. Zoalroshd, A note on isoperimetric inequalities for logarithmic potentials, J. Math. Anal. Appl. 437 (2) (2016) 1152–1158.