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Kinetic modelling of runaway in plasmas

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Abstract

The phenomenon of runaway occurs in plasmas in the presence of a strong electric field, which overcomes the collisional friction acting on the charged particles moving through the plasma. A subpopulation of particles can then be accelerated to energies significantly higher than the thermal energy. Such events are observed in both laboratory and space plasmas, and are of great importance in fusion-energy research, where highly energetic runaway electrons can damage the plasma-facing components of the reactor.

In this thesis, a series of papers are presented which investigate various aspects of runaway dynamics. The emission of synchrotron and bremsstrahlung radiation are important energy-loss mechanisms for relativistic runaway electrons. Photons emitted in bremsstrahlung radiation often have energy comparable to the energetic electrons, and we therefore use a Boltzmann transport equation in order to describe their effect on the electron motion. This treatment reveals that electrons can reach significantly higher energies than previously thought. In comparison, synchrotron radiation has lower frequency, and is well described by the classical electromagnetic radiation-reaction force. This loss mechanism, often dominant in laboratory plasmas, significantly alters the electron dynamics, and is found to produce non-monotonic features in the runaway tail.

A study is also presented of the related phenomenon of ion runaway acceleration, which differs from electron runaway due to their larger mass. Renewed interest in this topic has been sparked after recent observations of fast ions in various experiments. Finally a new method is explored to treat the non-linear Fokker-Planck equation which is commonly used to describe the collisional dynamics in a plasma. The new method is appealing for its physically intuitive description and analytic simplicity.

Keywords: plasma, runaway, Boltzmann equation, Fokker-Planck equation, bremsstrahlung, synchrotron radiation

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Contents

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Chapter 1

Introduction

A plasma is an ionized gas, sufficiently hot that the electrons have detached from the atoms that carried them. Because it consists of free charges, rather than neutral atoms, the plasma behaves differently to the familiar gases and fluids encountered in everyday life. Indeed, the addition of electric and magnetic forces between the particles creates a rich interplay, allowing a wide range of strange and wonderful phenomena to occur. Some of these are well-known to most: lightning, electric sparks, fluorescent lamps, the Sun and the stars, and even the *aurora borealis* – the northern lights – are examples of plasmas. In fact, a majority of the visible matter in the universe is in the plasma state. The study of plasmas is a huge field of research, ranging from academic contemplations in astrophysical research and space physics, through fusion-energy research to various industrial and medical applications.

Runaway is a phenomenon which occurs in any plasma in the presence of a sufficiently strong electric field. It is a process related to dielectric breakdown, which occurs when electric sparks are created. Runaway breakdown occurs in laboratory plasmas, such as those in tokamak fusion devices [1], as well as in lightning discharges during thunderstorms [2], and in astrophysical plasmas, such as solar flares [3]. In these scenarios, a subpopulation of particles — typically electrons, which are lightest — are accelerated by the applied field to energies significantly higher than the thermal energy, at which point they start emitting radiation.

The phenomenon of runaway can be understood by considering in detail the frictional drag force due to collisions which acts on a charged particle moving through a plasma which is near thermodynamic equilibrium. The friction is a non-monotonic function of speed: at low speed, the drag steadily grows in magnitude as the speed increases; however, above the thermal speed of the particles, the drag force will instead decrease in magnitude as the speed increases further. In the absence of an electric field, the friction force on the thermal particles will be balanced by velocity-space diffusion induced by collisions, which tends to increase the width of the velocity distribution. An equilibrium between friction and diffusion is reached when the distribution takes the Maxwellian form, $f_M = n(m/2\pi T)^{3/2}\exp(-mv^2/2T)$, where m, n and T are the mass, number density and temperature (in energy units, throughout this work) of the species, respectively, and v is the speed.

In the presence of an electric field which acts to accelerate charged particles, an electron with sufficiently high initial speed will experience an unbounded acceleration to highly relativistic energies, where the electrons move at nearly the speed of light. At these energies competing physical effects become important, such as radiation losses caused by the rapidly accelerated motion experienced by the particles when moving in electromagnetic fields (leading to synchrotron radiation) or in collisions (causing bremsstrahlung emission). Figure 1.1 schematically illustrates the forces which act on a runaway electron.

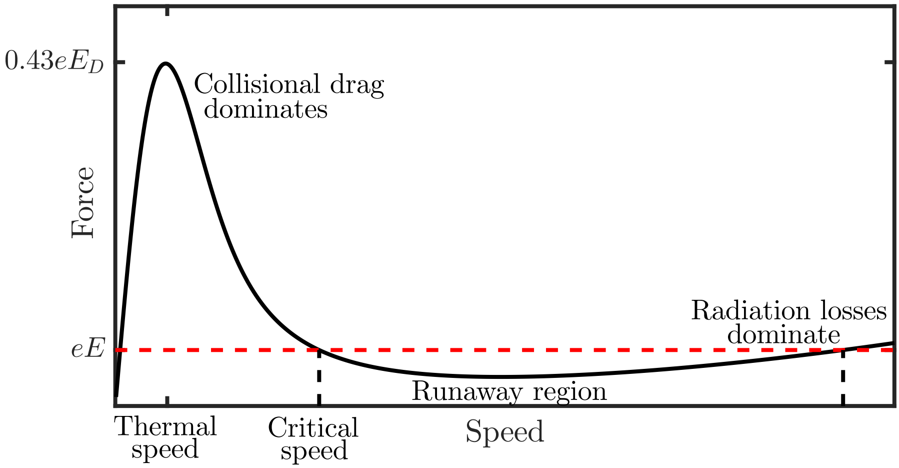

Figure 1.1: A schematic view of the speed-dependent force acting on a particle in a plasma, showing friction due to collisions and radiation (solid, black) and acceleration by an electric field (dashed, red). Not to scale (the speed where radiation losses become important can be thousands of times larger than the thermal speed).

1.1 Runaway generation

Historically, the runaway phenomenon in plasmas was first discussed by Dreicer [4] in 1959. He considered the total friction force between two Maxwellian particle species moving uniformly with a given speed relative to each other. When accelerated by a sufficiently strong electric field, greater than approximately 43% of the so-called Dreicer field $E_D = n_e \ln\Lambda e^3/(4\pi\varepsilon_0^2 T_e)$, the electric field overcomes the maximum frictional force, and he concluded that the particles would "run away" towards infinite energy (given infinite time). The Coulomb logarithm $\ln\Lambda$ is a plasma parameter which typically takes values 10-20 in the applications we consider [5].

In 1964, Kruskal and Bernstein [6] rigorously treated the runaway problem with an analytic solution to the kinetic equation (although using a simplified model for collisions). They solved the kinetic equation with an asymptotic technique, matching approximate solutions across five regions in momentum space, thereby obtaining expressions for the shape of the velocity distribution of runaway electrons and the rate at which new runaways are generated (here called the runaway growth rate, or runaway rate). It was found that all electrons moving with a velocity above a critical velocity v_c, defined approximately as that velocity above which the electric field becomes stronger than friction, will run away towards infinite energy. In addition, diffusion would supply the runaway region ($v > v_c$) with new particles from the bulk at a constant rate. This mechanism of runaway generation is referred to as primary generation, or Dreicer generation.

Early work on runaways considered primarily the initial generation of runaways at relatively low, non-relativistic speeds. A full description of electron runaway requires the use of a relativistic kinetic equation, a scenario which was first analyzed by Connor and Hastie [7] in 1975. They extended the method of Kruskal and Bernstein to account also for relativistic effects. Unlike the case of non-relativistic runaway, where the frictional force appears to tend towards zero for large speeds, the friction in the relativistic model attains a minimum value, corresponding to a critical electric field $eE_c = n_e \ln\Lambda e^4/(4\pi\varepsilon_0^2 m_e c^2)$. This is smaller than the Dreicer field by the factor $T/m_e c^2$. It is found that the runaway growth rate $\Gamma = \partial n_{\mathrm{RE}}/\partial t$ is exponentially sensitive to the electric field, scaling approximately as $\Gamma \propto \exp(-\lambda E_D/4E)$, with a (generally) small correction factor $\lambda(E)$ which forces the growth rate to zero as $E \to E_c$.

In a seminal paper by Rosenbluth and Putvinski [8] in 1997, the effect of large-angle collisions on the runaway rate was detailed. Runaway generation by large-angle collisions (also called knock-on or close collisions) is possible when runaway electrons already exist in the plasma. Then, in a single large-momentum-transfer collision event, an initially slow electron may obtain a velocity great enough to enter the runaway region where friction is smaller than the electric force. Thus, an initial runaway particle can become two after undergoing a knock-on collision, which can become four after further knock-on collisions, and so on. Therefore, the runaway rate due to knock-on collisions is proportional to the number of runaways already present, which causes an exponential growth of the runaway population and hence is referred to as a runaway avalanche. Large-angle collisions in plasmas will generally have a smaller effect on the evolution of the distribution than small-angle collisions by a factor of the inverse Coulomb logarithm $1/\ln\Lambda$. Since Dreicer generation is exponentially sensitive to the electric field, however, it will be completely negligible for small enough E/E_D. Because of this large-angle collisions can sometimes dominate the runaway rate even though they give a "small" modification to the equation (in the sense of being of order $1/\ln\Lambda$), with a contribution to the runaway rate that scales as $\Gamma \propto n_{\mathrm{RE}}(E/E_c - 1)$.

1.2 Runaway in tokamaks

Runaways are of particular interest in magnetic-fusion research, where they pose a great threat to the successful operation of tokamaks [9]. These are a promising concept for fusion-energy reactors, which confine the plasma by magnetic fields and heat it to several hundred million K. A magnetic field is partially generated by driving a strong current of several MA through the plasma, with the downside that this is then available for conversion into a runaway-carried current. The mechanism for this conversion is the runaway breakdown. This typically occurs during so-called disruptions, which are sudden unintentional events where heat confinement is lost. During these disruptions the plasma loses its energy and cools rapidly on a timescale of milliseconds [10], sometimes to one-thousandth of its original temperature. The temperature reduction is associated with a decrease in the electrical conductivity of the plasma, which causes the large plasma current to rapidly decay. By the induction equation (or Lenz's law), this induces an electric field in order to maintain the current, finally enabling runaway breakdown to occur.

It can be shown that avalanche runaway multiplication often dominates the runaway generation, although initiation of the avalanche requires the presence of an initial seed population of runaways. This can be provided either by Dreicer generation, as described above, or by so-called hot-tail generation [11, 12, 13, 14]. This third generation mechanism is enabled by the rapid temperature change that occurs during disruptions. If the cooling is sufficiently fast, the fastest particles in the tail of the thermal distribution (which experience a weaker drag force) will maintain their initial energy. During the cooling their velocities may at some point be greater than the critical velocity for runaway generation, and thus they can become runaways.

It can be shown that, if there is an initial seed population $n_{\mathrm{RE},0}$ of runaways, avalanche multiplication will increase this number to approximately $n_{\mathrm{RE}} \sim \exp{(2.5I[\mathrm{MA}])}\, n_{\mathrm{RE},0}$ before the electric field has decayed [9], where $I[\mathrm{MA}]$ is the original plasma current in MA. While the multiplication factor is fairly small in present-day experiments (of order 10^4 [9] in the JET tokamak [15], the biggest current experiment), in future tokamaks such as the international collaboration ITER [16] this implies a devastating number of 10^{16} or greater [9]. Because of this immense number, runaway-electron dynamics and disruption mitigation is a field of active study. Recent reviews can be found in Refs. [17, 18, 19, 20].

The qualitative features of the basic runaway phenomenon in plasmas can thus be summarized as:

- Runaway is only possible for electric fields exceeding the critical field, $E > E_c$.
- Primary (Dreicer) runaway generation is exponentially sensitive to electric field, and only gives an appreciable growth rate when $E \gtrsim 0.01 E_D$.
- Secondary (avalanche) runaway generation depends weakly on electric field and is caused by knock-on collisions, requiring a significant fraction of runaways to already be present in the plasma.
- Hot-tail runaway generation is caused by a rapid temperature drop, and describes the conversion of previously thermal electrons into runaways.

At highly relativistic (multi-MeV) energies, additional effects such as radiation losses become important for the dynamics of the fast electrons. These effects only weakly impact runaway generation, however, which typically occurs at low velocities. In Papers A and C we have investigated in detail the effect of the radiation losses by synchrotron radiation and bremsstrahlung emission. A significant fraction of the plasma energy can be emitted in the form of such radiation, and these effects will often have a strong impact on the motion of the electrons.

1.3 Ion runaway

Runaway acceleration of ions was first invoked in order to explain experimental observations at the Zeta device [21] in 1959. In 1972, Furth and Rutherford [22] used an asympotic technique similar to that used for electron runaway in order to obtain an analytic solution of the ion drift-kinetic equation. Their treatment provided only limited information about the runaway growth rate in most scenarios, however, due to the more complicated structure of the ion kinetic equation. A limited time-dependent solution of the ion kinetic equation was more recently developed in order to explain observations at the Mega Ampere Spherical Tokamak [23, 24, 25]. Simpler test-particle methods have also been used to study the ion runaway phenomenon in astrophysical contexts [26]. The lack of widely applicable analytic results has motivated a numerical study of the ion drift-kinetic equation, which is presented in Paper D. Experimentally observed ion acceleration in the Madison Symmetric Torus reversed-field pinch has lead to recent work where similar methods have been employed [27, 28].

Outline

Chapter 2 contains an introduction to the kinetic theory of plasmas, which describes the phase-space dynamics of charged particles. The theory presented here covers the physics required to understand the basic runaway phenomenon, but is also the foundation upon which further extensions of the theory can be developed. In chapter 3, we further develop the theory by describing a model for the effect of bremsstrahlung emission based on the Boltzmann collision operator, and its validity is investigated in detail. The theory for ion runaway requires a modified treatment compared to electron runaway; this theory is briefly summarized in chapter 4, and here we also derive improved analytic formulas for the runaway velocity and critical electric field. Finally the work is concluded in chapter 5.

Chapter 2

The kinetic equation

A detailed study of runaway particles requires the resolution of their momentum-space structure, accounting for the randomizing collisions in an accurate way. This is achieved using a kinetic equation, which provides a full description of the time evolution of the distribution function $f_a(t, \mathbf{x}, \mathbf{p})$ of a particle species a, where t is time, $\mathbf{x}$ is the particle position, $\mathbf{p} = m_a \mathbf{v}/\sqrt{1 - v^2/c^2}$ is the momentum and $\mathbf{v}$ is the velocity. The distribution function is the particle density function in phase space, defined such that $n_a(t, \mathbf{x}) = \int \mathrm{d}\mathbf{p}\, f_a(t, \mathbf{x}, \mathbf{p})$ is the number density, and $N_a(t) = \int \mathrm{d}\mathbf{x}\, n_a(t, \mathbf{x})$ is the total number of particles of species a. In the absence of collisions, the distribution function describes particles moving along trajectories $\mathbf{x} = \mathbf{x}(t)$ and $\mathbf{p} = \mathbf{p}(t)$, governed by the equations of motion for a charged particle

$$\frac{\mathrm{d}\mathbf{x}}{\mathrm{d}t} = \mathbf{v},$$
$$\frac{\mathrm{d}\mathbf{p}}{\mathrm{d}t} = q_a\big(\mathbf{E} + \mathbf{v} \times \mathbf{B}\big),$$

where q_a is the charge. The continuity equation in phase space is [29]

$$0 = \frac{\mathrm{d}f_a}{\mathrm{d}t} = \frac{\partial f_a}{\partial t} + \mathbf{v} \cdot \frac{\partial f_a}{\partial \mathbf{x}} + q_a\big(\mathbf{E} + \mathbf{v} \times \mathbf{B}\big) \cdot \frac{\partial f_a}{\partial \mathbf{p}}, \tag{2.1}$$

where the electric and magnetic fields are given by the charge and current distribution of the plasma according to Maxwell's equations,

$$\mathbf{E}(t,\mathbf{x}) = \frac{1}{4\pi\varepsilon_0}\int \mathrm{d}\mathbf{x}'\, \frac{\mathbf{x}-\mathbf{x}'}{|\mathbf{x}-\mathbf{x}'|^3}\rho(t,\mathbf{x}'),$$
$$\mathbf{B}(t,\mathbf{x}) = \frac{\mu_0}{4\pi}\int \mathrm{d}\mathbf{x}'\, \mathbf{j}(t,\mathbf{x}') \times \frac{\mathbf{x}-\mathbf{x}'}{|\mathbf{x}-\mathbf{x}'|^3},$$

and the charge and current are in turn determined by the distribution functions,

$$\rho(t,\mathbf{x}) = \sum_b e_b \int \mathrm{d}\mathbf{p}\, f_b(t,\mathbf{x},\mathbf{p}),$$
$$\mathbf{j}(t,\mathbf{x}) = \sum_b e_b \int \mathrm{d}\mathbf{p}\, \mathbf{v} f_b(t,\mathbf{x},\mathbf{p}),$$

with the sum taken over all particle species b present in the plasma. In order to obtain a useful kinetic equation, Eq. (2.1) needs to be ensemble-averaged over macroscopically equivalent systems. The distribution function will then be a smooth function, but the microscopic interactions between the discrete particles in the plasma will need to be accounted for by the addition of a new term [30], which is called the *collision operator* C, or the collision integral (as it generally takes the form of an integral operator). The kinetic equation then takes the form

$$\frac{\partial f_a}{\partial t} + \mathbf{v}\cdot\frac{\partial f_a}{\partial \mathbf{x}} + q_a\big(\mathbf{E} + \mathbf{v}\times\mathbf{B}\big)\cdot\frac{\partial f_a}{\partial \mathbf{p}} = \sum_b C_{ab}\{f_a, f_b\}, \tag{2.2}$$

where the electromagnetic fields denote the macroscopic fields, not including fluctuations caused by individual particles which are instead captured by the collision operator C. We shall focus on the simplest physical scenario that exhibits the runaway phenomenon: an infinite homogeneous plasma with an electric field in a constant direction. In this case we can suppress the space variables and write $f_a = f_a(t,\mathbf{p})$, and introduce a spherical momentum coordinate system (p,θ,φ), where the azimuthal angle φ is referred to as the gyroangle, the longitudinal angle θ is referred to as the pitch-angle, and we shall mostly use its cosine $\xi = \cos\theta$. The kinetic equation then becomes

$$\frac{\partial f_a}{\partial t} + q_a E\left(\xi\frac{\partial f_a}{\partial p} + \frac{1-\xi^2}{p}\frac{\partial f_a}{\partial \xi}\right) = \sum_b C_{ab}\{f_a, f_b\}. \tag{2.3}$$

This equation allows us to study the effects of various contributions in the collision operator C on the dynamics of the runaway particles. We do not at present take an interest in the dynamical evolution of the electric field but instead assume it to be some prescribed external field $E = E(t)$.

An essential part of the description of runaway electrons is the collision operator. This term in the kinetic equation describes the effect of microscopic particle-particle interactions, in contrast to the macroscopic interactions with the electromagnetic field set up by the charge distribution in the plasma. The collisions drive the particle distributions towards thermal equilibrium by always increasing entropy in the system, and this is the restoring effect which needs to be overcome by the electric field in order to generate runaway particles. Therefore the details of the collision operator can be expected to strongly influence the description of the runaway process.

In this chapter we will provide a detailed discussion of the collision operator, revealing a unified picture of small-angle collisions, knock-on collisions and bremsstrahlung radiation in the same framework. We shall begin by investigating in more detail how the collision operator can be obtained.

2.1 BBGKY hierarchy and the kinetic equation

A systematic framework for obtaining kinetic equations was initially developed by Bogolyubov, Born, Green, Kirkwood and Yvon (BBGKY) [31, 32, 33, 34, 35]. The starting point of their analysis is the Liouville theorem [36], which deterministically describes the time evolution of an N-body system according to Hamiltonian mechanics. The system is fully described by the phase space density function $f_N(t, \mathbf{x}_1, \mathbf{p}_1, ..., \mathbf{x}_N, \mathbf{p}_N)$ giving the location and momentum of all its constituents. A kinetic equation describes the time evolution of the distribution function, which is defined as $f(t, \mathbf{x}, \mathbf{p}) = \int \mathrm{d}\mathbf{x}_2 \mathrm{d}\mathbf{p}_2 \cdots \mathrm{d}\mathbf{x}_N \mathrm{d}\mathbf{p}_N \, f_N(t, \mathbf{x}, \mathbf{p}, \mathbf{x}_2, \mathbf{p}_2, ..., \mathbf{x}_N, \mathbf{p}_N)$. Note that, while this definition appears to single out the particle with subscript 1 as special, the particles described by the phase-space density function are identical, and hence it is symmetric in all indices. That is, non-identical particle species are each described by their own function.

The Liouville equation [37] for a species interacting pair-wise with a central potential $V_{ij} = V(|\mathbf{x}_i - \mathbf{x}_j|)$, with the force on particle i being $\mathbf{F}_i = -\sum_{(j\neq i)=1}^{N} \partial V_{ij}/\partial \mathbf{x}_i$ (for simplicity assuming a single species and no magnetic interaction, which would require a generalized form of the potential), is given by

$$\frac{\partial f_N}{\partial t} + \sum_{i=1}^{N} \frac{\mathbf{p}_i}{m_a} \cdot \frac{\partial f_N}{\partial \mathbf{x}_i} - \sum_{i=1}^{N} \sum_{(j \neq i)=1}^{N} \frac{\partial V_{ij}}{\partial \mathbf{x}_i} \cdot \frac{\partial f_N}{\partial \mathbf{p}_i} = 0.$$

By integrating over all but s particle coordinates, a *reduced* phase space density, or the *s-particle correlation function*, can be defined as $f_s = \int \mathrm{d}\mathbf{x}_{s+1}\mathrm{d}\mathbf{p}_{s+1} \cdots \mathrm{d}\mathbf{x}_N \mathrm{d}\mathbf{p}_N \, f_N$. Here, $s = 1$ gives the distribution function in which we are most interested, and $s = N$ returns the full N-particle phase space density. When performing such an integration over the Liouville equation, an equation for the time-evolution of the reduced distribution function is obtained; however, the equation for $\partial f_s/\partial t$ invariably contains f_{s+1} due to the pair-wise interaction term. Thus, the time-evolution of the distribution function depends on the two-particle correlation function f_2, which in turn is affected by f_3, and so on. This set of coupled partial differential equations is called the BBGKY hierarchy. A systematic approximation scheme to close this set of equations was developed by Frieman [38], Sandri [39] and collaborators, which takes the form of a perturbation expansion in two parameters μ and η. These appear naturally when normalizing the equation to characteristic values of particle separation r_0, velocities v_0 and interaction strength V_0, and are given by

$$\begin{aligned} \mu &= \frac{1}{n r_0^3}, \\ \eta &= \frac{V_0}{m v_0^2} \sim \frac{e^2}{4\pi\varepsilon_0 T r_0}. \end{aligned}$$

Here $1/\mu$ is the number of particles in the interaction region (defined by a characteristic range r_0), and η is a measure of the strength of the interaction (described by the potential function V_0) compared to the kinetic energy. There are three domains of primary interest [40, 41] which can be described as (1) "dilute, short-range", (2) "weak coupling" (small momentum transfer) and (3) "long-range". These, respectively, correspond to the choices (with ϵ a small expansion parameter)

$$\begin{aligned} &(1) \quad \mu = \mathcal{O}(\epsilon^{-1}), && \eta = \mathcal{O}(1), \\ &(2) \quad \mu = \mathcal{O}(1), && \eta = \mathcal{O}(\epsilon), \\ &(3) \quad \mu = \mathcal{O}(\epsilon), && \eta = \mathcal{O}(\epsilon). \end{aligned}$$

These lead, in turn, to the so-called Boltzmann equation, the Fokker-Planck equation and the Balescu-Lenard (or Bogolyubov-Lenard-Balescu) equation. Plasmas are particularly pathological, as no specific ordering applies to the entire phase space. The long-range Coulomb interaction allows for collisions where any of the orderings may apply, depending on the impact parameter.

An analysis shows that the Balescu-Lenard operator takes a similar form to the Fokker-Planck operator, but where the dielectric constant of the plasma appears in the collision integral. This factor accounts for *dynamical screening* in the plasma, which ensures that collisions with impact parameter of order the Debye length $\lambda_\mathrm{D} = \sqrt{\varepsilon_0 T/ne^2}$ or greater are exponentially damped. This effect demonstrates the well-known behavior of Debye screening [42], where the electric field from a point charge in a plasma will be exponentially damped on a length scale λ_D by the rearrangement of the surrounding plasma. The Fokker-Planck collision operator diverges in the contribution from large-impact-parameter collisions, but by following Landau's prescription from the original derivation [43] to cut the integration off at impact parameters λ_D (which can be motivated by the Balescu-Lenard equation), one obtains a convergent integral. The contribution from small-angle collisions in the Fokker-Planck operator is then found to be larger than the contribution from large-angle collisions in the Boltzmann operator by a factor $\ln n\lambda_\mathrm{D}^3 \simeq \ln\Lambda$, the so-called Coulomb logarithm.

In this chapter we will not pursue a detailed analysis of the BBGKY hierarchy of equations. Instead, we will derive the Boltzmann and Fokker-Planck collision operators with a heuristic argument, based on an analysis of binary collisions. This method gives the same result as the more rigorous derivation from first principles, and also provides some physical insight into how we may view collisions in a plasma.

2.2 The Boltzmann collision operator

The Boltzmann equation was originally derived by Ludwig Boltzmann in the late nineteenth century [44, 45] in order to study the dynamics of gases. As we indicated in the previous section, the Boltzmann equation for a plasma is valid when describing those large-angle collisions where the impact parameter is much smaller than the mean distance between particles in the plasma, that is for $\mu \ll 1$. The Boltzmann collision operator describes the rate-of-change of the distribution function due to binary collisions, and we shall briefly derive it here in a form that will be suited to our applications.

We describe a binary interaction with a differential cross-section $\mathrm{d}\sigma_{ab}(\mathbf{p}_1, \mathbf{p}_2;\, \mathbf{p}, \mathbf{p}')$ for particles a and b to be taken from initial momenta $\mathbf{p}$ and $\mathbf{p}'$, respectively, to final momenta $\mathbf{p}_1$ and $\mathbf{p}_2$, respectively. The cross-section is defined such that the total differential change of the phase-space particle density $\mathrm{d}n_a(t, \mathbf{x}, \mathbf{p}) = f_a(t, \mathbf{x}, \mathbf{p})\mathrm{d}\mathbf{p}$ due to these interactions in a time interval $\mathrm{d}t$ is

$$\begin{aligned}[\mathrm{d}n_a(\mathbf{p})]_{c,ab} &= f_a(\mathbf{p}_1)f_b(\mathbf{p}_2)g_{\o}(\mathbf{p}_1, \mathbf{p}_2)\mathrm{d}\sigma(\mathbf{p}, \mathbf{p}';\, \mathbf{p}_1, \mathbf{p}_2)\mathrm{d}\mathbf{p}_1\mathrm{d}\mathbf{p}_2\mathrm{d}t \\ &\quad - f_a(\mathbf{p})f_b(\mathbf{p}')g_{\o}(\mathbf{p}, \mathbf{p}')\mathrm{d}\sigma(\mathbf{p}_1, \mathbf{p}_2;\, \mathbf{p}, \mathbf{p}')\mathrm{d}\mathbf{p}\mathrm{d}\mathbf{p}'\mathrm{d}t, \qquad (2.4)\end{aligned}$$

where $\mathbf{p}_1$ and $\mathbf{p}_2$ are related to $\mathbf{p}$ and $\mathbf{p}'$ by the conservation of energy and momentum. The relativistic generalization of the relative speed $v_{\mathrm{rel}} = |\mathbf{v}-\mathbf{v}'|$, is the Møller relative speed $g_{\o}(\mathbf{p}, \mathbf{p}') = \sqrt{(\mathbf{v}-\mathbf{v}')^2 - (\mathbf{v}\times\mathbf{v}')^2/c^2}$ [46]. The collision operator can formally be defined as

$$\begin{aligned}C_{ab}\{f_a, f_b\} &\equiv \left(\frac{\partial^2 n_a}{\partial t \partial \mathbf{p}}\right)_{c,ab} \\ &= \int \mathrm{d}\mathbf{p}_1\, f_a(\mathbf{p}_1) \int \mathrm{d}\mathbf{p}_2\, f_b(\mathbf{p}_2)g_{\o}(\mathbf{p}_1, \mathbf{p}_2)\frac{\partial\sigma(\mathbf{p}, \mathbf{p}';\, \mathbf{p}_1, \mathbf{p}_2)}{\partial \mathbf{p}} \\ &\quad - f_a(\mathbf{p}) \int \mathrm{d}\mathbf{p}'\, f_b(\mathbf{p}')g_{\o}(\mathbf{p}, \mathbf{p}')\sigma(\mathbf{p}, \mathbf{p}'), \qquad (2.5)\end{aligned}$$

where the total cross-section $\sigma(\mathbf{p}, \mathbf{p}')$ is defined as

$$\sigma(\mathbf{p}, \mathbf{p}') = \int \mathrm{d}\mathbf{p}_1\, \frac{\partial\sigma(\mathbf{p}_1, \mathbf{p}_2;\, \mathbf{p}, \mathbf{p}')}{\partial \mathbf{p}_1}.$$

A symmetric form is obtained in the special case of elastic collisions by utilizing the *principle of detailed balance* [46], which is a symmetry relation for the cross-section stating that

$$g_{\o}(\mathbf{p}_1, \mathbf{p}_2)\mathrm{d}\sigma(\mathbf{p}, \mathbf{p}';\, \mathbf{p}_1, \mathbf{p}_2)\mathrm{d}\mathbf{p}_1\mathrm{d}\mathbf{p}_2 = g_{\o}(\mathbf{p}, \mathbf{p}')\mathrm{d}\sigma(\mathbf{p}_1, \mathbf{p}_2;\, \mathbf{p}, \mathbf{p}')\mathrm{d}\mathbf{p}\mathrm{d}\mathbf{p}'.$$

Using this relation, which is valid for classical particles interacting with a central potential and also in the first order spin-averaged Born approximation in quantum mechanics [47, 48], Eq. (2.4) leads to

$$\begin{aligned}C_{ab}\{f_a, f_b\} &= \int \mathrm{d}\mathbf{p}'\mathrm{d}\sigma(\mathbf{p}_1, \mathbf{p}_2;\, \mathbf{p}, \mathbf{p}')\, g_{\o}(\mathbf{p}, \mathbf{p}')\times \\ &\qquad \times \Big(f_a(\mathbf{p}_1)f_b(\mathbf{p}_2) - f_a(\mathbf{p})f_b(\mathbf{p}')\Big). \qquad (2.6)\end{aligned}$$

This is the operator which is typically known as the Boltzmann operator, although we shall apply the term more generally here to any integral operator of the form of Eq. (2.5). In Eqs. (2.4), (2.5) and (2.6) the first – the gain term – describes the rate at which particles a of initial momentum $\mathbf{p}_1$ are scattered into $\mathbf{p}$, while the second – the loss term – describes the rate at which particles scatter away from $\mathbf{p}$.

Two-dimensional form — The Boltzmann operator is inconvenient to work with because of the complicated relation between $(\mathbf{p}, \mathbf{p}')$ and $(\mathbf{p}_1, \mathbf{p}_2)$, combined with the singular nature of the Coulomb cross-section. However, it is significantly simplified under the conditions that
(i) the target particles are stationary, i.e. $f_b(\mathbf{p}) = n_b\delta(\mathbf{p})$;
(ii) the cross-section depends only on p_1, p and $\cos\theta_s = \mathbf{p}_1\cdot\mathbf{p}/p_1 p$, which excludes particles with internal degrees of freedom such as molecules, or spin-polarized plasmas;
(iii) and the distribution function is independent of gyroangle.
In this case, introducing again spherical coordinates $\mathbf{p} = (p, \cos\theta, \varphi)$ and $\mathbf{p}_1 = (p_1, \cos\theta_1, \varphi_1)$, the Boltzmann operator (2.5) takes the form

$$C_{ab}\{f_a, f_b\}(t, p, \cos\theta) = n_b \int \mathrm{d}\mathbf{p}_1\, v_1 f_a(t, p_1, \cos\theta_1)\frac{\partial\sigma}{\partial\mathbf{p}}(p;\, p_1, \cos\theta_s) \\ - n_b v f(t, p, \cos\theta)\sigma(p).$$

The angle $\cos\theta_s$ can be related to $\cos\theta_1$, $\cos\theta$, φ_1 and φ by

$$\cos\theta_s = \cos\theta_1\cos\theta + \sin\theta_1\sin\theta\cos(\varphi_1 - \varphi).$$

As we will now show, the Boltzmann operator for stationary targets is diagonalized by the Legendre polynomials, making this a particularly convenient representation of the distribution function for the runaway problem. By introducing

$$f_a(t, p, \cos\theta) = \sum_L f_L(t, p) P_L(\cos\theta),$$
$$C_{ab}(t, p, \cos\theta) = \sum_L C_L(t, p) P_L(\cos\theta),$$

an application of the addition theorem for spherical harmonics,

$$P_L(\cos\theta_s) = P_L(\cos\theta_1)P_L(\cos\theta) \\ + \sum_{m=1}^{L} \frac{(L-m)!}{(L+m)!}\cos\Big(m(\varphi_1 - \varphi)\Big)P_L^m(\cos\theta_1)P_L^m(\cos\theta),$$

yields

$$C_L(t,p) = n_b \int \mathrm{d}p_1\, p_1^2 v_1 f_L(p_1)\, 2\pi \int_{-1}^{1} \mathrm{d}\cos\theta_s\, P_L(\cos\theta_s) \frac{\partial\sigma}{\partial\mathbf{p}}(p;\, p_1,\, \cos\theta_s) - n_b v f_L(p)\sigma(p). \tag{2.7}$$

This form for the Boltzmann collision operator is particularly suitable for numerical calculations, as the angular part is encoded in the Legendre modes in a simple way; the general form for a linear operator takes the form $C(p,\, \cos\theta) = \sum_{L,L'} C_L\{f_{L'}\}(p)P_L(\cos\theta)$, but the Legendre polynomials diagonalize the collision operator in the sense that C_L depends only on the corresponding mode f_L of the distribution function. Numerically, this leads to a sparse matrix representation of the system, allowing a time and memory efficient treatment.

The Boltzmann operator given here is valid both for inelastic and elastic collisions, although in the elastic case $\partial\sigma/\partial\mathbf{p}$ will contain a delta function, as the kinematics then constrain $p_1 = p_1(\cos\theta_s)$.

Special case: Self-collisions — Equation (2.7) is also valid for self-collisions with a simple modification to the last term (the sink term); in this case, the assumption of stationary targets can be viewed as a linearization around a cold bulk population, $f_a = n_a\delta(\mathbf{p}) + f_{a1}$. This is a suitable description for scenarios where a relatively small population of energetic electrons is present, with momenta much greater than the thermal momentum. The collision operator then takes the form

$$C_L(t,p) = n_a \int \mathrm{d}p_1\, p_1^2 v_1 f_L(p_1)\, 2\pi \int_{-1}^{1} \mathrm{d}\cos\theta_s\, P_L(\cos\theta_s) \frac{\partial\sigma}{\partial\mathbf{p}}(p;\, p_1,\, \cos\theta_s) - \frac{1}{2} n_a v f_L(p)\sigma(p) - \frac{1}{2}\frac{\delta(p)}{p^2} n_a \int \mathrm{d}p'\, p'^2 v' \delta_{L,0} f_0(p')\sigma(p'). \tag{2.8}$$

The new term at the end describes the removal of the initially stationary particle, and the factors 1/2 are introduced to avoid double counting collisions. From this expression the collision operator for knock-on collisions (avalanche generation) can be derived. For example, the model by Rosenbluth and Putvinski [8] follows by setting $f_e(\mathbf{p}) = n_{\mathrm{RE}}\delta(\cos\theta - 1)\delta(p - p_0)/2\pi p_0^2$ and letting $p_0 \to \infty$, i.e. by assuming that all runaways have infinite energy and zero pitch-angle.

Special case: Heavy targets — Another interesting special case is given by elastic collisions with infinitely heavy targets. In that case, the energy of the light particle is conserved, and the cross-section takes the form $\partial\sigma/\partial\mathbf{p} = (\delta(p_1 - p)/2\pi p^2)\partial\sigma/\partial\cos\theta_s$. This leads to a collision operator

$$C_L(t,p) = -n_a v f_L(t,p)\int_{-1}^{1} \mathrm{d}\cos\theta_s \left[1 - P_L(\cos\theta_s)\right]\frac{\partial\sigma(p,\,\cos\theta_s)}{\partial\cos\theta_s}. \tag{2.9}$$

It is interesting to further reduce this in the case where the integral is dominated by the contribution from small scattering angles. We may then Taylor expand

$$1 - P_L(\cos\theta_s) = \frac{L(L+1)}{4}\theta_s^2 + \mathcal{O}(\theta_s^4),$$

which yields

$$C_L(t,p) = -n_a v f_L(t,p)\frac{L(L+1)}{4}\int_0^\pi \mathrm{d}\theta_s\,\theta_s^3\,\frac{\partial\sigma(p,\,\cos\theta_s)}{\partial\cos\theta_s}.$$

Although the angles are assumed to be small, the integration is extended to π as the contribution from large angles is assumed to be negligible. We identify this as a familiar eigenvalue equation for the Legendre polynomials, yielding the collision operator

$$C_{ab}(t,\mathbf{p}) = -n_b v\mathcal{L}\{f_a\}\int_0^\pi \mathrm{d}\theta_s\,\frac{\theta_s^3}{2}\,\frac{\partial\sigma(p,\,\cos\theta_s)}{\partial\cos\theta_s}, \tag{2.10}$$

where the Lorentz operator $\mathcal{L}$ is half the pitch-angle part of the Laplace operator $p^2\nabla_{\mathbf{p}}^2$:

$$\mathcal{L}\{f\} = \frac{1}{2}\frac{\partial}{\partial\xi}\left((1-\xi^2)\frac{\partial f}{\partial\xi}\right).$$

This is an energy-conserving diffusion operator in pitch-angle. It is interesting to note that, in the Legendre-polynomial representation, the Boltzmann operator (2.9) and the diffusion operator (2.10) have the same structure – the only difference is that the coefficients are given by different integral moments of the scattering cross-section. Therefore the small-angle approximation in this limit does not provide a significant decrease in computational cost compared to the Boltzmann operator here, unlike the general case which we shall now treat.

2.3 The Fokker-Planck collision operator

When the interaction distance is significant compared to the mean particle separation, but the interaction is weak, the appropriate collision term is the Fokker-Planck operator, rather than the Boltzmann operator. However, as we will now show, the Fokker-Planck operator can in fact be derived from the Boltzmann operator in the limit of small momentum transfers in the collisions. That the seemingly opposite description of weak interactions in the Fokker-Planck picture can be contained in the Boltzmann picture of binary collisions appears counter-intuitive. It can, however, be physically understood by the fact that the small momentum transfers described by the Fokker-Planck operator only negligibly change the particle momentum in a single collision; then the net effect of the many-body interaction can be viewed as a linear superposition of pairwise momentum transfers [49].

The procedure is as follows: a general integral moment of the Boltzmann operator is

$$J[\phi] = \int \mathrm{d}\mathbf{p}\,\phi(\mathbf{p})C_{ab} = \int \mathrm{d}\mathbf{p} \int \mathrm{d}\mathbf{p}'\mathrm{d}\sigma(\mathbf{p}_1,\mathbf{p}_2;\mathbf{p},\mathbf{p}')\,f_a(\mathbf{p})f_b(\mathbf{p}') \\ \times g_{\o}(\mathbf{p},\mathbf{p}')\Big[\phi(\mathbf{p}_1)-\phi(\mathbf{p})\Big],$$

which is most easily seen by integrating Eq. (2.4) and switching names of the dummy variables $\mathbf{p}_1$ and $\mathbf{p}_2$ in the first term with $\mathbf{p}$ and $\mathbf{p}'$, respectively. For convenience we will suppress the arguments of $\mathrm{d}\sigma$ and $g_{\o}$ as they will remain unchanged for the rest of the calculation. Here we introduce the small-momentum-transfer argument: the integral is assumed to be dominated by the contribution from $\mathbf{p}_1 \approx \mathbf{p}$. We then Taylor expand

$$\phi(\mathbf{p}_1)-\phi(\mathbf{p}) \simeq (\mathbf{p}_1-\mathbf{p})\cdot\frac{\partial\phi(\mathbf{p})}{\partial\mathbf{p}} + \frac{(\mathbf{p}_1-\mathbf{p})(\mathbf{p}_1-\mathbf{p})}{2} : \frac{\partial\phi(\mathbf{p})}{\partial\mathbf{p}\partial\mathbf{p}},$$

where we use dyadic notation such that the rank-2 tensor $\mathsf{T} = \mathbf{ab}$ has components $T_{ij} = a_i b_j$. By introducing the quantities

$$\Delta\mathbf{p} = \mathbf{p}_1 - \mathbf{p},$$

$$\mathbf{A} = \int \mathrm{d}\mathbf{p}'\, g_{\o} f_b(t,\mathbf{p}') \int \mathrm{d}\sigma\,\Delta\mathbf{p}, \tag{2.11}$$

$$\mathsf{D} = \int \mathrm{d}\mathbf{p}'\, g_{\o} f_b(t,\mathbf{p}') \int \mathrm{d}\sigma\,\Delta\mathbf{p}\Delta\mathbf{p}, \tag{2.12}$$

integrating by parts twice immediately yields

$$J[\phi] = \int \mathrm{d}\mathbf{p}\,\phi(\mathbf{p})\left[\frac{\partial}{\partial \mathbf{p}}\cdot\left(-\mathbf{A}(t,\mathbf{p})f_a(t,\mathbf{p}) + \frac{1}{2}\frac{\partial}{\partial \mathbf{p}}\cdot\Big[\mathsf{D}(t,\mathbf{p})f_a(t,\mathbf{p})\Big]\right)\right].$$

As this equality holds for any ϕ, the small-momentum-transfer assumption therefore leads to the well-known Fokker-Planck operator [50, 51]

$$C_{ab}\{f_a, f_b\} = \frac{\partial}{\partial \mathbf{p}}\cdot\left(-\mathbf{A}_{ab}(t,\mathbf{p})f_a(t,\mathbf{p}) + \frac{1}{2}\frac{\partial}{\partial \mathbf{p}}\cdot\Big[\mathsf{D}_{ab}(t,\mathbf{p})f_a(t,\mathbf{p})\Big]\right). \tag{2.13}$$

For collisions with infinitely heavy stationary targets, to leading order in scattering angle, this reduces to Eq. (2.10). For relativistic elastic electron-electron collisions, the Fokker-Planck operator was first given by Beliaev and Budker [52], with a direct derivation from Eqs. (2.11), (2.12) and (2.13) later given by Akama [53]. The Fokker-Planck operator is conveniently expressed in the form

$$C_{ab}\{f_e, f_e\} = \frac{\partial}{\partial \mathbf{p}}\cdot\int \mathrm{d}\mathbf{p}'\,\mathcal{E}\cdot\left(\frac{\partial f_e(\mathbf{p})}{\partial \mathbf{p}}f_e(\mathbf{p}') - \frac{\partial f_e(\mathbf{p}')}{\partial \mathbf{p}'}f_e(\mathbf{p})\right), \tag{2.14}$$

where the collision kernel $\mathcal{E}$ is the symmetric rank-2 tensor [54]

$$\mathcal{E} = 2\pi\left(\frac{e^2}{4\pi\varepsilon_0}\right)^2 \ln\Lambda \frac{\gamma'\gamma(1-\mathbf{v}'\cdot\mathbf{v}/c^2)^2}{c\{[\gamma'\gamma - \mathbf{p}'\cdot\mathbf{p}/(m_e c)^2]^2 - 1\}^{3/2}}\times\Bigg\{ \left[\left(\gamma'\gamma - \frac{\mathbf{p}'\cdot\mathbf{p}}{m_e^2c^2}\right)^2 - 1\right]\mathsf{I} - \frac{\mathbf{p}\mathbf{p} + \mathbf{p}'\mathbf{p}'}{m_e^2c^2} + \left(\gamma'\gamma - \frac{\mathbf{p}'\cdot\mathbf{p}}{m_e^2c^2}\right)\frac{\mathbf{p}'\mathbf{p} + \mathbf{p}\mathbf{p}'}{m_e^2c^2}\Bigg\},$$

where I is the unit tensor and $\gamma = \sqrt{1+(p/m_e c)^2}$ is the relativistic Lorentz factor. In this expression, only the leading-order term in $\ln\Lambda$ has been retained, which corresponds to the small-angle contribution to the integrals (2.11) and (2.12).

An approximate collision operator to study runaway electrons was developed in Ref. [55]. It is an asymptotic matching of the linearized Beliaev-Budker operator (2.14) in the high-energy limit with the non-relativistic collision operator [43, 56] (corresponding to Eq. (2.14) for $v \ll c$, linearized with a cold bulk of thermal velocity $v_{Te} = \sqrt{2T_e/m_e} \ll c$). The operator is

$$C = \frac{B(p)}{p^2}\mathcal{L}\{f_e\} + \frac{1}{p^2}\frac{\partial}{\partial p}\left[p^2\left(F(p)f_e + A(p)\frac{\partial f_e}{\partial p}\right)\right], \tag{2.15}$$

where

$$
\begin{aligned}
A(p) &= \frac{m_e^2 c^2}{\tau_c} \frac{c}{v} G\left(\frac{v}{v_{Te}}\right), \\
F(p) &= 2\frac{m_e c}{\tau_c} \frac{c^2}{v_{Te}^2} G\left(\frac{v}{v_{Te}}\right), \\
B(p) &= \frac{m_e^2 c^2}{\tau_c} \frac{c}{v} \left[Z_{\text{eff}} + \phi\left(\frac{v}{v_{Te}}\right) - G\left(\frac{v}{v_{Te}}\right) + \frac{1}{2}\frac{v_{Te}^2}{c^2}\frac{v^2}{c^2}\right].
\end{aligned}
$$

We have here introduced the collision time $\tau_c^{-1} = n_e \ln \Lambda e^4/(4\pi\varepsilon_0^2 m_e^2 c^3)$, the error function $\phi(x) = 2\pi^{-1/2} \int_0^x \mathrm{d}s\ \exp(-s^2)$ and the Chandrasekhar function $G(x) = (\phi(x) - x\phi'(x))/2x^2$. A term proportional to the plasma effective charge $Z_{\text{eff}} = \sum_i n_i Z_i^2/n_e$ (the sum taken over all ion species in the plasma) has been added to the pitch-angle scattering operator coefficient, which corresponds to the contribution from a set of stationary ion species, as in Eq. (2.10). For a non-relativistic bulk population, this collision operator has the correct asymptotic behaviour both as $v \to 0$ and $\gamma \to \infty$, although the expression is never exact.

2.4 Synchrotron radiation reaction

In this section we will show how the effect of synchrotron radiation losses can be accounted for. In paper C, this has been used to find the energy which electrons can be accelerated to by an electric field before radiation losses stop their acceleration, as illustrated in Fig. 1.1. The conditions for which a non-monotonic feature, a “bump”, can form in the tail of the runaway-electron distribution was also analyzed in the paper. The presence of such a feature can destabilize the runaway beam, and also effectively limits the maximum electron energy. This makes bump formation an interesting attribute of the solutions to the kinetic equation to study.

In the kinetic equation (2.2), it has been assumed that the only force acting on a particle is the Lorentz force $\mathbf{F} = q(\mathbf{E} + \mathbf{v} \times \mathbf{B})$ due to the presence of an electric field $\mathbf{E}$ or a magnetic field $\mathbf{B}$. However, this description can be generalized to account for a general force $\mathbf{F} = \mathbf{F}(\mathbf{p})$ by replacing [57]

$$
q(\mathbf{E} + \mathbf{v} \times \mathbf{E}) \cdot \frac{\partial f}{\partial \mathbf{p}} \mapsto \frac{\partial}{\partial \mathbf{p}} \cdot \Big[\mathbf{F}(\mathbf{p}) f(\mathbf{p})\Big]. \tag{2.16}
$$

With this replacement, Eq. (2.1) still takes the form of a continuity equation in phase space, and reduces to the familiar equation when the force $\mathbf{F}$ is chosen as the divergence-free Lorentz force.

The energy which runaway electrons lose by emitting radiation limits the maximum energy that they can reach [58], which shows that this effect must be carefully accounted for in order to understand the dynamics of runaway electrons. In the late nineteenth century, around the same time as Boltzmann derived his famous kinetic equation for dilute gases, it was discovered that – according to Maxwell's equations – a charged particle in accelerated motion will emit radiation [59]. Synchrotron radiation is the radiation emitted by a charged particle moving near the speed of light in a circular motion [60, 61]. In magnetized plasmas, synchrotron radiation is therefore emitted by runaway electrons due to their gyrating motion around the magnetic field, and also due to their orbit motion in a curved magnetic field. The synchrotron radiation reaction force can be understood classically by accounting for the electromagnetic radiation emitted by a particle in accelerated motion, which leads to the Abraham-Lorentz-Dirac (ALD) force [62, 63, 64]

$$\mathbf{F}_{\mathrm{ALD}} = \frac{q^2\gamma^2}{6\pi\varepsilon_0 c^3}\left[\ddot{\mathbf{v}} + \frac{3\gamma^2}{c^2}(\mathbf{v}\cdot\dot{\mathbf{v}})\dot{\mathbf{v}} + \frac{\gamma^2}{c^2}\left(\mathbf{v}\cdot\ddot{\mathbf{v}} + \frac{3\gamma^2}{c^2}(\mathbf{v}\cdot\dot{\mathbf{v}})\right)\mathbf{v}\right]. \quad (2.17)$$

This formula was simplified in Ref. [65] for the case of magnetized plasmas, where the motion is dominated by gyromotion such that $\mathbf{v}\cdot\dot{\mathbf{v}} \simeq 0$, and using the Landau method [61] of neglecting the acceleration by the ALD-force itself. This approximation yields the force components (in a spherical coordinate system, when averaged over the gyromotion)

$$\begin{aligned}
\left(\frac{\mathrm{d}p}{\mathrm{d}t}\right)_{\mathrm{ALD}} &= -(1-\xi^2)\frac{\gamma p}{\tau_r},\\
\left(\frac{\mathrm{d}\xi}{\mathrm{d}t}\right)_{\mathrm{ALD}} &= (1-\xi^2)\frac{1}{\gamma\tau_r},\\
\frac{1}{\tau_r} &= \frac{e^4B^2}{6\pi\varepsilon_0 m_e^3 c^3}.
\end{aligned}$$

Thus, a non-isotropic reaction force is produced by the emission of synchrotron radiation. At high energy, the retarding component is approximately $\mathrm{d}p/\mathrm{d}t \simeq p_\perp^2/m_e c\tau_r$, and increases rapidly in magnitude with perpendicular momentum. However, as the electric field only accelerates runaways in the parallel direction, force-balance alone cannot explain how synchrotron radiation losses limit the maximum runaway energy. It is the pitch-angle scattering due to collisions that increases the perpendicular momentum of the electrons, which together with the subsequent synchrotron emission causes all electrons to eventually reach a steady-state velocity distribution. This steady state often exhibits a “bump” in the tail of the distribution, beyond which the distribution decays exponentially with momentum, as demonstrated in paper C.

2.5 CODE

In order to study the momentum-space dynamics of runaway electrons, a numerical tool CODE (COllisional Distributions of Electrons [66]) has recently been developed. The code obtains solutions to the kinetic equation (2.3) with the collision operator (2.15), treated as an initial-value problem. By representing the distribution function in terms of Legendre polynomials in pitch-angle cosine and a finite-difference discretization of the momentum coordinate, a flexible and computationally efficient scheme is obtained. The model contains the essential physics needed in order to study a wide range of momentum-space runaway dynamics, making it highly suited for studies such as those presented in this thesis.

Various models for large-angle collisions are implemented in CODE, which can all be derived from the Boltzmann equation (2.8). As previously mentioned, the model by Rosenbluth and Putvinski [8] follows by assuming the runaway distribution function to be of the form

$$f_e(t,p,\xi,\varphi) = \frac{n_{\mathrm{RE}}(t)}{2\pi p^2}\delta(p-p_0)\delta(\cos\theta - 1), \tag{2.18}$$

and then taking the limit $p_0 \to \infty$ of the resulting collision operator. That is, all runaways are assumed to be infinitely energetic, and have no perpendicular momentum. The basic assumption here is that the runaways will have energies much larger than the thermal energy, and also larger than the knock-on particles – which predominantly have energies corresponding to the critical runaway speed (see Fig. 1.1). This model can be powerful for example in the later phase of a runaway discharge after a tokamak disruption, where most of the runaways have had time to be accelerated to highly relativistic ($\gamma \gg 1$) energies, but is not suited for studying the initial runaway phase where most electrons have near-thermal energies.

A more accurate model for knock-on collisions was derived by Chiu *et al.* [67] by relaxing some of the assumptions made in the preceding work, instead using the form

$$f_e(t, p, \xi, \varphi) = \frac{1}{2\pi} f_e(t, p)\delta(\cos\theta - 1) \tag{2.19}$$

for the distribution function. However, in their treatment they only follow the target particle, which in practice follows from neglecting the second term of (2.8) and integrating over only half the energy interval in the first term. While this still neglects the perpendicular momentum of the runaways and ignores the slowing-down of the runaway particle, it accounts fully for the energy distribution which makes it applicable for both low energies in the initial runaway phase, as well as for large runaway energies where it reduces to the Rosenbluth-Putvinski model.

These models for knock-on collisions have been compared in detail in paper B, using kinetic simulations with CODE to determine the differences they produce in avalanche runaway growth rate. Agreeing qualitatively with an approximate analytic model, it is shown that the difference can indeed be large when the runaway momentum does not far exceed the thermal momentum; sometimes the Chiu model yields higher avalanche runaway rates than the Rosenbluth-Putvinski model, and sometimes lower, depending on the strength of the electric field.

In paper B we also investigate the runaway growth by hot-tail generation, which occurs during a rapid temperature drop [11, 12, 13, 14]. Electrons which were initially in the far tail of the hot thermal population may become runaway due to their lower collisionality, allowing them to retain their energy during the temperature drop. In the paper, previous non-relativistic studies of hot-tail generation [68] were extended to the relativistic equation which CODE solves, and the effect of the electric field on the hot-tail generation was also investigated. Runaway growths of up to almost an order of magnitude larger were demonstrated in the more complete treatment achievable using CODE, compared to previous findings.

We have now briefly described the basic kinetic theory needed for electron-runaway investigations, introducing primarily well-known concepts and results. Based on this framework, we will in the following chapter develop a powerful method of accounting for the effect of bremsstrahlung emission.

Chapter 3

Bremsstrahlung

When charged particles collide, the resulting emission is referred to as *bremsstrahlung* (German for "braking radiation", as it causes the particles to decelerate) [69, 70]. In this chapter we will describe in detail how the effect of bremsstrahlung emission on the motion of runaway electrons can be accounted for in plasmas. This question has recently been investigated in the context of magnetic-fusion plasmas in Refs. [58, 71], where an approximate model for the bremsstrahlung losses was used in order to determine the maximum energy reached by runaway electrons. In paper A we have extended their work by introducing a more realistic framework for bremsstrahlung losses based on the Boltzmann operator, which produces a qualitatively different response of the electrons compared to the previous studies.

Unlike synchrotron radiation – where typical frequencies are low enough that the energies of individual photons can be ignored – it is found that at relativistic electron energies the frequency of the emitted bremsstrahlung radiation corresponds to photon energies comparable to the electron energy. Because of this, to describe the bremsstrahlung emission from highly relativistic runaways, a quantum-mechanical description is necessary. The quantum-mechanical treatment was first described in detail in an extensive 1934 paper by Bethe and Heitler [72]. In the quantum-mechanical picture, bremsstrahlung is the result of a binary interaction between two charged particles resulting in the emission (creation) of one or more photons. The analysis of Bethe and Heitler provides a differential cross-section for the process $\mathbf{p}_1 \mapsto \mathbf{p}, \mathbf{k}$, where $\mathbf{p}_1$ and $\mathbf{p}$ are the incident and outgoing electron momenta, respectively, and $\mathbf{k}$ is the photon momentum. The target is treated as a stationary scattering center, meaning that we neglect the recoil of the target which would generally cause modifications of order $\gamma m_e/M$ when M is the target mass.

This description of bremsstrahlung as a binary collision process allows us to describe its effect on the electron distribution with the Boltzmann collision operator. The differential cross-section $\partial\sigma/\partial\mathbf{p}$ (in the form that it appears in the collision integral (2.5)) for bremsstrahlung interactions in the Born approximation was originally published by Racah [73], with a crucial misprint that was corrected by McCormick *et al.* [74] 22 years later. The interactions, of the form $\mathbf{p}_1 \mapsto \mathbf{p} + \mathbf{k}$, satisfy conservation of energy $\sqrt{1 + p_1^2/m_e^2c^2} = \sqrt{1 + p^2/m_e^2c^2} + kc$, however, momentum is not conserved, as any amount can be transferred to the infinitely heavy scattering center (the nucleus). The cross-section formula is given by

$$
\begin{aligned}
\frac{\partial\bar{\sigma}_{ab}}{\partial\mathbf{p}} &= Z_b^2\alpha r_0^2 \frac{2k}{p_1\gamma} W(p;\, p_1,\, \cos\theta_s) \\
W(p;\, p_1,\, \cos\theta_s) &= \frac{2\gamma_1\gamma + (\gamma_1^2 + \gamma^2 - 1)\lambda - \lambda^2}{k^2\lambda^2\sqrt{\lambda(\lambda+2)}} \ln\left(1 + \lambda + \sqrt{\lambda(\lambda+2)}\right) \\
&\quad - \frac{2\gamma_1\gamma - \lambda}{k^2\lambda^2} - \frac{3(\gamma_1^2\gamma^2 - 1)^2}{\lambda^2 p_1^4 p^4} \\
&\quad + \frac{4(\gamma_1^2\gamma^2 - \gamma_1\gamma + 1) - \gamma_1\gamma(p_1^2 + p^2) + (\gamma_1^2 + \gamma^2 + \gamma_1\gamma - 1)\lambda}{2\lambda^2 p_1^2 p^2} \\
&\quad + \left(2\frac{\gamma_1\gamma - 1}{\lambda^3} - \frac{k^2}{\lambda^4}\right) \frac{2(\gamma_1^2 + \gamma^2 - \gamma_1\gamma)p_1^2p^2 + 3k^2(\gamma_1 + \gamma)^2}{p_1^4 p^4} \\
&\quad + \frac{l}{p^3}\left[\frac{\gamma + 2\gamma^3}{\lambda^2 p^2} + \frac{2\gamma^4 + 2p_1^2p^2 + \gamma_1(\gamma_1 + \gamma) - (\gamma_1\gamma + p^2)\lambda}{2k\lambda^2}\right. \\
&\quad \left. + \gamma\left(2\frac{\gamma_1\gamma - 1}{\lambda^3} - \frac{k^2}{\lambda^4}\right) \frac{2\gamma_1 p^2 - 3k\gamma^2}{kp^2}\right] \\
&\quad + \frac{l_1}{p_1^3}\left[\frac{\gamma_1 + 2\gamma_1^3}{\lambda^2 p_1^2} - \frac{2\gamma_1^4 + 2p_1^2p^2 + \gamma(\gamma_1 + \gamma) - (\gamma_1\gamma + p_1^2)\lambda}{2k\lambda^2}\right. \\
&\quad \left. - \gamma_1\left(2\frac{\gamma_1\gamma - 1}{\lambda^3} - \frac{k^2}{\lambda^4}\right) \frac{2\gamma p_1^2 + 3k\gamma_1^2}{kp_1^2}\right].
\end{aligned} \tag{3.1}
$$

In the expression for W, the electron momenta p_1 and p, and photon momentum k, have been normalized to $m_e c$ for clarity. The fine-structure constant is denoted $\alpha = e^2/(4\pi\varepsilon_0\hbar c) \approx 1/137$, and $r_0 =$

$e^2/(4\pi\varepsilon_0 m_e c^2) \approx 2.8 \cdot 10^{-15}\,\mathrm{m}$ is the classical electron radius. We have also introduced the auxiliary quantities

$$\begin{aligned} l &= \ln(\gamma + p), \\ l_1 &= \ln(\gamma_1 + p_1), \\ \lambda &= \gamma_1\gamma - p_1 p \cos\theta_s - 1, \end{aligned}$$

where the full angular dependence of the cross-section is captured in λ. This is the simplest bremsstrahlung formula that provides a complete and self-consistent description. The validity of the Born approximation is limited to $v/c \gtrsim Z\alpha$ and $v_1/c \gtrsim Z\alpha$, i.e. both the incident and outgoing electron must be sufficiently fast, otherwise the plane-wave assumption in the Born approximation will be violated. For runaways typically moving near the speed of light, in plasmas where $Z \ll 100$, this is well satisfied for the incident velocity. The condition on the outgoing velocity, however, puts an upper limit on photon energies for which the formula is valid. The correction when v/c is comparable to αZ can approximately be accounted for by multiplying the cross-section formula with the so-called Elwert factor [75]

$$F_{\mathrm{E}} = \frac{v}{v_1}\frac{1 - \exp(-2\pi\alpha Z c/v_1)}{1 - \exp(-2\pi\alpha Z c/v)}.$$

A thorough analysis of bremsstrahlung emission is given in Ref. [76] (and references therein), on which our current discussion is primarily based. More sophisticated models of bremsstrahlung can be obtained by numerical methods, of which a few examples are: a full partial-wave expansion solution of the Dirac equation; Elwert-Haug theory where the lowest-order wavefunction is taken as Coulomb-problem free states instead of plane waves; or accounting for screening in the Born approximation by including the effect of bound electrons through an atomic form factor. The latter can be studied analytically in the limit of complete screening [72], but an analytic expression for $\partial\sigma/\partial\mathbf{p}$ has, to our knowledge, not been published.

In addition, the formula given in Eq. (3.1) is only strictly valid for electron-ion bremsstrahlung; for electron-electron interactions, exchange and retardation effects are important, and complicate the analysis greatly. The full quadruply differential cross-section was originally given in a Ph.D. thesis [77], but a reprint can also be found in Ref. [76]. Despite a lengthy formula covering more than 5 full pages, it has been analytically integrated over electron and photon emission angles in Refs. [78, 79]. The expression for $\partial\sigma/\partial\mathbf{p}$ needed for the collision operator does, however, not appear to exist in the published literature. Because of this, we also apply Eq. (3.1) to electron-electron bremsstrahlung. Beyond the prohibitive complexity of the full formula, this can be further motivated by the fact that in the high-energy limit, the full electron-electron and electron-ion bremsstrahlung formulas produce the same total radiation cross-section, indicating that our choice is suitable for a first approximation.

3.1 Screening

In many scenarios of interest the ions in the plasma will not be fully ionized. For example, during tokamak disruptions the temperature may drop to a few electronvolts, which is lower than typical atomic ionization energies, and large quantities of high-atomic-number gases are sometimes injected as a disruption mitigation strategy [17, 19]. The presence of electrons which remain bound to the nuclei in a plasma have a so-called *screening* effect, as the electron cloud effectively cancels (partially or totally, depending on the degree of ionization) the charge of the nucleus, as seen from afar. The importance of screening can be estimated in the following way. The differential cross-section in the Born approximation involves the Fourier transform $\tilde{V}$ of the scattering potential V,

$$\tilde{V}(\mathbf{q}) = \int \mathrm{d}\mathbf{r}\, V(\mathbf{r}) e^{i\mathbf{q}\cdot\mathbf{r}/\hbar}.$$

Here, $\mathbf{q} = \mathbf{p}_1 - \mathbf{p} - \mathbf{k}$ is the momentum transferred to the nucleus, and by the conservation of energy $p = \sqrt{(\gamma_1 - k)^2 - 1} = \sqrt{p_1^2 - 2\gamma_1 k + k^2}$ (again using normalized units). The effect of screening will be important when significant contributions to $\tilde{V}$ originate from distances of the order of the atomic size or greater, i.e. near the Bohr radius $r \sim a_0 = \hbar/m_e c\alpha$ (with α again the fine-structure constant). For sufficiently large q, the exponential factor in the integrand will be rapidly oscillating and the contribution will vanish; therefore, the minimum value of q sets the length-scale that is probed in the interaction. For $q_{\min} \lesssim \hbar/a_0 = \alpha m_e c$ screening effects are important, as the bound electrons of the atom are then probed. The minimum momentum transfer (corresponding to a maximum probed radius) is given when the vectors $\mathbf{p}_1$, $\mathbf{p}$ and $\mathbf{k}$ are aligned, and has the value

$$q_{\min} = p_1 - p - k \approx \frac{k}{2\gamma_1\gamma}, \tag{3.2}$$

where in the last step large energies were assumed, $p_1 \gg m_e c$. Note the seemingly counter-intuitive behavior that the momentum transfer to the ion decreases with increasing electron energy, a direct consequence of the conservation laws involving the creation of a third particle. The threshold for the importance of screening effects is then given by the condition

$$\frac{k}{2\gamma_1(\gamma_1 - k)} \lesssim \alpha.$$

Approximately 80% of the contribution to the total radiative stopping power (see Eq. (3.3)) comes from photon energies greater than 10% of γ_1. Therefore, setting $k = 0.1\gamma_1$, we find the condition $\gamma_1 \gtrsim 7.6$ for screening effects to be important. This means that Eq. (3.1) must be modified for highly relativistic electron energies in the presence of ions which are not fully ionized. The modification will act to reduce the rate of bremsstrahlung interactions compared to when the full nuclear charge is inserted into the Bethe-Heitler formula (3.1). Here we will, however, restrict the study to fully ionized plasmas or sufficiently low electron energies where these effects can be neglected. For more details on how atomic screening effects collisions, a thorough analysis of screening in elastic collisions has recently been given in Ref. [80].

3.2 Low-energy photon contribution

Bremsstrahlung reactions resulting in the emission of photons with energies comparable to the incident-electron energy are responsible for the dominant contribution to the radiative stopping power, which is proportional to the energy moment of the cross-section [72];

$$\langle k\sigma \rangle = \int \mathrm{d}k\, k \frac{\partial \sigma}{\partial k}. \tag{3.3}$$

Indeed, for an incident electron of energy γ_1, the relative contribution to the integral from those photons with energy $k \leq k_0 \ll \gamma_1$ is of order k_0/γ_1. Therefore it is commonly argued that the low-energy photons contribute negligibly to the fast-electron dynamics [81]. However, the above argument only proves that the energy loss is small; the reactions are kinematically allowed to change the direction of the incident electron arbitrarily, as the nucleus acts as a momentum sink. As the cross-section for small k goes as $\mathrm{d}\sigma \propto 1/k$, it is clear that many reactions involving low-energy photons occur, indicating that this may be a significant effect.

The singularity in the cross-section also proves to be a challenge when numerically evaluating the Boltzmann operator, as the gain and loss terms are both infinitely large. However, with the method described below a simplified model can be derived, where the singularity is analytically resolved by considering the contribution from the low-energy photons separately, and utilizing the smallness of k/p_1, the ratio of photon to electron momentum.

To leading order in k/p_1, for large energies $\gamma_1 \gg 1$, the bremsstrahlung cross-section (3.1) takes the form

$$\frac{\partial\sigma}{\partial\cos\theta_s\partial k} = \frac{1}{k}W(\gamma_1,\gamma),$$

where $W(\gamma_1,\gamma) = W(\gamma,\gamma_1) = W(\gamma_1)$ is symmetric, and $k = \gamma_1 - \gamma$. We can rewrite the Boltzmann operator, Eq. (2.7), by writing

$$\frac{\partial\sigma}{\partial\mathbf{p}} = \frac{1}{2\pi}\frac{1}{p\gamma}\frac{\partial\sigma}{\partial k\partial\cos\theta_s},$$

and changing variables in the integral, $\mathrm{d}p_1 = (\partial p_1/\partial k)\mathrm{d}k = \mathrm{d}k/v_1$. This yields the equivalent form

$$\begin{aligned} C_L(t,p) = {}& n_b\int \mathrm{d}k\mathrm{d}\cos\theta_s\,\frac{p_1^2}{p\gamma}f_L(p_1)P_L(\cos\theta_s)\frac{\partial\sigma}{\partial k\partial\cos\theta_s} \\ & - n_b\int \mathrm{d}k\mathrm{d}\cos\theta_s\,vf_L(p)\frac{\partial\sigma}{\partial k\partial\cos\theta_s}, \end{aligned}$$

where the symmetry property of the cross-section was (indirectly) used in the bottom line. Now, to leading order in k/p_1 we also have $p_1 = p$, which finally yields the operator

$$C_L(t,p) = -n_b v f_L(p)\int \mathrm{d}\cos\theta_s\left[1 - P_L(\cos\theta_s)\right]\frac{\partial\sigma}{\partial\cos\theta_s}, \tag{3.4}$$

where

$$\frac{\partial\sigma}{\partial\cos\theta_s} = \int \mathrm{d}k\,\frac{\partial\sigma}{\partial k\partial\cos\theta_s} = 2\pi\gamma p\int \mathrm{d}k\,\frac{\partial\sigma(p;\,p_1,\,\cos\theta_s)}{\partial\mathbf{p}}.$$

The operator thus obtained is the stationary-target elastic-scattering operator of Eq. (2.9), which is expected as we have neglected the energy carried by the photons. For small angles $\theta \lesssim 1/\gamma_1$, the cross-section $\partial\sigma/\partial\cos\theta_s$ goes as $1/\theta^2$. This is in contrast with the cross-section for purely elastic Coulomb collisions, which goes as $1/\theta^4$, indicating that the low-energy photon contribution to bremsstrahlung will not be so dominated by small-angle collisions. Indeed, a significant contribution to the integrals in C_L are given by scattering angles $\theta_s \sim 1$, showing that a Fokker-Planck treatment is inadequate. The vanishing of the $L = 0$ term (corresponding to the spherically symmetric part of the operator, since $P_0(x) = 1$) ensures that these reactions contribute only to pitch-angle deflection, and not energy loss, consistent with the argument at the beginning of the section and the result obtained in Ref. [81].

To quantify the importance of the low-energy photon effect, we analytically evaluate the $L = 1$ term of Eq. (3.4) and compare it to the corresponding term of the elastic Coulomb-scattering operator. The latter is proportional to the term containing Z_{eff} in Eq. (2.15). In the limit $\gamma_1 \gg 1$ and $k/\gamma_1 \ll 1$, after a tedious integration of the full cross-section formula in Eq. (3.1), the ratio between the bremsstrahlung and Coulomb expressions is found to be

$$\frac{C_1^{\text{small-k}}}{C_1^{\text{elastic}}} = \alpha\frac{2}{\pi}\frac{\ln\Lambda_{\text{B}}}{\ln\Lambda}\left[\left(\ln\frac{2p}{m_e c} - 1\right)^2 + 1\right]. \tag{3.5}$$

We have introduced the bremsstrahlung logarithm $\ln\Lambda_{\text{B}}$, which arises in a manner analogous to the Coulomb logarithm for elastic collisions and is due to the logarithmic divergence with k of the bremsstrahlung cross-section. If the operator is constructed to account for all bremsstrahlung emissions of energy $k \leq k_0$, the bremsstrahlung logarithm is defined as $\ln\Lambda_{\text{B}} = \ln(k_0/k_c)$, where k_c is a lower cut-off in photon energy which has a physical origin, to be discussed below. The ratio in Eq. (3.5) decreases monotonically with L (which can be verified numerically by performing the integration in Eq. (3.4)), indicating that for the comparison it is sufficient to evaluate the $L = 1$ term.

3.3 Lower limit in photon energy

We will now discuss the integration limits in the evaluation of $\partial\sigma/\partial\cos\theta_s$. The operator (3.4) will be used to account for those bremsstrahlung emissions with photon energy $k \leq k_0$, where k_0 is some arbitrary cutoff satisfying $k_0 \ll \gamma_1 - 1$. However, the integral is singular, and needs to be cut off at some lower limit as well, which we denote k_c. There are three mechanisms which need to be considered in determining this lower limit:

1. Polarization of the background medium – in a semi-classical treatment, scattering in a dielectric medium can be modeled by multiplying the cross-section by a suppression factor depending on the dielectric constant of the medium. This will have the effect of effectively screening those interactions in which the radiation is emitted with frequency below the plasma frequency, thus effectively cutting off the integral.

2. Many-photon emissions – at low photon energies, many-photon emissions will become increasingly important for the reaction rate. It is known that this quantum-electrodynamical effect naturally resolves the singularity in the cross-section.

3. Stimulated emission and absorption – at sufficiently low photon energy, there will be a high enough number of photons that they will significantly interact with the plasma before escaping. This has the effect of limiting the validity of our equation to photon energies above some critical value.

We shall discuss these effects in turn, and investigate what the respective cutoff energies are.

Polarization of the background medium: The effect on the cross-section of the polarization of the background medium was originally treated in Ref. [82]; it was experimentally verified by the SLAC-E-136 accelerator [83], and the effect (together with related phenomena) has been discussed at length in a more recent review article [84]. The argument goes as follows: as we discussed in connection with the screening effect of bound electrons, there is a length-scale associated with a bremsstrahlung reaction, related to the momentum q transferred to the ion by $l = \hbar/q$ (which is the wavelength of the virtual photon exchanged between electron and nucleus). Here, however, we are interested in the parallel length scale $l_\mathrm{B} = \hbar/q_\|$, which is sometimes referred to as the *formation length*. The parallel momentum transfer is

$$q_\| = \frac{\mathbf{p}_1}{p_1} \cdot (\mathbf{p}_1 - \mathbf{p} - \mathbf{k}) = p_1 - p\cos\theta_s - k\cos\theta_1,$$

where $\cos\theta_s = \mathbf{p}_1\cdot\mathbf{p}/p_1p$ and $\cos\theta = \mathbf{p}_1\cdot\mathbf{k}/p_1k$. As we are now concerned about the contribution from photons carrying momentum much smaller than the electron momenta, we Taylor expand this formula to first order in k/p_1, using $p = \sqrt{(\gamma_1 - k)^2 - 1}$. This gives

$$q_{\|} = p_1(1 - \cos\theta_s) + k\left(\frac{1}{v_1}\cos\theta_s - \cos\theta_1\right).$$

If we here expand in large electron energy $\gamma_1 \gg 1$ and small angles $\theta_s \sim \theta_1 \ll 1$, we find

$$q_{\|} = \frac{k}{2\gamma_1^2} + \frac{\gamma_1}{2}(\theta_1^2 - \theta_s^2).$$

As expected, for vanishing scattering angles (when electron and photon momenta are aligned), we obtain again the minimum momentum transfer of Eq. (3.2). If we neglect the contribution due to the angular deviation, i.e. for $(\gamma_1\theta)^2 \ll k/\gamma_1$, we find a formation length

$$l_{\mathrm{B}} = \frac{2\hbar\gamma_1^2}{k}.$$

The analysis in Ref. [84] shows that the effect is covered by the introduction of a suppression factor S, defined so that

$$\frac{\partial\sigma}{\partial\mathbf{p}} = S\left(\frac{\partial\sigma}{\partial\mathbf{p}}\right)_0,$$

where $(\partial\sigma/\partial\mathbf{p})_0$ is the cross-section of Eq. (3.1) with polarization effects unaccounted for. The suppression factor S is given by the ratio between formation lengths when the photon energy k is replaced by $\sqrt{\epsilon}k$ (to account for the change in the speed of light of photons in the medium) and when it is left unchanged. That is, we evaluate

$$S = \frac{p_1 - \cos\theta_s\sqrt{(\gamma_1 - k)^2 - 1} - k\cos\theta_1}{p_1 - \cos\theta_s\sqrt{(\gamma_1 - k)^2 - 1} - \sqrt{\epsilon}k\cos\theta_1},$$
$$\epsilon(k) = 1 - \frac{\hbar^2\omega_p^2}{\omega^2},$$

where the high-frequency limit of the dielectric tensor ϵ is used. In the limit where angular deflection is ignored, we find a suppression factor

$$S(k) = \frac{1}{1 + (\gamma\hbar\omega_p/kc)^2},$$

indicating that the cross-section is effectively cut off at photon momenta $k = \gamma\hbar\omega_p/c$, or when the frequency of the radiation is $\gamma\omega_p$. However, as mentioned, this is valid only when $(\gamma_1\theta)^2 \ll k/\gamma_1$. In practice, the low photon-energy operator acquires significant contributions from k/γ_1 much smaller than 1%, and when $\gamma_1\theta$ is of order unity, or significantly larger. Therefore we are in fact often in the opposite limit, where the momentum transfer to the nucleus depends only on the scattering angle, and not on the photon energy. Hence the above argument (with an effective cut-off in parallel momentum transfer) will sometimes not cause a cut-off in photon energy, as the suppression factor stays close to unity.

However, if we anyway apply this model which has been commonly used in the literature and was experimentally observed, and since the final expression is only logarithmically sensitive to our choice, it is found that the effect of background polarization ensures that radiation with frequency below the plasma frequency $\omega_p = \sqrt{n_e e^2/m_e\varepsilon_0}$ will be suppressed. This corresponds to photon energies $k_c c = \hbar\omega_p$, or in normalized units

$$\frac{k_c}{m_e c} = \frac{\hbar\omega_p}{m_e c^2} = 7.3 \cdot 10^{-10}\sqrt{n_{20}},$$

where $n_{20} = n_e/(10^{20}\,\mathrm{m}^{-3})$. Introducing $\kappa_0 = k_0/m_e c$ as a normalized photon energy, the bremsstrahlung logarithm is given by

$$\ln\Lambda_\mathrm{B} = \ln\frac{\kappa_0 m_e c^2}{\hbar\omega_p} = \ln\frac{\sqrt{m_e\varepsilon_0}\kappa_0 m_e c^2}{\sqrt{n_e e^2}\hbar} \approx 21.0 - \ln\frac{\sqrt{n_{20}}}{\kappa_0}. \tag{3.6}$$

For typical runaway scenarios in fusion plasmas, where γ ranges from 10 to 100 and n_{20} ranges from 0.1 to 10, and if we choose $\kappa_0 = \gamma/1000$, the bremsstrahlung logarithm $\ln\Lambda_\mathrm{B}$ takes values in the range 15 to 20.

2. Many-photon emissions: We have discussed the bremsstrahlung cross-section in lowest-order perturbation theory, which corresponds to only accounting for bremsstrahlung processes where a single photon is emitted. This is known to break down (in the infamous infrared divergence of QED [85]) as the photon energies approach zero, where multi-photon processes contribute significantly. However, these multi-photon processes can be elegantly accounted for with a simple modification to the cross-section. To show this, we present an argument from Ref. [86]: assume that the cross-section is determined by a measurement device that can measure only photons with energy above some minimum energy E_l. Then, the observed cross-section is obtained by integrating over all multi-photon processes where photons of energy less than E_l are involved. This procedure yields the correction

$$\frac{\partial\sigma}{\partial\mathbf{p}} = F(q)\left(\frac{\partial\sigma}{\partial\mathbf{p}}\right)_0, \tag{3.7}$$

where $q^2 = (\mathbf{p}-\mathbf{p}_1)^2 - (\gamma-\gamma_1)^2$, and

$$\begin{aligned} F(q) &= \exp\left[-\frac{\alpha}{\pi} f_{\mathrm{IR}}(q^2)\ln\frac{q^2}{E_l^2}\right], \\ f_{\mathrm{IR}}(q^2) &= \int_0^1 \mathrm{d}x\,\frac{1+q^2/2}{1+q^2x(1-x)} - 1 \\ &= \frac{2+q^2}{q\sqrt{q^2+4}}\ln\frac{\sqrt{q^2+4}+q}{\sqrt{q^2+4}-q} - 1. \end{aligned}$$

The effect of soft-photon processes is then negligible whenever

$$\frac{\alpha}{\pi} f_{\mathrm{IR}}\left(q^2\right)\ln\left(\frac{q^2}{E_l^2}\right) \ll 1. \tag{3.8}$$

The left-hand side is monotonically increasing with q, indicating that we need to verify that the inequality (3.8) is well satisfied for the case where q is the greatest. In our case of low-energy photon emissions, $q \sim |\mathbf{p}-\mathbf{p}_1|$ is the momentum transferred to the nucleus, which is maximized by large-angle collisions where $q \sim p$. For $q \gg 1$, the asymptotic form of the infrared correction is $f_{\mathrm{IR}} = \log q^2$. We then obtain the condition

$$\frac{\alpha}{\pi}\ln\frac{p^2}{m_e^2c^2}\ln\frac{p^2c^2}{E_l^2} \ll 1.$$

Solving for p then yields

$$\frac{p}{m_ec} \ll \sqrt{\frac{E_l}{m_ec^2}}\exp\sqrt{\left(\log\sqrt{\frac{E_l}{m_ec^2}}\right)^2 + \frac{\pi}{4\alpha}}.$$

As we will consider “hard photons” (primary ones) emitted with frequencies as low as the plasma frequency, we will be interested in determining what the effect on the cross-section is of soft-photon emissions with lower energy. If we can demonstrate that the cross-section is negligibly affected in this worst-case scenario, we can safely ignore the effect entirely. Therefore, setting $E_l = \hbar\omega_p$ we obtain the condition

$$\begin{aligned}\frac{p}{m_e c} &\ll 2.7 \cdot 10^{-5}\sqrt{n_{20}}\exp\sqrt{\left(10.5 - \frac{1}{4}\ln n_{20}\right)^2 + 108} \\ &\simeq 70.1(n_{20})^{1/5.6} \quad (\text{for } \ln n_{20} \ll 40).\end{aligned}$$

Because of this, for typical runaway energies of order 10-100 MeV, we see that the soft-photon contribution will become important approximately at the plasma-frequency scale of photon energies, but only for the largest-angle reactions (where $q \sim p$). Note that the inequality is in fact logarithmic, indicating that this effect will also affect the process at lower energies. However, note that this is a pessimistic estimate. In fact, a large contribution to the collision operator is given by interactions with emission angles $\theta_s \sim 1/\gamma$, for which the momentum transfer q is of order $m_e c$. At this scale, the soft-photon correction will always be negligible in practical scenarios.

3. Stimulated emission and absorption: It is well known that existing photons in the plasma can interact with the electrons by the related processes of stimulated emission and absorption. These effects are proportional to the number of photons in the plasma, and will therefore be increasingly significant for low-energy photons, as the cross-section then grows as $1/k$. To ensure the validity of our proposed bremsstrahlung operator, we need to verify that these effects are small in the scenarios that we consider.

In Ref. [87] the electron bremsstrahlung Boltzmann operator is given, accounting for spontaneous and stimulated emission and absorption, as

$$\begin{aligned}C\mathrm{d}\mathbf{p} = n_i &\int \mathrm{d}\mathbf{p}'\mathrm{d}\nu\mathrm{d}\Omega\mathrm{d}\Omega_p\, f_e(\mathbf{p}')\tilde{\alpha}_\nu(\mathbf{p}';\mathbf{n},\mathbf{e})\left[1 + \frac{c^2}{2h\nu^3}I_\nu(\mathbf{n})\right] \\ &+ n_i \int \mathrm{d}\mathbf{p}'\mathrm{d}\nu\mathrm{d}\Omega\mathrm{d}\Omega_p\, f_e(\mathbf{p}')\tilde{\beta}_\nu(\mathbf{n},\mathbf{p}';\,\mathbf{e})I_\nu(\mathbf{n}) \\ &- n_i \int \mathrm{d}\mathbf{p}\mathrm{d}\nu\mathrm{d}\Omega\mathrm{d}\Omega_p'\, f_e(\mathbf{p})\tilde{\alpha}_\nu(\mathbf{p};\,\mathbf{n},\mathbf{e}')\left[1 + \frac{c^2}{2h\nu^3}I_\nu(\mathbf{n})\right] \\ &- n_i \int \mathrm{d}\mathbf{p}\mathrm{d}\nu\mathrm{d}\Omega\mathrm{d}\Omega_p'\, f_e(\mathbf{p})\tilde{\beta}_\nu(\mathbf{n},\mathbf{p};\,\mathbf{e}')I_\nu(\mathbf{n}).\end{aligned}$$

We here follow the original notation in [87], where the essential part is the specific intensity of radiation I_ν, which is related to the photon distribution function $\phi(\mathbf{r}, \mathbf{k}, t)$ by

$$\phi(\mathbf{r}, \mathbf{k}, t) = \frac{c^2}{h^4 \nu^3} I_\nu(\mathbf{n}, \mathbf{r}, t).$$

The emission and absorption coefficients are denoted $\tilde{\alpha}$ and $\tilde{\beta}$, respectively, and are related by the three-body version of the principle of detailed balance [48, 87]

$$\tilde{\beta}_\nu(\mathbf{n}, \mathbf{p};\, \mathbf{e}') = \frac{v'}{v} \frac{c^2}{2h\nu^3} \tilde{\alpha}_\nu(\mathbf{p}';\, \mathbf{n}, \mathbf{e}).$$

In this formula, the terms involving $\tilde{\alpha}$ but not I_ν correspond to spontaneous emission, as described by our operator, while the rest account for absorption and stimulated emission. Therefore, the relative importance of these effects compared to the spontaneous emission that we have considered is of order

$$\frac{\text{effect of absorption and stimulated emission}}{\text{effect of spontaneous emission}} \sim \frac{c^2}{2h\nu^3} I_\nu = \frac{h^3 \phi}{2}.$$

Thus we see that the effect is expected to be small when the mean phase-space volume occupied by photons is less than two Planck units, i.e. when

$$\frac{h^3}{2} \phi \ll 1.$$

To complete the argument we only need to find an estimate for the photon distribution function, ϕ. We can do so by using the following order-of-magnitude estimate:
The rate at which photons are created, per unit volume and momentum, is given by

$$\frac{\mathrm{d}n_\phi(t, \mathbf{r})}{\mathrm{d}t \mathrm{d}\mathbf{k}} = \sum_b n_b(t, \mathbf{r}) \int \mathrm{d}\mathbf{p}\, f_e(t, \mathbf{r}, \mathbf{p}) v \frac{\partial \sigma_{eb}}{\partial \mathbf{k}},$$

where $\partial \sigma_{eb} / \partial \mathbf{k}$ is the bremsstrahlung differential cross-section (with respect to emitted-photon energy and direction), which can be found for example in Ref. [88], and the sum is taken over all particle species in the plasma. Since the cross-section goes as $1/k$ for small k, it will be sufficient to show that our assumption is justified in the low-photon-energy limit where the photon distribution is expected to be the largest. The differential cross-section is

$$\frac{\partial\sigma}{\partial\mathbf{k}} \sim \frac{1}{4\pi k^2}\frac{\partial\sigma}{\partial k} \simeq \frac{4}{3\pi}\frac{\alpha Z^2 r_0^2}{k^3}\ln\frac{2p^2}{m_e ck},$$

as obtained from Eq. (3.1) integrated over electron angles (a calculation initially performed in Ref. [73]) in the low-k, high-p limit. Assuming a mono-energetic relativistic electron beam with momentum p, and the plasma to be confined in a volume $V = L^3$, the total number of photons created per second per momentum-space volume $\mathbf{k}$ is (summed over particle species)

$$\frac{\mathrm{d}N_\phi}{\mathrm{d}t\mathrm{d}\mathbf{k}} \approx \frac{4}{3}V n_e n_{\mathrm{RE}}\frac{(1+Z_{\mathrm{eff}})cr_0^2}{137\pi}\frac{1}{k^3}\ln\frac{2p^2}{m_e ck}.$$

Multiplying the expression by L/c yields the total number of photons in the volume, which divided by V gives the distribution function:

$$\phi(\mathbf{r},\mathbf{k},t) \sim \frac{4}{3}L n_e n_{\mathrm{RE}}\frac{(1+Z_{\mathrm{eff}})r_0^2}{137\pi}\frac{1}{k^3}\ln\frac{2p^2}{m_e ck}. \tag{3.9}$$

At the minimum value of $k = k_c = \hbar\omega_p/c$, we thus require the smallness of

$$\begin{aligned}\frac{h^3}{2}\phi &\sim \frac{16\pi^2}{3}\frac{1+Z_{\mathrm{eff}}}{137}\frac{Lr_0^2 n_e n_{\mathrm{RE}} c^3}{\omega_p^3}\left(2\ln\frac{p}{m_e c} - \ln\frac{\hbar\omega_p}{2m_e c^2}\right) \\ &\approx 4.6(1+Z_{\mathrm{eff}})\frac{L n_{\mathrm{RE},20}}{\sqrt{n_{20}}}\left(21.7 + 2\ln\frac{p}{m_e c}\right), \end{aligned}\tag{3.10}$$

where L is in meters and densities are in units of $10^{20}\,\mathrm{m}^{-3}$. As an upper limit for reasonable values of this quantity in laboratory plasmas, we take a post-disruption runaway scenario with $Z_{\mathrm{eff}} = 15$, $L = 10\,\mathrm{m}$, $n_{\mathrm{RE}} = 10^{-3}n_e$ and $n_e = 50\cdot 10^{20}\,\mathrm{m}^{-3}$, with $p = 200 m_e c$. Eq. (3.10) then takes the value 170, which means that there are many photons per Planck unit of phase-space volume, and absorption and stimulated emission can thus be expected to have a significant effect on those photons with the lowest energy. Note, however, that due to the $1/k^3$ sensitivity to photon energy in Eq. (3.9), less than an order of magnitude above $\hbar\omega_p$ the photon distribution will again be negligible. If we were to instead cut the integration off at the photon energy where $h^3\phi$ is small, $\ln\Lambda_{\mathrm{B}}$ would typically change by less than 10%, because of the weak logarithmic sensitivity to k_c in the bremsstrahlung logarithm.

In a more realistic scenario, one could have $L \sim 1\,\mathrm{m}$, $n_e \sim 10^{20}\,\mathrm{m}^{-3}$, $n_{\mathrm{RE}} \sim 10^{-4} n_e$ and $p \sim 50 m_e c$, giving $h^3\phi/2 \sim 0.2$. Then absorption and stimulated-emission effects become important close to the plasma frequency, making ω_p again a suitable cut-off point when these effects are accounted for.

In conclusion, it appears that in laboratory plasmas the lower limit k_c in photon energy is well described by considering only the screening effect of scattering in a dielectric medium, where the plasma frequency effectively cuts off the bremsstrahlung reaction rate. The QED effect of (soft) many-photon emissions typically becomes important at higher electron energies, or significantly lower plasma densities, while stimulated emission and absorption processes become important for larger systems, or higher runaway densities. Therefore the bremsstrahlung differential cross-section can be cut off near the plasma frequency, which introduces the bremsstrahlung logarithm $\ln \Lambda_{\mathrm{B}}$ from Eq. (3.6). This factor enhances the contribution to pitch-angle deflection from photons which carry energies much smaller than the electron energy.

Chapter 4

Ion runaway

In this chapter we will describe ion runaway, which is a process related to electron runaway. This will provide an introduction to the theory underlying the work presented in paper D, which numerically treats the ion kinetic equation to determine the conditions for ion runaway. For this purpose, the tool CODE for electron-runaway studies was extended to solve the ion kinetic equation, resulting in the new open-source code CODION (COllisional Distributions of IONs [89, 90]).

Ion runaway is a phenomenon similar to electron runaway in many ways. The initial runaway-generation mechanism is the same – a sufficiently strong electric field can overcome the collisional friction force of the thermal bulk and accelerate a subpopulation of ions to high energy. However, there are a few key differences between ion runaway and electron runaway:

1. The ions are not the lightest particle species in the plasma; the ion-electron collisions are qualitatively different to the electron-ion collisions.

2. The collisional friction force on a runaway ion is strongly non-monotonic, with the consequence that, unless the electric field is comparable to the Dreicer field, the kinetic energy of runaway ions is bounded from above because of electron friction

3. A non-relativistic treatment is sufficient, and radiation losses are negligible.

The kinetic description of runaway ions can be constructed in the following way. The particle species in a homogeneous plasma each satisfy a kinetic equation.

$$\frac{\partial f_a}{\partial t} + \frac{q_a}{m_a} E_\parallel \frac{\partial f_a}{\partial v_\parallel} = \sum_b C_{ab}\{f_a, f_b\}, \tag{4.1}$$

where the sum on the right-hand side is taken over all particle species b in the plasma. We will consider only that ion species which is accelerated at the highest rate, in order to be able to assume that the other ion species remain near equilibrium (i.e. that they are well described by a Maxwellian distribution function). However, this cannot be assumed for self-collisions of this species, nor for ion-electron collisions. Instead the self-collision operator can be linearized by writing $C_{aa}\{f_a, f_a\} \approx C_{aa}\{f_a, f_{Ma}\} + C_{aa}\{f_{Ma}, f_a\}$ where f_{Ma} is the Maxwellian distribution for particle species a with density n_a and temperature T_a, we obtain the kinetic equation

$$\begin{aligned}\frac{\partial f_a}{\partial t} + \frac{q_a}{m_a} E_\parallel \frac{\partial f_a}{\partial v_\parallel} =& C_{aa}\{f_{Ma}, f_a\} + C_{ae}\{f_a, f_e - f_{Me}\} \\ &+ \sum_b C_{ab}\{f_a, f_{Mb}\}.\end{aligned}$$

As argued in Ref. [57], the perturbed electron distribution $f_e - f_{Me}$ varies over velocities much greater than the ion velocity, and therefore yields a collision operator that describes a uniform friction force. For an ion species i, the ion-electron collision operator is

$$C_{ie}\{f_i, f_e - f_{Me}\} = -\frac{\mathbf{R}_{ie}}{m_i n_i} \cdot \frac{\partial f_i}{\partial \mathbf{v}},$$

where $\mathbf{R}_{ie} = \int \mathrm{d}\mathbf{v}\, m_i \mathbf{v} C_{ie}$ is the mean ion-electron friction force. This assumes that f_{Me} is chosen as a Maxwellian in the rest frame of the ions, otherwise the operator $C_{ae}\{f_a, f_{Me}\}$ will contribute additional friction that would need to be accounted for in the above term. Due to the conservation of momentum in collisions, the ion-electron friction force is related to the electron-ion friction by $\mathbf{R}_{ie} = -\mathbf{R}_{ei}$, and $\mathbf{R}_{ei}$ can readily be calculated from the electron kinetic equation.

Taking the momentum moment $\int \mathrm{d}\mathbf{v}\, m_e \mathbf{v} \ldots$ of the electron kinetic equation yields the force-balance equation

$$\frac{\partial (n_e m_e \mathbf{V}_e)}{\partial t} = -n_e e \mathbf{E} + \sum_j \mathbf{R}_{ej}, \tag{4.2}$$

where the sum is taken over all ion species j (the electron-electron contribution vanishing due to momentum conservation in collisions). Due to the large mass ratio m_j/m_e, the ions are generally much slower than the electrons. As a consequence, the dependence on ion species in the electron-ion friction is to leading order entirely captured in the collision frequency $\nu_{ej} \propto n_j Z_j^2$, yielding

$$\sum_j \mathbf{R}_{ej} = \mathbf{R}_{ei} \sum_j \frac{n_j Z_j^2}{n_i Z_i^2} = \frac{n_e Z_{\text{eff}}}{n_i Z_i^2} \mathbf{R}_{ei}.$$

The electrons equilibrate with the electric field on a time scale much shorter than that of the ion runaway process, due to their lower mass. Therefore, unless significant electron-runaway generation occurs which effectively acts as a momentum sink, the time-derivative can be neglected in the force-balance equation (4.2), yielding

$$\mathbf{R}_{ie} = -\mathbf{R}_{ei} = -\frac{Z_i}{Z_{\text{eff}}} n_i q_i \mathbf{E}.$$

In the ion kinetic equation this readily allows the ion-electron collision operator to be combined with the electric-field term. Equation (4.1) can then be written

$$\frac{\partial f_i}{\partial t} + \frac{q_i}{m_i} E^* \frac{\partial f_i}{\partial v_\parallel} = C_{ii}\{f_{Mi}, f_i\} + \sum_b C_{ib}\{f_i, f_{Mb}\},$$
$$E^* = \left(1 - \frac{Z_i}{Z_{\text{eff}}}\right) E_\parallel,$$

and describes a population being accelerated by the *effective* electric field E^*, and only experiencing collisions with Maxwellian background species (and the field-particle contribution from self-collisions in order to ensure the conservation of momentum and energy in such collisions). The effective electric field thus arises due to friction between ions and the electrons drifting in the electric field.

In a pure plasma, $Z_i = Z_{\text{eff}}$, and complete cancellation occurs between electric field and electron friction; no ion runaway is possible in this case. Only in impure plasmas does a finite effective electric field remain, and indeed, for an ion species with $Z_i \gg Z_{\text{eff}}$ the effective electric field can exceed the original electric field in magnitude, and is antiparallel to the electric field. In this scenario, runaway occurs due to acceleration by electron friction rather than by the electric field.

As we consider the initial-value problem starting from a thermal equilibrium, we will typically work in the initial rest frame. As the runaway-ion population builds up, the total friction against the Maxwellian electrons may end up being comparable to the friction $-(Z_i/Z_{\text{eff}})Z_i eE$ against the drifting electrons, at which point the model is no longer valid. This, together with the use of a linearized self-collision operator, puts an upper boundary on the ion runaway densities we may consider.

4.1 Ion friction-force estimates

Valuable physical insight into the ion runaway process can be obtained by considering the friction force acting on a test-ion moving through the plasma. Formally, the test-particle equations of motion can be obtained by considering velocity moments of the kinetic equation for a delta distribution $f_a = \delta(\mathbf{v} - \mathbf{u}(t))$ representing the test particle [91]. This method was pursued in Refs. [24, 26] where ion runaway in solar flares was considered, and later expanded upon in paper D to consider general plasma compositions. The momentum moment of the ion kinetic equation yields the test-particle equation of motion

$$\frac{\partial(m_i v)}{\partial t} = Z_i e E^* \xi - \frac{m_i v_{Ti}}{\tau_{ie}} \left(\frac{Z_{\text{eff}} + \bar{n}}{v^2/v_{Ti}^2} + \frac{4}{3\sqrt{\pi}} \sqrt{\frac{m_e T_i^3}{m_i T_e^3}} \frac{v}{v_{Ti}} \right),$$

where we use the pitch-angle cosine $\xi = v_\parallel/v$, the ion-electron collision frequency $\tau_{ie}^{-1} = n_e \ln\Lambda Z_i^2 e^4/(4\pi\epsilon_0^2 m_i^2 v_{Ti}^3)$, and introduced the quantity $\bar{n} = \sum_j n_j Z_j^2 m_j/(n_e m_i)$. Here explicit expressions for the collision operator with a Maxwellian background species [91] have been used, under the assumption that the velocities satisfy $v_{Tj} \ll v \ll v_{Te}$ for all ion species j. The term containing the parentheses represents collisional friction, in which the first term expresses ion-ion friction which decreases with velocity and dominates for low velocities, whereas the second term describing ion-electron friction increases with velocity and dominates at high velocities.

The solutions $\partial(m_i v)/\partial t = 0$ represent those velocities where electric-field acceleration exactly balances collisional friction. For electric fields

$$E^* > E^*_{\min} = 2\frac{m_i v_{Ti}}{Z_i e \tau_{ie}} \frac{T_i}{T_e} \left(\frac{3}{2\pi} \frac{m_e}{m_i} (Z_{\text{eff}} + \bar{n}) \right)^{1/3},$$

two solutions v_{c1} and v_{c2} exist which describe the runaway velocity above which an ion is accelerated by the electric field, and the maximum velocity before electron friction overcomes the electric field, respectively. Therefore, ions with velocity $v_{c1} < v < v_{c2}$ will be accelerated, before accumulating at v_{c2}.

However, it should be noted that the above test-particle equation of motion is not unique. By considering the energy moment $\int \mathrm{d}\mathbf{v}\,(m_i v^2/2)\ldots$ of the ion kinetic equation, one instead obtains

$$\frac{\partial(m_i v)}{\partial t} = Z_i e E^* \xi - \frac{m_i v_{Ti}}{\tau_{ie}}\left[\bar{n}\frac{v_{Ti}^2}{v^2} + \frac{4}{3\sqrt{\pi}}\sqrt{\frac{m_e T_i^3}{m_i T_e^3}}\left(\frac{v}{v_{Ti}} - \frac{3T_e}{T_i}\frac{v_{Ti}}{v}\right)\right].$$

If we assume that $(v/v_{Ti})^2 \gg 3T_e/T_i$, this reduces to the momentum-moment equation (4.2) with the simple exchange $\bar{n} \mapsto Z_{\text{eff}} + \bar{n}$. This equation may provide more accurate estimates of the critical velocities; using the same procedure to estimate the electron runaway velocity shows that the energy-balance equation yields the well-known formula $v_c/v_{Te} = \sqrt{E_D/2E}$, while the momentum-balance equation incorrectly gives a result which is larger by a factor $\sqrt{2}$. The discrepancy may be understood by the fact that pitch-angle scattering contributes to friction in the momentum-balance equation, but not in the energy-balance equation as it is an energy-conserving effect. The angular deflection will not stop a particle from running away (except sometimes indirectly), and hence the energy-balance equation provides a better estimate. The substitution $(Z_{\text{eff}} + \bar{n}) \mapsto \bar{n}$ may thus improve the results given in Refs. [24, 26] and paper D, although these estimates should perhaps primarily be viewed as a guide to interpret solutions of the kinetic equation, and to make qualitative predictions regarding the features of the ion runaway distribution.

Note finally the limits to the validity of the model described here. The linearization of the self-collision operator requires small runaway densities, corresponding to short times or electric fields $E^* \sim E^*_{\text{min}}$. Extending far above E^*_{min} requires the use of a non-linear self-collision operator. At the same time, the electric field must be sufficiently weak to avoid significant runaway-electron generation which would affect E^*, therefore requiring $E \lesssim 0.1 E_D$.

Chapter 5

Conclusions and outlook

Conclusions

Runaway is an important phenomenon, which occurs in both terrestrial and space plasmas. It is of particular interest in magnetic-fusion research where runaway electrons can strike the wall of the reactor after being accelerated to highly relativistic energies, at which point they can cause severe damage to plasma-facing components. Runaway is also of interest in space and astrophysical applications, where they may be responsible for observed gamma-ray emissions.

In this thesis, we have described a kinetic model of runaway in plasmas, accounting for acceleration by an electric field, elastic Coulomb collisions and dissipation by radiation. We have discussed how a combination of a Boltzmann collision operator and Fokker-Planck operator is needed in order to describe both the large-angle collisions which lead to runaway avalanches, as well as the small-angle collisions that otherwise dominate the collisional dynamics. The framework described here is applicable to model the runaway dynamics of both electrons and ions.

An existing numerical tool which solves the electron kinetic equation in a homogeneous plasma, CODE (COllisional Distribution of Electrons) [66], has been extended to include a range of new effects. The radiation reaction force due to synchrotron emission in a magnetized plasma was described in chapter 2 within the framework of the electromagnetic Abraham-Lorentz-Dirac force, and its effect on the dynamics of runaway electrons was considered in paper C. It is shown that the synchrotron reaction force has a tendency to form a non-monotonic "bump" in the runaway distribution, which effectively also limits the maximum energy of the electrons. A condition on plasma parameters for when such bumps can form – and at what energy – was derived, which is of importance in magnetic-fusion research as such features have the potential to destabilize the runaway beam.

In paper B the runaway dynamics was considered in plasmas where temperatures are rapidly varying in time. This causes hot-tail runaway generation to occur, and by generalizing previous studies to also account for relativistic corrections and the effect of an electric field, it was demonstrated that CODE is well-suited to study this phenomenon. In the same paper, a comparison was also performed between different knock-on collision operators that have been used in the literature. It was demonstrated that, while the more accurate model of Chiu *et al.* [67] reduces to the simpler Rosenbluth-Putvinski model [8] when the runaway energies are large, at lower energies the avalanche growth rates differ significantly. At low electric fields ($E \sim E_c$) the former model yielded runaway growth rates as small as an order of magnitude lower than the simpler model, while at high electric field ($E \gg E_c$) more than twice as large runaway rates were observed.

In chapter 3 the effect of bremsstrahlung emission was treated as a binary collision using the Boltzmann collision operator, which is necessary since the emitted photons often have energies comparable to the electron energy. The bremsstrahlung collision operator is explored in careful detail, as bremsstrahlung losses have previously only been accounted for in plasmas using approximate methods, where either photon energies or electron directions have been neglected. Accounting for both of these effects causes qualitatively new behavior of the electrons, such as enhanced pitch-angle scattering caused by the emission of low-energy photons.

In paper A the effect of bremsstrahlung is investigated with a numerical study of the kinetic equation, using CODE in which the bremsstrahlung collision operator has been implemented. It is revealed that broad runaway-electron energy distributions form, unlike in previous studies where all runaways accumulate at a well-defined energy where energy loss due to bremsstrahlung equals the gain from the electric field. In the new, more accurate model, a significant fraction of electrons are consistently found to be more than twice as energetic as predicted in the previous approximate models.

A sister code to CODE has been developed which solves the kinetic equation for runaway ions, using similar numerical methods. The new tool, CODION (COllisional Distribution of IONs) [90], includes the physics described in chapter 4 and is utilized in paper D to study runaway acceleration of ions, and under what conditions it occurs. The ion-runaway growth rate is investigated for various plasma compositions and electric fields for a solar-flare like scenario. It is also demonstrated that significant ion acceleration by the runaway mechanism alone is unlikely to occur during tokamak disruptions due to the large electric fields and long acceleration times required.

Papers A-D utilize a linearized equation, which limits the investigations to those scenarios where the electrons are sufficiently close to equilibrium. In order to study scenarios with large electric fields, or with rapidly time-varying plasma parameters, non-linear methods are required. In paper E, a novel method of numerically solving the non-linear Landau-Fokker-Planck equation is presented. By expressing the distribution function approximately as a finite sum of (non-orthogonal) Gaussian basis functions $f_n(\mathbf{p}) = c_n \exp\left[-(\mathbf{v}-\mathbf{v}_n)^2/2mT_n\right]$, the non-linear Fokker-Planck equation can be represented as a quadratic algebraic equation where the coefficients take the form of simple, analytically known functions. The conservation properties of density, momentum and energy of the numerical scheme is investigated, as well as the relaxation of an initial multi-peaked distribution into the Maxwellian equilibrium distribution.

Outlook

The Boltzmann transport equation has rarely been considered in the context of runaway modelling with continuum kinetic models (unlike Monte Carlo particle-following codes, where the corresponding processes are more naturally accounted for). We have presented a procedure to simplify the equation in the case of a cylindrically symmetric plasma near equilibrium, enabling accurate modelling of collisional processes with large momentum transfers in a computationally efficient way.

In the future this framework can be applied to investigate other runaway phenomena, which are not yet fully understood. For example, the models which have previously been used to consider knock-on collisions suffer from certain defects:

- The knock-on model by Rosenbluth and Putvinski [8] shows unphysical behavior, such as the creation of an infinite amount of momentum and energy per second in the plasma.

- The improved model by Chiu *et al.* [67], which was also used in paper B, resolves this issue but still breaks momentum and energy conservation by only following the target particle – the effect of the knock-on collisions on the incoming fast electron is ignored.

- The pitch-angle distribution of the runaway distribution is ignored in both models, and it is instead assumed that all electrons move purely in the parallel direction.

The Boltzmann operator described here is the necessary tool to develop a self-consistent and complete model for avalanche generation which suffers from none of these problems.

The impact on runaway-electron dynamics of partially ionized atoms in the plasma is another phenomenon which can be investigated using this framework. The screening effect of the bound electrons is analogous to the description in Section 3.1, which indicates that for momentum transfers greater than $q \sim \hbar/a_0$ the bound electrons may be ignored, and a runaway electron will feel the full charge of the nucleus. In a collision this implies that in such plasmas, large-angle electron-ion collisions will be enhanced relative to the small-angle collisions, hence a Boltzmann approach such as Eq. (2.9) is required in future studies in order to describe this effect accurately.

We thus see that there are several phenomena of importance to runaway where the methods developed in this work can be used to further improve current understanding.

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