THESIS FOR THE DEGREE OF DOCTOR OF PHILOSOPHY

Interacting particle systems for opinion dynamics: the Deffuant model and some generalizations

TIMO HIRSCHER

Division of Mathematics
Department of Mathematical Sciences
CHALMERS UNIVERSITY OF TECHNOLOGY
AND UNIVERSITY OF GOTHENBURG
Göteborg, Sweden 2016

Interacting particle systems for opinion dynamics:
the Deffuant model and some generalizations
Timo Hirscher
ISBN 978-91-7597-328-9

Doktorsavhandlingar vid Chalmers tekniska högskola
Ny serie nr 4009
ISSN 0346-718X

Department of Mathematical Sciences
Chalmers University of Technology
and University of Gothenburg
412 96 Göteborg
Sweden
Phone: +46 (0)31-772 1000

Printed in Göteborg, Sweden 2016

Interacting particle systems for opinion dynamics: the Deffuant model and some generalizations

Timo Hirscher

Department of Mathematical Sciences
Chalmers University of Technology
and University of Gothenburg

Abstract

In the field of *sociophysics*, various concepts and techniques taken from statistical physics are used to model and investigate some social and political behavior of a large group of humans: their social network is given by a simple graph and neighboring individuals meet and interact in pairs or small groups. Although most of the established models feature rather simple microscopic interaction rules, the macroscopic long-time behavior of the collective often eludes an analytical treatment due to the complexity, which stems from the interaction of the large system as a whole.

An important class of models in the area of opinion dynamics is the one based on the principle of *bounded confidence*: Individuals hold and share opinions with others in random encounters. Their mutual influence will lead to updated opinions approaching a compromise, but only if the distance of opinions was not too large in the first place. A much-studied representative of this class is the model, which was introduced by Deffuant et al. in 2000: Neighboring individuals meet pairwise and symmetrically move towards the average of the two involved opinions if their difference does not exceed a given threshold.

In the first paper of this thesis, we study the Deffuant model with real-valued opinions on integer lattices, using geometric and probabilistic tools as well as concepts from statistical physics. These prove to be very effective in the analysis of the model on the integer lattice in dimension 1, i.e. the two-sidedly infinite path $\mathbb{Z}$, and is adapted to give at least partial results for the lattice in higher dimensions as well as infinite percolation clusters. In papers 2 and 3, we stay on $\mathbb{Z}$ but consider a generalization of the model to higher-dimensional opinion spaces, namely vectors and absolutely continuous probability measures, as well as to more general metrics than the Euclidean, used to measure the distance between two opinions.

The last appended paper deals with "water transport on graphs", a new combinatorial optimization problem related to the possible range of opinions for a fixed individual given an initial opinion configuration. We show that on finite graphs, the problem is NP-hard in general and prove a dichotomy that is partly responsible for the fact that our methods used in the analysis of the Deffuant model are less effective on the integer lattice $\mathbb{Z}^d,\ d \geq 2$: If the initial values are i.i.d. and bounded, the supremum of values at a fixed vertex – achievable with help of pairwise interactions as in the Deffuant model – depends non-trivially on the initial configuration both for finite graphs and $\mathbb{Z}$, while it a.s. equals the essential supremum of the marginal distribution on higher-dimensional lattices.

Keywords: Deffuant model, bounded confidence, opinion dynamics, sociophysics, consensus formation, general opinion space, percolation, pumpless water transport.

List of Papers

A Olle Häggström and **Timo Hirscher**,
Further results on consensus formation in the Deffuant model,
Electronic Journal of Probability, Vol. 19, 2014.

B **Timo Hirscher**,
The Deffuant model on $\mathbb{Z}$ with higher-dimensional opinion spaces,
Latin American Journal of Probability and Mathematical Statistics, Vol. 11, 2014.

C **Timo Hirscher**,
Overly determined agents prevent consensus in a generalized Deffuant model on $\mathbb{Z}$ with dispersed opinions,
submitted to *Advances in Applied Probability*.

D Olle Häggström and **Timo Hirscher**,
Water transport on graphs,
submitted to *Networks*.

Paper not included in this thesis

E **Timo Hirscher** and Anders Martinsson,
Segregating Markov chains,
submitted to *Journal of Theoretical Probability*.

Acknowledgements

Provided you answer the question “What do you do for a living?” by saying “I’m doing a PhD in math.”, the most usual follow-up question is “How is it like to do a PhD in mathematics?”. Well, I always considered this question quite tricky since mathematics in general and PhD studies in particular are so varied that it is difficult to give a concise and yet satisfactory answer.

Last year in June, I hiked from the village of Kilpisjärvi to ‘Treriksröset’, the point where the borders of Finland, Norway and Sweden meet, and realized striking parallels between the 11km hike and the past nearly five years of study, making it an extremely suitable metaphor.

After having realized that there are no shortcuts (lake Kilpisjärvi was still partly frozen and the boat cutting the hike to 3km not operating yet), you set off for a journey with a clear goal in mind but little to no idea how the path leading there will look like. You know that others have done it and in the beginning you happily follow the trail. Every now and then you pass one of those little orange-tipped poles providing the affirmative feeling that you are still on track. After a while, the first snow fields appear and make you wonder if you started the journey being well-equipped or rather quite naive. Then within minutes, the sun disappears and snowfall sets in; your feet are all wet and you begin to fight the thought of turning around and heading back.

But then you remember your goal, realize that you cleared a considerable part of the way and keep going. Although the sun and the stunning view reward your decision, the rocky and snowy sections of the trail make it difficult not to loose orientation – until a tiny signpost in the distance makes you aware of the fact that you have gone astray and leads you back on track. Surprisingly enough, a Finnish mobile provider gives decent coverage all the way to the tripoint and an acceptable feeling of security that one could try and call for help in the worst case.

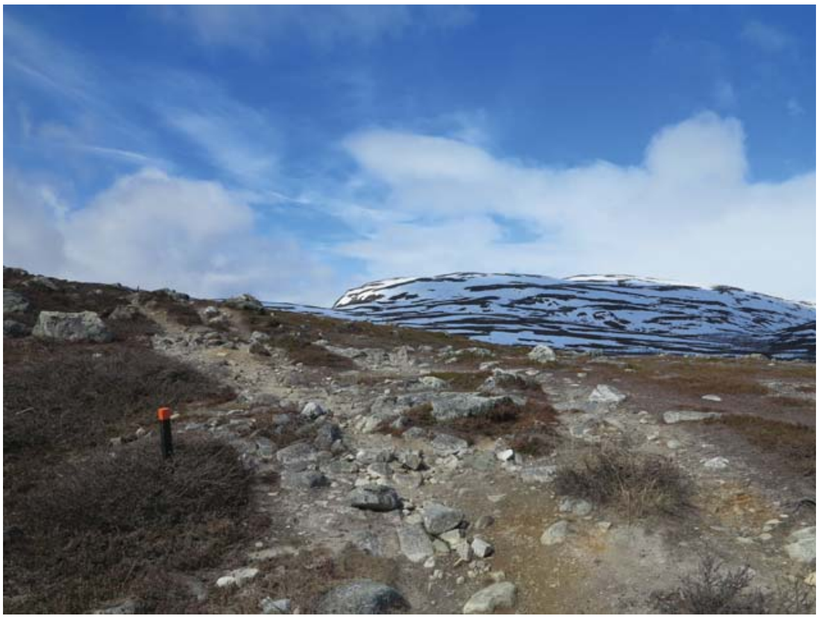
Besides all these similarities, there are a couple of notable dissimilarities: For instance, once the goal is reached, doing a PhD you have to take the next step in your life, not to hike 11km back to where you came from. Furthermore, during five years of study at Chalmers you meet more than two human beings. I now want to take the opportunity and give a few words of thanks to these people.

First and foremost, I would like to thank my advisor Olle Häggström for all his support, motivation and constructive criticism. Figuratively speaking, you have been both, the little poles marking the path and my mobile provider on this journey. Next I want to express my gratitude to Jeff Steif, for co-organizing the graduate course on 'Markov chains and mixing times' together with me, as this gave me the great opportunity to gain experience in and to get his advice on lecturing on graduate level. Furthermore, I would like to thank all my present and former colleagues at the mathematical department for creating this excellent working atmosphere and sharing both a laugh and some good advice in times when I desperately needed the one or the other. Without degrading others, I want to explicitely name Marie, my personal problem solver, as well as Dawan, Matteo and Peter whom I became close friends with.

Speaking of friends, there are a number of people outside the department contributing to this thesis by making my life in the past few years either easier or more exciting or both. I want to extend my deepest gratitude to Ewa, Gustav and Karin, Jan, Jörgen, Mareile, Mathias, Miri, Patricia, Sascha, Tino, Wiebke, all those who made Guldhedens Studiehem my home and many others whose names could make this list the major part of the thesis.

Finally, I want to thank my wonderful family, especially my parents, my sister and her family. You never questioned my decisions but provided me with constant love and support. There are no words adequately expressing my thankfulness to you.

Timo Hirscher
Göteborg, March 2016



"Whether you think you can, or you think you can't – you're right."
Henry Ford (1863 - 1947)

Contents

1
Introduction

Two friends, Jakob and Johan, meet by coincidence at Brunnsparken in central Gothenburg. They haven't seen each other in a long time, so they sit down in a café and have a chat. Since both of them are interested in new technologies, they soon start talking about the changes that the city planners intend to implement before Gothenburg's 400th anniversary in 2021. At some point, the question comes up how many of the busses will be running on renewable energies only by then. While Jakob is convinced that about 30 percent of the busses will be independent of fossil fuels, Johan is more pessimistic. His guess is that the fraction of local busses running on clean energy might be one tenth in 5 years from now. He points out that such changes are expensive and take time, especially in the public sector. Jakob argues that the pilot project ElectriCity in fact shows the city's effort towards such a change and that economic considerations could actually become a driving force away from fossil fuels in near future. During the exchange, they consider each other's arguments as well-founded and valid.

If confronted with the same question after their conversation, Jakob might have adapted his guess down to one fourth, Johan his instead up to 15 percent. Had Johan instead met an excessively optimistic Jakob claiming that all of Gothenburg's busses will be electric by 2021, both of them would have rated the view of the other as unrealistic, his arguments as not worth considering and hence left the café without updating their guesses.

This everyday phenomenon called *selective exposure* – people in general try to avoid new pieces of information likely to challenge their decisions and beliefs all too much – gained substantial attention in the field of psychology when Festinger [21] provided a solid theoretical framework in his book entitled "A Theory of Cognitive Dissonance", which was published in 1958. Following his pioneering work, a considerable number of experiments were conducted in order to describe, understand and explain this defensive behavior, that occasionally gets in the way when people actually try to form a knowledgeable opinion and in many cases accounts for the persistence of faulty beliefs. An extensive synopsis of these studies together with a thorough discussion of the area of conflict between curious open-mindedness and protective stubbornness in the process of information selection can be found in [34].

In an extremely simplified version, these competing principles are implemented in models for opinion formation based on bounded confidence (which will be reviewed in Section 4): On the one hand, people in general tend to assimilate, i.e. to adapt their points of view towards the opinion of others if confronted with their valid arguments. This process, on the other hand, only takes place if there is a certain trust in the position of one's discussion partner; if it is too far off our own standpoint, we are not willing to debate and re-evaluate our beliefs.

The idea to study opinion formation processes in a group of people using models with extremely simplified interaction rules is anything but new. The first attempts, however, were mere reinterpretations of mathematical models, used in statistical physics to describe interactions of elementary particles, and did not feature aspects of reflective behavior such as bounded confidence. Already in the 1930s, the theoretical physicist Ettore Majorana, a student of the famous Enrico Fermi, wrote an article titled "The value of statistical laws in physics and social sciences" [47]. It was originally supposed to be published in a sociology journal, hence to present the beneficial use of methods and ideas from statistics in physics to scholars of a different discipline and in this way to establish a connection between the two fields. This essay, however, was carelessly discarded and kept in a drawer until Majorana mysteriously disappeared on a boat trip from Palermo to Napels in 1938.

The manuscript was found by his brother and finally published in 1942, thanks to the efforts of Giovanni Gentile Jr., a former co-author and friend of Majorana. Despite its novel ideas, the fact that the paper was written in Italian and published posthumously limited its impact considerably. In fact, there was no translation into English until Mantegna [48] presented the article in the journal "Quantitative Finance" as recently as in 2005. Due to the fact that this last publication of Majorana received very little attention and therefore did not cause any notable further research efforts, it was not until the 1970s that theoretical physicists once more got interested in phenomena from social science and finally put Majorana's suggestion into practice: to see opinion dynamics in large groups as interacting particle systems and then exploit the fact that these are amenable to a rigorous mathematical modelling and an analysis based on statistical laws.

As a first step, statistical models – originally designed to describe the dynamic development of an ensemble of interacting particle spins on atomic level – were used to model the opinion formation in a social group of individuals mutually influencing each other. One of the major aims was to reinterpret known phenomena from physics, such as phase transitions or ordered and disordered states, in the new sociological context and by that to relate purely mathematical aspects of the model's dynamics to common social phenomena in group behavior.

During the last two decades more and more physicists and mathematicians started similar attempts to understand the opinion dynamics in a large group of individuals by using simplistic interaction models and to analyze them by applying qualitative and quantitative methods from statistical physics. The fact that new social phenomena which arose with the advancement of the internet, like e-mail correspondences for example, feature large groups of individuals, simple interactions and allow for a computational treatment of the corresponding large datasets contributed substantially to this evolution.

2

Can elementary magnets go on strike? – A historical account

2.1 Statistical mechanics and the Ising model

Taking into consideration that the research area of opinion dynamics is rooted in the discipline of physics, the story really began in the second half of the 19th century, when James Clerk Maxwell, Ludwig Boltzmann and Josiah Willard Gibbs elaborated the ideas of Daniel Bernoulli to describe the kinetic dynamics in gases statistically and in this way launched the branch of statistical mechanics. Their pioneering idea was not to focus on each single particle and its individual movements, but to characterize the whole system with a set of parameters and their distributions among the possible states of the system, the so-called *statistical ensemble*.

The starting point of opinion dynamics based on statistical physics, a field that later became labelled as *sociophysics*, was however not the branch of thermodynamics but the closely related field dealing with ferromagnetism. Just like water changing its state of matter depending on the temperature, ferromagnetic material undergoes a phase transition in the sense that macroscopic properties of the matter are changed. Well above a certain critical temperature, the ferromagnetic material is unmagnetic on a macroscopic scale (if not exposed to a strong external magnetic field); well below this temperature however, a phenomenon that is called *spontaneous magnetization* occurs: the microscopic magnetic dipole moments, originating from atomic spins, start to align and turn the material into a magnet – even in the absence of an external field.

Already in 1907, Pierre Weiss [66] tried to explain this behavior, building on earlier work by Pierre Curie. He used an approach that became known as *mean field theory*: In a large statistical system, the effects of all other particles on one fixed particle is replaced by their statistical average. This approximation turns a many-body problem with interactions, which in general is very difficult to solve exactly, into a one-body problem with external field. Clearly, this is a rather crude simplification as the fluctuating interaction of the considered particle with the rest of the system is approximated by a time-independent effective field. Nevertheless, it made the spin problem tractable and allowed Weiss to draw conclusions explaining the two different phases of ferromagnetic material. The mean field theory approximation is however only qualitatively accurate and fails to give satisfactory answers to questions about the behavior near the phase transition. For temperatures near the critical one, the actual local magnetic fields are rapidly varying in time and consequently turn their statistical average into a quite poor representation of their effect.

A slightly different approach to explain ferromagnetic behavior was the following theoretical model that physicist Wilhelm Lenz invented in 1920 and proposed to his student Ernst Ising for further studies two years later: A collection of n atoms is arranged to form a regular atomic lattice. Their elementary magnetic dipoles, often simply called *spins*, can be either in the state "up" or "down", represented by the numerical values $+1$ and -1 respectively. All spins taken together form what is called a spin configuration $\sigma \in \{-1, +1\}^n$. If we assume that neighboring spins interact with a certain coupling strength J and that the material is exposed to an external magnetic field h, the configuration σ

is attributed a total energy given by the *Hamiltonian function*

$$H(\sigma) = -J \sum_{\langle i,j \rangle} \sigma_i \sigma_j - \mu\, h \sum_i \sigma_i, \tag{2.1}$$

where the first sum is taken over all pairs $\langle i, j \rangle$ of nearest neighbors in the atomic lattice and μ denotes the magnetic moment. While the minus sign of the second term is mere convention (as the magnetic moment actually is antiparallel to the spin), $J > 0$ corresponds to a ferromagnetic interaction. Thus, in the ferromagnetic case, the energy of the configuration decreases with both the number of nearest neighbor pairs having spins pointing into the same direction and spins aligned in accordance with the external field.

Following a basic physical principle, the system will act in a way to minimize the free energy, which makes states of low energy more probable in thermal equilibrium. This is captured by the so-called Gibbs measure attributing probability

$$\mathbb{P}(\sigma) = \frac{1}{Z_\beta}\, \mathrm{e}^{-\beta H(\sigma)} \tag{2.2}$$

to a fixed spin configuration σ, with the *partition function* $Z_\beta = \sum_\sigma \mathrm{e}^{-\beta H(\sigma)}$ being the appropriate normalizing constant. The model parameter β, called the *inverse temperature*, is given by $\beta = \frac{1}{k_\mathrm{B}\, T}$, where k_B denotes a (positive) physical constant, the so-called *Boltzmann constant*, and T is the temperature (in degree Kelvin). If we consider the case with no external field (i.e. $h = 0$), it is intuitively obvious from (2.2) that for high temperature all possible configurations nearly have the same probability, while for low temperature configurations with high energy (i.e. many opposing nearest neighbor pairs) are almost excluded.

For a finite system, this transition happens smoothly and a phase transition in the sharp (mathematical) sense can only be observed in the case of infinitely many particles, commonly known as *thermodynamic limit*. On the infinite d-dimensional grid $\mathbb{Z}^d$, we can consider the spatial average of spins which is called *magnetization* of the material and defined by

$$\langle\, \sigma \,\rangle = \lim_{n \to \infty} \frac{1}{|\Lambda_n|} \sum_{i \in \Lambda_n} \sigma_i, \tag{2.3}$$

where $\Lambda_n = \{-n, \ldots, n\}^d$. With this notion in hand, we can distinguish between a paramagnetic, disordered phase in which the magnetization is almost surely 0 and a ferromagnetic, ordered phase in which non-zero magnetization has positive probability.

In his PhD thesis, Ising [37] analyzed the one-dimensional case and found that the correlation of spin values decays exponentially with the distance of two sites, which implies that the magnetization equals 0. He erroneously concluded that the model does not feature any phase transition even in higher dimensions. This claim was proven wrong by Rudolf Peierls [54] about one decade later. He investigated the two-dimensional zero-field Ising model (i.e. on the square-lattice $\mathbb{Z}^2$ with $h = 0$) and proved that it has a non-zero magnetization at sufficiently low temperatures. As the model without external field must have zero magnetization at sufficiently high temperatures, he was the first to show that a model from statistical mechanics exhibits a phase transition. A few years later, Lars Onsager [53] computed the critical temperature for the zero-field Ising model on the square-lattice rigorously and found it to be

$$T_{\mathrm{c}} = \frac{2\,J}{k_{\mathrm{B}} \cdot \ln(1+\sqrt{2})}.$$

The Ising model on the square-lattice still is one of the simplest mathematical models that does feature the phenomenon of a phase transition.

To simulate a configuration of the Ising model on a finite graph with given external parameters (T and h), the standard approach is to use the Monte Carlo method based on the well-known algorithm by *Metropolis–Hastings*. In this rejection sampling algorithm, applied to the Ising model, one starts with a random configuration and then performs single spin updates according to the following rule: Pick a site uniformly at random and flip its spin with probability $\min\{\mathrm{e}^{-\beta\,\Delta H}, 1\}$, where ΔH is the invoked change of the total energy. In the ferromagnetic regime without external field, flipping the spin at a chosen site might be rejected only if the majority of its neighbors agrees with the current spin as this implies $\Delta H > 0$. Evidently, a low temperature will considerably favor flips decreasing the energy over flips increasing it and therefore drive the system towards more ordered states with growing patches of aligned spins.

A different way to incorporate the microscopic evolution in a ferromagnet at a fixed temperature with help of the Ising model is the so-called *Glauber dynamics*. In this algorithm, to flip the randomly chosen spin has probability $\frac{1}{1+\mathrm{e}^{\beta\,\Delta H}}$. In contrast to the Metropolis–Hastings algorithm, here even transitions to lower energy states might be rejected, but the tendency to order remains as updates towards lower energy have probability larger than $\frac{1}{2}$, towards higher energy less than $\frac{1}{2}$.

In a long chain of atoms, these alignments at low temperature do take place as well, but for any temperature above absolute zero, thermal fluctuations will consistently break the aligned parts of the chain and in this way prevent a global alignment of the system. This is the reason why the model on $\mathbb{Z}$ does not achieve a global magnetization even for low temperatures. A quite comprehensive expositon of the early years of statistical physics including a more detailed discussion of mean field theory and the Ising model from a slightly more physical point of view can be found in [38].

2.2 Sociophysics

Even though the proposal by Majorana to start treating social phenomena by using statistical models of reduced complexity and to focus on how microscopic interaction rules entail macroscopic properties of the system, that can be compared to global observables, went more or less unheard by the social sciences, the striking similarity between interacting elementary magnets and simplified processes of group behavior led physicists about 30 years later to finally establish this connection.

In a colloquium in 1969, physicist Wolfgang Weidlich suggested to compare the interactions within a group of individuals holding opposing attitudes towards a given yes-no question with ferromagnetism, more precisely the dynamics of the Metropolis–Hastings algorithm applied to the Ising model. Two years later, he published this idea in the article *'The statistical description of polarization phenomena in society'* [64] in which he elaborated how the mathematical model intended to describe and explain ferromagnetism with help of statistical mechanics can be put into a sociological context: In the sociological reinterpretation, the interaction strength J corresponds to the willingness of an individual to adopt the attitude of the majority among its neighbors and the temperature as a model parameter for the social pressure exerted on each individual (low temperature corresponding to high social pressure). An external magnetic field (i.e. $h \neq 0$) is understood to shape some preference of one attitude over the other, shared by all individuals. Weidlich derived the stationary distributions for different values of h and J and even included a section in which a possible comparison between model and real data is discussed. Furthermore, he already suggested natural extensions of this initial link between social dynamics and statistical physics: More than two possible attitudes could be considered, h and J could be replaced by sets of parameters $\{h_i\}$ and $\{J_{ij}\}$ (i.e. chosen to be depending on the individuals and nearest-neighbor pairs respectively) and letting the transition probability to flip the spin at a given site depend not only on the current configuration of its neighbors, but also on its own history could introduce a sense of tradition or stubbornness.

In 1982, Galam et al. [27] used the Ising model on K_n, the complete graph on n vertices, to describe the collective behavior in a plant where dissatisfied workers might start a strike. Using a mean field theory approach, they rediscovered the phase transition described in the foregoing section and interpreted the regime of high temperature as an individual phase (mutual influences are very limited) and low temperature as a collective one (the group behaves coherently), separated by a critical phase in which small changes in the system can lead to drastic changes in the group. In contrast to the physical application of the Ising model, where a collection of atoms is forming a regular lattice, it is reasonable to consider the underlying interaction network among workers in a small plant to be all-to-all, meaning that every worker can actually influence all his fellow workers.

Following these seminal papers, an increasing number of related models were introduced, motivated and analyzed – in the past two decades predominantly with the help of computer simulation. The principle interaction rules diverged slowly but surely from particle physics and today the area of sociophysics comprises an abundance of models for opinion dynamics in groups. The most noted among these will be reviewed in the following chapters.

Just as in any cross-disciplinary application, the question has to be adressed whether these interacting particle systems are suitable to model human group behavior or not. Interestingly enough, already Weidlich [64] and Galam et al. [27] tried to survey the advantages as well as limitations of and possible objections against applying a simplified model from statistical mechanics in a sociological context. Apparently, there are glaring differences between the two fields of application. Possibly most important is the contrasting complexity of the elementary components: In physics, the systems consist of relatively simple objects, usually atoms and molecules, the behavior of which is relatively well understood; hence the complex evolution of the collective arises from the interaction patterns. In social science, however, the collective consists of a large number of human beings and the behavior of each single individual is already the outcome of a complex interplay between physiology and psychology of which only very little is understood. Especially the fact that in all common models for opinion dynamics the individuals are presupposed to behave adaptively (i.e. reacting to external influences) and not strategically (i.e. following a certain plan they have in mind) seems to be an unrealistic assumption. Apart from that, one has to admit that humans differ a great deal from one another in many aspects while it is rather safe to consider atoms of the same kind as perfectly identical. It is doubtful whether the few parameters needed to capture the state of a physical system are sufficient to describe the properties of a collection of human beings.

In a nutshell, the reduction of humans to identical and simplistic elements in a large system is a quite controversial issue and critics might come to the conclusion that reducing the complexity on microscopic level to such an extent that the system makes a treatment using tools from statistical physics possible without changing the essential macroscopic phenomenology is a hopeless task. One could even take this one step further and claim that researchers were tempted by the substantial progress in the study of collective phenomena in the field of physics to apply these models in other contexts, such as social behavior in groups, and established this connection at any sacrifice.

Nevertheless, one cannot deny the fact that there are certain phenomena in the dynamics of group behavior (both animal and human), that show striking structural similarities to ferromagnetism and suggest a meaningful relation between the two. Just like the spins in an ensemble of atoms, the individuals might be in a chaotic state at first – meaning that no large scale structure exists – but then gradually align and finally undergo a transition from disorder to order in the sense that the system exhibits large scale regularities, which in the physical context correspond to a state of low energy. In their article "A theory of social imitation", Callen and Shapero [7] name the collective movement in a school of fish or a flock of birds, the synchronous flashing of fireflies as well as temporary fashion styles as prominent examples: Without any leader, the group becomes increasingly homogeneous through local interaction and alignment until a consistent collective is formed – similarly to spontaneous magnetization of ferromagnetic matter not exposed to an external field.

For prey, being a part of a homogeneous group provides a certain degree of safety against predator attacks. In the context of social interactions and opinion formation in groups, the drive towards order is due to the tendency of interacting individuals to become more alike, an effect called *social influence*. This effect is often intensified by the known psychological phenomena of *selective attention* and *pleasure of recognition*: Our brain is geared towards filtering out relevant information, giving an advantage to things we can relate to. The idea of a selective internal filter was originally proposed by Broadbent [5] in 1958 and later refined and elaborated with help of various experiments investigating human habits and capabilities of handling information input (see also [20]). The pleasure of recognition (which incidentally is an important aspect in the composition of musical and literary work, see [57]) as well as the phenomenon of selective exposure, mentioned in the introduction, are closely intertwined with the inclination of people in general to meet and interact with others that resemble themselves in various aspects and share central attitudes, a behavior referred to as *homophily*. This term was introduced by Lazarsfeld and Merton [42], who considered two forms of homophily: *value homophily*, based on shared values and beliefs, as well as *status homophily*, based on a similar cultural background. The form that is most relevant in the context of opinion dynamics, *induced homophily*, which is based on similarity emerging from regular contact and mutual influence, was added and studied later (see for instance [50]). In this form it is most obvious how homophily can lead to a self-enhancing process and play a central role in the homogenization of a social group.

If we stick to the metaphor, ordered low energy states in statistical mechanics correspond to consensus or uniformity in the context of opinion dynamics and disordered states of higher energy in turn to fragmentation or disagreement. One of the main questions in social dynamics is – similarly to the situation in statistical physics – under which circumstances the microscopic interactions will lead to such a transition, since if there were no interactions, in both contexts heterogeneity would prevail.

Apart from this rather heuristic relation, there are other important arguments that alleviate the problem of reducing humans to elementary particles: In statistical physics most of the qualitative properties of a larger-scale system do not depend on the microscopic details of the dynamics but instead on global properties like symmetries, dimensionality or conservation laws. Diverse models exhibit essentially similar phenomena (e.g. phase transitions) despite their different rules and patterns, making these features in some sense model-invariant, a concept called *universality*. In this respect it is at least justifiable that modelling a few of the most important properties of single individuals will capture the essential driving forces of the evolution and thereby give meaningful results when it comes to qualitative features of the model's large scale behavior. In addition to that, just as many other complex systems, the opinion formation in a large group of humans is of statistical nature, i.e. a large number of comparable microscopic elements compose a macroscopic object, which has properties that are formed by the collective but the contribution of any individual particle is negligible. A statistical approach therefore seems to be quite reasonable. In fact, this argument was brought up already by Majorana [47] in the 1930s.

The lack of analytical means that could be applied to the common models for social dynamics as well as the increasing computational power resulted in numerous simulation-based analyses beginning in the 1990s. On the one hand they surely complement the analytical study of such models based on tools from statistical physics, on the other hand simulation-approaches are limited to a rather small number of individuals. Even if it seems to be sufficient for an examination of the opinion formation in social groups, as mentioned before, the concept of order-disorder phase transitions is rigorously defined only in the limit of a system with infinitely many particles. A number of individuals that is not sufficiently large might therefore cause finite size effects that invalidate conclusions drawn from a comparison with analog systems in physics, in which the number of interacting particles is commonly by far larger than in a social group. In this respect it is of vital importance to be able to figure out which macroscopic features are robust with respect to changes in the number of interacting individuals by analyzing the used model for different orders of magnitude of the system's size.

3
Opinion dynamics

Since there are many situations in everyday life where it is necessary for a group of people to form a point of view with majority appeal in order to make a shared decision (especially in a democratic framework, as discussed in [4]), it has always been a major focus of social science to understand the opinion formation process in a larger group of socially interacting individuals (for a broader introduction of the concept of 'public opinion' and an overview of some early efforts of social scientists in this area of research, see [14]). Inspired by statistical mechanics, in particular Weidlich's sociological reinterpretation of the Ising model for ferromagnetism, various models for opinion dynamics arose in the sequel.

In this chapter, we will shortly introduce a number of models used in the field of opinion dynamics that are either based on or very similar to interacting particle systems from statistical physics. First, we will list commonly used network structures and opinion spaces, then describe the characteristic interaction rules of the most common models. Before we engage in this review, it should be mentioned that not all of the models which appeared in the early years of opinion dynamics were inspired by statistical mechanics.

In 1974, for instance, DeGroot [17] presented a different approach to describe the dynamics of an opinion formation process, reminiscent of a finite Markov chain. In his model, n individuals update their opinions in rounds and compose their new ones as a weighted average of all current opinions:

$$\eta_{t+1}(i) = \sum_{j=1}^{n} p_{ij}\, \eta_t(j), \tag{3.1}$$

where $\eta_t(i)$ is the opinion of individual i after round t and p_{ij} is the weight it attributes to the opinion of individual j. In the definition of the model, DeGroot does however not specify (deliberately) which convex set the initial opinions belong to; could be real numbers, vectors or probability distributions. He considers the weights, which form a row-stochastic matrix $P = (p_{ij})_{i,j}$, to be time-independent. This allows to transfer standard results about the asymptotics of time-homogeneous finite Markov chains to the model: A consensus is reached (starting from a general set of initial opinions), in the sense that all opinions converge to a common limit, if and only if the matrix P, taken as one-step transition matrix, corresponds to a Markov chain in which all recurrent states belong to the same aperiodic communication class. Then the unique stationary distribution gives the weights according to which the common limiting opinion is composed.

Note that the iterated matrix products that represent the array of opinions at later times are multiplications from the left (as apposed to multiplications from the right in the case of a Markov chain). A stochastic process of this kind is commonly known as *repeated averaging*. A few years later, Chatterjee and Senata [11] addressed the more general case in which the weights depend on time. They establish sufficient conditions on the sequence of weight matrices for the opinions to converge to a common limit.

3.1 Underlying social network structures

No matter if we consider the model of DeGroot based on repeated averaging or interacting particle systems based on models from statistical mechanics, the following is apparently true for opinion dynamics in general: When it comes to the question whether the individual opinions will converge to a common limit or not, it is a very important aspect, between which of the individuals there is a potential for mutual influence – in other words the topology of the interaction network. We think of the individuals as nodes that form a social network (given by a simple graph) in which a connection between two individuals that enables them to influence each other is represented by an edge.

Under the assumption that the interaction is all-to-all, often termed *complete mixing*, the mean field approximation becomes particularly useful. In most cases it makes an analytical treatment possible in the sense that solving the corresponding differential equations will give insights about the long-term behavior. However, already in today's globalized companies this assumption is hardly realistic – not to mention the extremely sparse networks of e-mail correspondences and the like. For this reason, all of the models we are about to review were mainly considered on much sparser networks than the complete graph.

Finite graphs

Clearly, all simulation-based analyses are confined to opinion dynamics on finite social networks. A particularly simple example is that of a finite square lattice: It features two dimensions (which as we know from the Ising model can make a crucial difference to dimension 1) and still has comparably few edges. The necessary compromise between the efforts to keep both computation time and boundary effects to a minimum, led to samples comprising a number of individuals roughly ranging from $n = 10^2$ to $n = 200^2$. In some simulations (e.g. in [2], [22] and [49]), the boundary conditions were taken to be periodic in order to remediate their negative impact on the homogeneity of the network. In [16], where both a complete graph and a finite square lattice were used to represent the underlying social network, the authors accentuated the fact that a grid features many short cycles (measured against the relatively small number of edges) just like real social networks do. In respect of its striking regularity it might however be questioned if this makes a square lattice an appropriate candidate to model social relations.

More sophisticated choices for the interaction network that have been studied, among others, are realizations of random graph models such as the three introduced by Erdős–Rényi, Barabási–Albert and Watts–Strogatz: The so-called *Erdős–Rényi graph*, often simply denoted by $G(n, p)$, is a random graph on n nodes, in which each of the $\binom{n}{2}$ possible edges is independently chosen to be present with probability p. If the size of this network is varied, it might be suitable to choose $p = \frac{c}{N-1}$ in order to keep the average degree constant (at the chosen value c).

The *Barabási–Albert model* is one of the most popular algorithms for generating random scale-free networks, i.e. graphs with a degree distribution that follows a power law (at least in the tail)

$$N(d) \sim d^{-\gamma},$$

where $N(d)$ is the fraction of nodes with degree d and γ a parameter typically valued in the range $[2, 3]$.

The model is based on a principle called *preferential attachment*: The network is built incrementally from a core of m fully connected individuals by adding new nodes one by one, each choosing m older nodes to connect to with a probability proportional to their degree. Scale-free networks proved to be realistic models for e-mail networks or friendship graphs, both popular objects of study in the branch of social network analysis.

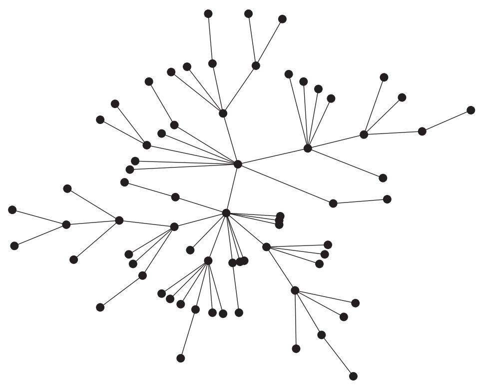


Figure 3.1: *A typical Barabási–Albert network, for $m = 1$, of comparatively small size ($n = 70$).*

Lastly, the algorithm proposed by Watts and Strogatz generates a simple random graph that has two main features found in real social networks: local, strongly connected clusters and short average path lengths. Graphs of this kind are called *small-world networks*. The algorithm features three parameters (the number of nodes n, the mean degree $2m$ as well as the rewire probability β) and proceeds as follows: Given the set of nodes $\mathbb{Z}_n = \{0, \dots, n-1\}$ placed on a circle, first, a directed ring lattice is constructed by including an arrow from each node i to its m immediate successors, i.e.

$$\vec{E} = \{(i,j);\ i,j \in \mathbb{Z}_n,\ 1 \leq j - i \pmod{n} \leq m\}.$$

Then, all of these directed edges are processed in lexicographical order and replaced by undirected ones: With probability $1 - \beta$, the arrow (i, j) simply gets transformed into the edge $\langle i, j \rangle$. With probability β, however, it gets rewired and instead the edge $\langle i, k \rangle$ is included, where k is picked uniformly at random from the elements of $\mathbb{Z}_n \setminus \{i, j\}$, that are currently not linked to i (neither by an arrow nor by an undirected edge).

In this way, for β positive but small, a few of the local connections get replaced by long-range relations and a small-world network is formed. For extreme choices of β, this is not the case: $\beta = 0$ corresponds to the regular ring lattice with degree $2m$ and for $\beta = 1$, the algorithm returns a graph with average degree $2m$ in which all edges were placed randomly, see Figure 3.2.

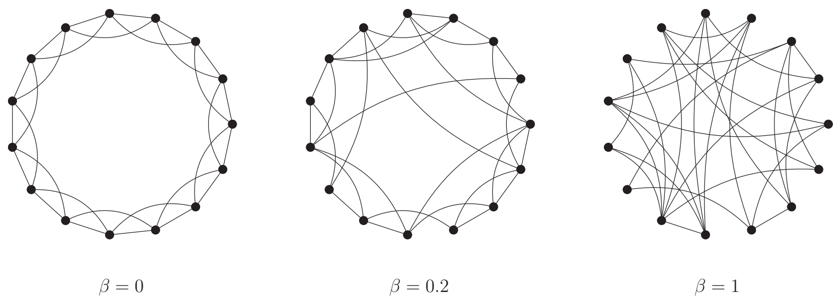


Figure 3.2: *Output of the Watts–Strogatz algorithm for $n = 15$, $m = 2$ and different values of β.*

The same idea can of course be applied to square lattices etc. as well.

It should be mentioned that there have been various efforts to implement opinion dynamics on adaptive random networks. Gil and Zanette [29], for example, proposed a model in which the social network is given by the complete graph initially, but whenever two agents meet and fail to agree on one opinion, the link in between them is deleted with a certain probability. This procedure leads to a gradual thinning of the network until only homogeneous opinion clusters remain.

Although certainly more realistic, the coevolution of opinions and relations adds substantially to the complexity of the problem. A different approach to implement homophily is the one of *bounded confidence*: While the network stays unchanged, neighboring agents only interact if their opinions are reasonably close. Models of this kind are reviewed in Section 4.1.

Infinite graphs

In a probabilistic analysis of opinion formation processes, as opposed to studies that are simulation-based, considering infinite networks becomes feasible and in fact, it often makes both the arguments and results more elegant: Tools like the law of large numbers or ergodicity might be applied and turn phenomena that occur with high probability on finite networks into almost sure events. Apart from that, infinite systems can serve as idealized approximations to finite but very large systems.

Major parts of this thesis deal with opinion dynamics on the, in a way, simplest infinite network: the two-sidedly infinite path. To be more precise, we consider the graph with vertex set $\mathbb{Z}$ and edge set $E = \{\langle v, v+1\rangle;\ v \in \mathbb{Z}\}$, see Figure 3.3 below for an illustration.

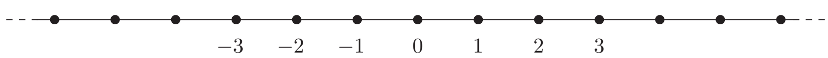


Figure 3.3: *A section of the two-sidedly infinite path* $\mathbb{Z}$.

Since it marks a natural next step, we also looked at its higher-dimensional equivalent: the d-dimensional lattice, i.e. the graph $G = (V, E)$ with $V = \mathbb{Z}^d$ and $E = \{\langle u, v\rangle;\ u, v \in V,\ \|u - v\|_2 = 1\}$, where $d \geq 2$ and $\|\,.\,\|_2$ denotes the Euclidean norm.

Additionally, we investigated opinion dynamics on the infinite cluster of supercritical *i.i.d. bond percolation* on the lattice $\mathbb{Z}^d,\ d \geq 2$, a standard representative for the class of infinite random graphs. The concept of i.i.d. bond percolation is in effect nothing else but the formal procedure to get the Erdős–Rényi graph from the complete graph K_n as described above – applied to more general graphs, in our case the integer lattice: For every edge, we decide independently if it is kept (with probability p) or removed (with probability $1 - p$). A maximal set of vertices linked by kept edges is called a *cluster*. For a more extensive introduction of the model, we refer to the book by Grimmett [31].

Broadbent and Hammersley [6] introduced this model in 1957 and proved that for all $d \geq 2$, there exists a critical probability p_c (depending on and monotonically decreasing with d) that marks a phase transition in the following sense: For $p < p_c$ there will almost surely be only finite clusters, while for $p < p_c$ a.s. a unique infinite cluster exists.

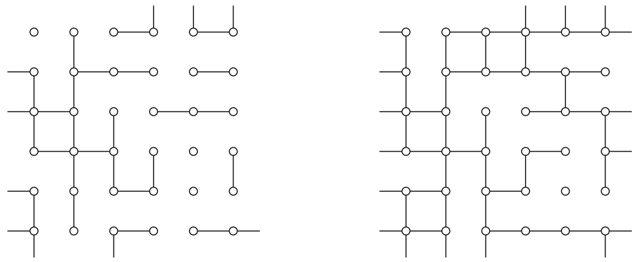


Figure 3.4: *A segment of i.i.d. bond percolation on the square lattice, with parameter $p = 0.4$ to the left and $p = 0.6$ to the right.*

In 1980, Kesten [39] proved that $p_c = \frac{1}{2}$ for the square lattice, completing an earlier result by Harris [33], who established $p_c \leq \frac{1}{2}$ by showing that there is a.s. no infinite cluster for bond percolation on the square grid with parameter $p = \frac{1}{2}$.

3.2 Opinion spaces

Just as DeGroot noted in the penultimate section of [17], when it comes to the mathematical modeling of opinions there are no rigid limits: They could be represented by numbers, vectors or even probability distributions. One only has to make sure that the opinion space is geared towards the interaction rule of the model, i.e. that it is closed with respect to all possible opinion updates.

Adopted from the Ising model, the first attempts to study opinion dynamics based on statistical mechanics featured $\{+1, -1\}$-valued opinions. As long as the evolution of attitudes towards a single yes-no question is to be modelled, this might seem sufficient, but already allowing an agent to be in the state ‘irresolute’ makes it necessary to include more than two opinion values i.e. to depart from binary variables. As counterpart to discrete-valued opinions, normally used to represent choices, over time there appeared models featuring opinion variables, continuously distributed on $[0, 1]$ or even the whole set of non-negative real numbers. Besides the fact that in many situations, e.g. estimating a certain unknown, a continuous opinion space is more natural, it simplifies to implement compromising behavior of interacting agents holding different opinions: The restriction to discrete opinions sometimes forces *imitating behavior* (one agent takes on the exact opinon of another).

Actually, there is a rather crucial downside to interaction rules of this kind: During the updates, the aggregate value of opinions changes, which violates the idea of *(mass) conservation* found in many physical systems. Surely, this is not a natural property in a social science setting, where mutual influences in general are asymmetric. However, as mentioned before, global properties of interacting particle systems (like conservation laws) play an important role, not least in the mathematical analysis. As a consequence, updates based on imitation – which are simple taken by themselves but render it impossible to adopt arguments using the principle of mass conservation – potentially make a model more involved from a technical point of view. This is one reason why considering continuous opinions can be quite different; the fact that a concept like 'majority opinion' does not have an equivalent in the continuous setting is another.

Accompanying the advances in the field of opinion dynamics, a growing interest in the natural extension to vector-valued opinions arose. In 1997, Axelrod [1] was one of the first to publish an article focussed on higher-dimensional opinions as opposed to earlier publications considering opinions to be scalar variables. He coined the notion of *cultural dynamics* interpreting the opinion vector as 'culture' of an individual, comprising "the set of individual attributes that are subject to social influence". In his original model, the mindset of an agent comprises 5 features which can take on any one of 10 traits. In short, the opinion space is given by $\{0, 1, \ldots, 9\}^5$. Due to the reasons named above, it didn't take long until variants with continuous higher-dimensional opinion spaces emerged.

The border between cultural and scalar opinion dynamics is not sharp and many similarities exist. However, there are models featuring multidimensional opinions that do not have counterparts with scalar opinions and are therefore qualitatively different. In addition to that, as soon as the distance between two opinions matters (as is the case for bounded confidence models, see Section 4.1), the geometry comes into play. Regardless of the fact that there are many more standard metrics to choose from in higher dimensions, there is one very important difference even in Euclidean geometry: Consider a set A, its convex hull $\overline{A}$ and a point $x \notin \overline{A}$. In dimension $d \geq 2$, the distance of x to $\overline{A}$ is in general strictly less than the distance to the set A itself, see Figure 3.5 for an illustration. This is not true for $d = 1$ and makes compromising in some sense more powerful in higher dimensions when it comes to bridging gaps in between different opinions.

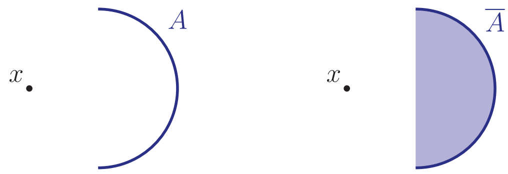


Figure 3.5: *Forming convex combinations can crucially reduce gaps – yet only in dimension $d \geq 2$.*

There have in fact been very few attempts to represent opinions by probability distributions, although this can be seen as a very natural way of modelling indetermination. In 2008, Martins [49] proposed a model in which the individuals are given two choices and internally hold a distribution embodying their preference. When they interact, they only tell each other which of the two options they would prefer and then update their probabilities according to the information received. From a mathematical point of view, a distribution on a finite probability space is nothing but a vector from the simplex of corresponding dimension, hence the opinion space still finite-dimensional. In this thesis, even infinite-dimensional opinion spaces, more precisely a model in which opinions are given by absolutely continuous distributions on $[0, 1]$, will be considered.

3.3 Interaction rules

In what follows, we are going to list some of the standard models used in sociophysics. All of them share similar ideas and they were studied with the common aim to define opinion states of the whole population (e.g. consensus or disagreement) and to determine if and how the range of the model's parameters splits up into different regimes, according to the long-time behavior of the model: In most of the cases, the dynamics tends to reduce the variability compared to the initial opinion values, a trend that can lead to a state of consensus in the long run, depending on the model specifications.

This subsection is dedicated to models that are still very close to those used in statistical mechanics, while in the next chapter we will review models that include rational behavior which might not have a counterpart in elementary physics. For further references and a more detailed discussion of the listed models, we refer the reader to the comprehensive survey article *'Statistical physics of social dynamics'* [9] by Castellano, Fortunato and Loreto.

(a) Voter model

Shortly after Weidlich's sociological reinterpretation of the Ising model, in 1973, this interacting particle system was introduced by Clifford and Sudbury [12] as a model for two spatially competing species and later named for its natural interpretation in the context of opinion dynamics among voters. Its definition is very simple: Each individual holds an opinion given by a $\{-1, +1\}$-valued variable. At every time step, one individual is selected at random and will then adopt the opinion of another agent, picked uniformly among its neighbors.

On regular lattices the evolution of this model is to some extent similar to the Ising model – in one dimension, that is on the two-sidedly infinite path $\mathbb{Z}$, it actually corresponds exactly to the limiting case of the Ising model with zero temperature. Based on well known results about random walks on grids, Clifford and Sudbury were able to conclude that on the integer lattice in dimension $d \in \{1, 2\}$ any fixed finite subset of agents will a.s. finally agree (on one of the two opinions), while this does not hold for $d \geq 3$. This behavior comes from the fact that a simple random walk on the lattice is recurrent (i.e. will a.s. return to its starting point) in dimension 1 and 2, but transient (i.e. the event that there is no return to the starting point has non-zero probability) in dimension 3 and higher. A more exhaustive analysis including ergodic theorems and a complete description of all invariant measures was done by Holley and Liggett [36] in 1975. Later, the voter model was studied on various other social networks and qualitatively different behavior was found also on small-world networks for instance (see [10]).

Variants of the model include the *multitype voter model* (introduced by Spitzer [59]), in which more than two opinion values are possible, as well as the *constrained voter model* (introduced by Vazquez et al. [63]) which is defined as follows: Each agent is in one of three states ('left', 'right' or 'center') and interactions as described above can only occur involving at least one centrist (as the extremists, 'left' and 'right', are assumed to ignore each other). This behavior is a discrete analog of the so-called *bounded confidence* principle (see Chapter 4).

(b) Majority rule model

A finite collection of n individuals is considered, a fraction p_+ of which initially holds opinion $+1$, all others the opinion -1. The interaction rule is reminiscent of the one in the voter model, however agents do not necessarily meet in pairs: At each iteration a random group of individuals is chosen, and all group members then adopt the majority opinion inside the group. In the simplest version, the size of the chosen groups is a fixed odd number. But there are various variants with random size and different ways to resolve a tie in a group consisting of an even number of individuals. The model was introduced by Galam [26] and proposed to describe public debates.

Another model based on the majority rule is the so-called *majority-vote model*. Just like in the Ising model, spins are updated one at a time. At each step, the spin to be updated takes on the value of the majority of its neighbors with probability $1 - q$, the minority value with probability q and is chosen uniformly from $\{-1, +1\}$ if there is a tie. For $q = 0$ this corresponds to the Metropolis–Hastings kinetics for the zero-field Ising model at zero temperature (except for the fact that given a tie, the Metropolis–Hastings algorithm will perform a flip with probability 1), for $q = \frac{1}{2}$ to the Glauber dynamics at infinite temperature. The majority-vote model was introduced by Liggett [44], however slightly different from what became standard as he considered an individual to be part of its own neigborhood. Based on simulations, de Oliveira [18] showed that the model, considered on the square lattice, exhibits an order-disorder phase transition when q is increased. More recent studies verified this property also for small-world networks [8] and the Erdős–Rényi graph [55].

(c) Hierarchical majority rule model

A structurally different model based on the majority rule was proposed by Galam [25]: A group of $n = r^k$ individuals ($r, k \in \mathbb{N}$) equipped with identically distributed $\{-1, +1\}$-valued opinions is considered, but no social network is specified. Let p_0 denote the probability for the opinion to be $+1$.

Instead of forming a consensus by interacting, they iteratively elect group-representatives: In the first round, all individuals are randomly divided into groups of size r. In every group a representative is chosen among the members sharing the majority opinion of the group – uniformly among all members if r is even and there is a tie. This procedure is then iterated among the elected representatives until a single leader is chosen in the kth round. If p_i denotes the probability that a representative on hierarchical level i holds opinion $+1$, the recursion is given by

$$p_{i+1} = \sum_{l=\frac{r+1}{2}}^{r} \binom{r}{l} p_i^l (1-p_i)^{r-l} \quad \text{if } r \text{ is odd and}$$

$$p_{i+1} = \frac{1}{2}\binom{r}{\frac{r}{2}} p_i^{\frac{r}{2}} (1-p_i)^{\frac{r}{2}} + \sum_{l=\frac{r}{2}+1}^{r} \binom{r}{l} p_i^l (1-p_i)^{r-l} \quad \text{if } r \text{ is even.}$$

(d) Sznajd model

There are different versions of this model sharing the same basic interaction principle. The following is not the one originally introduced by Sznajd-Weron and Sznajd [62] although the most popular variant. The individuals are again considered to occupy the sites of a graph (forming the interaction network) and to hold $\{-1,+1\}$-valued opinions. A pair of neighboring agents is picked and if they agree, all their neighbors adopt this opinion as well (illustrated in Figure 3.6 below). If they disagree, however, nothing happens.

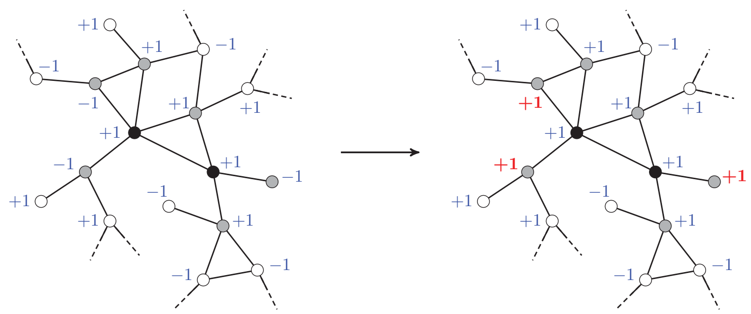


Figure 3.6: *Update rule in the Sznajd model: If the two neighbors picked (black) agree, they impose their opinion on all other individuals linked to them (gray).*

The Sznajd model is designed to incorporate the typical human behavior to be influenced more easily by a group of people that agree on a certain topic, compared to the influence of single individuals. Variants of the model have in fact been applied in order to describe and analyze voting behavior in elections.

(e) CODA model

In 2008, Martins [49] presented a new model featuring binary choices (between options A and B say, again represented by a spin $\sigma \in \{+1, -1\}$) that is based on **c**ontinuous **o**pinions and **d**iscrete **a**ctions (CODA) in the following sense: An opinion is in fact given by a probability distribution (more precisely the odds $\frac{p}{1-p}$ are considered, where p denotes the probability the considered individual attributes to option A being the better choice, $1-p$ consequently the probability for the complementary option B). When agents interact, they only tell each other their preference (i.e. $\sigma = +1$ corresponding to $p > \frac{1}{2}$ or $\sigma = -1$ corresponding to $p < \frac{1}{2}$) but not the precise value of p.

From this piece of information, the opinions are updated with a Bayesian reasoning: Let $\alpha := \mathbb{P}(\sigma = +1 \,|\, A)$ denote the probability that an agent believes in A if that actually is the better choice and $\beta := \mathbb{P}(\sigma = -1 \,|\, B)$ the analog for B in place of A. Assuming rational agents, one might think of α and β to be larger than $\frac{1}{2}$. When individuals i and j meet and share their preferences, σ_i and σ_j, the prior odds $\frac{p_i}{1-p_i}$ of agent i get updated to

$$\frac{\mathbb{P}(A \,|\, \sigma_j = +1)}{\mathbb{P}(B \,|\, \sigma_j = +1)} = \frac{p_i}{1-p_i} \cdot \frac{\alpha}{1-\beta},$$

if $\sigma_j = +1$ and to $\frac{p_i}{1-p_i} \cdot \frac{1-\alpha}{\beta}$ otherwise.

These interaction rules make the model distinct from the ones introduced so far in two different ways: On the one hand, despite binary choices, the agents hold continuous opinions and as a consequence hold back some information when they interact. On the other, despite pairwise interactions, the model does equip the agents with a certain memory of the past, which is normally not the case for adaptive behavior in this setting. Both features can be seen to incorporate traits of human behavior.

4

Incorporation of selective exposure

There are many phenomena in opinion formation processes in groups, that can not be captured by the models based on or closely related to the Ising model. Although contrarian behavior can be incorporated into the Ising model by considering antiferromagnetic material (i.e. $J < 0$), as discussed already by Callen and Shapero at the end of [7], this again leads to conformity even though only on antiparallel sublattices. In order to include phenomena like homophily or individual strong-willed behavior and persisting extremism, additional concepts had to be implemented, such as *bounded confidence* for instance.

As alluded to in the introduction, models that incorporate this principle involve in their interaction rules the mental defense mechanism known as *selective exposure*, a psychological phenomenon which can not be found in the interplay of physical particles: When two individuals meet, they will only influence each another if their current opinion values are not too far apart from each other. More precisely, in most of the models there exists a parameter $\theta \geq 0$ shaping the *tolerance* of the individuals: If the current opinion value of an agent is η, other agents holding opinions at a distance larger than θ from η will just be ignored.

Many of the bounded confidence models listed below have also been reviewed in [45]. Let us continue the list from the foregoing chapter with the interacting particle system that is the core theme of this thesis.

4.1 Bounded confidence models

(f) Deffuant model
Besides the aforementioned tolerance θ, this model features another parameter, $\mu \in (0, \frac{1}{2}]$, that embodies the willingness of an individual to approach the opinion of the other in a compromise. Encounters always happen in pairs, so if agents u and v meet at time t, holding opinions $a, b \in \mathbb{R}$ respectively, the update rule reads as follows:

$$(\eta_t(u), \eta_t(v)) = \begin{cases} (a + \mu(b-a), b + \mu(a-b)) & \text{if } |a-b| \leq \theta, \\ (a, b) & \text{otherwise,} \end{cases}$$

where $\eta_t(u)$ denotes the opinion of agent u at time t. The idea behind this is simple: When two individuals interact and discuss the topic in question, they will only rate the opinion encountered as worth considering if it is close enough to their own personal belief. If this is the case, however, they will have a constructive debate and their opinions will symmetrically get closer to each other – in the special case $\mu = \frac{1}{2}$, they will separate having come to a complete agreement at the average of the opinions they hold before the interaction.

In this manner, groups of compatible agents concentrate more and more around certain opinion values (their initial average) and once each such cluster of individuals is sufficiently far from neighboring ones, the final opinions are formed and all groups will from then on only become more homogeneous by internal interactions.

When Deffuant, Neau, Amblard and Weisbuch introduced this model in [16] (some authors refer to it as Deffuant–Weisbuch model), it was considered on a finite number of agents having i.i.d. initial opinions, distributed uniformly on $[0, 1]$. As social network they chose the complete graph and a finite square lattice respectively. The encounters occurred in discrete time by picking at each time step a pair of agents uniformly at random from the edge set of the underlying interaction network graph. Depending on the values of the model parameters, θ and μ, in their simulation-studies they observed one of the following two long-time scenarios: Either the agents ended up in one opinion cluster (corresponding to a consensus) or split into several clusters (fragmentation or disagreement). A controversial point in this context is the size threshold beyond which a small number of outliers are considered minor clusters.

Stauffer et al. [61] introduced a discretized version of the model, in which the opinions can take on values from the set $\{1, 2, \dots, q\},\ q \in \mathbb{N}$, and are rounded to the nearest integer after an update of the form described above. There have also been attempts to analyze the model with the tolerance parameter θ varying from individual to individual, revealing that in such a generalization it is the individuals with largest tolerance that ultimately determine the system's behavior.

In a recent publication [58], the idea of variable confidence bounds $\theta_t(v)$ that depend on the current opinion values has been presented: the more extreme the opinion $\eta_t(v)$ of an agent v, the smaller the corresponding value of θ. This extension of the Deffuant model bears resemblance to the relative agreement model (see below).

(g) Hegselmann–Krause model

The model introduced in [35] is quite similar to the Deffuant model, only the rule for encounters (which again happen in discrete time) is different: Given a network graph, at every time step each individual interacts with all its compatible neighbors at once and takes the average as its new opinion. If we let $\sim$ denote the reflexive adjacency relation, i.e. $u \sim v$ if $u = v$ or u and v are neighbors in the graph, and $\eta_t(u)$ once more the opinion of agent u at time t, we can write the update rule as follows:

$$\eta_{t+1}(v) = \frac{1}{N_t(v)} \sum_{\substack{u \sim v \\ |\eta_t(u) - \eta_t(v)| \leq \theta}} \eta_t(u) \quad \text{for all } v, \tag{4.1}$$

where the sum runs over the set of agents that consists of v plus its compatible neighbors and $N_t(v) = \big|\{u;\ u \sim v,\ |\eta_t(u) - \eta_t(v)| \leq \theta\}\big|$ is the size of this set at time t. Note that in contrast to the Deffuant model, the mean opinion is not conserved over time.

When it comes to simulations of the model, the major disadvantage of the Hegselmann–Krause model compared to the one introduced by Deffuant et al. is that for a dense interaction network averages of large groups of agents have to be calculated. This makes the running time until a meaningful pattern – allowing to decide whether the system approaches consensus or fragmentation – emerges rather long. However, for a finite number of individuals the system converges to a stable state in finite time: Once the opinion clusters are formed and all agents in one fixed cluster are compatible with one another, they will completely agree after one more time step making further changes of their opinions impossible.

The two models for opinion dynamics introduced by Deffuant et al. as well as Hegselmann and Krause, as described above, can be transferred to higher-dimensional opinions without any further changes – only the notion of distance has to be specified: We need to replace the absolute value by a suitable metric, which then determines the confidence ranges around a given opinion.

The vectorial version of both models was studied in [24] for instance – on the complete graph with opinion vectors from the unit square $[0,1]^2$. Both the Euclidean and the supremum norm (i.e. $||\,.\,||_2$ and $||\,.\,||_\infty$) were used as distance metric, shaping circular and square confidence ranges respectively. The generalization of the Deffuant model on the two-sidedly infinite path $\mathbb{Z}$ to higher-dimensional opinion spaces is the object of investigation in two of the appended papers (see below): While in Paper B, vector-valued opinions are considered and the Euclidean as well as other metrics used as notions of distance, Paper C deals with the case of opinions given by absolutely continuous probability measures on $[0,1]$ and the total variation as distance metric.

(h) Axelrod model

The model proposed by Axelrod [1] in 1997 was actually the first one introducing the concept of bounded confidence. However here, rather than having a sharp threshold, the probability of interaction decays gradually with respect to the distance of the two opinions involved: Think of the individuals again as nodes of a network. Every single one of them is endowed with an opinion vector in $\{1,2,\dots,q\}^d$, each coordinate of which is understood to represent one of d cultural features and q is the number of possible traits per feature. In that sense, the opinion vector $\boldsymbol{\eta}(i) = (\eta_1(i),\dots,\eta_d(i))$ is modelling the current beliefs and attitudes of agent i with respect to d interrelated topics.

In an elementary step of the dynamics an individual i and a neighboring one, say j, are randomly selected and interact with probability

$$p_{i,j} = \frac{1}{d} \sum_{k=1}^{d} \mathbb{1}_{\{\eta_k(i)=\eta_k(j)\}}, \tag{4.2}$$

which is scaling with the number of shared attitudes. If they interact, one of the features in which they disagree (i.e. k such that $\eta_k(i) \neq \eta_k(j)$) is chosen uniformly at random and individual j assumed to be convinced by the arguments of i, in other words $\eta_k(j)$ is set to be equal to $\eta_k(i)$, just like in the multitype voter model.

The Axelrod model became quite popular among social scientists for the fact that it includes two principles (which we mentioned earlier) that are considered to be typical in cultural assimilation: *social influence*, i.e. interacting makes people more alike, and *homophily* – humans tend to interact more frequently with others that share essential beliefs, attitudes and behaviors. Obviously, this model also features two kinds of absorbing states: Either all opinions are the same (consensus) or any two neighboring opinions do not share one single trait (disagreement).

Following the seminal paper of Axelrod [1] – who focussed on i.i.d. initial opinion vectors being uniform on $\{1, 2, \ldots, q\}^d$ and finite square lattices as network – several analyses based on numerical simulations have been performed and show that the value of q determines whether the final state reached will be consensus or disagreement, for different networks and initial distributions.

In the original model, the actual values of the coordinates are mere labels: It does not make a difference if two neighbors have traits that differ by 1 or $q-1$. In [19] a more metric variant of the model has been considered in the sense that the interaction probability in (4.2) is changed to

$$p_{i,j} = \frac{1}{d} \sum_{k=1}^{d} \Big(1 - \frac{|\eta_k(i) - \eta_k(j)|}{q-1}\Big).$$

A further variant of the Axelrod model was suggested in the paper by Deffuant et al. [16] as a multidimensional counterpart of the Deffuant model: They considered the traits to be binary variables (corresponding to $q = 2$

above) and neighbors interact only if the number of features they disagree on does not exceed a given threshold. So the interaction probability becomes a step function at some given confidence bound. Also the interaction rule itself was defined slightly different: Once the random feature the individuals i and j disagree on is selected, j is not convinced of $\eta_k(i)$ by default but adapts with probability $\mu \in (0, \frac{1}{2}]$.

(i) Relative agreement model

Shortly after introducing the Deffuant model, the authors Deffuant, Amblard, Weisbuch and Faure [15] came up with yet another model shaping opinion formation under bounded confidence: In the so-called *relative agreement model*, the mindset of an individual is characterized not only by a continuous real-valued opinion, but also by an associated uncertainty. Agents start from i.i.d. opinions, uniformly distributed on $[-1, 1]$, and the interaction rules are as follows: Individuals meet pairwise and when agent i (holding opinion x_i and uncertainty u_i) encounters agent j (opinion x_j, uncertainty u_j), they interact only if the intervals $[x_i - u_i, x_i + u_i]$ and $[x_j - u_j, x_j + u_j]$ overlap. Under this premise, let

$$h_{ij} = \min\{x_i + u_i, x_j + u_j\} - \max\{x_i - u_i, x_j - u_j\}$$

denote the overlap. If $h_{ij} > u_j$, i.e. $x_j \in [x_i - u_i, x_i + u_i]$, the opinion and uncertainty value of agent i get updated from (x_i, u_i) to

$$\big(x_i + \mu \cdot (\tfrac{h_{ij}}{u_j} - 1) \cdot (x_j - x_i), u_i + \mu \cdot (\tfrac{h_{ij}}{u_j} - 1) \cdot (u_j - u_i)\big)$$

and analogously for agent j. The parameter $\mu \in (0, \frac{1}{2}]$ plays essentially the same role as in the Deffuant model. Besides the fact that the relative agreement model (just like the Axelrod model) implements a more gradual decay of confidence with distance of opinions, there is another feature that makes it a less idealized simplification of real-life opinion dynamics: the asymmetry in its interactions. Even if $h_{ij} > \max\{u_i, u_j\}$, implying that both agents update their opinion and uncertainty when they meet, the amount of influence agents have on each other differs. Individuals with low uncertainty influence others more compared to those with high uncertainty value.

In [15], the model was simulated with the complete graph as interaction network. In a later work [2], Amblard and Deffuant studied the model additionally on both a regular grid and a small-world network.

Having listed some of the most common models for opinion dynamics, which incorporate the idea of bounded confidence, it should be mentioned that in recent years, there have been first attempts to apply these interacting particle systems to areas outside the field of opinion formation in groups. For example, Morărescu and Girard [51] used a variant of the weighted Hegselmann–Krause model to define a randomized algorithm designed to detect communities in networks: Given the network $G = (V, E)$ and opinion processes $\{\eta_t(v)\}_{v \in V}$, they considered the confidence bound to be decreasing in time – to be more precise, they set $\theta_t = R\,\rho^t$ for appropriately chosen constants $R > 0$ and $\rho \in (0, 1)$ – and defined the set of active links at time t as

$$E(t) = \{(u, v) \in E;\ |\eta_t(u) - \eta_t(v)| \leq \theta_t\}.$$

The weighted version is a generalization of the original Hegselmann–Krause model in the sense that the arithmetic mean in (4.1) gets replaced by a weighted convex combination (to account for the fact that the influences of compatible neighbors contributing to the updated opinion might not be equally strong). Together with the time-dependent confidence bound, the update rule thus reads

$$\eta_{t+1}(v) = \sum_{\substack{u \sim v \\ |\eta_t(u) - \eta_t(v)| \leq \theta_t}} p_t(v, u)\, \eta_t(u) \quad \text{for all } v.$$

The authors chose the weights to be given by doubly stochastic invertible matrices $P(t) = \big(p_t(u, v)\big)_{u,v \in V}$, that depend on $E(t)$ only. They showed that under these technical assumptions, the model started with absolutely continuous opinions converges almost surely in finite time and their algorithm then returns the stable opinion clusters as communities of the graph.

The idea behind it is easy to grasp: Strongly connected local clusters of the graph perform enough updates to become more alike before the confidence bound gets so small that it prevents further assimilation. Sparsely connected parts of the network instead, will most likely not manage to homogenize fast enough and thus freeze with multiple opinions. In fact, this community detection algorithm performed quite well when tested on standard benchmark graphs and compared to more established algorithms in this field, such as the traditional methods of graph partitioning and spectral clustering or the popular one based on edge centrality, which was proposed by Girvan and Newman in 2002. For a detailed introduction to the topic of community detection in graphs as well as a presentation of the standard techniques just named, we refer to [23].

4.2 Disagreement versus consensus – earlier investigations of the Deffuant model

Now that we have put the model, which Deffuant et al. proposed first, into the broader context of other common models for opinion formation processes, we want to give a short overview of the results that have been achieved in earlier analyses of the Deffuant model.

The findings in the original paper [16] were threefold. Starting from i.i.d. initial opinions, uniformly distributed on $[0, 1]$, the authors simulated various configurations in order to understand the influence of the parameters θ and μ as well as the underlying network topology in respect of the model's long-term behavior.

For the complete mixing case with $n = 1000$ individuals (i.e. the interaction network is the complete graph K_{1000}), Deffuant et al. noted that a confidence parameter $\theta = \frac{1}{2}$ most likely leads to consensus (pretty much at the expectation $\frac{1}{2}$), whereas $\theta = \frac{1}{5}$ causes a fragmentation into two finally homogeneous groups (with opinion values roughly at $\frac{1}{4}$ and $\frac{3}{4}$ respectively). Besides this dichotomy of regimes, by keeping θ fixed they found that the convergence parameter μ and the model size n influence the convergence time only, not the qualitative behavior, which as a consequence primarily depends on θ. The persistent opinions were arranged equidistantly and their number scaled roughly like $\lfloor \frac{1}{2\theta} \rfloor$.

When they tracked the opinion evolution of single agents from their initial opinions to one of the several persistent ones in the fragmentation case, μ turned out to be influential after all: They observed that the overlap of ranges (in terms of initial opinions) that finally led to one of the persistent opinions strongly depends on μ. For $\mu = \frac{1}{2}$ agents holding initial opinions in regions between two persistent ones could basically end up in either of the two groups, while for smaller values (e.g. $\mu = \frac{1}{20}$) almost every individual joined the cluster, whose final opinion was closest to its initial opinion value. So in a certain sense, the parameter μ determines how conservative the individuals are – both in the microscopic interactions and overall.

In addition to that, they simulated the model also for agents occupying the sites of a square lattice (of size 29×29). Here, essentially the same qualitative behavior was found: for $\theta > 0.3$ a large group consensus around $\frac{1}{2}$ with few extrem-valued outliers and no consensus for smaller values of θ. In the fragmentation case, however, the variety of scattered opinions was way bigger than in the setting of complete mixing, as clusters of individuals that are compatible in terms of opinion values can be separated spatially and in this way be prevented from interacting.

In another article published by almost the same group of authors [65], an investigation concerning heterogeneous confidence bounds was added: They simulated the complete mixing case on 200 individuals with confidence bound $\theta = \frac{1}{5}$, except for 8 individuals among them featuring a larger value ($\theta = \frac{2}{5}$). It is important to note that individual θ-values in the Deffuant model in general violate mass conversation: An encounter of two agents, whose difference in opinions lies in between their different values of θ, leads to the situation where the one with larger θ performs an opinion update, the other one does not.

Nevertheless, an interesting combination of the fragmentation and consensus case over the course of time could be observed in the simulations: In the short run clustering depends on the lower confidence bound, in the long run it depends on the higher bound. First, the majority of agents formed two incompatible opinion clusters at a distance larger than $\frac{1}{5}$, then the few 'open-minded' agents started to act as mediators between these groups and slowly but steadily brought them within talking distance of each other, which finally led to a global consensus – not at $\frac{1}{2}$ though, as asymmetric interactions are not average preserving and can cause such a drift. The transition time from one regime to the other depended very much on the proportion of individuals with larger confidence bound.

In addition to it, Deffuant et al. simulated the model with confidence bounds decreasing in time (which can be seen to describe the reasonable process of positions hardening in the course of time). In the simplest fragmentation case this led to major opinion clusters at values of about 0.60 and 0.42 – closer to each other than in the case of constant confidence bounds. Clearly, this arises from the fact that the opinions gather in a convergence movement first, before the confidence bound becomes too small and they split into two incompatible groups.

A completely different approach to the original model with fully mixed population, i.e. everybody interacts with everybody else, was pursued by Ben-Naim et al. [3]. They did not run any computer simulations of the agent based model, but considered a density based model instead (assuming that the number of individuals is large – a method termed *thermodynamical limit* in statistical physics):

If $P(x,t)$ denotes the density of agents having opinion x at time t and μ is fixed to be $\frac{1}{2}$, the following rate equation arises:

$$\frac{\partial}{\partial t} P(x,t) = \iint\limits_{|x_1-x_2|\leq\theta} P(x_1,t)\, P(x_2,t) \Big[\delta(x - \tfrac{x_1+x_2}{2}) - \delta(x-x_1)\Big]\, \mathrm{d}x_1\, \mathrm{d}x_2,$$

where $\delta(.)$ denotes the Dirac delta function.

Given i.i.d. $\mathrm{unif}([0,1])$ initial opinions and $\theta = 1$ (i.e. no bounded confidence restriction), they showed that the density converges to a delta function at the initial mean $\frac{1}{2}$. In the non-trivial cases ($\theta < 1$), however, the rate equation is no longer analytically solvable. Ben-Naim and his co-workers solved it numerically (after having discretized the opinion space into $\frac{200}{\theta}$ equally spaced states) and discovered some further interesting facts about the persistent opinion clusters: In the long term, the density converges to a finite weighted sum of delta functions, i.e.

$$P(x,\infty) = \sum_{i=1}^{r} m_i\, \delta(x - x_i),$$

where $x_1, \dots, x_r$ are the persistent opinions and $m_i,\ 1 \leq i \leq r$, the masses of (that means the fraction of agents ending up in) the corresponding clusters. The conservation laws (for mass and mean) obviously force

$$\sum_{i=1}^{r} m_i = 1 \quad \text{as well as} \quad \sum_{i=1}^{r} m_i\, x_i = \tfrac{1}{2}.$$

As could be expected, the behavior in the case of absent confidence restriction (namely $r = 1,\ x_1 = \frac{1}{2}$) was also found for values $\theta > \frac{1}{2}$, while for $\theta < \frac{1}{2}$ the number of clusters (at pairwise distance larger than θ) is larger than 1, in fact $r \geq 3$. In addition to that, they also found that there occur three types of persistent opinion clusters: major (mass $> \theta$), minor (mass $< \frac{\theta}{100}$) and a central cluster located at opinion value $\frac{1}{2}$. All of them are generated (and the central cluster annihilated) in a periodic sequence of bifurcations as θ is decreased. The first major clusters appear for $\theta < \frac{1}{4}$, which coincides well with the findings of Deffuant et al. who only considered major clusters and disregarded single outliers sticking to extrem opinions. Actually Ben-Naim et al. considered $\theta = 1$ to be fixed, the initial opinions instead to be i.i.d. $\mathrm{unif}([-\Delta, \Delta])$ with variable Δ, but a simple rescaling translates their results to the original model.

The heuristics they used and implemented, inspired by the methods in statistical physics, were more rigorously applied and verified in a rather recent work by Gómez-Serrano et al. [30]. They motivate the mean-field approach mathematically and prove that the long-term behavior of the limiting case (infinitely many particles) is similar to that of the model with a very large but finite number of completely mixing agents.

Laguna et al. [40] discovered another feature of the long-term behavior in the Deffuant model with complete mixing which is governed by the convergence parameter μ: The fraction of agents that end up in the two most extreme opinion clusters (which Ben-Naim et al. already showed to be minor but of larger order compared to the other minor clusters) is scaling with μ. For $\theta < \frac{1}{2}$ and larger values of μ, the formation of central opinions is fast enough to seclude many agents holding extreme initial opinions from the unification process. If μ is comparatively small, however, those extremists have enough time to become more moderate in order to be included in one of the major opinion clusters later on. In this sense, even if it may sound counterintuitive, for $\theta < \frac{1}{2}$ the formation of a partial consensus in the population actually benefits from a slower pace in the dynamics.

Stauffer and Meyer-Ortmanns [60] were among the first ones to follow up on the idea by Deffuant et al. to consider the model with an interaction topology other than the complete graph. They used random graphs generated by the Barabási–Albert model as underlying network – the usual undirected version (introduced in Section 3.1) as well as a directed one. The results of their computer simulations suggest that the transition from fragmentation to consensus happens for the value of θ being about 0.4 (on both the directed and undirected network). Unlike the case of a fully mixed population, the number of persistent opinions in the non-consensus case not only depends on θ but also on n, the number of individuals (for the same reason as in the case of a square lattice). The dependence of the number of clusters on n (with θ fixed) was estimated to be linear.

In 2004, Fortunato [22] investigated the threshold for a *complete consensus* among the agents – as opposed to previous notions of consensus describing the formation of a widely adapted main stream opinion neglecting some few outliers (in other words: only one major cluster). He simulated the Deffuant model on a complete graph, a square lattice with perodic boundary conditions as well as two random graphs – those originating from the Barabási–Albert and the Erdős–Rényi model. In the latter, he chose to adapt the probability p (with which an edge is kept) to the number n of agents in such a way that the average degree, $(n-1)p$, stays roughly constant.

Fortunato made two central observations: Firstly, the critical value for θ above which a complete consensus is formed equals $\frac{1}{2}$ in all four social topologies, irrespectively of μ. Secondly, on each of the four networks the probability of complete consensus converges to a step function at the threshold $\theta = \frac{1}{2}$ when the number of individuals is increased.

It has to be mentioned at this point that he performed update steps as ordered sweeps over the population (for the sake of simplicity): In each round every individual gets – one after the other – the opportunity to interact with a randomly selected neighbor. This is different from the original update rule where the edge along which the next potential interaction takes place is picked uniformly at random. For large regular systems, however, this seems unlikely to matter.

The first result for the Deffuant model considered on an infinite graph was published by Lanchier [41] in 2011. He studied the standard Deffuant model (i.i.d. $\mathrm{unif}([0,1])$ initial opinions) on the two-sidedly infinite path $\mathbb{Z}$ using the following geometric idea: Instead of analyzing the opinion profiles $\{\eta_t(v)\}_{v\in\mathbb{Z}}$ directly, where $\eta_t(v)$ denotes the opinion of individual v at time t, he considered what he calls their *broken line representation*, i.e. $\{\xi_t(v)\}_{v\in\mathbb{Z}}$ with

$$\xi_t(0) = 0 \quad \text{and} \quad \xi_t(v) = \begin{cases} \sum\limits_{0\leq u\leq v-1} (2\eta_t(u)-1), & \text{if } v>0, \\ \sum\limits_{v\leq u\leq -1} (2\eta_t(u)-1), & \text{if } v<0. \end{cases}$$

Using quite intricate geometric arguments and the concentration inequality due to Azuma–Hoeffding, he verified a set of properties for this concatenation of two symmetric random walks (one evolving forwards, one backwards in time; both starting at the origin) which allowed to prove the following result:

Theorem 4.1. *Consider the Deffuant model on the graph $G=(V,E)$, where $V=\mathbb{Z}$ and $E=\{\langle v,v+1\rangle;\ v\in\mathbb{Z}\}$. If $\mu\in(0,\frac{1}{2}]$ is arbitrary but fixed, the initial opinions are i.i.d.* $\mathrm{unif}([0,1])$ *and $\{\eta_t(v)\}_{v\in\mathbb{Z}}$ denotes the opinion profile at time t, then the following holds:*

(i) For $\theta>\frac{1}{2}$, all neighbors are eventually compatible in the sense that for all $v\in\mathbb{Z}$:

$$\lim_{t\to\infty} \mathbb{P}(|\eta_t(v)-\eta_t(v+1)|\leq\theta) = 1.$$

(ii) For $\theta < \frac{1}{2}$, with probability 1 there will be infinitely many $v \in \mathbb{Z}$ with

$$\lim_{t\to\infty} |\eta_t(v) - \eta_t(v+1)| > \theta.$$

One thing that is quite remarkable about this phase transition in the behavior of the Deffuant model is the fact that it already occurs for the one-dimensional lattice – in marked contrast to the Ising model.

Häggström [32] used different techniques to reprove and slightly sharpen this result – showing that in the consensus regime (i), all opinions actually converge almost surely to the mean $\frac{1}{2}$ of the initial distribution. The crucial idea in his proof resides in the connection of the opinion dynamics of the Deffuant model to a non-random interaction process, which he proposed to call *Sharing a drink* (SAD). The SAD-procedure is dual to the opinion formation in the sense that it keeps track of the opinion genealogy of an individual, i.e. the contributions of all initial opinions to the current composition of its opinion.

This idea could in fact be employed to generalize the result for the Deffuant model on $\mathbb{Z}$ to initial opinion configurations other than i.i.d. $\mathrm{unif}([0,1])$, as was done in Paper A (see below) and by Shang [56] simultaneously and independently.

5

Extreme opinions and water transport

In their analyses of the Deffuant model on $\mathbb{Z}$ featuring i.i.d. $\mathrm{unif}([0,1])$ initial opinions, both Lanchier [41] and Häggström [32] singled out agents that are cast-iron centrists. These agents start with an opinion value close to the mean $\frac{1}{2}$ and will never move far away from it (irrespectively of future interactions), due to the fact that the influences they can possibly be exposed to are – loosely speaking – either close to the mean as well or marginal. The opinion $\eta_t(v)$, of an agent $v \in \mathbb{Z}$ at a later time $t > 0$, is a convex combination of all initial opinions and the maximally possible contributions on $\mathbb{Z}$ decay inversely proportional to the graph distance. Hence, the initial opinion profile $\{\eta_0(v)\}_{v\in\mathbb{Z}}$ can be such that agent v sits well-shielded in a large section of individuals equipped with initial opinions close to $\frac{1}{2}$ and all individuals holding more extreme opinions are too far away to have a significant influence on v.

With this idea in mind (leaving aside the fact that the bounded confidence restriction might actually eliminate possible influences), obvious candidates for vertices of this kind are what Häggström [32] calls two-sidedly ε-flat vertices and Lanchier [41] denotes by the random set

$$\Omega_0 = \Big\{ v \in \mathbb{Z};\ \tfrac{1}{2} - \varepsilon < \tfrac{1}{n+1} \sum_{u=v}^{v+n} \eta_0(u),\ \tfrac{1}{n+1} \sum_{u=v-n}^{v} \eta_0(u) < \tfrac{1}{2} + \varepsilon,\ \forall\, n \geq 0 \Big\}.$$

If the initial opinions are i.i.d. $\mathrm{unif}([0,1])$, it can be verified that the set Ω_0 is almost surely non-empty (in fact of infinite cardinality) for all $\varepsilon > 0$, see Prop. 1.1 in [41] or Lemma 4.3 in [32], and that the opinion at two-sidedly ε-flat vertices will be confined to the interval $[\frac{1}{2} - 6\varepsilon, \frac{1}{2} + 6\varepsilon]$ for all times, see Lemma 6.3 in [32]. This consideration, however, is adjusted to the geometry of the underlying network $\mathbb{Z}$ and does not answer the question whether on more general graphs as well (e.g. higher-dimensional grids), we can find vertices whose opinions are constrained to stay close to the mean by the initial profile already.

In the standard Deffuant model, the existence of agents that will hold an opinion close to the mean $\frac{1}{2}$, no matter how the random interactions take place, force a supercritical behavior of the system (for θ sufficiently large) as they will always be at speaking terms with the whole range of opinions $[0,1]$ then. Lorenz and Urbig [46] addressed the question, for which values of θ an asymptotic consensus on K_n can be enforced (alternatively prevented) if the interactions are not random but chosen in an elaborate succession, i.e. the agents follow a predefined communication plan, adjusted to the initial opinion profile. More precisely, Lorenz and Urbig define θ_{low} (resp. θ_{high}) as the infimum (resp. supremum) of confidence bounds, such that returning to random encounters after an appropriately chosen finite succession of interactions will lead to consensus (disagreement) with probability 1, and prove

$$\max_{1 \leq k \leq n-1} \Delta x_k \leq \theta_{\mathrm{low}} \leq \max_{1 \leq k \leq n-1} \sum_{j=0}^{k-1} \mu^j\, \Delta x_{k-j} \qquad \text{as well as}$$

$$\theta_{\mathrm{high}} = \max_{1 \leq k \leq n-1} \left(\frac{1}{n-k} \sum_{i=k+1}^{n} x_i - \frac{1}{k} \sum_{j=1}^{k} x_j \right),$$

where $\Delta x_i = x_{i+1} - x_i$, for $1 \leq i \leq n-1$, and $(x_1, \ldots, x_n)$ denotes the vector of initial opinions $\{\eta_0(i)\}_{i=1}^n$ in increasing order. These results are verified by exhibiting a communication plan that circumvents (resp. aims for) the creation of large gaps in the opinion range.

In the same way as for gaps, one can try to manipulate the interaction scheme in such a way that the opinion of one fixed agent gets as extreme as possible (which might then answer the question if there are nodes stuck with an opinion close to the mean for all times right from the beginning). If we drop the bounded confidence restriction, this combinatorial optimization problem can be seen as the task of transporting water on a graph without pumps: Agents are reinterpreted as identical water barrels on a plane, their social network as a system of (locked, water-filled) pipes connecting them and the opinion values as the corresponding water levels. Opening pipe $\langle u, v\rangle$ will lead to an update $(\eta(u), \eta(v)) \mapsto ((1-\mu)\,\eta(u) + \mu\,\eta(v), (1-\mu)\,\eta(v) + \mu\,\eta(u))$, where $\mu \in (0, \frac{1}{2}]$ can be chosen arbitrarily. We then want to maximize the water level in a fixed barrel (*target vertex*) by opening and closing the locks in an appropriate order.

If we disregard the option to close locks, the problem turns into finding a connected subset of nodes including the target vertex with maximal average; a concept known as greedy lattice animal, which will be introduced and reviewed in the next section. Its relation to the water transport problem, which is relevant in the analysis of the Deffuant model as outlined above, will be discussed in Section 5.2.

5.1 Greedy lattice animals and site percolation

In 1993, Cox, Gandolfi, Griffin and Kesten [13] introduced the notion of greedy lattice animals: They considered an i.i.d. family of positive random variables $\{X_v;\ v \in \mathbb{Z}^d\}$ and the set of connected subsets comprising n vertices of the grid including the origin, $\Xi_{\mathbf{0}}(n) := \{\xi \subseteq \mathbb{Z}^d;\ \mathbf{0} \in \xi,\ |\xi| = n,\ \xi \text{ is connected}\}$. A set $\xi \in \Xi_{\mathbf{0}}(n)$ with maximal weight $\sum_{v\in\xi} X_v$ is called a *(vertex) greedy lattice animal* (of size n), its weight denoted by N_n.

With respect to the common marginal distribution, represented by $X_{\mathbf{0}}$, the random variable associated to the origin, they established the following asymptotic bound (where $\log^+(x)$ is a short notation for the positive part of the logarithm, i.e. $\max\{\log(x), 0\}$):

Theorem 5.1. *If for some $a > 0$,*

$$\mathbb{E}\big(X_{\mathbf{0}}^d\,(\log^+ X_{\mathbf{0}})^{d+a}\big) < \infty, \tag{5.1}$$

then there exists a constant $M \in \mathbb{R}$ such that

$$\limsup_{n\to\infty} \frac{N_n}{n} \le M \quad \textit{almost surely.}$$

In a subsequent publication, Gandolfi and Kesten [28] improved this result by verifying that the moment condition for the marginal distribution in fact implies a.s. linear growth of N_n in n: They showed that given (5.1), there exists a constant $N \in \mathbb{R}$ such that $\lim_{n\to\infty} \frac{N_n}{n} = N$ almost surely and in L^1.

Recall that in the model of i.i.d. *bond* percolation, which was introduced in Subsection 3.1, we take a graph and toss independent p-biased coins to decide which of the edges are kept and removed respectively. Applying the same thinning procedure not to the edges but to the vertices of a graph instead – i.e. independently, each vertex is chosen to be kept (with probability p) or erased along with all edges it is incident to (with probability $1-p$) – is called i.i.d. *site* percolation. Similarly as for bond percolation, in dimension $d \geq 2$, there exists a critical probability $p_c \in (0,1)$ for i.i.d. site percolation on $\mathbb{Z}^d$ marking the emergence of an infinite cluster. Note that the critical probabilities for bond and site percolation on the integer lattice of dimension at least 2 are related but not equal. For further details we refer once again to Grimmett [31].

Relating both concepts, greedy lattice animals and site percolation, Lee [43] proved among other things the following:

Theorem 5.2. *Fix $d \geq 2$ and consider an i.i.d. family of positive bounded random variables $\{X_v;\ v \in \mathbb{Z}^d\}$, the sets $\Xi_{\mathbf{0}}(n)$ and random variables N_n, for $n \in \mathbb{N}$, as above. Let p_c denote the critical probability of i.i.d. site percolation on $\mathbb{Z}^d$, $R := \inf\{r \in \mathbb{R};\ X_{\mathbf{0}} \leq r$ almost surely$\}$ be the essential supremum of the marginal distribution and N the almost sure limit of $\frac{N_n}{n}$. Then the following holds:*

(i) If $\mathbb{P}(X_{\mathbf{0}} = R) < p_c$, then $N < R$.

(ii) If $\mathbb{P}(X_{\mathbf{0}} = R) \geq p_c$, then $N = R$.

The case $\mathbb{P}(X_{\mathbf{0}} = R) > p_c$ is particularly easy and exhibits the connection to site percolation most obviously: If we disregard all nodes but those $v \in \mathbb{Z}^d$ with $X_v = R$, with probability 1 an infinite cluster remains. The origin can be connected to this cluster through finitely many other nodes, which guarantees a nested sequence $\{\xi_n\}_{n\in\mathbb{N}}$ of connected sets containing the origin with $\lim_{n\to\infty} \frac{1}{n} \left|\{v \in \xi_n;\ X_v = R\}\right| = 1$.

Apart from these results, the idea of a vertex greedy animal (as defined above) can of course be applied to more general graphs than integer lattices.

5.2 Optimizing pumpless water transport

In the context of making a fixed agent's opinion (respectively the water level at a target vertex) most extreme, we don't care about the number of involved vertices, hence with respect to greedy lattice animals the following definition is most appropriate:

For a fixed graph $G = (V, E)$, target vertex v and water levels $\{\eta(u)\}_{u\in V}$, let us call a finite set $C \subseteq V$ a *lattice animal (LA)* for v if C contains v and is connected. C is a *greedy lattice animal* for v if it maximizes the average of water levels over such sets, i.e. if its average equals the value

$$\mathrm{GLA}(v) := \sup_{C \text{ LA for } v} \frac{1}{|C|} \sum_{u\in C} \eta(u).$$

Note that with this altered definition, a greedy lattice animal need not necessarily exist for infinite graphs, as $\mathrm{GLA}(v)$ might not be attained.

If $\kappa(v)$ denotes the supremum of water levels attainable at v by opening *and* closing locks, $\mathrm{GLA}(v)$ can be used as a lower bound on $\kappa(v)$ only. As a consequence, for i.i.d. $\mathrm{unif}([0, 1])$ initial water levels, we can not conclude from Theorem 5.2 and $\mathbb{P}(\eta(\mathbf{0}) = 1) = 0$ that on $\mathbb{Z}^d$, the highest possible water level at the origin **0** is bounded away from 1 with positive probability.

In fact, the two problems – greedy lattice animals and water transport – as related as they might seem, are quite different from a technical point of view: The option to shut open locks introduces a temporal dimension and makes it crucial, which moves are performed first. To get the idea, consider the elementary example depicted in Figure 5.1 below.

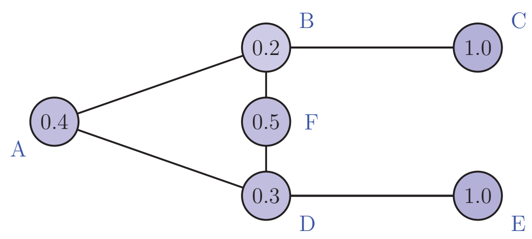


Figure 5.1: *A simple water transport instance on 6 nodes.*

If A is chosen to be the target vertex, the greedy lattice animal is given by the set $\{A, B, C, D, E\}$ with a value of $\mathrm{GLA}(A) = 0.58$. Vertex F can however be used to improve the two bottlenecks B and D. This is done most beneficially, if the pipe $\langle D, F\rangle$ is opened first (until the two water levels have balanced out completely at 0.4), then closed and thereafter the same procedure repeated for the edge $\langle B, F\rangle$, leading to $\kappa(A) = 0.62$. Further, this simple water transport instance exemplifies the enhanced structural complexity of the problem: While it is sufficient to consider spanning trees when searching for a greedy lattice animal, additional edges forming circles might become important in the corresponding water transport problem, as it is the case here.

In Paper D (see below), we address the water transport problem both on finite and infinite graphs and consider its complexity. It turns out that one does not gain from opening several pipes simultaneously or chosing the mixture parameter μ in a move to be less than $\frac{1}{2}$, i.e. closing a pipe before the contents of the two connected barrels have levelled completely. Furthermore, we found that in dimension $d \geq 2$ and given i.i.d. $\mathrm{unif}([0, 1])$ initial opinions, the water level of a fixed vertex of the integer lattice $\mathbb{Z}^d$ can almost surely be raised as close to 1 as desired – in contrast to both greedy lattice animals and dimension $d = 1$.

This fact is one of the main obstacles when trying to generalize the results established for the Deffuant model on the two sidedly-infinite path $\mathbb{Z}$ to higher dimensions, as it invalidates one of the most central arguments.

6

Summary of appended papers

Paper A:
Further results on consensus formation in the Deffuant model

(co-authored with Olle Häggström)

The contribution of this paper to the analysis of long-term behavior in the Deffuant model featuring real-valued opinions on infinite graphs can be broken down into three parts.

The first one – as alluded to in Section 4.2 – is the extension of the statement from Theorem 4.1 to more general initial distributions. As was done in [41] and [32], we consider the model on $\mathbb{Z}$ with i.i.d. initial opinions, this time however distributed according to a general law $\mathcal{L}(\eta_0)$ in place of $\mathrm{unif}([0,1])$. Building on the ideas from [32], we were able to settle all cases in which the mean $\mathbb{E}\,\eta_0$ of the initial marginal distribution is well-defined: If $\mathcal{L}(\eta_0)$ is bounded, there exists a critical value θ_c for the parameter θ that marks a sharp phase transition in the long-term behavior from almost sure disagreement (the agents split into finite, incompatible but finally homogeneous segments) to a.s. complete consensus (all opinions converge to the mean $\mathbb{E}\,\eta_0$). The value of θ_c depends on two characteristics of the distribution $\mathcal{L}(\eta_0)$: its mean and its support. More precisely, the critical value turns out to be

$$\theta_c = \max\{\mathbb{E}\,\eta_0 - \text{essinf}(\eta_0), \text{esssup}(\eta_0) - \mathbb{E}\,\eta_0, h\}, \tag{6.1}$$

where the essential infimum and supremum mark the extreme ends of the support and h denotes the length of its largest gap (which means the largest subinterval $I \subseteq [\text{essinf}(\eta_0), \text{esssup}(\eta_0)]$ with $\mathbb{P}(\eta_0 \in I) = 0$). In the case of an unbounded initial distribution – under the assumption that not both $\mathbb{E}\,\eta_0^+$ and $\mathbb{E}\,\eta_0^-$ are infinite – the model a.s. behaves subcritically (disagreement) for any choice of $\theta > 0$. Note that this matches the statement for the bounded case, since θ_c as defined in (6.1) becomes infinite for an unbounded initial distribution.

In addition to that, we point out how these results can be transferred to special cases of dependent initial opinions. For the arguments used to be valid, it is sufficient that the initial configuration is ergodic and fulfils an additional requirement, that is called *finite energy condition* in percolation theory (and was introduced by Newman and Schulman [52]).

In the second part, the model is considered on higher-dimensional integer lattices $\mathbb{Z}^d,\ d \geq 2$. Although the central ideas of proof from dimension one do not transfer to higher dimensions, elaborating some of the arguments allows us to prove at least the following partial result: If the marginal distribution of the i.i.d. initial configuration is bounded and θ sufficiently large (strictly larger than $\frac{3}{4}$ in the case of $\text{unif}([0, 1])$ initial opinions for example), the opinion of every agent will still almost surely converge to the mean of the initial distribution. In addition to this, on the one hand we show that the opinions converge in distribution for any value of θ and on the other hand discuss a generalization to transitive, amenable graphs.

In the last part, we consider the Deffuant model on the infinite cluster of supercritical i.i.d. bond percolation on $\mathbb{Z}^d,\ d \geq 2$. In this setting one can retrieve the results derived for the full grid and on top of that, we were able to show that for small values of θ, the opinions of the agents belonging to the infinite cluster cannot converge to one fixed value. Neighboring individuals could, however, still come to a complete agreement in the long run without their opinions converging to a deterministic limit (corresponding to the type of consensus, which Lanchier [41] formulated in part (i) of Theorem 4.1).

Paper B:
The Deffuant model on $\mathbb{Z}$ with higher-dimensional opinion spaces

As mentioned in Section 3.2, this paper deals with the generalization of the Deffuant model on $\mathbb{Z}$ to vector-valued opinions. In the first part, we generalize the findings for univariate opinions from Paper A to multivariate opinions and stick to the Euclidean norm as natural replacement for the absolut value (which was taken to measure the distance between two opinions in the case of real-valued opinions). Using geometric arguments, that are considerably more involved than in the case of scalar opinions, we manage to verify properties of the support of the opinion distribution $\mathcal{L}(\eta_t)$ for times $t > 0$, depending on the initial distribution $\mathcal{L}(\eta_0)$. Especially the notion of a gap in the support of $\mathcal{L}(\eta_0)$ has to be properly defined and analyzed in higher dimensions in order to play the same role as for univariate distributions.

In the second part, we allow for more general metrics ρ to be employed as measures of distance – determining if the opinions of two agents are close enough for them to interact. We are able to transfer the results from the Euclidean case, given that ρ satisfies appropriate extra conditions: weak convexity, local domination by the Euclidean distance and sensitivity to unbounded coordinates. Through several examples, the necessity of these additional assumptions is verified.

Paper C:
Overly determined agents prevent consensus in a generalized Deffuant model on $\mathbb{Z}$ with dispersed opinions

The generalization of the original Deffuant model in terms of opinion spaces is taken one step further in this paper: We consider the model on $\mathbb{Z}$, in which opinions are represented by absolutely continuous probability distributions on $[0, 1]$. In comparison to finite-dimensional opinions, the expectation of $\mathcal{L}(\eta_0)$ corresponds to the so-called *intensity measure* in the context of random probability distributions.

For the sake of concreteness, we consider a model in which the initial opinions are given by symmetric triangular distributions: Initially, for each agent $v \in \mathbb{Z}$ independently, a vector (U, V) from the uniform distribution on $[0, 1]^2$

is drawn. Then v gets attributed the random measure given by the density that is 0 outside $[m, M]$ and linear on both $[m, \frac{m+M}{2}]$ and $[\frac{m+M}{2}, M]$, where $m := \min\{U, V\}$ and $M := \max\{U, V\}$. This way of representing opinions can be seen as an improvement over real-valued opinions introducing the idea of uncertainty (around a favored value).

For this model, we calculate the intensity measure and verify that extremely determined agents (i.e. $|U - V|$ very small) will a.s. prevent consensus for any $\theta \in (0, 1)$, given that the total variation is used to measure the distance between two opinions.

If the determination is bounded, in the sense that the random vector (U, V) is taken from $\mathrm{unif}([0, 1]^2)$ conditioned on $|U - V| \geq \gamma$, for a fixed constant $\gamma \in (0, 1)$, the picture changes. The phase transition in the long-term behavior from a.s. disagreement to a.s. consensus, known from the investigations dealing with finite-dimensional opinion spaces, reappears and we are able to calculate the precise threshold value θ_c.

Paper D:
Water transport on graphs

(co-authored with Olle Häggström)

Incited by the impossibility of transferring the ideas used in the analysis of the Deffuant model on $\mathbb{Z}$ to higher-dimensional grids, we defined and analyzed a combinatorial optimization problem that can be seen as pumpless water transport on a graph: The agents holding different opinion values are reinterpreted as identical water barrels that are filled to different levels, the interactions (still taking place along the edges of the network) as opening the lock in the pipe between the two nodes for a certain time span. In this manner, we essentially consider the same interacting particle system and only think of converging water levels instead of compromising individuals, but we drop the randomness of encounters and the confidence bound restriction.

Asking for the maximal amount of water that can be accumulated in a fixed target barrel by opening and closing the locks in an elaborate succession is closely related to the question of how extreme the opinion of an agent possibly can become depending on the initial configuration. First, we provide some tools to describe and analyze optimal strategies to maximize the water in a given barrel and solve the optimization problem for different types of finite graphs.

Then, we consider the problem's complexity and prove by a polynomial reduction of the satisfiability problem 3-SAT to a suitably chosen instance of the water transport problem that the latter is NP-hard.

Finally, we verify a fact that in a manner of speaking accounts for the different challenges faced in the analysis of the Deffuant model on the integer lattice $\mathbb{Z}^d$, depending on the dimension d: Given i.i.d. $\mathrm{unif}([0,1])$ initial water levels, the highest achievable amount in a fixed barrel depends on the initial configuration in a non-deterministic way both for finite graphs and the two-sidedly infinite path $\mathbb{Z}$. For all other quasi-transitive infinite graphs, however, the level can a.s. be increased to a value as close to 1 as desired by opening (and closing) the locks in an appropriate order. The crucial feature of the underlying graph turns out to be, whether or not the graph contains a neighbor-rich half-line, i.e. an infinite path with sufficiently many extra vertices attached to it.

References

[1] AXELROD, R. (1997), *The dissemination of culture: A model with local convergence and global polarization,* The Journal of Conflict Resolution, Vol. 41 (2), pp. 203-226.

[2] AMBLARD, F. and DEFFUANT, G. (2004), *The role of network topology on extremism propagation with the relative agreement opinion dynamics,* Physica A: Statistical Mechanics and its Applications, Vol. 343, pp. 725-738.

[3] BEN-NAIM, E., KRAPIVSKY, P.L. and REDNER, S. (2003), *Bifurcations and patterns in compromise processes,* Physica D: nonlinear phenomena, Vol. 183, pp. 190-204.

[4] BERELSON, B. (1952), *Democratic theory and public opinion,* Public Opinion Quarterly, Vol. 16 (3), pp. 313-330.

[5] BROADBENT, D.E. (1958), "Perception and communication", Pergamon Press.

[6] BROADBENT, S.R. and HAMMERSLEY, J.M. (1957), *Percolation processes. I. Crystals and mazes,* Mathematical Proceedings of the Cambridge Philosophical Society, Vol. 53 (3), pp. 629-641.

[7] CALLEN, E. and SHAPERO, D. (1974), *A theory of social imitation,* Physics Today, Vol. 27 (7), pp. 23-28.

[8] CAMPOS, P.R., DE OLIVEIRA, V.M. and MOREIRA, F.G. (2003), *Small-world effects in the majority-vote model,* Physical Review E: statistical, nonlinear, biological and soft matter physics, Vol. 67 (2).

[9] CASTELLANO, C., FORTUNATO, S. and LORETO, V. (2009), *Statistical physics of social dynamics,* Reviews of Modern Physics, Vol. 81, pp. 591-646.

[10] CASTELLANO, C., VILONE, D. and VESPIGNANI, A. (2003), *Incomplete ordering of the voter model on small-world networks,* Europhysics Letters, Vol. 63 (1), pp. 153-158.

[11] CHATTERJEE, S. and SENETA, E. (1977), *Towards consensus: Some convergence theorems on repeated averaging,* Journal of Applied Probability, Vol. 14, pp. 89-97.

[12] CLIFFORD, P. and SUDBURY, A. (1973), *A model for spatial conflict,* Biometrika, Vol. 60 (3), pp. 581-588.

[13] COX, J.T., GANDOLFI, A., GRIFFIN, P.S. and KESTEN, H. (1993), *Greedy lattice animals I: upper bounds,* The Annals of Applied Probability, Vol. 3 (4), pp. 1151-1169.

[14] DAVISON, W.P. (1958), *The public opinion process,* Public Opinion Quarterly, Vol. 22 (2), pp. 91-106.

[15] DEFFUANT, G., AMBLARD, F., WEISBUCH, G. and FAURE, T. (2002), *How can extremism prevail? A study based on the relative agreement interaction model,* Journal of Artificial Societies and Social Simulation, Vol. 5 (4).

[16] DEFFUANT, G., NEAU, D., AMBLARD, F. and WEISBUCH, G. (2000), *Mixing beliefs among interacting agents,* Advances in Complex Systems, Vol. 3, pp. 87-98.

[17] DEGROOT, M.H. (1974), *Reaching a consensus,* Journal of the American Statistical Association, Vol. 69 (345), pp. 118-121.

[18] DE OLIVEIRA, M.J. (1992), *Isotropic majority-vote model on a square lattice,* Journal of Statistical Physics, Vol. 66 (1-2), pp. 273-281.

[19] DE SANCTIS, L. and GALLA, T. (2009), *Effects of noise and confidence thresholds in nominal and metric Axelrod dynamics of social influence,* Physical Review E: statistical, nonlinear, biological and soft matter physics, Vol. 79 (4).

[20] DEUTSCH, J.A. and DEUTSCH, D. (1963), *Attention: Some theoretical considerations,* Psychological Review, Vol. 70 (1), pp. 80-90.

[21] FESTINGER, L. (1957), “A Theory of Cognitive Dissonance”, Stanford University Press.

[22] FORTUNATO, S. (2004), *Universality of the threshold for complete consensus for the opinion dynamics of Deffuant et al.,* International Journal of Modern Physics C – Computational Physics and Physical Computation, Vol. 15 (9), pp. 1301-1307.

[23] FORTUNATO, S. (2010), *Community detection in graphs,* Physics Reports, Vol. 486, pp. 75-174.

[24] FORTUNATO, S., LATORA, V., PLUCHINO, A. and RAPISARDA, A. (2005), *Vector opinion dynamics in a bounded confidence consensus model,* International Journal of Modern Physics C – Computational Physics and Physical Computation, Vol. 16 (10), pp. 1535-1551.

[25] GALAM, S. (1990), *Social paradoxes of majority rule voting and renormalization group,* Journal of Statistical Physics, Vol. 61 (3-4), pp. 943-951.

[26] GALAM, S. (2002), *Minority opinion spreading in random geometry,* The European Physical Journal B - Condensed Matter and Complex Systems, Vol. 25 (4), pp. 403-406.

[27] GALAM, S., GEFEN, Y. and SHAPIR, Y. (1982), *Sociophysics: A new approach of sociological collective behaviour. I. Mean-behaviour description of a strike,* Journal of Mathematical Sociology, Vol. 9, pp. 1-13.

[28] GANDOLFI, A. and KESTEN, H. (1994), *Greedy lattice animals II: linear growth,* The Annals of Applied Probability, Vol. 4 (1), pp. 76-107.

[29] GIL, S. and ZANETTE, D.H. (2006), *Coevolution of agents and networks: Opinion spreading and community disconnection,* Physics Letters A, Vol. 356 (2), pp. 89-94.

[30] GÓMEZ-SERRANO, J., GRAHAM, C. and LE BOUDEC, J.-Y. (2012), *The bounded confidence model of opinion dynamics,* Mathematical Models and Methods in Applied Sciences, Vol. 22 (2).

[31] GRIMMETT, G. (1999), "Percolation" (2nd edition), Springer.

[32] HÄGGSTRÖM, O. (2012), *A pairwise averaging procedure with application to consensus formation in the Deffuant model,* Acta Applicandae Mathematicae, Vol. 119 (1), pp. 185-201.

[33] HARRIS, T.E. (1960), *A lower bound for the critical probability in a certain percolation process,* Mathematical Proceedings of the Cambridge Philosophical Society, Vol. 56 (1), pp. 13-20.

[34] HART, W., ALBARRACÍN, D., EAGLY, A.H., BRECHAN, I., LINDBERG, M.J. and MERRILL, L. (2009), *Feeling validated versus being correct: A meta-analysis of selective exposure to information,* Psychological Bulletin, Vol. 135 (4), pp. 555-588.

[35] HEGSELMANN, R. and KRAUSE, U. (2002), *Opinion dynamics and bounded confidence: models, analysis and simulation,* Journal of Artificial Societies and Social Simulation, Vol. 5 (3).

[36] HOLLEY, R.A. and LIGGETT, T.M. (1975), *Ergodic theorems for weakly interacting infinite systems and the voter model,* The Annals of Probability, Vol. 3 (4), pp. 643-663.

[37] ISING, E. (1925), *Beitrag zur Theorie des Ferromagnetismus,* Zeitschrift für Physik, Vol. 31 (1), pp. 253-258.

[38] KADANOFF, L.P. (2009), *More is the same; phase transitions and mean field theories,* Journal of Statistical Physics, Vol. 137 (5-6), pp. 777-797.

[39] KESTEN, H. (1980), *The critical probability of bond percolation on the square lattice equals $\frac{1}{2}$,* Communications in Mathematical Physics, Vol. 74 (1), pp. 41-59.

[40] LAGUNA, M.F., ABRAMSON, G. and ZANETTE, D.H. (2004), *Minorities in a model for opinion formation,* Complexity, Vol. 9 (4), pp. 31-36.

[41] LANCHIER, N. (2012), *The critical value of the Deffuant model equals one half,* Latin American Journal of Probability and Mathematical Statistics, Vol. 9 (2), pp. 383-402.

[42] LAZARSFELD, P.F. and MERTON, R.K. (1954), *Friendship as a social process: a substantive and methodological analysis,* pp. 18-66 in "Freedom and Control in Modern Society", ed. Berger, M., Van Nostrand.

[43] LEE, S. (1993), *An inequality for greedy lattice animals,* The Annals of Applied Probability, Vol. 3 (4), pp. 1170-1188.

[44] LIGGETT, T.M. (1985), "Interacting Particle Systems", Springer.

[45] LORENZ, J. (2007), *Continuous opinion dynamics under bounded confidence: A survey,* International Journal of Modern Physics C – Computational Physics and Physical Computation, Vol. 18 (12), pp. 1819-1838.

[46] LORENZ, J. and URBIG, D. (2007), *About the power to enforce and prevent consensus by manipulating communication rules,* Advances in Complex Systems, Vol. 10 (2), pp. 251-269.

[47] MAJORANA, E. (1942), *Il valore delle leggi statistiche nella fisica e nelle scienze sociali,* Scientia, Ser. 4, pp. 58-66.

[48] MANTEGNA, R.N. (2005), *Presentation of the English translation of Ettore Majorana's paper: The value of statistical laws in physics and social sciences,* Quantitative Finance, Vol. 5 (2), pp. 133-140.

[49] MARTINS, A.C. (2008), *Continuous opinions and discrete actions in opinion dynamics problems,* International Journal of Modern Physics C – Computational Physics and Physical Computation, Vol. 19 (4), pp. 617-624.

[50] MCPHERSON, M., SMITH-LOVIN, L. and COOK, J.M. (2001), *Birds of a feather: Homophily in social networks,* Annual Review of Sociology, Vol. 27, pp. 415-444.

[51] MORĂRESCU, I.-C. and GIRARD, A. (2011), *Opinion dynamics with decaying confidence: Application to community detection in graphs,* IEEE Transactions on Automatic Control, Vol. 56 (8), pp. 1862-1873.

[52] NEWMAN, C.M. and SCHULMAN, L.S. (1981), *Infinite clusters in percolation models,* Journal of Statistical Physics, Vol. 26 (3), pp. 613-628.

[53] ONSAGER, L. (1944), *Crystal Statistics. I. A two-dimensional model with an order-disorder transition,* Physical Review, Vol. 65 (3-4), pp. 117-149.

[54] PEIERLS, R. (1936), *On Ising's model of ferromagnetism,* Mathematical Proceedings of the Cambridge Philosophical Society, Vol. 32 (3), pp. 477-481.

[55] PEREIRA, L.F. and MOREIRA, F.G. (2005), *Majority-vote model on random graphs,* Physical Review E: statistical, nonlinear, biological and soft matter physics,, Vol. 71 (1).

[56] SHANG, Y. (2013), *Deffuant model with general opinion distributions: First impression and critical confidence bound,* Complexity, Vol. 19 (2), pp. 38-49.

[57] SMITH, A.B. (1932), *The pleasures of recognition,* Music & Letters, Vol. 13 (1), pp. 80-84.

[58] SOBKOWICZ, P. (2015), *Extremism without extremists: Deffuant model with emotions,* Frontiers in Physics, Vol. 3, no. 17, pp. 1-12.

[59] SPITZER, F. (1981), *Infinite systems with locally interacting components,* The Annals of Probability, Vol. 9 (3), pp. 349-364.

[60] STAUFFER D. and MEYER-ORTMANNS, H. (2004), *Simulation of consensus model of Deffuant et al. on a Barabási–Albert network,* International Journal of Modern Physics C – Computational Physics and Physical Computation, Vol. 15 (2), pp. 241-246.

[61] STAUFFER D., SOUSA, A. and SCHULZE, C. (2004), *Discretized opinion dynamics of the Deffuant model on scale-free networks,* Journal of Artificial Societies and Social Simulation, Vol. 7 (3).

[62] SZNAJD-WERON, K. and SZNAJD, J. (2000), *Opinion evolution in closed community,* International Journal of Modern Physics C – Computational Physics and Physical Computation, Vol. 11 (6), pp. 1157-1165.

[63] VAZQUEZ, F., KRAPIVSKY, P.L. and REDNER, S. (2003), *Constrained opinion dynamics: freezing and slow evolution,* Journal of Physics A: Mathematical and General, Vol. 36 (3), pp. L61-L68.

[64] WEIDLICH, W. (1971), *The statistical description of polarization phenomena in society,* British Journal of Mathematical and Statistical Psychology, Vol. 24 (2), pp. 251-266.

[65] WEISBUCH, G., DEFFUANT, G., AMBLARD, F. and NADAL, J.-P. (2002), *Meet, discuss, and segregate!,* Complexity, Vol. 7 (3), pp. 55-63.

[66] WEISS, P. (1907), *L'hypothèse du champ moléculaire et la propriété ferromagnétique,* Journal de Physique Théorique et Appliquée, Vol. 6 (1), pp. 661-690.