Fundamental Directivity Limitations of Dense Array Antennas: A Numerical Study Using Hannan´s Embedded Element Efficiency

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Per-Simon Kildal, *Fellow, IEEE*, Abbas Vosoogh, and Stefano Maci, *Fellow, IEEE*

***Abstract*—Hannan introduced in 1964 an embedded element efficiency concept that explains the so-called element-gain paradox in antenna arrays, i.e. that the array gain always is smaller than the sum of the element gains. In the present paper we show for the first time the usefulness of his approach by evaluating directivities and aperture efficiencies of an array of open-ended waveguides by commercial full-wave EM solvers for a large range of element spacing. The results show also that by using embedded element analysis, the realized gain of regular arrays actually becomes equal to the sum of the realized gains of the embedded elements. Thus, the embedded element efficiency is more practical to use in design and numerical analysis than the more commonly used active element pattern approach. We also show that the embedded element efficiency can be approximated by a simple formula when the element spacing is smaller than half wavelength.**

***Index Terms*—array antenna, embedded element efficiency, fundamental limitations**

I. Introduction

THIS paper deals with fundamental gain and directivity limitations of dense arrays. Dense arrays also suffer from strong mutual coupling among the ports of the neighboring elements. The overall effect of the mutual coupling is characterized by the radiation efficiencies of the embedded elements of the array, and this is the essential degrading performance parameter of multi-port antennas for digital MIMO enabled communications systems [1]. The present paper aims at showing that the embedded element pattern efficiency also is essential for understanding the behavior of dense regular arrays. It has already proven to be an important performance parameter of dense focal plane arrays [2].

Stein stated in 1962 that the mutual coupling represents a limitation in multi-beam antennas [3], later referred to as Stein´s limit [4]. Hannan defined the general embedded element efficiency already in 1964 (he called it element efficiency) [5]. He claimed that it will be very low for small element spacing and thereby explains the so-called element gain paradox. i.e., that the array gain always is smaller than the sum of the isolated element gains. This paradox was recently also pointed out by Kahn [6]. Hannan's formulations have not been used much in theoretical and numerical works. The active element pattern approach [8]-[9] is more common.

We should emphasize that the name "active element pattern" is used in [9] and by some authors to identify the gain of an array element when the others are terminated by matched loads. We use here instead the term embedded element pattern to avoid confusion with the concepts of "active impedance" and "active Green's function" in which the term "active" is used to describe that all the array elements are simultaneously excited. This is in agreement with the IEEE Standards for definition of terms for antennas of 2013.

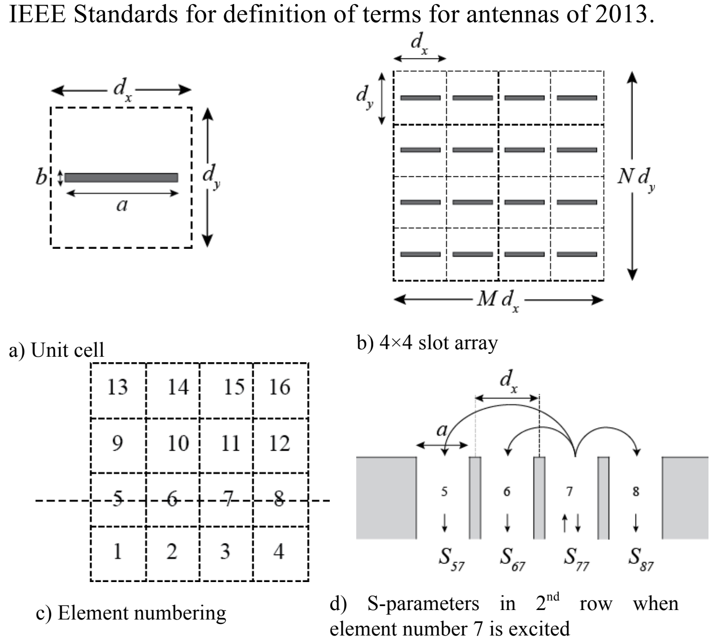

Fig. 1. Definition of a planar array of rectangular open-ended waveguides, illustrated by thick solid lines. The unit cells are illustrated by dashed squares. The lower sketches illustratess the definition of the element numbering.

Moreover, in agreement with this we will use the "active element" approach to describe an analysis method in which all

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elements are excited by certain amplitude and phase. This method as well as the term is widely accepted and very useful in array theory, especially for very large arrays. However, the approach is more theoretical than practical and not applicable to modern adaptive beam-forming and MIMO arrays with independent usages of each elements. On the other hand, the embedded element pattern is directly measurable at its port and automatically includes the low element efficiency characterizing dense arrays. The present paper will illustrate by simulations that the gain and directivity variations versus element spacing of a uniformly-excited broadside open-ended waveguide array is fully explained by the gain of the embedded element. Similar results are not previously available except for the analysis of the two-element array in [10].

II. MULTI-PORT ANTENNAS

The embedded element efficiencies needed to characterize multi-port antennas represent a fundamental limitation caused by mutual coupling. The simplest expression for the embedded radiation efficiency of port j in a lossless antenna array with MN elements is [1][2][5][6]

$$(e_{emb})_j = 1 - \sum_{i=1}^{MN} \left|S_{ij}\right|^2 \qquad (1)$$

where S_{ij} is the S-parameter between ports i and j. Note that the numbering of the elements are done successively as shown in Fig. 1 for a 4x4 array with $MN=16$. Thus, $i=mn$ corresponds to element number (m,n) in the more common numbering with $m=1,2,..M$ and $n=1,2,.N$. Thus, the embedded element efficiency is seen to represent the sum of the powers lost in the matched source impedance on the excited port as well as in the matched loads on all unexcited ports.

The efficient in (6) is particularly useful in small arrays for diversity and MIMO systems, which may suffer from strong mutual coupling among the antenna elements. They are different from classical arrays in the sense that the phase and amplitude excitations (both in transmitting and receiving mode) of the elements are dynamically adjusted to match the statistical field variations in the environment. As a result, they cannot be impedance matched for one specific excitation.

Such fundamental and limiting radiation efficiencies are also present in large classical arrays for producing narrow beams. The embedded efficiency becomes very low only if the mutual coupling S_{ij} is large, which happens for very small element spacing, such as in dense arrays for multiple beams.

Let us now consider the regular open-ended waveguide array in Fig. 1. The embedded element efficiency can be seen as the ratio between the realized gain per element of the array and the directivity of one array element [5]. For small element spacing the radiation intensity of the embedded element is known to have a $\cos\theta$ shape [5] (due to the projection of the aperture). With this variation of the radiation intensity, the integral representing the total radiated power becomes π, giving a directivity of $D_{emb}=4$ *(=6dBi)*.

The maximum available gain of a large aperture of area $A=MNd_xd_y$ is $G_{max}=4\pi A/\lambda^2$. Then, the maximum available gain per element becomes $G_{emb}=G_{max}/MN=4\pi(d_x/\lambda)(d_x/\lambda)$. This means that the embedded element will have a maximum available total radiation efficiency of

$$e_{emb} = \frac{G_{emb}}{D_{emb}} = \pi \frac{d_x}{\lambda}\frac{d_y}{\lambda} \qquad (2)$$

in a dense array, which is valid under the assumption D_{emb}=6 *dBi*. This result can be inferred from the text of Hannan's paper [5], although it is not explicitly formulated there. The derivation is also discussed in [7]. The embedded efficiency in (2) is an asymptotic value valid when both d_x and d_y are small, and we will refer to it as *Hannan's asymptote*. It represents the maximum available embedded element efficiency for a dense array. The term embedded *efficiency* applied to (2) is justified as long as $e_{emb}<1$, corresponding to both d_x and d_y being smaller than $1/\sqrt{\pi}\lambda$=0.56λ, at the same time.

For periodic infinite array, the embedded element efficiency does not depend on the position of the element. In this case there is a relationship between the so-called active reflection coefficient and the embedded element pattern (see eq. (14) in [5]). The active power reflection coefficient R(γ_x, γ_y) is defined as the reflection coefficient at the single port when all the array elements are simultaneously excited with phasing (γ_x, γ_y). From [5, eq. (14)] (1) can be calculated as follows:

$$e_{emb} = 1 - \sum_{p,q=1}^{\infty} \left|S_{pq}\right|^2 = 1 - \frac{1}{\pi^2}\int_0^{\pi}\int_0^{\pi} |R(\gamma_x,\gamma_y)|\, d\gamma_x d\gamma_y \qquad (3)$$

which is approximately valid also for very large truncated arrays for elements far from the edges. It should be emphasized however that (3) can easily induce mistakes, if erroneously applied, especially for dense arrays. As a matter of fact, in dense arrays there is a region of the (γ_x, γ_y) integration space where the array does not radiate, and therefore the active array elements are mismatched, giving $|R|$ close to 1. The integration over this region gives a fundamental contribution to reducing the embedded element efficiency. Integrating only over the region *where the dense array radiates* – which is sometimes done - is therefore wrong. Furthermore, we emphasize that (3) becomes invalid for medium size to small arrays as well as to sparse arrays, where indeed (1) is still applicable. It is also clear that Hannan´s asymptote in (2) can be derived from (3) by assuming that R is 0 in real space and 1 outside real space. Thus, Hannan´s asymptote is an upper limit. It will degrade dependent on how the elements are matched.

The embedded element efficiency has been verified by measurements for a singly-excited element in a dense focal-plane array, see Fig. 11 in [2]. It becomes very small for small element spacing, and is already e_{emb} *= π/4 = 79%* for planar arrays with 0.5λ element spacing. When many ports are excited, the embedded element efficiency concept evolves naturally into the decoupling efficiency concept [2].

III. NUMERICAL CALCULATION OF REALIZED GAIN

The numerical example is an array of 32 × 32 open-ended waveguides, each with aperture dimensions a *= 0.505λ* and b

= *0.067λ* along x-and y-directions, respectively. Thus, $M = 32$ and $N = 32$ in Fig. 1. Since a>>b , the apertures are actually slots. The element spacing is fixed to d_x =0.67λ in x-direction (H-plane) and it varies from 0.1λ to 10λ in y-direction (E-plane). Thus, when the element spacing is 0.1λ, parallel slots are extremely close to each other, and when the element spacing is 10λ they are very far apart.

The maximum available gain of the array is plotted as the straight solid diagonal line in Fig. 2 marked "maximum available". The realized gains of the whole array and of its elements have been found by three different full-wave numerical approaches:

a) *Infinite array approach:* This consists of simulating a unit cell of the array with periodic boundary conditions, and corresponds to exciting all waveguide elements with the same amplitude and phase. The actual finiteness of the arrays due to the 32 × 32 elements is taken into account by a windowing of the infinite array, thereby neglecting additional fringe effects due to the actual truncation which change the embedded efficiency close the edges. This realized gain is plotted as the curve marked "*infinite array approach*" in Fig. 2.

b) *Embedded element approach:* An element in the center of the array is simulated when all the other elements are present and terminated. Our elements are rectangular waveguides, so a termination means that there is an ideally matched load at the end of the waveguide. The results of the simulation are the far-field function, directivity, and realized gain G_{emb} of the embedded element. The realized gain of the total array is $G_{arr} = MNG_{emb}$. This is plotted as the curved marked "*MN × embedded element gain*" in Fig. 2. The curve is seen to be almost identical to the "*infinite array approach*", as expected. The discrepancies when d_y is between 0.3λ and 0.8λ are due to the numerical accuracy.

c) *Isolated element approach:* This is one open-ended waveguide in an infinite ground plane. The results are its input S-parameter, far-field function, and realized gain G_{iso}. An approximate array gain can be obtained from $G_{arr} = MNG_{iso}$. This result is shown as the curve called "*asymptote from isolated element gain*" in Fig. 2. The directivity of the isolated element can in our case also be found analytically to be 5.2 dBi from the analytical far-field function of a single slot [11].

Fig. 2 shows that the gain simulated with the infinite array method approaches the isolated element asymptote in a slowly oscillating manner for large d_y. The slow convergence is due to all the grating-lobes appearing with periodic intervals of 1λ. They have a large effect in E-plane because the isolated element pattern of a slot is omnidirectional in E-plane. Therefore, the graph shows dips that appear with regular intervals when d_y increases, corresponding to the sudden appearance of a new grating-lobes along the array in E-plane.

The effect on the array gain due to the presence of grating-lobes can easily be modeled by correcting the maximum available realized gain by the grating efficiency e_{grt}, as defined in Chapter 10 in [11], i.e.,

$$G_{arr} = \frac{4\pi MNd_x d_y}{\lambda^2} e_{grt} . \quad (4)$$

When evaluating the grating efficiency we need to know the far field pattern of the element. Here, we simply use the fact that the far-field function of an isolated slot is uniform in E-plane, and then the grating efficiency expression becomes completely analytical. The result is the curve marked "*maximum with grating efficiency*" in Fig. 2. We see that this is able to model the periodic variation of the realized gain very well, except precisely at d_y/λ where grating-lobes emerge, i.e., at each multiple of wavelengths. Therefore, the grating efficiency in [11] gives a good understanding of losses in directivity due to grating-lobes. A complete numerical study of the grating efficiency is given in [12].

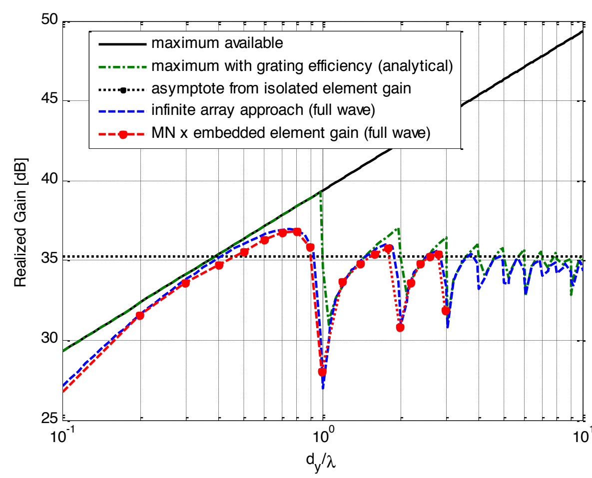

Fig. 2. Realized gain of 32 × 32 element regular array of open-ended waveguides in infinite ground plane when the element spacing in H-plane is dx=0.67λ for different element spacings in E-plane, by different methods.

IV. Calculation of Embedded Element Efficiencies

Let us look more carefully at the realized gain for small element spacing d_y/λ, or rather the related embedded element efficiency. Fig. 3 shows the embedded element efficiencies evaluated in different ways. The continuous straight black line shows Hannan's asymptotic formula in (2). The continuous blue line with squares is obtained by using the definition of the embedded element efficiency for a lossless multi-port antenna in (1). The latter is evaluated numerically from all S-parameters obtained from the embedded element simulations described in b) in Sec. III, so the curve is marked "*from all S-parameters*". Note that this approaches -1 dB for large element spacing due to the mismatch factor of the isolated element. The dash-dotted black line shows the embedded element efficiency when correcting Hannan's asymptote by the mismatch factor of the fully-excited array, i.e., the mismatch factor when all elements are equally excited. Finally, the dotted red curve marked "*gain per element minus 6 dB*"., is obtained as follows: taking the result from the infinite array approach, dividing it with *MN* to get the realized gain per element, and removing the 6 dB directivity of a single embedded element in a dense array, see explanation to (2).

The results show that Hannan's asymptote represents the highest efficiency for all d_y/λ. Thus, it is the fundamental limiting factor describing the fact that the embedded element will have low radiation efficiency in dense arrays. When we correct this with the mismatch factor we get exactly the same result as obtained from the full wave simulation of "gain per element minus 6 dB". This shows that it is very easy to correct for Hannan's asymptote to get the actual realized gain in dense regular arrays, but we need then to know the S_{11} of the array elements when all the elements are excited. Finally, we see that the actual embedded element efficiency evaluated from all S-parameters using (1) is lower than the four other results, and approaches them for small d_y/λ.

Eq. (1) can never be larger than unity (0 dB) by definition. However, the three other curves can be larger than unity when the assumptions $d_x << \lambda$ and $d_y << \lambda$ for which they are evaluated, are not satisfied. This assumption is implicit also in the full wave efficiencies when assuming that the directivity of the embedded element is *6 dBi*. If the directivity is larger, the computed value of the embedded element efficiency will be lower and thereby satisfy the physical requirement of $e_{emb} \leq 1$. Thus, it is reasonable to believe that embedded elements in dense arrays always will have directivities equal to or larger than 6 *dBi*. The latter is a minimum directivity limit.

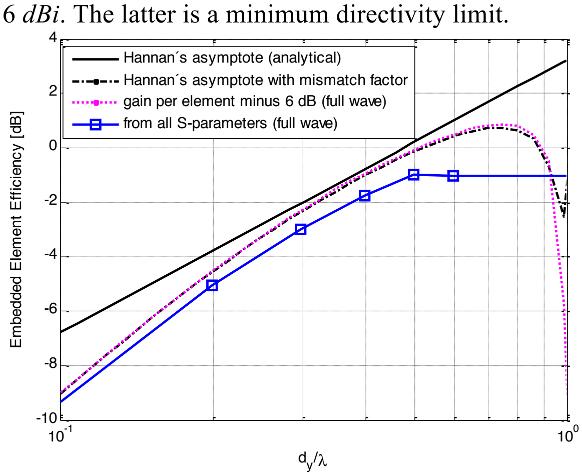

Fig. 3. Embedded element efficiency of the same 32x32 array as in Fig. 2, evaluated by different accurate and approximate methods.

V. Conclusion

We have shown by a numerical analysis that the embedded element efficiency fully explains the so-called element-gain paradox as theoretically predicted by Hannan [5] and Kahn [6]. Thus, when we have MN equal array elements, the array gain can be found as MN times the realized gain of the embedded element when the embedded element efficiency is included properly in the realized embedded element gain.

We have also shown that the embedded element efficiency becomes very low in dense arrays with element spacing smaller than *0.5* λ. It can for this case be approximated well by Hannan's asymptote in (2), and even very well if this is corrected by the mismatch factor with all elements excited.

The numerical study has been done for an infinite array of open-ended waveguides in an infinite ground plane, in which each element of a central subarray of 32×32 elements have been excited with incident waves of the same amplitude and phase. Still, we believe that the conclusions are generally valid also for other types of arrays. In fact, the theory behind the embedded element approach is general and very simple and easy to understand. The elements of the all-excited 32×32 array are generally not matched in our study. Therefore, we have corrected Hannan´s asymptote with the mismatch factor, thus leading to an excellent agreement with the full-wave simulations. As a result, if the waveguides were matched for each d_y/λ, the embedded element efficiency will follow Hannan´s asymptote without any correction, for small d_y/λ.

It should be noted that the 32x32 element array get a width $w=32\ d_y/\lambda$ in H-plane. In E-plane the width is only 3.2λ at the smallest value of $d_y/\lambda=0.1$ used in Figs. 2 and 3. The realized gain of such physically small array may be slightly reduced by the general decoupling efficiency in [2] due to losses in the dummy surrounding non-excited elements. This will decrease with width w for small d_y/λ, but the effect must be small because it is not yet visible in the curves.

The embedded element efficiency is therefore the major factor contributing to the realized gains of dense array antennas. Unfortunately, this is not so well known, although the understanding of this fundamental limitation dates back to Hannan in 1964 [4]. To account correctly for it, arrays must always be simulated with source impedances on excited ports and loads on non-excited ports.

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