Robust Gain-Scheduled PSD Controller Design from Educational Perspective

Robust Gain-Scheduled PSD Controller Design from Educational Perspective ⋆

Adrian Ilka ∗, **Vojtech Veselý** ∗∗, **Tomas McKelvey** ∗

∗ *Department of Signals and Systems, Chalmers University of Technology, Göteborg SE-412 96, Sweden, (e-mails: adrian.ilka@chalmers.se, tomas.mckelvey@chalmers.se)*
∗∗ *Institute of Robotics and Cybernetics, Faculty of Electrical Engineering and Information Technology, Slovak University of Technology in Bratislava, Ilkovičova 3, 812 19 Bratislava (e-mail: vojtech.vesely@stuba.sk)*

Abstract: Nowadays, the demand for qualified engineers in novel model-based controller design techniques is rapidly increasing. Therefore, it is necessary to include these up-to-date techniques in control theory courses. We are confident that this paper provides a motivation for embedding new linear/bilinear matrix inequality (LMI/BMI) -based approaches in topic-related courses and student projects. Within this paper a systematic design framework for robust discrete-time gain-scheduled proportional, summation, and difference (PSD) controller design for uncertain linear parameter-varying systems is presented. Conditions to satisfy robust stability and performance requirements are translated to an optimization problem subject to LMI/BMI constraints which students can easily solve using commercial as well as free and open source software tools and solvers. Two student project examples are given to illustrate and validate the proposed methodology.

Keywords: Robust control, Gain scheduling, Linear parameter-varying systems, LMI, BMI, Lyapunov theory of stability, Guaranteed cost, Matlab, Octave, Scilab/Xcos.

1. INTRODUCTION

It is well known that Proportional, Integral, and Derivative (PID) and discrete-time Proportional, Summation, and Difference (PSD) controllers are extensively used in industry (Åström and Murray, 2011). However, in the past few years the interest in advanced PID/PSD controller design techniques has increased gradually. Therefore, today's industry need more and more qualified engineers trained specifically in working with these new techniques. In order to meet these new demands, it is necessary to include up-to-date controller design techniques in control theory courses.

The robust control theory (Veselý et al., 2015) is well established for linear systems but almost all real processes are more or less nonlinear. If the plant's operating region is small, one can use the robust control approaches to design a linear robust controller where the nonlinearities are treated as model uncertainties. However, for nonlinear processes, where the operating region is large, the above mentioned controller synthesis may be inapplicable. For this reason, the controller design for nonlinear systems is nowadays a very important field of research. Gain scheduling (GS) is one of the most commonly used controller design approaches for nonlinear systems and has a wide range of use in industrial applications. Many of the early publications were associated with flight control (Adams et al., 1992; Nichols et al., 1993) and aerospace (Hyde and Glover, 1993). Then, gradually, this approach has been used almost everywhere in control engineering, which was greatly helped with the introduction of linear parameter-varying (LPV) systems. LPV systems were introduced first by Jeff S. Shamma in 1988 to model gain-scheduling. Today the LPV paradigm has become a standard formalism in systems and controls with many publications devoted to analysis, controller design and system identification of these models (Shamma, 2012).

Robust and robust gain-scheduled control belongs to important research topics at our department (Veselý and Ilka, 2013, 2015; Ilka and Veselý, 2015). Research results have always been included in topic-related courses and student projects. In this paper, we present a robust discrete-time gain-scheduled PSD controller design approach for uncertain LPV systems which guarantees the closed-loop stability and guaranteed cost for a prescribed rate of change of scheduled parameters. Students can easily design a robust gain-scheduled PSD controller using commercial as well as free and open source software tools like MATLAB (The Mathworks, Inc., 2014), Octave (Octave community, 2015), SciLab (Scilab Enterprises, 2012), and powerful linear matrix inequality (LMI) (LMILAB (Gahinet et al., 1994), SeDuMi (Sturm, 1999), SPDT3 (Toh et al., 1999)) and bilinear matrix inequality (BMI) (PENBMI (Henrion et al., 2005), PenLab (Fiala et al., 2013)) solvers with efficient programming interface YALMIP (Löfberg, 2004).

⋆ This work has been sponsored by Chalmers Area of Advance Transportation, by Vinnova under the FFI project MultiMEC and by Vinnova under FFI project VCloud II.

YALMIP is a modelling language for advanced modelling and solution of convex and nonconvex optimization problems. It is a free toolbox for MATLAB and Octave. Simulations can be done in MatLab/Simulink or in Scilab/Xcos. We are confident that this paper provides a motivation for embedding up-to-date LMI/BMI-based approaches in control education programs for which free and open source program tools and solvers can also be used.

The rest of the paper is organized in four sections. Introduction is followed by robust discrete-time GS controller design in Section 2, where the key notions are introduced and the robust discrete-time GS controller design problem is articulated with robust stability and performance conditions. Examples of student projects are given in Section 3, where the proposed methodology is validated using commercial as well as free and open source program tools and solvers. Concluding remarks close the paper in Section 4.

The mathematical notations of the paper are as follows. Given a symmetric matrix $P = P^T \in \mathbb{R}^{n\times n}$, the inequality $P > 0$ ($P \geq 0$) denotes the positive definiteness (semi-definiteness) of the matrix. Matrices, if not explicitly stated, are assumed to have compatible dimensions.

2. ROBUST DISCRETE GS CONTROLLER DESIGN

2.1 Preliminaries and problem formulation

The following class of discrete-time linear parameter varying systems is considered through the paper:

$$\begin{aligned} x(k+1) &= A(\theta(k))x(k) + B(\theta(k))u(k), \\ y(k) &= Cx(k), \end{aligned} \tag{1}$$

where $x(k) \in \mathbb{R}^n$, $u(k) \in \mathbb{R}^m$, and $y(k) \in \mathbb{R}^l$ are the state, control input, and the measured output vectors, respectively. The matrix functions $A(\theta(k)) \in \mathbb{R}^{n\times n}$ and $B(\theta(k)) \in \mathbb{R}^{n\times m}$ are assumed to depend on the scheduling variable $\theta(k) \in \langle \underline{\theta}, \overline{\theta} \rangle \in \Omega$. This variable $\theta(k) = [\alpha_i, \dots, \alpha_{N_p}, \beta_1, \dots, \beta_{Nu}]$ can be split to a part where it is assumed that the scheduling parameters $\alpha_i(k)$ $i = 1, 2, \dots, N_p$ are constant or time varying and can be measured or estimated (therefore used in the controller), and to a part, where the scheduling parameters $\beta_j(k)$, $j = 1, 2, \dots, N_u$ are constant or time varying but unknown (uncertain) parameters.

$$\begin{aligned} L(\theta(k)) &= L_0 + \sum_{i=1}^{N_p} L_i\alpha_i(k) + \sum_{j=1}^{N_u} L_{N_p+j}\beta_j(k) \\ &= L_0 + \sum_{i=i}^{p} L_i\theta_i(k) \end{aligned} \tag{2}$$

with $L(\theta(k)) = \{A(\theta(k)),\ B(\theta(k))\}$. In addition A_0, B_0, A_i, B_i, $i = 1, 2, \dots, p$, and C are constant matrices with appropriate dimensions.

The output feedback gain-scheduled control law is considered for discrete-time PSD controller in the form:

$$\begin{aligned} u(k) = \Big(& K_P(\theta(k))e(k) + K_S(\theta(k))\sum_{i=0}^{k} e(i) \\ &+ K_D(\theta(k))(e(k) - e(k-1))\Big), \end{aligned} \tag{3}$$

where $e(k) = y(k) - w(k)$ is the control error, $w(k)$ is the reference signal, and matrices $K_P(\theta(k))$, $K_S(\theta(k))$, $K_D(\theta(k))$ are controller gain matrices in the form (2) with $L(\theta(k)) = \{K_P\theta(k),\ K_S\theta(k),\ K_D\theta(k)\}$ (for SISO systems they are scalars). Note that the number of controller gain matrices is only N_p, the rest N_u are equal to zero.

With the assumption that the reference signal $w(k)$ is bounded, the control law for $w(k) = 0$ can be rewritten as follows:

$$\begin{aligned} u(k) = \Big(& K_P(\theta(k))y(k) + K_S(\theta(k))\sum_{i=0}^{k} y(i) \\ &+ K_D(\theta(k))(y(k) - y(k-1))\Big), \end{aligned} \tag{4}$$

We can extend the system with two state variables (Veselý and Rosinová, 2013):

$$z_1(k) = \textstyle\sum_{i=0}^{k-2} y(i), \quad z_2(k) = \sum_{i=0}^{k-1} y(i), \tag{5}$$

furthermore, substituting expressions $y(k-1) = z_2(k) - z_1(k)$ and $\sum_{i=1}^{k} y(k) = z_2(k) + y(k)$ to the control law (4), one can obtain:

$$\begin{aligned} u(k) = \Big(& (K_P(\theta(k)) + K_S(\theta(k)) + K_D(\theta(k)))\, y(k) \\ &+ K_S(\theta(k))z_2(k) - K_D(\theta(k))(z_2(k) - z_1(k))\Big). \end{aligned} \tag{6}$$

The control law (6) can be transformed to the following state space matrix form:

$$u(t) = F(\theta(k))\tilde{y}(k), \tag{7}$$

where $\tilde{y} = [y(k),\ z_1(k),\ z_2(k)]^T$ is the extended measured output vector and

$$F(\theta(k))^T = \begin{bmatrix} K_P(\theta(k)) + K_S(\theta(k)) + K_D(\theta(k)) \\ K_D(\theta(k)) \\ K_S(\theta(k)) - K_D(\theta(k)) \end{bmatrix}.$$

Substituting the control law (7) to the system (1), the following closed-loop system is obtained:

$$\tilde{x}(k+1) = A_{cl}(\theta(k))\tilde{x}(k), \tag{8}$$

where $\tilde{x}(k) = [x(k), z_1(k), z_2(k)]^T$ and

$$A_{cl}(\theta(k)) = A_r(\theta(k)) + B_r(\theta(k))F(\theta(k))C_r(\theta(k)),$$

$$A_r(\theta(k)) = \begin{bmatrix} A(\theta(k)), 0, 0 \\ 0, \quad 0, I \\ C, \quad 0, I \end{bmatrix},\ C_r(\theta(k)) = \begin{bmatrix} C, 0, 0 \\ 0, I, 0 \\ 0, 0, I \end{bmatrix},$$

$$B_r(\theta(k)) = [B(\theta(k)), 0, 0]^T.$$

2.2 Sufficient Stability Conditions

Quadratic Stability. To ensure quadratic stability the following Lyapunov function has been chosen:

$$V(k) = \tilde{x}(k)^T P\tilde{x}(k). \tag{9}$$

Definition 1. (Apkarian et al., 1995) The linear closed-loop system (8) for $\forall\theta \in \Omega$ is quadratically stable if there exist a symmetric positive definite matrix $P > 0$ and for the first difference of the Lyapunov function (9) along the trajectory of closed-loop system (8) holds:

$$\begin{aligned} \Delta V(\theta(k)) = \tilde{x}(k)^T \Big(& A_{cl}(\theta(k))^T P A_{cl}(\theta(k)) \\ &+ P\Big)\tilde{x}(k) < 0 \end{aligned} \tag{10}$$

Affine Quadratic Stability. To ensure affine parameter-dependent quadratic stability (Gahinet et al., 1996), the following Lyapunov function has been chosen:

$$V(\theta(k)) = \tilde{x}^T(k)P(\theta(k))\tilde{x}(k). \tag{11}$$

The first difference of the Lyapunov function (11) is given as follows:

$$\Delta V(\theta(k)) = \tilde{x}^T(k+1)\, P(\theta(k+1))\, \tilde{x}(k+1) - \tilde{x}^T(k)\, P(\theta(k))\, \tilde{x}(k), \tag{12}$$

where

$$P(\theta(k)) = P_0 + \textstyle\sum_{i=1}^{p} P_i\theta_i(k). \tag{13}$$

Substituting $\theta(k+1) = \theta(k) + \Delta\theta(k)$ to $P(\theta(k+1))$, one obtains the following result:

$$P(\theta(k+1)) = P_0 + \sum_{i=1}^{p} P_i\theta_i(k) + \sum_{i=1}^{p} P_i\Delta\theta_i(k), \tag{14}$$

where if assuming that $\Delta\theta_i \in \langle \Delta\underline{\theta_i}, \Delta\overline{\theta_i}\rangle \in \Omega_t$, $i = 0, 1, \ldots, p$, and $\max|\Delta\theta_i| < \rho_i$, one can write

$$P(\theta(k+1)) \leq P_0 + \sum_{i=1}^{p} P_i\theta_i(k) + P_\rho = P_\rho(\theta(k)), \tag{15}$$

where $P_\rho = \sum_{i=1}^{p} P_i\rho_i$.

Based on equations (12), (13), and inequality (15) the following definition can be formulated:

Definition 2. The linear closed-loop system (8) for $\theta(k) \in \Omega$ and $\Delta\theta(k) \in \Omega_t$ is affinely quadratically stable if $p+1$ symmetric matrices $P_0, P_1, \ldots, P_p$ exists such that $P(\theta(k))$ (13), $P_\rho(\theta(k))$ (15) are positive defined and for the first difference of the Lyapunov function (12) along the trajectory of closed-loop system (8) it holds:

$$\Delta V(\theta(k)) = \tilde{z}^T(k) \begin{bmatrix} V_{11}(\theta(k)), V_{12}(\theta(k)) \\ V_{12}^T(\theta(k)), V_{22}(\theta(k)) \end{bmatrix} \tilde{z}(k) \leq 0, \tag{16}$$

where $\tilde{z}^T(k) = \left[\tilde{x}^T(k+1), \tilde{x}^T(k)\right]$, furthermore

$$V_{11}(\theta(k)) = P_\rho(\theta(k)) + N_1 + N_1^T,$$
$$V_{12}(\theta(k)) = N_2 - N_1^T A_{cl}(\theta(k)),$$
$$V_{22}(\theta(k)) = -P(\theta(k)) - N_2^T A_{cl}(\theta(k)) - A_{cl}^T(\theta(k)) N_2,$$

where $N_1, N_2 \in \mathbb{R}^{n\times n}$ are auxiliary matrices.

2.3 Performance Quality

To assess the performance quality with possibility to obtain different performance quality in each working point, a quadratic cost function described in our previous paper (Ilka and Veselý, 2014) is used:

$$J_{df}(\theta(k)) = \sum_{k=0}^{\infty} \Big(\tilde{x}(k)^T Q(\theta(k))\, \tilde{x}(k) + u(k)^T R\, u(k) + \Delta\tilde{x}(k)^T S(\theta(k))\, \Delta\tilde{x}(k)\Big) = \sum_{k=0}^{\infty} J_d(\theta(k)), \tag{17}$$

where

$$Q(\theta(k)) = Q_0 + \sum_{i=1}^{p} Q_i\theta_i(k),\; S(\theta(k)) = S_0 + \sum_{i=1}^{p} S_i\theta_i(k),$$
$$Q_i = Q_i^T \geq 0, \quad S_i = S_i^T \geq 0, \quad R > 0,$$

furthermore, Q_0, Q_i, S_0, $S_i \in \mathbb{R}^{(n+2l)\times(n+2l)}$, $R \in \mathbb{R}^{m\times m}$ are symmetric positive definite (semidefinite) and definite matrices, respectively.

Definition 3. Consider the system (1) with the control algorithm (7). If a control law u^* and a positive scalar J_d^* exist such that the closed-loop system (8) is stable and the value of closed-loop cost function (17) satisfies $J_d \leq J_d^*$, then J_d^* is said to be a guaranteed cost and u^* is said to be a guaranteed cost control law for the system (1).

2.4 Robust gain-scheduled PSD controller design

The robust gain-scheduled PSD controller design is based on the following lemmas:

Lemma 1. Consider the closed-loop system (8). Closed-loop system (8) is quadratically/affinely quadratically stable with guaranteed cost if the following inequality holds:

$$B_e(\theta(k)) = \max_u \{\Delta V(\theta(k)) + J_d(\theta(k))\} \leq 0, \tag{18}$$

for $\forall\theta(k) \in \Omega$ and $\Delta\theta(k) \in \Omega_t$. For proof see (Kuncevic and Lycak, 1977).

Lemma 2. Consider a scalar quadratic function of $\theta \in \mathbb{R}^p$.

$$f(\theta_1, \ldots, \theta_p) = a_0 + \sum_{i=1}^{p} a_i\theta_i + \sum_{i=1}^{p}\sum_{j>i}^{p} b_{ij}\theta_i\theta_j + \sum_{i=1}^{p} c_i\theta_i^2,$$

and assume that $f(\theta_1, \ldots, \theta_p)$ is multi-convex, that is $\frac{\partial^2 f(\theta)}{\partial\theta_i^2} = 2c_i \geq 0$ for $i = 1, 2, \ldots, p$. Then $f(\theta)$ is negative for all $\theta \in \Omega$ if and only if it takes negative values at the corners of θ. (Gahinet et al., 1996)

Using *Lemmas 1* and *2* the following theorems are obtained:

Quadratic Stability.

Theorem 1. Closed-loop system (8) is quadratically stable with guaranteed cost if there exist a positive definite matrix $P > 0$ for all $\theta(k) \in \Omega$, a positive semidefinite symmetric matrices $G_i \geq 0$, $i = 1, \ldots, p$, matrices $Q_i, R, i = 0, 1, 2, \ldots, p$ (with assumption that for quadratic stability $S_i = 0$), and gain-scheduled controller matrices K_{P_i}, K_{S_i}, K_{D_i}, $i = 0, 1, \ldots, p$ satisfying:

$$M(\theta(k)) = M_0 + \sum_{i=1}^{p} M_i\theta_i(k) + \sum_{i=1}^{p}\sum_{j>i}^{p} M_{ij}\theta_i(k) + \sum_{i=1}^{p} M_{ii}\theta_i^2(k) \leq 0; \quad \forall\theta(k) \in \Omega, \tag{19}$$

$$M_{ii} \geq 0; \quad i = 1, 2, \ldots, p, \tag{20}$$

where $M_{ii} = M_{ij} + G_i,\; i = j$, and

$$M_0 = \begin{bmatrix} -P + Q_0, C_r^T F_0^T, & A_{cl_0}^T \\ F_0 C, \quad -R^{-1}, & 0 \\ A_{cl_0}, \quad 0, & X^{-1}(P - X)X^{-1} - X^{-1} \end{bmatrix},$$

$$M_i = \begin{bmatrix} Q_i, & C_r^T F_i^T, A_{cl_i}^T \\ F_i C_r, & 0, \quad 0 \\ A_{cl_i}, & 0, \quad 0 \end{bmatrix},\; M_{ij} = \begin{bmatrix} 0, & 0, A_{cl_{ij}}^T \\ 0, & 0, \; 0 \\ A_{cl_{ij}}, 0, & 0 \end{bmatrix},$$

$$A_{cl0} = A_{r0} + B_{r0}F_0C_r,\; A_{clij} = B_{ri}F_jC + B_{rj}F_iC_r,$$
$$A_{cli} = A_{ri} + B_{ri}F_0C_r + B_{r0}F_iC_r,\; A_{clii} = B_{ri}F_jC_r.$$

Proof. Proof is based on *Lemmas 1* and *2*. From Bellman-Lyapunov function (18) we can obtain:

$$W(\theta(k)) = A_{cl}(\theta(k))^T P A_{cl}(\theta(k)) + P + Q + C_r^T F(\theta(k))^T R F(\theta(k)) C_r \leq 0. \tag{21}$$

Using Schur complement we obtain:

$$W(\theta(k)) = \begin{bmatrix} W_{11}, W_{21}^T, W_{31}^T \\ W_{21}, W_{22}, W_{32}^T \\ W_{31}, W_{32}, W_{33} \end{bmatrix} \leq 0, \tag{22}$$

$$W_{11} = -P + Q(\theta(k)), W_{22} = -R^{-1},$$
$$W_{21} = F(\theta(k))C_r, \quad W_{32} = 0,$$
$$W_{31} = A_{cl}(\theta(k)), \quad W_{33} = -P^{-1}.$$

One can linearise the nonlinear part as follows:

$$lin(-P^{-1}) \leq X^{-1}(P-X)X^{-1} - X^{-1}, \tag{23}$$

where in each iteration holds $X_i = P_{i-1}$ (i – actual iteration step). After we extend (22) to affine form we can obtain (19) and (20) by relaxing the multi-convexity requirement:

$$M(\theta(k)) = W(\theta(k)) + \sum_{i=1}^{p} G_i\theta_i^2 \geq W(\theta(k)) \tag{24}$$

where $G_i \geq 0$, $i = 1,\dots,p$ are symmetric semi-definite auxiliary matrices.

Affine Quadratic Stability.

Theorem 2. Closed-loop system (8) is affinely quadratically stable with guaranteed cost if $p+1$ symmetric matrices $P_0, P_1, \dots, P_p$ exists such that $P(\theta(k))$ (13), $P_\rho(\theta(k))$ (15) are positive definite for all $\theta(k) \in \Omega$ with pre-defined maximal rate of change ρ_i of scheduled parameters θ_i, furthermore a positive semi-definite symmetric matrices $G_i \geq 0$, $i = 1,\dots,p$, weighting matrices Q_i, S_i, R, $i = 0,2,\dots,p$, and controller gain matrices K_{P_i}, K_{S_i}, K_{D_i}, $i = 0,1,\dots,p$ satisfying:

$$\begin{aligned} M(\theta(k)) = M_0 + &\sum_{i=1}^{p} M_i\theta_i(k) + \sum_{i=1}^{p}\sum_{j>i}^{p} M_{ij}\theta_i(k) \\ &+ \sum_{i=1}^{p} M_{ii}\theta_i^2(k) \leq 0; \quad \forall\theta(k) \in \Omega, \end{aligned} \tag{25}$$

$$M_{ii} \geq 0; \quad i = 1,2,\dots,p, \tag{26}$$

where $M_{ii} = W_{ii} + G_i$, furthermore

$$M_0 = \begin{bmatrix} W_{110}, W_{120} \\ W_{120}^T, W_{220} \end{bmatrix},\ M_i = \begin{bmatrix} W_{11i}, W_{12i} \\ W_{12i}^T, W_{22i} \end{bmatrix},$$

$$M_{ij} = \begin{bmatrix} W_{11ij}, W_{12ij} \\ W_{12ij}^T, W_{22ij} \end{bmatrix},\ W_{ii} = \begin{bmatrix} W_{11ii}, W_{12ii} \\ W_{12ii}^T, W_{22ii} \end{bmatrix},$$

$$\begin{aligned}
W_{110} &= P_0 + N_1 + N_1^T + S_0 + P_\rho, \\
W_{11i} &= P_i + S_i, \qquad W_{11ij} = W_{11ii} = 0, \\
W_{120} &= N_2 - N_1^T A_{cl_0} - S_0, \quad W_{12i} = -N_1^T A_{cl_i} - S_i, \\
W_{12ij} &= -N_1^T A_{cl_{ij}}, \qquad W_{12ii} = -N_1^T A_{cl_{ii}}, \\
W_{220} &= Q_0 + S_0 - P_0 - N_2^T A_{cl_0} - A_{cl_0}^T N_2 \\
&\quad + C_r^T F_0^T R F_0 C_r, \\
W_{22i} &= -P_i - N_2^T A_{cl_i} - A_{cl_i}^T N_2 + C_r^T F_0^T R F_i C_r \\
&\quad + C_r^T F_i^T R F_0 C_r + Q_i + S_i, \\
W_{22ij} &= -N_2^T A_{cl_{ij}} - A_{cl_{ij}}^T N_2 + C_r^T F_i^T R F_j C_r \\
&\quad + C_r^T F_j^T R F_i C_r, \\
W_{22ii} &= -N_2^T A_{cl_{ii}} - A_{cl_{ii}}^T N_2 + C_r^T F_i^T R F_i C_r, \\
A_{cl0} &= A_{r0} + B_{r0} F_0 C_r,\ A_{cl\,ij} = B_{ri} F_j C + B_{rj} F_i C_r, \\
A_{cl\,i} &= A_{ri} + B_{ri} F_0 C_r + B_{r0} F_i C_r,\ A_{cl\,ii} = B_{ri} F_j C_r.
\end{aligned}$$

Proof. Substituting the control algorithm (7) to the quadratic cost function (17) one can obtain:

$$J_d(\theta(k)) = \tilde{z}^T \begin{bmatrix} J_{11}(\theta(k))\ J_{12}(\theta(k)) \\ J_{12}^T(\theta(k))\ J_{22}(\theta(k)) \end{bmatrix} \tilde{z}, \tag{27}$$

$$J_{11}(\theta(k)) = S(\theta(k)), \quad J_{12}(\theta(k)) = -S(\theta(k)),$$
$$J_{22}(\theta(k)) = S(\theta(k)) + Q(\theta(k)) + C_r^T F(\theta(k))^T R F(\theta(k))\, C_r$$

If one substitutes the obtained cost function (27) and the first difference of Lyapunov function (16) to the Bellman-Lyapunov function (18), after some manipulation, using *Lemma 2* and by relaxing the multi-convexity requirement as in *Theorem 1*, we obtain (25) and (26) which proves *Theorem 2*.

2.5 Application in Student Projects

The application of the above described robust discrete-time gain-scheduled PSD controller design in student projects allows reaching a reasonable balance between classical and modern concepts of control theory. These projects demonstrate that control design is a complex, interactive and multi-stage process.

The controller synthesis can be done in a computationally tractable and systematic way using commercial or free and open source program tools (Matlab, Octave, Scilab, NSP), powerful LMI solvers (LMILAB, SDPT3, SeDuMi, Mosek), BMI solvers (PENBMI, PenLab) and efficient interfaces provided by YALMIP praser.

Theorems 1 and *2* allow for the designer (in our case for the student):

- to design a robust controller for uncertain discrete-time linear systems,
- to design a gain-scheduled or robust gain-scheduled controller for uncertain discrete-time linear parameter-varying systems,
- to define and set up different performance quality in each equilibrium/working point,
- to learn about and use free and open source program tools and solvers besides the commercially available ones.

3. STUDENT PROJECT EXAMPLES

In order to show the viability of the previous proposed method in student projects, the following 2 examples have been chosen.

Example 1. Consider a simple uncertain linear parameter-varying plant (Stewart, 2012):

$$\begin{aligned} \dot{x}(t) &= a(\gamma_1)x(t) + b(\gamma_1,\gamma_2)u(t), \\ y(t) &= x(t), \end{aligned} \tag{28}$$

where $\gamma_1 \in \langle 0, 100\rangle$ is a known, and $\gamma_2 \in \langle 0.9, 1.1\rangle$ is an unknown (uncertain) parameter that changes the parameters of the plant as follows:

$$a(\gamma_1) = -6 - \tfrac{2}{\pi}\arctan\left(\tfrac{\gamma_1}{20}\right), \tag{29}$$

$$b(\gamma_1,\gamma_2) = \tfrac{1}{2}\gamma_2 + \tfrac{5}{\pi}\arctan\left(\tfrac{\gamma_1}{20}\right). \tag{30}$$

The task is to design a robust PSD and robust gain-scheduled PSD controller which guarantees the closed-loop stability and guaranteed cost for the LPV plant (28) for all $\gamma_1 \in \langle 0, 100\rangle$, and $\gamma_2 \in \langle 0.9, 1.1\rangle$ using the affine quadratic stability (*Theorem 2*).

The system (28) can be transformed into the form (1) with sample time $T_s = 0.01\,s$ and rescaled scheduling parameters:

$$\theta_1 = \tfrac{\arctan(\frac{\gamma_1}{20}) - a_0}{a_1} \in \langle -1, 1\rangle,\ \theta_2 = \tfrac{\gamma_2 - b_0}{b_1} \in \langle -1, 1\rangle,$$

where the coefficients a_0, a_1, b_0, and b_1 are calculated so as to maintain the scheduling parameters in the range $\langle -1, 1\rangle$:

$$a_0 = \tfrac{\max\left(\arctan(\frac{\gamma_1}{20})\right) + \min\left(\arctan(\frac{\gamma_1}{20})\right)}{2},$$

$$a_1 = \tfrac{\max\left(\arctan(\frac{\gamma_1}{20})\right) - \min\left(\arctan(\frac{\gamma_1}{20})\right)}{2},$$

$$b_0 = \tfrac{\max(\gamma_2) + \min(\gamma_2)}{2},\ b_1 = \tfrac{\max(\gamma_2) - \min(\gamma_2)}{2}.$$

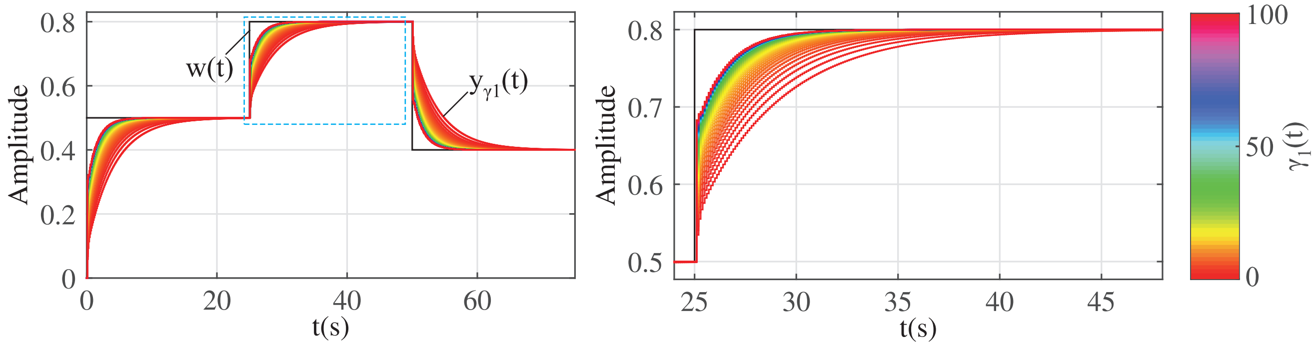

Fig. 1. Simulation results with robust PSD controller (31)

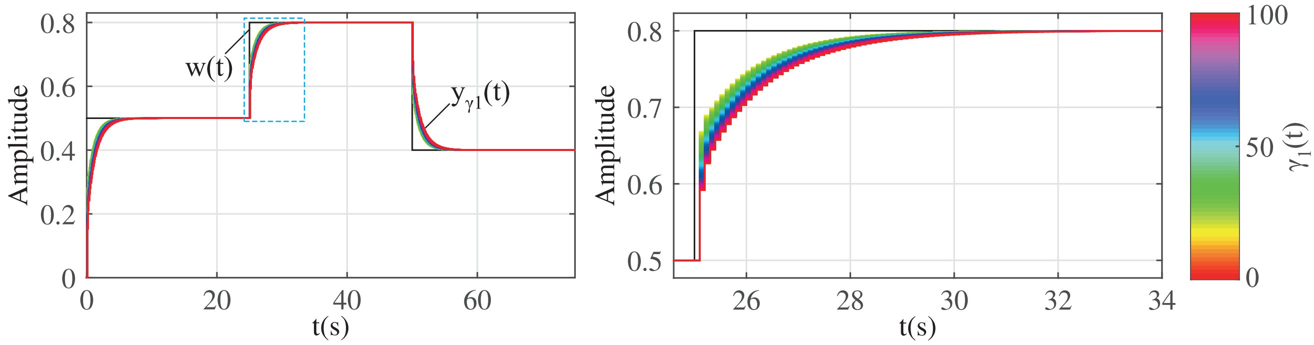

Fig. 2. Simulation results with robust gain-scheduled PSD controller (32)

The discrete-time linear parameter-varying system's matrices/scalars are as follows:

$$A_0 = 0.5258,\ A_1 = -0.0230,\ A_2 = 0,\ C = 1,$$
$$B_0 = 0.1159,\ B_1 = 0.0783,\ B_2 = 1.8079 \times 10^{-4}.$$

Using *Theorem 2* with weighting matrices $Q_i = q_i I$, $q_0 = 0.001$, $q_1 = q_2 = 0$, $S_i = s_i I$, $s_1 = 0.1$, $s_2 = s_3 = 0$, $R = rI$, $r = 0.001$, $\rho_i = 0$, $i = 1, 2$, and $\xi_L \leq P(\theta) \leq \xi_U$, $\xi_L = 1 \times 10^{-9}$, $\xi_U = 1 \times 10^3$ we obtained a robust discrete-time PSD controller in the form (3), where

$$\begin{aligned} K_P = -3.262562,\ K_S = -3.494751, \\ K_D = 1.228968 \times 10^{-1}. \end{aligned} \tag{31}$$

Using *Theorem 2* with weighting matrices $Q_i = q_i I$, $q_0 = 0.0505$, $q_1 = -0.0495$, $q_2 = 0$, $S_i = s_i I$, $s_1 = 0.0505$, $s_2 = 0.0495$, $s_3 = 0$, $R = rI$, $r = 0.001$, $\rho_i = 0$, $i = 1, 2$, and $1 \times 10^{-9} \leq P(\theta) \leq 1 \times 10^3$ we obtained a robust gain-scheduled PSD controller in the form

$$\begin{aligned} K_P(\theta) &= -4.876325 + 3.290914\,\theta_1, \\ K_S(\theta) &= -7.304019 + 4.538988\,\theta_1, \\ K_D(\theta) &= +0.044032 - 0.031514\,\theta_1. \end{aligned} \tag{32}$$

Numerical solution has been carried out by PENBMI 2.1 solver under MATLAB 2014b using YALMIP R20150918. The simulations were done via SIMULINK. Simulation results for $\gamma_2 = 1$ with the robust PSD controller (31) are shown in Fig. 1 and with the robust gain-scheduled PSD controller (32) are shown in Fig. 2.

Example 2. The following simple nonlinear system with an unstable equilibrium point has been chosen to demonstrate the robust gain-scheduled PSD controller design with quadratic stability (*Theorem 1*), using only free and open source program tools and solvers:

$$\begin{aligned} \dot{x}(t) &= -x(t)|x(t)|\gamma + u(t), \\ y(t) &= x(t), \qquad -0.5 \leq u(t) \leq 0.5, \end{aligned} \tag{33}$$

where $\gamma \in \langle 0.95, 1.05 \rangle$ is unknown (uncertain) parameter. The task is to stabilize the system under the given control input constraints. The system (33) can be transformed into the following form:

$$\begin{aligned} \dot{x}(t) &= -a(\theta(t))x(t) + bu(t), \\ y(t) &= cx(t), \qquad -0.5 \leq u(t) \leq 0.5, \end{aligned} \tag{34}$$

where $a(\theta(t)) = a_0 + a_1\theta_1(t) + a_2\theta_2(t)$, $b = 1$, $c = 1$, and $\theta_2 \in \langle -1, 1 \rangle$ is unknown (uncertain) variable, furthermore,

$$\theta_1 = \tfrac{|y| - a_0}{a_1} \in \langle -1, 1 \rangle.$$

The coefficients a_0 and a_1 were calculated so as to maintain the scheduling parameter θ_1 in the range $\langle -1, 1 \rangle$:

$$a_0 = \tfrac{\min(|y|) + \max(|y|)}{2};\quad a_1 = \tfrac{\min(|y|) - \max(|y|)}{2}.$$

With the assumption that $y \in \langle -0.5, 0.5 \rangle$, the $\max(|y|) = 0.5$ and $\min(|y|) = 0$ and it follows that $a_0 = 0.25$ and $a_1 = -0.25$. The parameter $a_2 = 0.1$ (computed from γ). The obtained LPV system (34) was transformed to discrete-time with sample time $T_s = 0.01\,s$ to obtain the model for controller design in the form (1). Using *Theorem 1* with weighting matrices $Q_i = q_i I$, $q_0 = 0.1$, $q_1 = q_2 = 0$, $R = rI$, $r = 5.5 \times 10^5$, and $\xi_L \leq P(\theta) \leq \xi_U$, $\xi_L = 0$, $\xi_U = 8 \times 10^5$ we obtained a robust gain-scheduled PSD controller in the form (3), where

$$\begin{aligned} K_P(\theta) &= -1.418184 - 9.364592 \times 10^{-3}\,\theta_1, \\ K_S(\theta) &= -9.273503 \times 10^{-3} - 4.006072 \times 10^{-5}\,\theta_1, \\ K_D(\theta) &= -3.011434 \times 10^{-3} - 1.173246 \times 10^{-5}\,\theta_1. \end{aligned} \tag{35}$$

Numerical solution has been carried out by SeDuMi 1.32 solver under Octave 4.0.0 using YALMIP R20150918. The simulations were done via Scilab 5.2.2/Xcos. Simulation results with the robust gain-scheduled PSD controller (35) are shown in Fig. 3. In the simulation $y_\gamma(t)$, $u_\gamma(t)$, and $\theta_{1,\gamma}(t)$ are the system output, controller output, and the

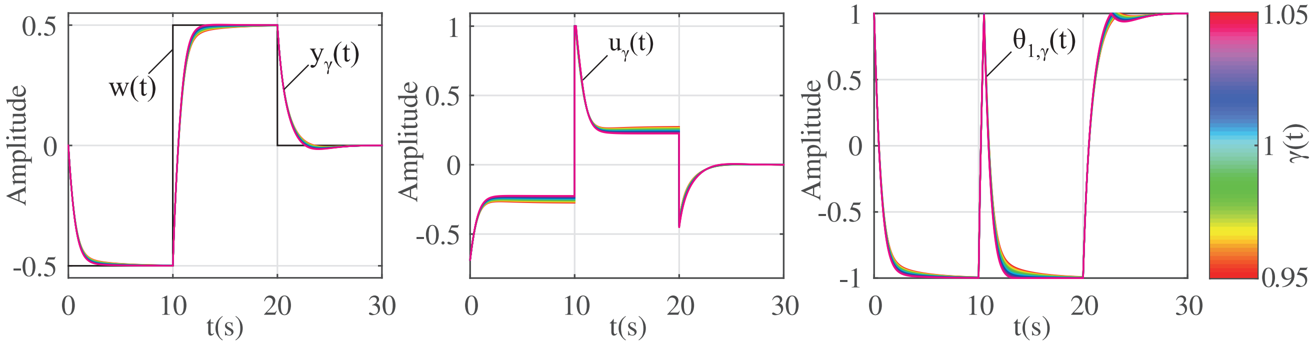

Fig. 3. Simulation results with robust gain-scheduled PSD controller (35)

calculated scheduled parameter θ_1, for different values of the unknown (uncertain) parameter $\gamma \in \langle 0.95, 1.05 \rangle$.

4. CONCLUSION

A novel methodology is presented in the paper for robust discrete-time gain-scheduled PSD controller design for uncertain linear parameter-varying systems from educational perspective. The proposed approach ensures the robust quadratic or affine quadratic stability and guaranteed cost for all scheduled parameters and their prescribed maximal rate of change. The controller synthesis can be done in a computationally tractable and systematic way. The proposed LMI/BMI-based controller design methodology is validated on student project examples using commercial as well as free and open source program tools and solvers.

REFERENCES

Adams, R., Sparks, A.G., and Banda, S. (1992). A gain scheduled multivariable design for a manual flight control system. In *First IEEE Conference on Control Applications*, 584–589.

Apkarian, P., Gahinet, P., and Becker, G. (1995). Self-Scheduled H_∞ Control of Linear Parameter-Varying Systems: A Design Example. *Automatica*, 31(9), 1251–1261.

Fiala, J., Kocvara, M., and Stingl, M. (2013). Penlab - a solver for nonlinear semidefinite programming. Submitted to Mathematical Programming Computation in October 2013.

Gahinet, P., Apkarian, P., and Chilali, M. (1996). Affine parameter-dependent Lyapunov functions and real parametric uncertainty. *IEEE Transactions on Automatic Control*, 41(3), 436–442.

Gahinet, P., Nemirovskii, A., Laub, A., and Chilali, M. (1994). The LMI control toolbox. In *Proceedings of the 33rd IEEE Conference on Decision and Control*, volume 3, 2038–2041.

Henrion, D., Löfberg, J., Kočvara, M., and Stingl, M. (2005). Solving polynomial static output feedback problems with PENBMI. In *44th IEEE Conference on Decision and Control, 2005 and 2005 European Control Conference. CDC-ECC '05.*, 7581–7586.

Hyde, R. and Glover, K. (1993). The application of scheduled H_∞ controllers to a VSTOL aircraft. *IEEE Transactions on Automatic Control*, 38(7), 1021–1039.

Ilka, A. and Veselý, V. (2014). Gain-Scheduled Controller Design: Variable Weighting Approach. *Journal of Electrical Engineering*, 65(2), 116–120.

Ilka, A. and Veselý, V. (2015). Robust switched controller design for linear continuous-time systems. *Archives of Control Sciences*, 25(4), 401–416.

Kuncevic, V.M. and Lycak, M.M. (1977). *Control systems design using Lyapunov function approach*, volume in Russian. Nauka, Moscow.

Löfberg, J. (2004). YALMIP : A Toolbox for Modeling and Optimization in MATLAB. In *Proceedings of the CACSD Conference*. Taipei, Taiwan.

Nichols, R.A., Reichert, R.T., and Rugh, W.J. (1993). Gain Scheduling for H_∞ Controllers: A Flight Control Example. *IEEE Transactions on Control Systems Technology*, 1, 69–79.

Octave community (2015). GNU Octave. URL `www.gnu.org/software/octave/`.

Åström, K. and Murray, R. (2011). *Feedback Systems: An Introduction for Scientists and Engineers*. Princeton University Press.

Scilab Enterprises (2012). *Scilab: Free and Open Source software for numerical computation*. Scilab Enterprises, Orsay, France. URL `http://www.scilab.org`.

Shamma, J.S. (2012). *Control of Linear Parameter Varying Systems with Applications*, chapter An overview of LPV systems, 3–26. Springer.

Stewart, G.E. (2012). A Pragmatic Approach to Robust Gain Scheduling. In *7th IFAC Symposium on Robust Control Design*, volume 7, 355–362. Aalborg Congress & Culture Centre, Denmark.

Sturm, J. (1999). Using SeDuMi 1.02, a MATLAB toolbox for optimization over symmetric cones. *Optimization Methods and Software*, 11–12, 625–653.

The Mathworks, Inc. (2014). *MATLAB R2014b*. The Mathworks, Inc., Natick, Massachusetts.

Toh, K.C., Todd, M., and Tütüncü, R.H. (1999). SDPT3 – a MATLAB software package for semidefinite programming. *OPTIMIZATION METHODS AND SOFTWARE*, 11, 545–581.

Veselý, V. and Ilka, A. (2013). Gain-scheduled PID controller design. *Journal of Process Control*, 23(8), 1141–1148.

Veselý, V. and Ilka, A. (2015). Design of robust gain-scheduled PI controllers. *Journal of the Franklin Institute*, 352(4), 1476 – 1494.

Veselý, V. and Rosinová, D. (2013). Robust PID-PSD Controller Design: BMI Approach. *Asian Journal of Control*, 15(2), 469–478.

Veselý, V., Rosinová, D., and Kozáková, A. (2015). *Robust Controller Design*. Felia s.r.o., Bratislava, Slovak Republic.