THESIS FOR THE DEGREE OF DOCTOR OF PHILOSOPHY

LOST IN LOCALISATION

searching for exact results in supersymmetric gauge theories

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Abstract

This thesis deals with one of the very basics of theoretical physics: computing observable quantities. In the language commonly used to describe the subatomic world, gauge theories, this problem is far from trivial as the observables are expressed in terms of infinite-dimensional integrals. This holds true even in supersymmetric gauge theories, but in some cases, this additional symmetry may be used to reduce the infinite-dimensional integrals to finite-dimensional ones − which naturally simplifies the expressions significantly. This thesis revolves around one of these techniques: *Localisation.*

In general, this poses strict requirements on the theory as well as the manifold on which the theory is placed. However, by first twisting the theory so as to obtain a topological field theory, localisation can be carried out on any background, whereas one otherwise is confined to manifolds with a large amount of symmetry such as for example d-dimensional spheres. The explicit calculation of the path integral is nonetheless in general still complicated even after localisation, and it is only in certain limits that it may be computed exactly. For example, simplifications often occur in the limit of infinitely many colours (the large N limit).

Of the five papers appended to this thesis, the first three deal with topological twists of maximally supersymmetric Yang-Mills theory and (2,0) theory, whereas the last two revolve around the behaviour of the free energy of massive ABJM theory in the large N limit.

Keywords: Localisation, Supersymmetric Gauge Theory, Topological Field Theory, Extended Supersymmetry, Large N-techniques, Matrix Models

This thesis is based on the work contained in the following papers:

Paper I:

Tunneling solutions in topological field theory on $R \times S^3 \times I$
Louise Anderson and Måns Henningson
Journal of High Energy Physics, vol 1202.

Paper II:

Five-dimensional topologically twisted maximally supersymmetric Yang-Mills theory
Louise Anderson
Journal of High Energy Physics, vol 1302.

Paper III:

The trouble with twisting (2,0) theory
Louise Anderson and Hampus Linander
Journal of High Energy Physics, vol 1403.

Paper IV:

Quantum phase transitions in mass-deformed ABJM matrix model
Louise Anderson and Konstantin Zarembo
Journal of High Energy Physics, vol 1409.

Paper V:

ABJM theory with mass and FI deformations and quantum phase transitions
Louise Anderson and Jorge G. Russo
Journal of High Energy Physics, vol 1505.

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Contents

"There is no such thing as 'the unknown',
only things temporarily hidden,
temporarily not understood"

– Captain James T. Kirk

Part I

INTRODUCTORY CONCEPTS

Chapter 1:

Introduction

Elementary particles. The smallest constituents of matter as we know it: three generations of quarks (up and down, charm and strange, and finally, top and bottom), three generations of leptons (the electron, the muon and the tauon, together with their corresponding neutrinos), and their anti-particles. Everything around us is made up of these tiny particles, known as *fermions*. They interact with one another through four fundamental forces: gravity, electromagnetism, weak- and strong interaction, which are mediated by force-bearing particles known as *bosons*: the graviton, the photon, the W- and Z-bosons of the weak interaction and the gluons which mediate the strong interaction. And finally, the now most well-known of them all, the Higgs boson, representing the interaction with the Higgs field through which all of the elementary particle masses are created.

The ingredients appear simple, yet we still do not fully understand the recipe for how they all interact. If we for the moment neglect gravity, these basic constituents of nature are described to an amazingly good extent by the Standard Model of Particle Physics. However, there are still many things which remain unknown. For example, we do not know what makes up 85% of our universe (this is commonly known as *dark matter*[i]), or, even amongst our "ordinary matter", as soon as things become both small and heavy, we do not know how they work. The most obvious examples of this are probably black holes. How do they work? Do they have an internal structure, and what does that look like? These truly are exotic objects where our intuition fails us gravely, and creating apparent paradoxes involving them appear to be a favourite pass-time amongst physicists, if the number of them that have been conceived is any indication.

So we truly live in exciting times with a lot of unanswered questions, and it is clear that we need new theories to be able to fully describe the world. The best candidate for such a "theory of everything" which we currently have is *string theory*, or rather, M-theory − an incredibly complex theory containing (amongst other things) something known as *supersymmetry*; a symmetry relating bosons to fermions, and vice versa. This inevitably introduces more particles to the theory, which for our purposes is very good - our current theories obviously do not contain everything we see in nature.

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[i]If one also include "dark energy", the percentage of unknown constituents of the world rises to 95%, but we do not know that this "dark energy" actually exist as anything more than an extra mathematical term in Einsteins equations. Dark matter is however another kind of "physical matter", which just happens to interact only through gravity.

However, M-theory, or even any of the five kinds of string theories that exist, is not the focus of this thesis. This rather deals with a closely related set of theories: *supersymmetric field theories*. These can be thought of as some low-energy limits of string theory, and understanding these theories well is a necessary step on the way to fully understand the fundamental "theory of everything" one day.

1 From determinism to probabilities to this thesis

In classical physics, we are often faced with some system in a certain initial state, and our goal is to give a prediction for what will happen. Not *may* happen, *will* happen, with definite certainty. If we let go of a pendulum in the highest position, it will fall and start to oscillate with a fixed period[ii]. This nice and deterministic behaviour that we are used to from the macroscopic world doest not appear to be the way the microscopic world works.

In quantum theories, which we must use to describe the fundamental particles and interactions, we cannot say anything for certain, rather only give *probabilities* for a specific outcome. This is not just a case of our theories not being good enough, or that we are not able to specify initial conditions with enough accuracy. This really is the way nature works. If two particles are scattered off each other (i.e. forced to collide with one another), we cannot with certainty say what the products of that reaction will be, only with what *probability* the end result will be particles x and y. The mathematical object encoding the probabilities for these different outcomes is known as the S-*matrix*, and it depends on the details of the theory in question. These details may be specified by a Hamiltonian density, or, more commonly used, a Lagrangian density [1]. The integral of the latter is known as *the action* of the theory, and in classical field theory, the trajectory of a particle is given by the path that minimises this action.

This is no longer true in quantum mechanics. Rather, *all* trajectories are taken, and any observable is then computed by summing up the contributions to it from all possible paths. It is just that those that contribute the most are those close to the classical path. This is known as the path integral formalism, where (expectation values of) observables are written as:

$$\int \mathcal{D}\Phi \; \mathcal{O} \; e^{\,-i\mathcal{S}(\Phi)/\hbar}, \tag{1.1}$$

where Φ denotes all fields in the theory, and depending on which observable one wishes to compute, one may insert different operators $\mathcal{O}$. It is in this way the observables of quantum field theories are encoded: infinite-dimensional integrals. As such, even if we are able to figure out enough about the theory that we are able to write down the partition function (i.e. the simplest observable, given by (1.1) with $\mathcal{O} = \mathbb{1}$), in general, these integrals are not solvable. Some methods which may be used to simplify these − some

[ii] Granted, this applies for small angles in the absence of friction.

times to the extent that we are actually able to perform explicit calculations – will be the subject of this thesis.

This is the very core of theoretical physics: to compute quantities from theory that may serve as a guide for experimentalists on how to design experiments, and hopefully, these theoretical predictions will then be tested against the experimental results. In the best of worlds, the results will agree with one another, and if they do not, it forces theory (if the measurement is correct) to change, expanding the scientific knowledge of humanity. In some cases, the distance between theory and experiments is too vast to be overcome in the near future. For example, it took nearly 50 years from when the Brout-Englert-Higgs-mechanism was suggested in the two 1964-papers [2, 3], until it was experimentally observed with the discovery of the Higgs boson in 2012.

It is a fact that experiments dealing with the very smallest constituents and the fundamental laws of the world ironically enough require huge equipment, and they are immensely intricate and thus may require more time to set up than what the theorists may need to develop new mathematics. This is the situation we are in today: any prediction made from string theory is expected to be experimentally testable at energy scales close to the Planck scale – 10^{14} times the energy accessible at the Large Hadron Collider (LHC) at CERN. We may thus have to wait a long time for experimental observations of strings. The lightest supersymmetric particles in supersymmetric extensions of the Standard Model could however, be visible within the next year if we are lucky.

The theories considered in this thesis will not directly give rise to any predictions which may be experimentally tested, now or in the future. They are toy models which provide us with a playground where we may test and develop techniques for, amongst other things, computing the complicated expressions which describe the observable quantities. The techniques used in this thesis are all related, in one way or another, to the concept of *localisation*, where symmetry is used to freeze out modes of the path integral, and, in a manner of speaking, making it infinitely simpler, reducing it to a finitely-dimensional integral instead. It far from solves all of our problems, but it allows us to expand our knowledge one step further. It allows us to make the image on the puzzle piece we are currently looking at slightly sharper – it does not allow us to fully finish the puzzle. But than again, being a physicist would be terribly boring if the puzzle would ever be finished, so making the pieces a little clearer is a good enough reason for me to want to study it. And, making the individual pieces clearer may help us realise which ones of them should actually fit together, and thus it takes us one small step towards understanding it all.

2 Outline

Before we can properly explain how localisation works, however, we must introduce the language in which theoretical physicists like to describe the world: in terms of quantum field theories, which was briefly mentioned above, and symmetries. This latter concept is vital for the formulation of our theories, and for localisation especially, so it will be our starting point. After this is introduced, we will move on to introducing the quantum field theories which have been investigated in the papers appended to this thesis: maximally supersymmetric Yang-Mills theory, the six-dimensional (2,0) theory, and the three-dimensional ABJM theory. In the final part of this thesis, the techniques used to investigate these different theories will be presented: topological twisting, localisation, as well as some comments on techniques which becomes available in the large N limit.

Papers I − III all deal with topological twists of the two first theories mentioned, which give rise to topological field theories. One of the major motivations for studying such theories is that the path integral in these cases may be localised and in this sense, they are much simpler than in the original theories. In the two last papers, (IV and V), we approach localisation in another way. We here place the theory on a particular manifold with enough symmetry, (in this case $\mathbb{S}^3$), and use this to reduce the infinite-dimensional path integral to a matrix model. If one then lets the number of colours tend to infinity (i.e. the large N limit), this matrix model may then be computed exactly in a certain limit.

Chapter 2:

Symmetries

In the previous section, we said that localisation uses symmetry to freeze out modes of the path integral. But what do we mean by “symmetry”? And when do we have “enough” in the sense that it suffices to simplify the partition function?

On any quantum field theory, we have some fundamental restrictions for it to be a physically meaningful theory. For example, if we observe some apparent symmetries in the world around us, we must require that our theory respects these symmetries. This means that we require that the Lagrangian (or Hamiltonian, or S-matrix, or whatever language one chooses to write the theory in) should be invariant under these. Some of these symmetries we recognise from our everyday life, such as rotational- or translational symmetry, whereas some are more unintuitive (such as Lorentz invariance or even supersymmetry). In the sections below, we shall attempt to give a brief introduction to all of these.

1 Space-time symmetries

The outcome of an experiment does not depend on where it is carried out. Or naturally, if the experiment is of the type “measure if it rains today”, the result of such an experiment may vary with location, but if we want to know something more fundamental, say, “what is the speed of light?” or “what is the mass of the electron?” (or something more macroscopic, like “how much energy is required to heat one litre of water from 0°C to 25°C?”), none of these experiments should depend on if they are carried out here in Gothenburg, in my childhood home at the island of Öckerö, or halfway across the world. This seems obvious to most people. Physics would indeed not be a very good description of the world if the very fundamental description of the world varied as you moved around the Earth, and it would certainly not be as useful. However, this property of the world, that *the physical laws appear invariant under translations and rotations*, is actually a very deep statement. Furthermore, it is quite intuitive that the outcome of experiments should not depend on if they are carried out now, tomorrow, or six months from now (or for that matter, ten thousand years from now); we still expect the same result. That is, *physical laws are invariant under time translations.*

From a mathematical perspective, these invariances may be stated by requiring that the equations should not change under transformations

$$\begin{aligned} x^i &\to x^i + a^i && \text{(spatial translations)} \\ x^i &\to \Lambda^i{}_j x^j && \text{(spatial rotations)} \\ t &\to t + a^0 && \text{(time translations)}, \end{aligned} \tag{2.1}$$

where x is a spatial vector with components x^i, i taking the values 1,2 and 3, and a^i denoting the amount each coordinate changes. Λ here is a matrix with elements $\Lambda^i{}_j$, given by the parameters of the rotation. We may note the similarity of the two translations, and by grouping the time and space coordinates together into one *four-vector*, x^μ, (where $\mu \in \{0, \dots 3\}$, and the 0-component denotes the time coordinate), these may be written as

$$x^\mu \to x^\mu + a^\mu. \tag{2.2}$$

These transformations are often considered on an infinitesimal level, where they are said to be generated by some g (known as the *generator*) together with a *parameter* a. The generators of the symmetry then form a Lie algebra $\mathfrak{g}$ with the corresponding group G, which is known as the *symmetry group*.

For the translations, the generators g are often denoted P_μ, and, in a coordinate representation, they are proportional to the derivative and given by $-i\partial_\mu$ in Planck units[i]. The infinitesimal change δ is in general given by $ia \cdot g$, and so in the case of translations, we find

$$\delta x^\mu = i\left[-ia^\nu \partial_\nu, x^\mu\right] = (a^\nu \partial_\nu)x^\mu - x^\mu a^\nu \partial_\nu = a^\nu(\partial_\nu x^\mu) = a^\mu, \tag{2.3}$$

just as expected from equation (2.2). The generators P_μ all commute with one another, so the symmetry group of translations, $G_{\text{tr.}}$, is *Abelian.* For example, in four-dimensional (Minkowski) space-time, which we denote by $\mathbb{E}^{1,3}$, the translation group is given by $\mathbb{R}^{1,3}$.

Similarly, for spatial rotations, we may consider the infinitesimal change in x^i by decomposing the matrices $\Lambda^i{}_j = \delta^i_j + \omega^i{}_j$, where δ^i_j is simply the identity (Kroenecker delta) and ω^{ij} is an anti-symmetric matrix, (which follows from the fact that the matrix Λ^i_j must preserve lengths[ii]). The generators of rotations around the k-axis are denoted by J_k, and, again, in a coordinate basis, may be written $J^k = -\epsilon^{ijk} J_{ij} = \epsilon^{ijk} x_{[i} P_{j]}$, where ϵ^{ijk} is the totally antisymmetric Levi-Civita symbol, P_j is the generator of translations defined above, and the bracket in the subscripts denotes antisymmetrisation[iii]. From this, it is clear that the generators of rotations in the i,j-plane, J_{ij}, are antisymmetric in i,j.

[i] Notice that this gives us the well-known commutation relations $[x_\mu, P_\nu] = i\delta_{\mu\nu}$, which will be convenient to have as we move on to rotations (our units are chosen such that $\hbar = 1$).

[ii] That is, $\delta(x_i x^i)$ must vanish. Explicitly writing this out, we find that it leads to $(\omega^{ij} + \omega^{ji})x_j x_i = 0$, or equivalently: ω_{ij} must be antisymmetric.

[iii] The convention used herein will be $a_{[i}b_{j]} = \frac{1}{2}(a_i b_j - a_j b_i)$.

This gives us the infinitesimal change in x^i as:

$$\begin{aligned} \delta x^i = i[\omega^{jk} J_{jk}, x^i] &= -i\, \omega^{jk}\, [x_{[j} P_{k]}, x^i] \\ &= -\, \omega^{jk}\, x_{[j} \delta^i_{k]} = \, \omega^{ij}\, x_j, \end{aligned} \tag{2.4}$$

which is again just what we expected. Furthermore, the generators of spatial rotations satisfy the commutation relations

$$[J_i, J_j] = i\epsilon_{ijk} J^k, \tag{2.5}$$

which is the Lie algebra of the group $SO(3)$. This is the group of orthogonal 3-dimensional matrices with positive determinant − which indeed are three-dimensional rotations. Hence the symmetry of spatial rotations is just $SO(3)$.

2 Lorentz invariance

However, there are more symmetries in the world around us. Imagine yourself sitting on a train about to depart. There are other trains around you in the station, and in those first few moments, it is very difficult to determine if you are aboard the moving train or if your train is stationary, and the train next to you is the one moving. If the train started moving very rapidly, that is, with a large acceleration, you would probably be able to feel it and determine that it was indeed your train that was moving, but when you are at constant velocity, it actually is impossible to carry out any experiment at all to determine if it is you that are moving, or the other train[iv]. This is known as the principle of relativity: the outcome of an experiment does not depend on if it is carried out in a laboratory stationary on the Earth, or aboard a train moving with constant velocity. Or, in more mathematical terms: *the laws of physics are the same in all inertial frames*. This was first stated by Galilei in 1632, but its implication was not understood until the early 1900's, when Albert Einstein formulated the special theory of relativity. If there are no experiments you can carry out to determine whether you are moving with constant velocity, or if you are stationary and your surroundings are moving, then, what would happen if you tried measuring the speed of light in any of these reference frames? Einstein's answer to this was that the speed of light is a universal constant, and you should get the same answer regardless of in which inertial frame you make the measurement. Taking this, together with the principle of relativity, as our basic axioms, it inevitably forces time and space to behave in rather contraintuitive ways as one travels at speeds approaching the speed of light.

Suppose we have two observers, say Alice and Bob. Alice moves with the uniform velocity v along the x-axis relative Bob, and at time $t = 0$, the origins of the two reference frames overlap. At this time, a spherical wavefront is emitted from the origin (of the two

[iv] At a train station, the acceleration is often so small that constant velocity is not a completely unreasonable approximation.

frames). Since we require Alice and Bob to both measure the speed of light to c, this forces the coordinates in the two inertial frames, x_{Alice}, t_{Alice} and x_{Bob}, t_{Bob}, to be related by a *Lorentz transformation*, which may be written as:

$$\begin{aligned} t_{Alice} =& \gamma\left(t_{Bob} - \frac{v\, x_{Bob}}{c^2}\right) \\ x_{Alice} =& \gamma\left(x_{Bob} - v\, t_{Bob}\right), \end{aligned} \tag{2.6}$$

where $\gamma = \frac{1}{\sqrt{1-v^2/c^2}}$. This was one of the greatest achievements of the last century, and by this rather simple thought, that there is no privileged inertial frame, Albert Einstein united the ideas of space and time into one single concept: *spacetime*.

In general, the Lorentz transformations can be seen as spacetime rotations, and they had actually previously been observed as a peculiar symmetry of Maxwell's equations of electrodynamics by Hendrik Lorentz and Henri Poincaré. It was not until the special theory of relativity however, that their implications were properly understood. We will from now on again use Planck units where $c = 1$, and in these units, we may write the Lorentz transformations as:

$$x^\mu \to \Lambda^\mu{}_\nu\, x^\nu, \tag{2.7}$$

where the matrices $\Lambda^\mu{}_\nu$ are, just as for spatial rotations, subject to certain constraints to force them to be "good" rotations. In the case of spatial rotations, this constraint was that they needed to leave distance invariant. In this case, the transformation must leave a "four-distance", known as proper time, invariant. They may again be decomposed into $\delta^\mu_\nu + \omega^\mu{}_\nu$, where $\omega^\mu{}_\nu$ is an infinitesimal transformation. The generators of this symmetry, $J_{\mu\nu}$ may again be defined in an equivalent fashion as for the spatial rotations, that is:

$$J_{\mu\nu} = ix_{[\mu}\partial_{\nu]}, \tag{2.8}$$

which satisfies the commutation relations

$$[J_{\mu\nu}, J_{\kappa\lambda}] = \eta_{\kappa[\nu}J_{\mu]\lambda} - \eta_{\lambda[\mu}J_{\nu]\kappa}. \tag{2.9}$$

The only difference from the commutation relations of generators of spatial rotations are that the metric here is not simply the identity matrix, but rather the Minkowski metric $\eta_{\mu\nu} = \text{diag}(-1,1,1,1)$. This means that the $J_{\mu\nu}$ here generate the non-compact group $SO(1,3)$, which is known as the Lorentz group. Note that spatial rotations are contained as a subgroup of this group.

We have now seen that the laws of physics are invariant under both *spacetime translations* and *spacetime rotations*. Together, these form the *Poincaré group*, which is defined as the semi-direct product of space-time translations and the Lorentz group. All meaningful physical theories must be left invariant by this group, otherwise, well, an experiment aboard a train leaving Gare-du-Nord in Paris tomorrow would not give the same results as if it was carried out here, right now. And that would obviously be bad. Furthermore, this invariance will have deep consequences for the theory. The invariance under the Poincaré group gives rise to the conservation of energy, momentum, and angular momentum of the theory. This is known as Noether's theorem: For every continuous symmetry of the theory, there is a conserved current, with a corresponding conserved charge [4].

It should here be pointed out that in the special case where the theory does not contain any scale, there is a more general possibility known as *Conformal invariance*. This preserves angles, but not lengths. After this, that is it. In the sixties, Sidney Coleman and Jeffrey Mandula proved a no-go theorem: any quantum field theory with a "nice" S-matrix can at most have a symmetry described by a Lie algebra which is the direct sum of the Poincaré algebra (or, in the absence of scales, the conformal algebra) with an internal symmetry algebra [5]. If the algebra was any larger, this would over-constrain the S-matrix, allowing for scatterings to take place only in certain angles for example, which would obviously be unphysical.

Naturally, the fields of the theory transform under these symmetry transformations, but they will transform in different ways depending on their properties. For example, a scalar field, corresponding to a particle with spin 0, does not mix with a spin 1/2-field (spinor).

Consider a general state $|\Psi\rangle$ with some spin s, and let us express this as a linear combination of some basis, given by the set $\{|\Psi_i\rangle\}$. Let us now perform a Poincaré transformation. This acts on the basis states as

$$|\Psi_i\rangle \ \rightarrow \ |\Psi_i'\rangle = \mathcal{P}_i{}^j|\Psi_j\rangle, \tag{2.10}$$

and takes the state $|\Psi\rangle$ to some $|\Psi'\rangle$. This new, transformed state $|\Psi'\rangle$ can naturally be expanded in terms of the basis $\{|\Psi_i'\rangle\}$. If the two sets of basis states, $\{|\Psi_i\rangle\}$ and $\{\mathcal{P}_i{}^j|\Psi_j\rangle\}$ span the same space for all $\mathcal{P}_i{}^j$ in the Poincaré group, we say it forms a *representation* of the group. If this is true, it means that our spin-s state will still belong to the same representation after the Poincaré transformation, and we may as well express it in terms of the original basis, $\{|\Psi_i\rangle\}$. If no non-trivial subspace of span$\{|\Psi_i\rangle\}$ closes into itself under the action of the group in question, this representation is furthermore said to be *irreducible*.

Different kinds of fields transform in different representations of the group, and the different irreducible representations of the symmetry group therefore contains all the necessary information to determine the possible field content of the theory. In 1939, Wigner brought some order to chaos when he classified all irreducible representations of the Poincaré group by using his now-famous *little group*. In four dimensions, these representations may be fully specified by the mass, $m \geq 0$, and the spin, $j \in \{0,1/2,1,2/3,\dots\}$ (or in the case of $m = 0$, helicity $j \in \{0, \pm 1/2, \pm 1, \pm 2/3, \dots\}$) [6].

3 Superspace symmetries

However, as opposed to what the Coleman-Mandula theorem would make us believe, the game does not end here: there are theories with yet another kind of symmetry, one which relates fermions to bosons and vice versa. That this sort of symmetry is allowed is actually highly non-trivial, but there is a small loophole in the Coleman-Mandula theorem: supersymmetry is generated by *fermionic operators*, spinors rather than Lorentz tensors, and the result is that they do not form a Lie algebra, but rather a *super-Lie algebra*. So, we still have a chance to move onwards from here[v]. However, after this, it appears to end for real. Haag, Lopuszanski and Sohnius proved that the most general form of a graded Lie algebra is that where the fermionic generators have spin 1/2, that is, supersymmetry [7].

The generators of this fermionic symmetry, Q_α, together with their conjugates $Q^\dagger_\beta$, are said to generate the *supersymmetry algebra*. The anticommutator of $Q, Q^\dagger$ must be given by some conserved, bosonic vector quantity. Such a quantity is highly restricted by the Coleman-Mandula theorem, and the only thing we have is the total momentum of the theory P_μ, such that:

$$\{Q_\alpha, Q^\dagger_\beta\} = -\frac{1}{2} P_M \left(\Gamma^M \Gamma^0\right)_{\alpha\beta}. \tag{2.11}$$

Γ_μ are the gamma matrices generating the appropriate Clifford algebra[vi] in d-dimensional Minkowski space (M,$N \in \{0, \dots, d\}$):

$$\{\Gamma^M, \Gamma^N\} = 2\eta^{MN}. \tag{2.12}$$

The absence of any conserved quantities of higher spin for the right-hand side of equation (2.11) is precisely what rules out supersymmetry generators with higher spin than 1/2.

In the most general form of a supersymmetric theory, there may be several supersymmetry generators, $Q^i, Q^{i\dagger}$, where $i \in \{1, \dots, \mathcal{N}\}$. For $\mathcal{N} > 1$, we say we have *extended supersymmetry*. The symmetry group which acts on the Q^i's is called the R-symmetry group. For $\mathcal{N} = 1$, this is simply given by $U(1)_R$ (since the supersymmetry algebra is left invariant by multiplication of the supercharges by a phase), but for extended supersymmetry, this group may be more complicated.

The algebra of (2.11) tells us more: that the right-hand side of the equation acts on the total momentum of the theory means that it involves all states. Thus, so must the left-hand side. That means that supersymmetry must act on all particles in the

[v] Though any traces of supersymmetry has yet to be detected in the physical world, it is my sincere hope that nature has found this loophole in the Coleman-Mandula theorem too, and has decided to put it to good use.

[vi] We shall herein restrict ourselves to the situation where there are no central charges in the anticommutation relation (2.11). Our conventions for the gamma matrices in d dimensions are, in addition to the anticommutation relation above, $(\Gamma^M)^\dagger = \Gamma^0\Gamma^M\Gamma^0$ and, for even dimensions, a chirality matrix given by $\Gamma_* = i^{d/2-1}\prod_{i=0}^{d-1}\Gamma_i$.

theory, pairing to each boson a fermion of the same mass, and vice versa. This endows supersymmetric theories with several seemingly magical properties. For example, if the vacuum state in a theory is supersymmetric, its energy must be precisely zero (fermionic and bosonic contributions to the zero-point energy cancel exactly!). The renormalisation properties of supersymmetric theories are also significantly simplified by seemingly magical cancellations of divergencies.

Even though supersymmetry in the form described here is not present in the world around us (we know for example that there is no boson with the same mass as the electron present in nature), it may be spontaneously broken. That is, the Lagrangian description of the theory may be left invariant by the symmetry, but not the vacuum state. This would mean that the underlying theory could still contain supersymmetry, and testing models of this type is the next big mission for the LHC.

Part II

THEORIES

Chapter 3:

Abelian gauge theories

The properties of a theory are encoded into the Lagrangian, $\mathcal{L}$. If we have this, we can compute observables, find equations of motion and investigate the symmetry properties of the theory. However, creating such a Lagrangian is not a trivial task. It must be invariant under all symmetries we wish our theory to have, such as for example the Lorentz symmetry we described in chapter 2. The theories describing our world turn out to not only be invariant under global symmetries such as those which have been introduced so far, but also under a kind of *local* symmetries, known as *gauge transformations* (or gauge symmetries). We shall see that such symmetries arise whenever there is a massless vector field present, which is just the situation in the world around us. Actually, we have four kinds of so-called "gauge bosons" originating from such fields: the gluons, the photon, and the $W^{\pm}$ and Z-bosons, but three of them are massive due to spontaneous symmetry breaking − the Higgs mechanism. The result is that we only see the photon as massless (and gluons, but we do not observe them at all in our everyday lives). Our goal in this chapter is to review how we may explicitly write down Lagrangians and find equations of motion for these kind of theories, known as *gauge theories*.

1 *Lagrangian for a massive vector field*

As mentioned in section 2 in the chapter on symmetries, the representations of the Poincaré group contain vital information of the properties of the theory. Spin one particles are mathematically described by vector fields, which, unsurprisingly, transform in the vector representation of the Poincaré group. There are however two parts to such a vector field: One degree of freedom which describes a scalar, and three which describe a spin 1-field. This can be seen since we can write some parts of A_μ as $\partial_\mu\phi$, which naturally transforms as a vector, but whose only degree of freedom is that of a scalar field ϕ.

Let us attempt to write a consistent Lagrangian for a four-vector A_μ. The possible terms we can create are a mass-term, $\frac{m^2}{2}A_\mu A^\mu$, and two different kinetic terms, $A_\mu\Box A^\mu$, $A^\mu\partial_\mu\partial_\nu A^\nu$. The most general Lagrangian density we can create from these is given by

$$\mathcal{L}_{\text{vector}} = c_1 A_\mu\Box A^\mu + c_2 A^\mu\partial_\mu\partial_\nu A^\nu + \frac{m^2}{2}A_\mu A^\mu, \tag{3.1}$$

where c_1, c_2 are some numbers. Notice that if c_2 were to vanish, this would look like the action of four scalars, where three of them would behave normally, but the action of the fourth would be multiplied by an overall minus sign:

$$\mathcal{L}_{\text{scalars}} = \sum_{i=1}^{3} \phi_i \left(c_1\Box + \frac{m}{2}\right)\phi_i - \phi_4 \left(c_1\Box + \frac{m}{2}\right)\ \phi_4. \tag{3.2}$$

This would mean that the energy of ϕ_4 would have a different sign than that of the other three scalars, and we would have negative norm states in our theory. To evade this problem with negative norm states, we require $c_2 \neq 0$. We then find our equations of motion by varying $\mathcal{L}_{\text{vector}}$ with respect to A_μ and forcing this to vanish, which gives us:

$$c_1\Box A^\mu + c_2\partial^\mu\partial_\nu A^\nu + m^2 A^\mu = 0. \tag{3.3}$$

In general, this has four propagating modes of freedom (three for spin 1, and one for spin 0). How do we remove the scalar degree of freedom?

Consider the spin 0-part of our vector field, where the degrees of freedom arise from an underlying scalar, that is, take $A_\mu = \partial_\mu\phi$. The equations of motion then become:

$$\Big((c_1 + c_2)\Box + m^2\Big)\partial^\mu\phi = 0. \tag{3.4}$$

By choosing $c_1 = -c_2 = 1/2$, this equation forces ϕ to be constant (for $m \neq 0$), thus effectively removing the scalar degree of freedom as desired. By introducing the *field strength*

$$F_{\mu\nu} = 2\partial_{[\mu}A_{\nu]}, \tag{3.5}$$

the Lagrangian may be written in the more well-known form:

$$\mathcal{L}_{\text{vector}} = -\frac{1}{4}F_{\mu\nu}F^{\mu\nu} + \frac{m^2}{2}A_\mu A^\mu, \tag{3.6}$$

with the corresponding equation of motion for A_μ as:

$$\Box A^\mu - \partial^\mu\partial_\nu A^\nu + m^2 A^\mu = 0. \tag{3.7}$$

By contracting the equation of motion with another partial derivative, the two terms with three derivatives cancel, and we are left with the condition $\partial_\mu A^\mu = 0$ (for $m \neq 0$). This will be used to create explicit solutions to (3.7). One way to do this would be to go to the rest frame of the particle, and solve the equations of motion there. However, this approach is not valid when the vector field becomes massless (since there is no rest frame of such a particle), and since our goal is to eventually generalise our reasoning to the case of $m = 0$, we will herein take the more general approach and expand $A_\mu(x)$ in terms of plane waves:

$$A_\mu(x) = \int \frac{d^4p}{(2\pi)^4}\epsilon_\mu(p)e^{ipx}, \tag{3.8}$$

where ϵ_μ is known as the polarisation vector, (conventionally normalised to satisfy $\epsilon^\mu \bar{\epsilon_\mu} = 1$). The condition $\partial_\mu A^\mu = 0$ in momentum space gives us $\int \frac{d^4p}{(2\pi)^4} p^\mu \epsilon_\mu(p)\, e^{ipx} = 0$, which means that we must have $p^\mu \epsilon_\mu = 0$. We now wish to define a set of linearly independent polarisation vectors ϵ^μ. For some momentum $p^\mu = (E, \mathbf{p})$ (satisfying $-E^2 + |\mathbf{p}|^2 = m^2$), we have two transversal polarisations:

$$\epsilon^i = (0, \mathbf{e}^i), \qquad \mathbf{e}^i \cdot \mathbf{p} = 0, \qquad i \in \{1,2\}, \tag{3.9}$$

where we choose to let $\mathbf{e}$ be such that $\epsilon^\mu \bar{\epsilon_\mu} = 1$. There is also one longitudinal polarisation, which when normalised to unity takes the form:

$$\epsilon^{\text{long.}} = \frac{1}{m} \left(|\mathbf{p}|, E\, \hat{\mathbf{p}}\right), \tag{3.10}$$

where $\hat{\mathbf{p}}$ denotes a unit vector along the direction of $\mathbf{p}$. Without loss of generality, we can take $\hat{\mathbf{p}}$ along the z-axis. It is obvious that the longitudinal polarisation vector is invariant under rotations around z, whereas the two transversal polarisations rotate into one another. It is sometimes convenient to take the two basis vectors of these as the *helicity eigenvectors*, which we denote by $\epsilon^\pm$. These have eigenvalues $j_z = \pm 1$ under rotations around z, and are given by:

$$\epsilon^{\pm 1} = \frac{1}{\sqrt{2}} \left(0, 1, \mp i, 0\right). \tag{3.11}$$

2 Massless vector fields and gauge invariance

But, what happens if the field is massless? An obvious guess would be to take the $m \to 0$ limit of the case above, but this limit is ill-defined for the polarisation vector of the state with $j_z = 0$. What happens to this degree of freedom? Well, the answer is that it is unphysical, which we will see in this section. If we study the Lagrangian for a massless vector field,

$$\mathcal{L} = \frac{1}{4} F_{\mu\nu} F^{\mu\nu}, \tag{3.12}$$

we see that this is invariant under the transformation

$$A_\mu \to A_\mu + \partial_\mu \Theta(x), \tag{3.13}$$

where $\Theta(x)$ is some parameter, since such a transformation does not change $F_{\mu\nu}$. This is known as *gauge invariance*, or *gauge symmetry.* So instead of having a Lagrangian which is invariant under only Lorentz transformations, the theory is also invariant under some *local transformations.*

To show that the longitudinal polarisation indeed is an unphysical degree of freedom, and that the dependence of A_μ on it may be taken to vanish by an appropriate gauge transformation, we once more consider the equations of motion and solve these in momentum space. For a massless vector field, the equations of motion from (3.12) are given by:

$$\partial^\nu (\partial_\nu A_\mu - \partial_\mu A_\nu) = 0. \tag{3.14}$$

However, given a solution to this equation, due to gauge invariance, we can always construct a new solution by adding a term $\partial_\mu \Theta(x)$ to A_μ. In momentum space, such a gauge transformation amounts in an arbitrary shift of $A_\mu(p)$ in the direction of p_μ. We can further expand $A_\mu(p)$ into the orthogonal basis given by our two helicity eigenvectors $\epsilon_\mu^\pm$ together with the two linearly independent (lightlike) four-momenta, $p_\mu = (|\mathbf{p}|,\ \mathbf{p}), k_\mu = (-|\mathbf{p}|, \mathbf{p})$[i], such that:

$$A_\mu(p) = c_+\epsilon_\mu^+ + c_-\epsilon_\mu^- + \tilde{c}_p p_\mu + \tilde{c}_k k_\mu. \tag{3.15}$$

However, $\tilde{c}_k$ must vanish due to the equations of motion, and by a gauge transformation, we can arbitrarily shift the value of $\tilde{c}_p$. Thus the only two physical degrees of freedom for a massless vector field is given by the two transversal polarisations: the helicity eigenstates.

As opposed to the previous symmetries that have described in this thesis, the gauge symmetry actually relates physically equivalent states, and so the notion of "symmetry" is slightly misleading. However, any theories hoping to describe our universe must inevitably admit this sort of local symmetry, since we cannot deny the fact that the photon is indeed a massless spin-1 particle, and very present in the world around us. All theories we will study in this thesis will be theories of this kind − *gauge theories*− but for the most part, the gauge group will be non-Abelian. This will be further explained in the sections that follow.

3 Interactions with the gauge field

If we wish to add any term to the Lagrangian not only containing the field strength, but rather the vector field A_μ itself coupled to some quantity J^μ, we must also require it to be invariant under the gauge transformations of (3.13). The infinitesimal change of this term under a gauge transformation is given by:

$$\delta (A_\mu J^\mu) \ = \ \partial_\mu \Theta(x) J^\mu + A_\mu \delta J^\mu. \tag{3.16}$$

However, it follows from the equations of motion for A_μ, which now takes the form $\partial_\mu F^{\mu\nu} = J^\nu$, that J^μ must be left invariant under gauge transformations. This means that we only need to require $\partial_\mu \Theta(x) J^\mu$ to vanish. By assuming the parameter Θ to vanish at the boundary, we may use integration by parts to rewrite this as:

$$\Theta(x)\partial_\mu J^\mu \ = \ 0, \tag{3.17}$$

[i]Again, we can take $\hat{\mathbf{p}}$ to lie along z.

which tells us that J^μ must be some conserved current of the theory, and so the reason for introducing the interaction terms with the notation $A_\mu J^\mu$ should be obvious. This current has a corresponding conserved charge given by $Q = \int d^3x J^0$. In the simplest example of a gauge theory, that is, electrodynamics, this conserved charge is simply the electric charge. The fact that we can indeed have interactions with A_μ, and not only with $F_{\mu\nu}$ is lucky, since it is only by interactions with the gauge field we can create forces with an $1/r^2$ fall-off, such as the Coulomb force. Since the field strength contains derivatives of A_μ, forces originating from interactions with this will have a more rapid fall-off.

A similar reasoning applies to gravity, where we have a massless spin 2 particle − the graviton. In analogy with the field strength of the spin 1-field, there is now a four-tensor $R_{\mu\nu\rho\sigma}$, but to maintain Lorentz invariance, the terms involving only the metric must be of the form $g_{\mu\nu}T^{\mu\nu}$, where $T^{\mu\nu}$ is some conserved quantity. Thankfully, we have such a quantity: the energy-momentum tensor, arising from the symmetry of general covariance.

4 Adding matter

What if we wish to add some matter fields to the Lagrangian? We already know that any interaction terms containing A_μ must be of the form $A_\mu J^\mu$, but what if we wish to add a spinor field (and its conjugate) for example, λ (and $\bar{\lambda}$)? What are the requirements on this new field?

Well, these new terms in the Lagrangian must naturally preserve this gauge symmetry. In general, a kinetic term for some spinor,

$$\bar{\lambda}\Gamma^\mu \partial_\mu \lambda \tag{3.18}$$

is invariant under the global transformation $\lambda \to e^{i\alpha}\lambda$ (which can be thought of as rotating the components of the spinor into one another). However, the gauge symmetry is a local symmetry. Can terms involving fermions be made invariant under a local symmetry as well? The answer to this is yes. By making α dependent on space, we find that the kinetic term transforms as

$$\bar{\lambda}\Gamma^\mu \partial_\mu \lambda \to \bar{\lambda}\Gamma^\mu \left[\partial_\mu + i\left(\partial_\mu \alpha(x)\right)\right]\lambda, \tag{3.19}$$

and so the problems with such a term under gauge transformations lies in the derivative. This is quite natural, since the derivative involves taking the limit $\delta x \to 0$ of the difference of the field λ at two different points, x and $(x + \delta x)$. But, since we may multiply λ at each one of these points with a different phase due to gauge invariance, this limit is not well-defined. If we instead try to build a kinetic term out of the *covariant derivative*[ii],

$$D_\mu = \partial_\mu \; - \; iA_\mu, \tag{3.20}$$

[ii]The convention for the definition of the covariant derivative differs depending on the subject studied. This follows the conventions of [8], and is the choice used in the two last papers appended to this thesis. However, in the first three papers, the covariant derivative is rather defined as $D_\mu = \partial_\mu \; + \; A_\mu$.

we find that such a term transforms as:

$$\bar{\lambda}\Gamma^{\mu}D_{\mu}\lambda \to \bar{\lambda}\Gamma^{\mu}\left[D_{\mu} + i\partial_{\mu}\left(\alpha(x)\ +\ \Theta(x)\right)\right]\lambda. \tag{3.21}$$

And so, if the spinor λ transforms as $\lambda \to e^{-\,i\Theta(x)}\lambda$ under gauge transformations, the Lagrangian

$$\mathcal{L} = -\frac{1}{4}F_{\mu\nu}F^{\mu\nu}\ +\ i\bar{\lambda}\Gamma^{\mu}D_{\mu}\lambda\ +\ \left(m\bar{\lambda}\lambda\right) \tag{3.22}$$

is left invariant under simultaneous transformations of the fields by

$$\begin{aligned} A_{\mu}\ &\to A_{\mu} + \partial_{\mu}\Theta(x) \\ \lambda\ &\to e^{-\,i\Theta(x)}\lambda. \end{aligned} \tag{3.23}$$

(We did not mention the last term in equation (3.22), but such a mass term for the fermions is obviously gauge invariant and should thus be included in the general case.)

By the same reasoning as for the spinor field, we may add terms involving a complex scalar field, ϕ, again transforming as $\phi \to e^{-\,i\Theta(x)}\phi$ to our Lagrangian:

$$\mathcal{L}^{\text{scalars}}\ =\ |D_{\mu}\phi|^2\ +\ V\left(|\phi|\right), \tag{3.24}$$

where $V\left(|\phi|\right)$ is some potential. We may instead consider the kinetic term for ϕ as an "ordinary" kinetic term and an interaction term with the gauge field. This gives us:

$$\mathcal{L}^{\text{scalars}}\ = |\partial_{\mu}\phi|^2\ -\ iA_{\mu}\left(\bar{\phi}\partial_{\mu}\phi - \phi\partial_{\mu}\bar{\phi}\right)\ +\ |\phi|^2A^{\mu}A_{\mu}\ +\ V\left(|\phi|\right), \tag{3.25}$$

where we have obtained a term which looks like a mass term for the gauge field, with mass $m = |\phi|$. So, if the scalar field has a non-zero vacuum expectation value, the gauge field appears massive. This is an example of the Higgs mechanism. Naturally, if we have both a scalar field and spinors, there may be interaction terms between these as well, but we shall not go into further details on such terms here.

5 The gauge field, connections, and parallel transport

There is a natural geometrical interpretation of Yang-Mills theory in terms of principal fibre bundles over some base manifold $\mathcal{M}$, where the gauge field defines a connection on this bundle, E. All fields of the theory are then thought of as forms on the manifold, taking their values in the vector bundle $\text{ad}(E)$ associated to the gauge bundle via the adjoint action of the gauge group G. In this language, a scalar field ϕ corresponds to a zero-form on the vector bundle, a vector field A_{μ} corresponds to a one-form, an antisymmetric rank-two tensor a two-form, and so on. In a more concise notion, we may say that a n-form on $\text{ad}(E)$ belongs to $\Omega^n(\mathcal{M}, \text{ad}(E))$. This is a beautiful way to think of the theory where the analogs to general relativity become more apparent, but for the purpose of this work, we shall not go into this in much depth. The interested reader is referred to [9] for a review of these concepts.

In the previous section, we saw that the pairs $(\psi\ ,\ A_\mu)$ and $\left(e^{-i\Theta(x)}\psi\ ,\ A_\mu + \partial_\mu\Theta(x)\right)$ represent physically equivalent solutions. This can be thought of in the following way: the change in phase of the field ϕ amounts to a local change of basis in the internal space[iii], and the covariant derivative describes parallel transport in this space. That is, the covariant derivative is said to be a connection on this space, which allows us to compare fields at different points in space despite their arbitrary local phases.

Instead taking equation (3.5) as our definition of the field strength, we can define it in terms of covariant derivatives. Notice that:

$$[D_\mu, D_\nu]\psi = [\partial_\mu, \partial_\nu]\psi \ +\ i[\partial_\mu, A_\nu]\psi \ +\ i[A_\mu, \partial_\nu]\psi - [A_\mu, A_\nu]\psi = \ i\psi F_{\mu\nu}, \tag{3.26}$$

for any field ψ. Thus, from now on we will take this as the definition of $F_{\mu\nu}$

$$F_{\mu\nu} = -\,i\,[D_\mu, D_\nu]\,, \tag{3.27}$$

and so the field strength can be thought of as the *curvature* in the internal space. This definition will hold through the generalisation to non-Abelian gauge theories as well.

This language may be familiar from the theory of general relativity, where the Christoffel connection allows us to compare fields at different points in space-time in spite of the fact that we may choose our basis in different ways on different patches. That is, instead of gauge invariance, we here have an invariance under local coordinate transformations. The analog of the curvature $F_{\mu\nu}$ is here given by the Riemann tensor, $R^\lambda{}_{\mu\nu\kappa}$.

[iii]For example in electromagnetism, the basis according to which we measure the charge of the particle.

Chapter 4:

(Maximally supersymmetric) Yang-Mills theory

So far, we have considered a theory with one gauge field A_μ and remaining fields transforming as $\psi \to e^{i\Theta}\psi$. This is simply a $U(1)$ transformation of the field ψ, but what if we instead had some other, non-Abelian, gauge group G?[i] Such a non-abelian gauge theory is known as a *Yang-Mills theory.*

To eventually be able to formulate Yang-Mills theory in a nice fashion, let us first start by considering some basic group theory. Take t_a to be the generators of some representation of the Lie algebra of G, $\mathfrak{g}$. Any such t_a's must satisfy the commutation relations

$$[t_a, t_b] = i f_{ab}{}^c t_c, \tag{4.1}$$

where the $f_{ab}{}^c$'s are known as the *structure constants* of the group. They can be used to obtain the generators of the adjoint representation of $\mathfrak{g}$, T_a:

$$(T_a)^b{}_c \;=\; -i f^b{}_{ca}. \tag{4.2}$$

The Yang-Mills gauge field $\mathcal{A}_\mu$ can in general be thought of as taking values in the adjoint representation, where it is given by the matrix:

$$\mathcal{A}_\mu = A^a_\mu T_a. \tag{4.3}$$

In the case $G = U(1)$, the adjoint representation is one-dimensional (as are all the other irreducible representations), and we find $\mathcal{A}_\mu \propto A_\mu$, as expected (any proportionality constant simply amounts to a field redefinition). However, in more general cases, we have several gauge fields, A^a_μ, that transform into one another under a gauge transformation with some matrix-valued parameter $\Theta(x)$ [10, 11]. We view this parameter as taking values in the adjoint of $\mathfrak{g}$ as well, such that:

$$\Theta(x) \;=\; \Theta^a(x) T_a. \tag{4.4}$$

How do we go about creating a Lagrangian for such a theory? The biggest obstacle we encountered in the Abelian case was to find a well-defined way of comparing fields at

[i] G is here taken to be a semi-simple Lie group.

different points in space-time, or, equivalently: to find a covariant derivative. So let us start by generalising the concept of a covariant derivative to the non-Abelian case.

Assume that we have some Lagrangian which is left invariant under some infinitesimal transformations of the fermion field λ:

$$\delta\lambda \;=\; -\, i\;\Theta\;\lambda, \tag{4.5}$$

where it is implied that Θ is taken in the same representation as λ, and the appropriate notion of multiplication for that representation is used. (For simplicity, we will no longer explicitly write out the space-time dependence of the parameters Θ.)

By using equation (4.5), we may compute how $\partial_\mu\lambda$ transform under a gauge transformation, and we find:

$$\delta\,(\,\partial_\mu\lambda\,) \;=\; -\,i\Big((\partial_\mu\lambda)\Theta +\; \lambda\,\partial_\mu\Theta\Big). \tag{4.6}$$

The last term above clearly spoils the transformation properties of $\delta\,(\,\partial_\mu\lambda\,)$, and so we need to define a covariant derivative which behaves nicely under gauge transformations. Take this to be given by:

$$D_\mu\lambda = \partial_\mu\lambda \;+\; i\;\mathcal{A}_\mu\lambda, \tag{4.7}$$

and the infinitesimal variation of the gauge field $\delta\mathcal{A}_\mu$ as

$$\delta\mathcal{A}_\mu = D_\mu\Theta. \tag{4.8}$$

This gives us precisely what we need, namely that the covariant derivative of λ transform just as λ itself does under gauge transformations:

$$\delta\,(D_\mu\lambda) = \; i\;\Theta\;D_\mu\lambda. \tag{4.9}$$

By taking the definition of the field strength to be given by equation (3.27), $F_{\mu\nu}$ will no longer be invariant under gauge transformations, but rather transform in a covariant way:

$$\delta F_{\mu\nu} \;= i\;\Theta\;F_{\mu\nu}. \tag{4.10}$$

However, the kinetic terms of the Lagrangian are still gauge invariant. Since $\mathcal{A}_\mu$ can be thought of as taking values in the Lie algebra, so can the curvature $F_{\mu\nu}$. It therefore contains the matrices T^a, and we need to form a scalar from these somehow before we can integrate the Lagrangian to form a nice action. All invariant bilinear forms on the Lie algebra are proportional to the trace, and so we will use this to create a scalar quantity. Thus it is easily realised that we can cyclically permute the factors in the Lagrangian without altering the action. This gives us that the variation of $F_{\mu\nu}F^{\mu\nu}$ vanishes. In this way, terms which are bilinear in quantities which transform covariantly under gauge transformations are always gauge invariant. Thus, in general, Yang-Mills theory may be described by the Lagrangian

$$\mathcal{L}_{YM} = \mathrm{Tr}\Big(-\frac{1}{4} F^{\mu\nu} F_{\mu\nu} \;+\; \mathcal{L}_{\text{matter}} \Big), \tag{4.11}$$

where

$$F_{\mu\nu} = -\, i[D_\mu, D_\nu] = \partial_\mu \mathcal{A}_\nu - \partial_\nu \mathcal{A}_\mu \;+\; i[\mathcal{A}_\mu, \mathcal{A}_\nu], \tag{4.12}$$

and the equation of motion for the gauge field takes the form:

$$D^\mu F_{\mu\nu} = 0. \tag{4.13}$$

1 Interactions

The matter-terms can now be formulated in precisely the same way as we did in section 4 of chapter 3, using the covariant derivative of equation (4.7). However, there is a difference when it comes to the interaction terms. Consider a term of the form $\mathcal{A}_\mu J^\mu$ (with J^μ taking values in the adjoint of $\mathfrak{g}$, that is, $J^\mu = J^{a\mu} T_a$). Adding such a term to the Lagrangian modifies the equations of motion for the gauge field by adding a source, just as in the Abelian case:

$$D^\mu F_{\mu\nu} \;=\; J_\nu. \tag{4.14}$$

However, since $F_{\mu\nu}$ is no longer invariant under a gauge transformation, nor can J_ν be. This gives us that $\delta J_\mu = i\,\Theta J_\mu$ (since it is equal to a covariant derivative of a field which transforms in this way). Thus we find that the infinitesimal change in the Lagrangian due to the term $\mathcal{A}_\mu J^\mu$ under a gauge transformation is equal to:

$$D_\mu \Theta J^\mu + i \mathcal{A}_\mu \Theta J_\mu \;= \Theta\, \partial_\mu J^\mu, \tag{4.15}$$

where integration by parts has been used to obtain the final expression, (which indeed is allowed since we assume that the parameter Θ vanishes at the boundary). If the current J^μ is conserved in the ordinary sense, such interaction terms will be gauge invariant. However, $\partial_\mu J^\mu$ is not a covariant expression, and cannot therefore be allowed. So, we must exclude any couplings of the form $\mathcal{A}_\mu J^\mu$ where J^μ is an external source. However, the gauge field will indeed couple to matter fields, but only through the covariant derivative. In this case, the variation of $A_\mu J^\mu$ would be cancelled by the variation of the kinetic terms for J^μ. Notice that the equations of motion (4.14) give us that any such currents coupling to A_μ must be *covariantly conserved*, which is the analog of the requirement $\partial_\mu J^\mu = 0$ in the Abelian case [12].

We may conclude this section by noticing that the generalisation of gauge theories to non-Abelian gauge groups, though complicated, leaves us with objects that we are quite used to handling from the Abelian case. We have the gauge field, $\mathcal{A}_\mu$, field strength, $F_{\mu\nu}$ and the covariant derivative, D_μ, which we all recognise from the Abelian case. However, we must keep in mind that these are objects that take values in $\mathfrak{g}$ (i.e. they are matrices) instead of ordinary numbers, but we may use them in the same way as their counterparts from the Abelian case to construct Lagrangians and find equations of motion. This is precisely what we do in the papers appended to this thesis (at least papers I and II). From now on, we will return to denoting gauge fields by A_μ, and they will always be considered as taking values in $\mathfrak{g}$.

The first time this type of theory was written down was in 1954, when Yang and Mills's attempted to find an explanation for the strong interaction [13]. In their work, they used the gauge group $SU(2)$, but what was really needed to give an accurate description of this was $SU(3)$. Such a theory was eventually constructed successfully almost 20 years later when quantum chromodynamics was born [14, 15]. Yang and Mills original work with $G = SU(2)$ instead turned out to be relevant for the theory of electroweak interaction [16, 17, 18]. The Standard Model of Particle Physics which we currently use also belongs to this class of theories: it is a Yang-Mills theory with gauge group $SU(3) \times SU(2) \times U(1)$.

2 Maximally supersymmetric Yang-Mills theory

Even though Yang-Mills theory has been amazingly successful at describing nature, there are still a lot of unanswered questions and open problems which lead us to consider extensions of this. One of the most widely studied classes of such extended models is supersymmetric Yang-Mills theories.

In a supersymmetric theory, there are fermionic symmetry generators present, which form the supersymmetry algebra, introduced in section 3 of chapter 2. This algebra is obviously dimension-dependent since the formulation of it contains the Γ-matrices, whose properties depend on the number of dimensions. Thus different dimensions allow for different amounts- and kinds of supersymmetry [19]. The minimal supersymmetric Yang-Mills theory one could imagine is simply a massless theory with one massless vector field, A_μ, and its fermionic superpartner λ, described by the Lagrangian density

$$\mathcal{L} = -\frac{1}{4} F_{\mu\nu}\, F^{\mu\nu} + i\bar{\lambda}\Gamma^\mu D_\mu \lambda. \tag{4.16}$$

In order for this action to have a chance of being supersymmetric as it stands, the number of bosonic- and fermionic degrees of freedom must be equal. In dimension d, the number of degrees of freedom for a massless vector field is given by $d-2$, and the number of independent components of a general spinor is $2^{d/2}$ for d even, and $2^{(d-1)/2}$ for d odd. These will however never agree for any d. But, for massless fermions in even dimensions, the degrees of freedom actually are a lot fewer than those of a Dirac spinor [20]. In all even dimensions, we may impose a chirality condition on our spinor λ:

$$\lambda = \pm \overbrace{i^{d/2-1} \prod_{\mu=0}^{d} \Gamma_\mu}^{\Gamma^*} \lambda \;=\; \pm\Gamma^* \lambda, \tag{4.17}$$

where the factor of $i^{d/2-1}$ is a convenient choice of normalisation such that $(\Gamma^*)^2 = 1$. However, in an odd number of dimensions, there is no notion of chirality since the product of the d independent Γ-matrices is proportional to the identity matrix.

In certain dimensions, one can also enforce a reality condition on the spinor, relating the conjugate to the spinor itself through the charge conjugation matrix, C,

$$\lambda = C\bar{\lambda}^T. \tag{4.18}$$

In Minkowski signature, this condition can be applied in 2,3,4 and 10 dimensions [21]. It is furthermore compatible with the Weyl-condition in dimensions 2 and 10, where we can have Majorana-Weyl-spinors[ii]. (For six dimensions, a similar symplectic-Majorana-Weyl-condition may be imposed.) These statements also hold true for some higher dimensions, but since it is impossible to construct super Yang-Mills theories in dimensions higher than 10, we will limit ourselves to $d \leq 10$ here. Each of these conditions decreases the number of degrees of freedom by half, and the total number of degrees of freedom for (Majorana and/or Weyl) spinors as well as massless vector fields are summarised in table 2:

d	Vector	Dirac	Weyl	Majorana	M-W
3	**1**	2	-	**1**	-
4	**2**	4	**2**	**2**	-
6	**4**	8	**4**	-	-
10	**8**	32	16	16	**8**

Thus we find that the minimal action in (4.16) may be supersymmetric in d = 3,4 by imposing the Majorana condition (which in four dimensions is equivalent to the Weyl-condition), in $d = 6$ while imposing the (symplectic) Majorana condition, and finally in $d = 10$ when imposing both Majorana and Weyl-conditions simultaneously. In [23], it was shown that these are indeed the only possible dimensions where this minimal action is supersymmetric.

3 *Ten-dimensional super Yang-Mills*

We will now take a closer look at the ten-dimensional case, which is the highest dimension where we may have a supersymmetric Yang-Mills theory. In this case, there is actually one

[ii]For a review, see [22].

unique theory, due to the fact that we do not allow any fields with spin higher than one. Any maximally supersymmetric Yang-Mills theory in lower dimensions may be obtained from this ten-dimensional theory by using a process known as dimensional reduction, where the fields are simply assumed not to depend on the $(10-d)$ last coordinates, giving rise to a d-dimensional theory. This is the approach that was taken in papers I and II to obtain the four- and five-dimensional theories considered therein.

Before moving on to these lower-dimensional theories, let us show that there is indeed only one possibility for a supersymmetric Yang-Mills theory in $d=10$. We will do this by considering how the supersymmetry generators act on a massless state Ψ with some ten-momentum K^M (capital roman letters will denote the ten-dimensional indices). We now use Wigners' classification where states are classified according to the representation they furnish of the little group, which in ten dimensions is given by $SO(8)$. To proceed, we take a hint from the more familiar four-dimensional setting, where little group is simply given by $SO(2)$. All irreducible representations of this are one-dimensional and may be labelled by the helicity. Even though the irreducible representations of $SO(8)$ are more complicated, we can still label them by a "spin", which we denote by j. We define this spin to be the maximum absolute value of any generator J_{mn} [24].

To start with, we assume that the space spanned by the supersymmetry generators, Q_α have n (real) degrees of freedom, and recall that the symmetry algebra was given by

$$\{Q_\alpha, Q^\dagger_\beta\} = -\frac{1}{2}P_M\left(\Gamma^M\Gamma^0\right)_{\alpha\beta}. \tag{4.19}$$

Without loss of generality, we can take the direction of the spatial part of the ten-momentum K^M to lie in the one-direction, giving us $K^M=(E,E,0,\ldots,0)$. This allows us to rewrite the right hand side of the equation above (4.19) as

$$\{Q_\alpha, Q^\dagger_\beta\} = \frac{1}{2}E\,\left(\mathbb{1}+\Gamma^1\Gamma^0\right)_{\alpha,\beta}. \tag{4.20}$$

We can use $\Gamma^0\Gamma^1 = \Gamma^*\Gamma^{\text{transversal}}$, where Γ^* is the chirality operator and $\Gamma^{\text{transversal}} = \prod_{i=2}^{9}\Gamma^i$. This means that, for massless states, the anticommutator $\{Q_\alpha, Q^\dagger_\beta\}$ is proportional to a projection operator which projects out half the spinor space, so half of the supersymmetry generators must annihilate physical states. The remaining $n/2$ ones form a Clifford algebra, where half of them may be considered as "creation operators" and the others as "annihilation operators" depending on if they raise- or lower the eigenvalue of J_{mn}, where the mn-plane is taken to be transversal to the spatial part of K^M.

Consider now any representation of the little group with spin j, and let Ψ be an eigenstate of J_{mn} with eigenvalue j_{mn}. Any state which takes the maximum eigenvalue j for J_{mn} will be of this form. When we act with one of our annihilation operators on Ψ, we lower j_{mn} by 1/2, and so in total, we can lower this value by $n/8$. By requiring that we do not have any particles with $|j|>1$, this gives us that $n<16$. Hence, we cannot have more than 16 real, or equivalently, 8 complex, components of the supersymmetry generators Q without including gravity. This is precisely the number of independent components of *one* ten-dimensional Majorana-Weyl spinor, and the only supersymmetric theory we can have is the $\mathcal{N} = 1$ super Yang-Mills theory with the minimal amount of supersymmetry in $d = 10$.

Thus, the ten-dimensional Lagrangian of equation (4.16),

$$\mathcal{L} = -\frac{1}{4} F_{MN}\, F^{MN} + i\bar{\lambda}\Gamma^M D_M \lambda, \tag{4.21}$$

will be our starting point for obtaining the lower-dimensional theories which are the subject of papers I and II. The supersymmetry transformations which leaves the Lagrangian of equation (4.21) invariant are given by

$$\delta A_M = i\bar{\varepsilon}\Gamma_M \lambda \qquad , \qquad \delta\lambda = \frac{1}{2} F_{MN}\Gamma^{MN}\varepsilon, \tag{4.22}$$

where $\Gamma^{MN} = \Gamma^{[M}\Gamma^{N]}$ and ε is the supersymmetry parameter transforming in the same representation as the fermions λ of the Lorentz group.

4 ***Super Yang-Mills on*** $M_5 = M_4 \times I$

The super Yang-Mills theory which has been considered in the papers appended to this thesis is as mentioned not the ten-dimensional theory, but rather a lower-dimensional maximally supersymmetric theory: five-dimensional super Yang-Mills on the manifold $M_5 = M_4 \times I$ (M_4 being a four-manifold of Euclidean signature and I an interval). To obtain this five-dimensional theory from the ten-dimensional one, we use dimensional reduction, meaning we assume that the fields only depend on the first five coordinates. This splits the vector field A_M into a five-dimensional vector, $A_{M=0,\dots,4}$ and five scalar fields, $A_{M=5,\dots,9}$. It also decomposes the Lorentz group Spin(1,9) into Spin(1,4)$\times SO_R(5)$, where the first factor denotes the holonomy group of the space M_5, and the second one the R-symmetry group of the five-dimensional theory, which acts on the five scalar fields. In the special case $M_5 = I \times M_4$, we can take the time-like direction to lie along the interval, which allows us to consider a compact subgroup of the Lorentz group of M_5 which is isomorphic to Spin(4).

Let the coordinates along M_4 be denoted by indices $\mu, \nu, \dots$, the direction along the interval be denoted by y, and the indices in the remaining directions be denoted by I,J,.... In this language, the decomposition of the ten-dimensional gauge field becomes:

$$A_M \to A_\mu, A_y, \Phi_I, \tag{4.23}$$

corresponding to a four-dimensional gauge field on M_4, a gauge field along I, and five scalar fields. The fermionic fields will decompose as two copies of two four-dimensional spinors of opposite chirality, where the two copies are distinguished by their charge under the R-symmetry group. Notice that the ten-dimensional supersymmetry parameters are also affected by the dimensional reduction, and have decomposed in the same fashion as the fermions: into four different (constant) spinors, two of each four-dimensional chirality, which are distinguished by their R-symmetry charge. This theory is therefore said to have $\mathcal{N} = 4$ *supersymmetry*, which is the maximum possible in four dimensions[iii].

Let us spend a brief moment on this wondrous four-dimensional theory before moving on. The $\mathcal{N} = 4$ Yang-Mills theory actually enjoys a conformal symmetry, making the theory one of few known superconformal theories, which amongst other things means that it is ultra-violet finite. The large amount of symmetry also allows for explicit calculations, and $\mathcal{N} = 4$ super Yang-Mills has thus been extensively used as a playground for studying higher-dimensional, or less supersymmetric, theories. The $\mathcal{N} = 4$ was for a long time believed to be the highest-dimensional superconformal field theory without gravity in existence, but as we shall see in the next section, this is not the case (though it still appears to be the highest-dimensional superconformal field theory with a classical limit where it behaves as we expect a classical field theory to do).

[iii]Even though we are actually considering the theory on a five manifold, $M_5 = M_4 \times I$, the maximal amount of supersymmetry is the same as in four dimensions due to the product structure of M_5.

Chapter 5:

(2,0) Theory

In the last section, we claimed that all lower-dimensional super Yang-Mills theories could be obtained from the ten-dimensional one, but what about theories which are not Yang-Mills theories? Do they also have their origin in the ten-dimensional super Yang-Mills theory somehow, or are there indeed any other, fundamentally different, supersymmetric quantum field theories (not including gravity) out there?

Well, the answer is that there appear to be fewer than one might expect. Through correspondences and accidents, evidence is being gradually uncovered indicating that there might just be a few "fundamental" theories out there, and that all others may, in one way or another, be obtained from these theories. The ten-dimensional super Yang-Mills theory appears to be one of these mother theories. Another one, or rather, *the* other one, appears to be a much more mysterious entity: a six-dimensional theory known as the *(2,0) theory*.

This theory has never really been the main focus in any of the works appended to this thesis, and I shall not claim, nor attempt, to give a complete description of it in this chapter. Rather, I will only give a brief overview of that which is needed in order to be able to consider topological twists of the free tensor multiplet which was the goal of paper III. For a more detailed review, I refer the reader to [25], [26].

The (2,0) theory is a six-dimensional, superconformal field theory, where the name refers to the amount of supersymmetry of the theory: it contains two chiral supercharges, (and no anti-chiral). In 1977, Nahm classified all supersymmetries and their representations [19], and proved that superconformal symmetry is only possible in dimensions $d \leq 6$. However, it was not until much later that the full impact of this result was clear: such six-dimensional theories actually exist, with the two possibilities being the (2,0) theory and its truncation to (1,0).

On a quantum mechanical level, the existence of (2,0) theory has been established beyond doubt by string theory arguments. There are several ways of seeing this, either by considering a compactification of type IIB string theory on a four-dimensional hyper-Kähler manifold known as a K3 surface with certain singularities [25], which results in a six-dimensional field theory with string-like objects. Or, one may instead view the (2,0) theory as describing the world-volume of n parallel M5-branes, where these string-like object would correspond to M2-branes ending on these M5-branes [27]. It is worth pointing out that in no description or view of (2,0) theory are there any free parameters, no "coupling" in which one could perform a perturbative expansion, adding to the mystery surrounding this fascinating object.

However, the biggest mystery of them all is probably this: there is no known classical formulation of the theory, and no way to write down a Lagrangian that satisfactorily describes it. This may be seen as follows: It is known that compactification of (2,0) on a circle gives us five-dimensional super Yang-Mills theory [28], which, in turn, has a nice five-dimensional Lagrangian with a coupling constant g^2 (a factor $\propto 1/g^2$ multiplies the Lagrangian). This coupling constant will be proportional to the radius of the circle used in the compactification. However, if there was a nice Lagrangian for the six-dimensional theory too, compactification would simply amount to integrating over the fibre, giving us a multiple of the area of this fibre. In this case, this is simply the circumference of the circle. Thus, the five-dimensional Lagrangian one would obtain in this way would have a coupling $g^2 \propto R^{-1}$, never proportional to R which we know to be the correct result [26]. So we clearly have a problem with the very basics of the formulations of the theory in the language which we commonly use to this end.

Let us for a moment restrict ourselves to the non-interacting version of the theory, i.e. the theory describing the world-volume theory on a single $M5$-brane. The field content of this is known as *the free tensor multiplet of (2,0) theory*, and consists of five scalars, Φ_K[i], one chiral spinor, Ψ, obeying a symplectic Majorana-Weyl condition (giving us in total eight fermionic degrees of freedom) and one self-dual three-form field strength, H_{MNP} [29]. This last field is slightly similar to the field strength $F_{\mu\nu}$ of the gauge field A_μ in four-dimensional Yang-Mills. In this case, F may be seen as a middle-dimensional form (in four dimensions, that is a two-form), just as H is a middle-dimensional form in the six-dimensional theory. However, when $d = 4k + 2$ (and the space has Minkowski signature), such a middle-dimensional form may satisfy a *self-duality condition*, that is, $\star H = H$ (or equivalently in index notation: $H_{MNP} = \epsilon_{MNPQRS} H^{QRS}$). This means that there is no way to write down a Lorentz-covariant term for this H, since any term similar to "H^2" identically vanishes simply due to the self-duality condition. So, even for the free theory, a Lagrangian description is impossible.

This forces us to carry out all calculations on the level of equations of motion. For the case of the free tensor multiplet, these are in flat space given by:

$$\begin{aligned} \partial^M \partial_M \Phi &= 0 \\ dH &= 0 \\ \Gamma^M \partial_M \Psi &= 0 \end{aligned} \tag{5.1}$$

The supersymmetry transformations then induce an isomorphism on the solution space

[i] K being an index of the vector representation of the R-symmetry group $SO(5)$.

of these equations, and are explicitly given by:

$$\begin{aligned}
\delta H_{MNP} &= 3\partial_{[M}\left(\bar{\Psi}_\alpha \Gamma_{NP]}\varepsilon^\alpha\right)\\
\delta\Phi_K &= 2\left(\gamma^R_K\right)_{\alpha\beta}\bar{\Psi}^\alpha\varepsilon^\beta\\
\delta\Psi^\alpha &= \frac{i}{12}H_{MNP}\Gamma^{MNP}\varepsilon^\alpha + iM_{\beta\gamma}\partial_M\left(\Gamma^R_K\right)^{\alpha\beta}\Phi^K\Gamma^M\varepsilon^\gamma,
\end{aligned} \tag{5.2}$$

where M,N,P are six-dimensional Lorentz indices, α, β spinor indices, and K an index in the R-symmetry group vector representation. Γ are the gamma-matrices of the Clifford algebra of the Lorentz group, and Γ^R the same for the R-symmetry group. ε as usual denotes the supersymmetry parameter. These equations are taken as the starting point for carrying out a topological twist in paper III.

1 Relation to lower-dimensional theories and correspondences

Even though little is known about the full (2,0) theory, it is believed to give rise to a vast variety of lower-dimensional theories and correspondences between these. As mentioned in the previous section, compactification on a circle leads to five-dimensional super Yang-Mills [28], whereas compactification of (2,0) on a Riemann surface C gives rise to four-dimensional $\mathcal{N} = 2$ super Yang-Mills. In the special case of $C = T^2$, the remaining four-dimensional theory is the famous $\mathcal{N} = 4$ super-Yang Mills theory [30].

Since the late nineties, when Maldacena presented a conjecture between gauge theories on the boundary and gravity duals in the bulk of Anti-deSitter space-time [31], correspondences have been a very active area of theoretical physics. The original so-called AdS/CFT correspondence has proven itself invaluable, since it allows for computations of quantities in the strongly-coupled regime in a theory on one end of the correspondence by translating the problem to the weakly-coupled regime in the dual theory. Furthermore, it allows for a non-perturbative formulation of certain string theories in terms of the gauge theory on the boundary.

Thus, correspondences have opened new ways of gaining insight into strongly-coupled theories, and several new ones have been discovered in recent years. A whole class of such correspondences relate theories in d and $(6-d)$ dimensions, and they may have their origins in six dimensions and the (2,0) theory. The first one of this new class of correspondences is the AGT correspondence, first presented by Alday, Gaiotto and Tachikawa in 2009 [32]. This relates the correlation functions of two-dimensional Liouville theory to the Nekrasov partition function [33, 34] of some $\mathcal{N} = 2$ Yang-Mills theories, known as class $\mathcal{S}$-theories [35, 36]. We may very briefly summarise the idea behind this correspondence as follows:

The class $\mathcal{S}$ theories may be obtained by compactifying the (2,0) theory on a Riemann surface with genus g and n punctures [37]. Such a Riemann surface may be constructed by a process known as “sewing together pairs of pants”. From the four-dimensional perspective, different ways of sewing together the same Riemann surface results in different Lagrangian descriptions, and the operations relating two different sewings corresponds to S-duality transformations between these different Lagrangians. So, in some way, the four-dimensional theories must encode the information on the possible ways to sew this Riemann surface used in the compactification.

In two dimensions, the Liouville correlation functions in conformal field theory may also be defined in terms of ways to sew together the Riemann surface on which the theory is defined. As such, it appears that there may be a correspondence between the four-dimensional class $\mathcal{S}$-theories and the two-dimensional conformal field theories. For general Riemann surfaces (with general g and n) this has however not been proven yet, though extensive tests have been carried out.

The belief is that this correspondence may have its home in six dimensions, something which may find its explanation in a twisted form of (2,0) theory. This is the subject of paper III, and shall be further discussed in section 2 of the next chapter. In general, by compactifying on a d dimensional manifold, we may expect to find a correspondence between d and $(6-d)$-dimensional theories. For example, for $d=3$, this has been done with great success, and correspondences between different three-dimensional theories have been discovered [38, 39, 40]. Only time will tell what other nice tools the mysterious (2,0) theory will provide us with.

Chapter 6:

Chern-Simons matter theories

Chern-Simons theory is a three-dimensional topological field theory of Schwarz-type[i], specified by choice of gauge group G and coupling k, (which is often referred to as the *level* of the theory). These are often written together in the more concise manner of G_k. In its minimal form, Chern-Simons theory contains only one gauge field, A_μ, which, just as in the Yang-Mills case, takes its value in the Lie algebra of G. We can describe the bosonic Chern-Simons theory by the Lagrangian[ii]

$$\mathcal{L}_{CS} \;=\; \frac{ik}{4\pi}\,\epsilon^{\mu\nu\rho}\,\mathrm{Tr}\left(A_\mu\partial_\nu A_\rho + \frac{2\,i}{3}A_\mu A_\nu A_\rho\right), \tag{6.1}$$

where μ, ν, ρ are three-dimensional indices and $\epsilon^{\mu\nu\rho}$ the totally antisymmetric Levi-Civita symbol (and the relative factor i between the two terms originates from the choice of convention for the covariant derivative). If the gauge group is Abelian, the trilinear term in the gauge field vanishes due to the antisymmetry of $\epsilon^{\mu\nu\rho}$.

Under an infinitesimal variation in the gauge field, δA_μ, the variation in the Chern-Simons-Lagrangian (6.1) is given by:

$$\delta\mathcal{L}_{CS} \;= \frac{ik}{4\pi}\,\epsilon^{\mu\nu\rho}\,\mathrm{Tr}\left(\delta A_\mu F_{\nu\rho}\right), \tag{6.2}$$

where $F_{\nu\rho}$ is the field strength of A_μ, defined in equation (4.12). This straight away gives the equations of motion as:

$$F_{\mu\nu} \;=\; 0. \tag{6.3}$$

If this is the form of the source-free equation of motion, at first glance, this appears to be an utterly boring theory. So why do we bother studying it, if all physical connections must be “flat connections”, or equivalently, “pure gauges”? Well, the answers to this question are many. For one, we may couple the theory to some conserved current J^μ, which would give rise to a source term in the equations of motion, just as in the Yang-Mills case. By choosing this current in a clever way, the resulting theory can have applications in a wide variety of fields such as condensed matter or three-dimensional gravity. Another

[i] This will be further explained in chapter 7.

[ii] This is more commonly written using differential forms, where the action takes the form $A \wedge \mathrm{d}A + \frac{2\,i}{3} A \wedge A \wedge A$

interesting thing one can do, which is perhaps the most common, is to consider the theory on a background with non-trivial topology. It will then have many interesting mathematical applications, such as for example providing an intrinsic three-dimensional interpretation [41, 42] of the Jones polynomial of a knot [43][iii].

We have already determined that the action behaves nicely under infinitesimal variations of the fields (6.2), but what if we consider a variation in $A_\mu$which is not infinitesimal? This is equivalent to considering the action on A_μ by some element g in the gauge group, which amounts to taking

$$A_\mu \to g^{-1} A_\mu g \; - \; i \, g^{-1} \partial_\mu g. \tag{6.4}$$

On an infinitesimal level, this is equivalent to the familiar expression $\delta A_\mu = D_\mu \Theta$ if g can be continuously deformed to the identity, that is, if it can be written as $g^{i\Theta}$. However, if g takes its values in some non-connected part of G, we must use the expression in equation (6.4) instead. Such transformations are often known as a "large gauge transformation", and acts on the Chern-Simons Lagrangian by:

$$\mathcal{L}_{CS} \;\to\; \mathcal{L}_{CS} + \frac{ik}{4\pi} \epsilon^{\mu\nu\rho} \, \mathrm{Tr} \left(i \, \partial_\nu (A_\mu \partial_\rho g \, g^{-1}) + \, \frac{1}{3} \, g^{-1} \partial_\mu g \, g^{-1} \partial_\nu g g^{-1} \partial_\rho g \right). \tag{6.5}$$

The first term is only a total derivative, which will not effect the action, whereas the last term is proportional to the *winding number density*, w. With appropriate boundary conditions, this satisfies:

$$\int w = \frac{1}{24\pi^2} \int \epsilon^{\mu\nu\rho} \, \mathrm{Tr}\Big(g^{-1} \partial_\mu g g^{-1} \partial_\nu g g^{-1} \partial_\rho g \Big) \; = \; N, \tag{6.6}$$

where N is an integer. Hence, a large gauge transformation results in a shift in the Chern-Simons action given by:

$$\mathcal{S}_{CS} \; \to \mathcal{S}_{CS} - \; 2\pi i k N. \tag{6.7}$$

To ensure that there is no change in physical observables under such a gauge transformation (i.e. that $\exp(\mathcal{S}_{CS})$ remains invariant), we require k to be an integer. Thus the coupling of Chern-Simons theory is very different from the Yang-Mills coupling, since g^2_{YM} is a contnuous parameter, whereas k is discrete. To facilitate calculations however, generic values of $k \in \mathbb{R}$ are often considered, and sometimes, (as done in papers IV and V), even analytic continuations to complex values of the level is used.

Herein, we will restrict ourselves to $\mathcal{N} = 2$ supersymmetric Chern-Simons theory with gauge group and level $U_k(N)$. The $\mathcal{N} = 2$ theory contains a vector multiplet which can be

[iii]This was the first time the definition of this fundamentally three-dimensional, mathematical object could actually be given in precisely three dimensions, without having to use a two-dimensional projection of the knot in question. It is all a beautiful story of a mathematical problem finding its solution in physics, but as so many other things, it unfortunately lies outside the scope of this thesis.

thought of as the dimensional reduction of the vector multiplet of four-dimensional $\mathcal{N}=1$ super Yang-Mills theory. In three-dimensional notation, this contains the following fields:

$$A_\mu, \sigma, \lambda, \bar{\lambda}, D, \tag{6.8}$$

where A_μ is a gauge field, σ is an auxiliary scalar field, $\lambda, \bar{\lambda}$ are two-component complex Dirac spinors, and D is an auxiliary scalar.

The supersymmetry transformations for these fields are parametrised by two independent (two-component) complex spinors $\varepsilon, \bar{\varepsilon}$, and are given by:

$$\begin{aligned}
\delta A_\mu &= \frac{i}{2}(\bar{\varepsilon}\gamma_\mu\lambda - \bar{\lambda}\gamma\varepsilon) \\
\delta\sigma &= \frac{1}{2}(\bar{\varepsilon}\lambda - \bar{\lambda}\varepsilon) \\
\delta\lambda &= -\frac{1}{2}\gamma^{\mu\nu}\varepsilon F_{\mu\nu} - D\varepsilon + i\gamma^\mu\varepsilon D_\mu\sigma + \frac{2i}{3}\sigma\gamma^\mu D_\mu\varepsilon \\
\delta D &= -\frac{i}{2}\Big(\bar{\varepsilon}\gamma^\mu D_\mu\lambda + D_\mu\bar{\lambda}\gamma^\mu\varepsilon\Big) + \frac{i}{2}\Big([\bar{\varepsilon}\lambda,\sigma] + [\bar{\lambda}\varepsilon,\sigma]\Big) - \frac{i}{6}\Big(D_\mu\bar{\varepsilon}\gamma^\mu\lambda + \bar{\lambda}\gamma^\mu D_\mu\varepsilon\Big).
\end{aligned} \tag{6.9}$$

The transformation rule for $\bar{\lambda}$ is simply obtained by Dirac conjugating the transformation rule for λ in equation (6.9).

From the supersymmetry algebra, we know that the anticommutator of two successive variations with different parameters gives the same result as acting with a bosonic symmetry, that is, it should be equal to a combination of a translation and a gauge transformation. Consider this condition on the auxiliary scalar D. For it to be true even when the equations of motions are not satisfied, we find a condition on the spinors ε and $\bar{\varepsilon}$: they must satisfy the Killing spinor equation[iv]. On the three-sphere, this takes the form:

$$D_\mu\varepsilon = \frac{i}{2r}\gamma_\mu\varepsilon \qquad\qquad D_\mu\bar{\varepsilon} = \frac{i}{2r}\gamma_\mu\bar{\varepsilon}, \tag{6.10}$$

where r is the radius of the sphere.

The action of the Chern-Simons vector multiplet is given by:

$$\mathcal{S}_{CS} = \int d^3x\sqrt{g}\,\mathrm{Tr}\Big(\epsilon^{\mu\nu\rho}\Big(A_\mu\partial_\nu A_\rho + \frac{2i}{3}A_\mu A_\nu A_\rho\Big) - \bar{\lambda}\lambda + 2D\sigma\Big), \tag{6.11}$$

with the trace taken in the fundamental representation [44]. For convenience, we here do not take the factor $\frac{ik}{4\pi}$ to be included in the action, but rather in the expression for the partition function, which therefore reads:

$$Z = \int \mathcal{D}\varphi\; e^{-\frac{ik}{4\pi}\mathcal{S}_{CS}}, \tag{6.12}$$

where $\mathcal{D}\varphi$ denotes the integration over all fields.

[iv]This means that the spinor must satisfy $\nabla_x\varepsilon = ex\cdot\varepsilon$ for all tangent vectors x on the manifold in question (e being some constant).

Furthermore, since the three-dimensional $\mathcal{N}=2$ vector multiplet may be obtained simply by dimensional reduction from the four-dimensional $\mathcal{N}=1$ vector multiplet, we may write the Yang-Mills Lagrangian for this multiplet as a total super-derivative:

$$\bar{\varepsilon}\varepsilon\mathcal{L}_{YM} = \delta_{\bar{\varepsilon}}\delta_{\varepsilon}\ \mathrm{Tr}\Big(\frac{1}{2}\bar{\lambda}\lambda - 2D\sigma\Big). \tag{6.13}$$

This will be important later on when using localisation to compute the path integral of these theories, but we shall return to that in chapter 8. For the moment, let us consider what else we may do with these three-dimensional theories.

1 Adding matter

We may add even more fields to the action of equation (6.11), for example matter fields in the form of a chiral multiplet Φ in some representation R of the gauge group. This multiplet consists of

$$\phi, \bar{\phi}, \psi, \bar{\psi}, F, \bar{F}, \tag{6.14}$$

and has the Lagrangian

$$\begin{aligned}\mathcal{L}_{\text{matter}} =& D_\mu\bar{\phi}D^\mu\phi + \bar{\phi}\sigma^2\phi + \frac{i(2\Delta-1)}{r}\bar{\phi}\sigma\phi + \frac{\Delta(2-\Delta)}{r^2}\bar{\phi}\phi + i\bar{\phi}D\phi + \bar{F}F \\ &- i\bar{\psi}\gamma^\mu D_\mu\psi + i\bar{\psi}\sigma\psi - \frac{2\Delta-1}{2r}\bar{\psi}\psi + i\bar{\psi}\lambda\phi - i\bar{\phi}\bar{\lambda}\psi.\end{aligned} \tag{6.15}$$

Δ denotes the dimension of ϕ, which in the classical case is given by $\frac{1}{2}$, but quantum corrections to this may occur in theories with $\mathcal{N}\leq 3$ supersymmetry.

The supersymmetry transformation of the matter fields are given by

$$\begin{aligned}\delta\phi =& \bar{\varepsilon}\psi \\ \delta\psi =& i\gamma^\mu\varepsilon D_\mu\phi + i\varepsilon\sigma\phi + \frac{2\Delta i}{3}\gamma^\mu D_\mu\varepsilon\phi + \bar{\varepsilon}F \\ \delta F =& \varepsilon\Big(i\gamma^\mu D_\mu\psi - i\sigma\psi - i\lambda\phi\Big) + \frac{i}{3}\Big(2\Delta-1\Big)D_\mu\varepsilon\gamma^\mu\psi,\end{aligned} \tag{6.16}$$

where $\varepsilon, \bar{\varepsilon}$ again are two independent complex spinors. However, the restriction obtained by studying two variations of the scalar field D in the last section naturally still holds, so both $\varepsilon, \bar{\varepsilon}$ must thus satisfy the Killing spinor equation. The transformation rules for $\bar{\phi}, \bar{\psi}, \bar{F}$ may be obtained straight-forwardly by Dirac conjugation of the relevant equations in (6.16).

Under these supersymmetry transformations, the matter-part of the Lagrangian (6.15) may be written as a total superderivative:

$$\bar{\varepsilon}\varepsilon\mathcal{L}_{\text{matter}} = \delta_{\bar{\varepsilon}}\delta_{\varepsilon}\Big(\bar{\psi}\psi - 2i\bar{\phi}\sigma\phi + \frac{2(\Delta-1)}{r}\bar{\phi}\phi\Big). \tag{6.17}$$

2 ABJM theory

ABJM theory is a three-dimensional superconformal theory with gauge group (and levels) $U_k(N) \times U_{-k}(N)$ with matter in the bifundamental representation. k (and $-k$) denotes the Chern-Simons levels of the two $U(N)'s$ respectively [45]. This is the theory of interest in the two last papers appended to this thesis. The field content of ABJM theory is given by two copies of the Chern-Simons vector multiplet of equation (6.8), (one for each gauge group), and four copies of the supersymmeric matter multiplet, Φ_i, $i \in \{1,2,3,4\}$. The two $U(N)$'s may also be taken to have different numbers of colours, that is, we may take the gauge group as $U(N_1) \times U(N_2)$. This small generalisation is known as ABJ theory, and it is very helpful in some cases when doing calculations. However, it comes at the cost of losing conformal invariance, which only exists for massless matter multiplets and equal numbers of colours.

In the strong-coupling limit, with $k = 1$, ABJM theory is conjectured to describe the world-volume of N intersecting $M2$-branes, and for general values of k, to describe type IIA string theory on $AdS_4 \times \mathbb{CP}^3$. Thus computations on the field theory side of coupling-independent quantities may be compared to geometrical analyses on the string theory side. In such a way, this allows for non-trivial tests of the conjectured AdS/CFT duality between the two theories, which was briefly mentioned in the previous chapter. Thus, like in the case of (2,0) theory, the ABJM theory has some very interesting applications in the context of dualities.

However, in order to carry out these tests, we need to be able to explicitly compute quantities such as for example the free energy of the theory. This means that we must be able to evaluate the partition function

$$Z = \int \mathcal{D}\varphi \; e^{-\frac{ik}{4\pi}\left(\mathcal{S}_{CS_+} - \mathcal{S}_{CS_-} + \mathcal{S}_{\text{matter}}\right)}, \tag{6.18}$$

where $\mathcal{S}_{CS_\pm}$ denotes the Chern-Simons action for the two gauge groups $U_k(N)$, $U_{-k}(N)$ respectively. In general, this is a complicated problem, which may be solved using the technique of localisation. This will be described in chapter 8.

Part III

TECHNIQUES

Theory / Technique	Super Yang-Mills	(2,0)	Chern-Simons + matter
Topological Twisting	X	X	-
Localisation	X	-	X
Large N	-	-	X

We have now arrived at a point where we have introduced all of our main characters: the supersymmetric theories studied in the papers appended to this thesis. We may now start to explore the various things we can do with our stars, that is, the techniques we may use to extract information from them[v]. The areas of study in the appended papers are summarised in the table above.

[v]No theories were harmed during the construction of this thesis.

Chapter 7:

Topological twisting

In previous chapters, the action functionals – or in the case of (2,0) theory: the equations of motion – were presented for some supersymmetric theories. In this chapter, a method for constructing theories with another kind of symmetry, so-called *topological field theories*, from the supersymmetric theories previously introduced will be reviewed. This method is known as *topological twisting.*

In all of the theories considered in the previous chapters (with the exception of pure Chern-Simons theory), the observables contain a metric dependence, and so these theories are Riemannian dependent on the manifold on which they are placed. For example, a general manifold does not admit any supersymmetry since it does not have any (constant) Killing spinors. However, the situation is different for topological field theories. In such a theory, the only dependence on the manifold is on the topological invariants of it rather than the metric. This greatly facilitates calculations in the theory as for example allowing for the use of localisation (which will be described in the next chapter).

Herein, we shall start by giving a brief introduction to two different classes of topological field theories, and a technique for constructing theories belonging to the second one of these classes, after which we shall review two explicit examples of topological twists that are the subjects of papers I – III.

There are two ways in which the metric dependence may be absent from the action of the theory, defining two classes of topological field theories. These are:

- ⋆ Schwartz-type theories [46], where the formulation of the action simply does not contain the metric.
- ⋆ Cohomological-type theories, with a much more subtle metric independence.

The Chern-Simons theory described by the action of equation (6.1) is the classical example of a topological field theory of Schwartz-type [41], simply because there is no metric dependence whatsoever in (6.1). In the case of theories of cohomological-type, their metric independence takes its form in that the variation of the action with respect to the metric, i.e. the stress tensor of the theory, *vanishes in the cohomology of a certain nilpotent scalar charge Q.*

Let us rewind a little: In all topological field theories of this second class, there exists a symmetry generator which squares to zero (up to a gauge transformation), and we define the topological field theory by restricting ourselves to only consider physical observables (such as for example correlation functions) lying in the cohomology of this operator; that is, requiring the observables to satisfy $\{Q,\Psi\} = 0$. Any two states of the theory will thus be considered equivalent if they differ by a quantity of the form $\{Q, \dots\}$. Such a quantity is said to be Q-exact. The variation of the observables with respect to the metric will be proportional to the stress tensor of the theory, and if this is Q-exact, this metric-variation vanishes in cohomology. Hence, even though the metric may be present in the formulation of the theory, the actual observables are independent of it.

So, how can we construct such theories? Well, starting from a theory with enough supersymmetry[i], there is a way to construct a topological field theory of cohomological type through a process known as *topological twisting*. In short, this is a way of constructing a suitable nilpotent symmetry generator by creating a scalar supercharge, which then, by the supersymmetry algebra, must square to zero up to a gauge transformation. To create such a charge, we first notice that the supersymmetries transform under both the Lorentz- and R-symmetry groups. By choosing a suitable diagonal subgroup of these two, under which the spinor representation decomposes to give at least one scalar representation, a supersymmetry charge with the desired properties may be obtained. Thus the twisting may be defined by the particular homomorphism used to embed the Lorentz- in the R-symmetry group. The idea is schematically illustrated in figure 7.1.

Figure 7.1: *Schematic picture of the idea of topological twisting.*

The attentive reader may here notice that this inevitably forces topological twisting to be indigenous to Euclidean space, since we must require the Lorentz group to be compact to be able to embed it into the compact R-symmetry group. In certain cases, however, the structure of the manifold allows for a time-like direction, but the topological twist must nonetheless be done only along the Euclidean directions. Examples of this are discussed in papers I – III.

In the case of a flat background, this recipe of topological twisting is simply a relabelling of the fields in the theory, but when the background is curved, one in addition may be required to add curvature terms both to the action, equations of motion, as well as the supersymmetry transformation rules. These terms must be added manually, and to my knowledge, no universal recipe exists. Precisely how this is done may best be described by examples, and we shall below consider twists of (four- and) five-dimensional super

[i]The precise meaning of "enough" will be illustrated in the examples.

Yang-Mills theory as well as of the free tensor multiplet of (2,0) theory on a manifold with a product structure.

1 The GL-twist of $\mathcal{N} = 4$ super Yang-Mills

The first example of topological twisting we shall consider is the geometric Langlands twist of $\mathcal{N} = 4$ super Yang-Mills theory [47, 48] and its five-dimensional analog [49]. It is mainly the five-dimensional theory which has been the focus of papers I and II, but it is so closely related to the more well-known four-dimensional theory that no explanation of the five-dimensional theory would be complete without also mentioning the four-dimensional twist.

1.1 Twisting the four-dimensional theory

$\mathcal{N} = 4$ super Yang-Mills on an Euclidean four-manifold M_4 (which was introduced in section 4 of chapter 4), has the Lorentz group $Spin(4)$, and the R-symmetry group $Spin(6)_R$. The Lorentz group may be embedded in the R-symmetry group in three different fashions, giving rise to three inequivalent topological twists of the theory [47, 48, 50, 51]. We shall focus on the so-called geometric Langlands twist [47, 48].

Since the twisting procedure amounts to finding a suitable diagonal subgroup of the Lorentz- and R-symmetry groups, it is convenient to start by noticing the following group theoretical relations:

$$\begin{aligned} Spin(4) &\simeq SU(2)_l \times SU(2)_r \\ Spin(6) &\simeq SU(4) \supset SU(2)_R^l \times SU(2)_R^r \times U(1)_R. \end{aligned} \tag{7.1}$$

We can chose to identify the two $SU(2)_l$'s and $SU(2)_r$'s, and replace them with diagonal subgroups $SU(2)_l'$ and $SU(2)_r'$ respectively, after which we restrict ourselves to the part of the theory which is invariant under the product of these new symmetry groups, $SU(2)_l' \times SU(2)_r' \times U(1)_R$.

Let us study what happens to the supersymmetries under this choice of diagonal subgroup. The spinor representation of $Spin(4)$ decomposes as two copies of sums of two chiral spinors under $SU(2)_l \times SU(2)_r$ with opposite chirality under the $SU(2)_R^l \times SU(2)_R^r$'s, where the charge under the R-symmetry $U(1)_R$ is fixed by requiring them to have the same ten-dimensional chirality (in the sense that the fields may be viewed as originating from the ten-dimensional Yang-Mills theory by dimensional reduction).

Under the twisted symmetry group, we find that the spinor representation decomposes into (amongst others) two scalar representations with opposite charges under the remaining $U(1)$. We can take these to be generated by some constant spinors $e^{\pm}$. The twisted theory is then invariant under any linear combination of these two scalar supersymmetries, that is, invariant under a scalar supersymmetry with parameter

$$\varepsilon = ue^+ + ve^-. \tag{7.2}$$

Through this twist, we have therefore found a family of topological field theories, parametrised by the quotient u/v. If our manifold has a boundary, the conditions on this may be used to fix this quotient. For example, for half-BPS[ii] boundary conditions, this fixes $u = \pm v$ [49].

This twisting procedure is illustrated in table 7.1, where the representations of the $SU(2)$'s are denoted by bold faced numbers and the $U(1)_R$-charges are denoted by superscripts.

	$SU(2)_l \times SU(2)_r \times SU(2)_l^R \times SU(2)_r^R \times U(1)_R$	$\xrightarrow{twist}$	$SU(2)_l' \times SU(2)_r' \times U(1)$	
Spinor rep.	$(\mathbf{2},\mathbf{1},\mathbf{2},\mathbf{1})^{+1/2} \oplus (\mathbf{2},\mathbf{1},\mathbf{1},\mathbf{2})^{-1/2}$ $\oplus(\mathbf{1},\mathbf{2},\mathbf{2},\mathbf{1})^{+1/2} \oplus (\mathbf{1},\mathbf{2},\mathbf{1},\mathbf{2})^{-1/2}$		$(\mathbf{1},\mathbf{1})^{\pm 1/2} \oplus (\mathbf{2},\mathbf{2})^{\pm 1/2}$ $\oplus(\mathbf{3},\mathbf{1})^{+1/2} \oplus (\mathbf{1},\mathbf{3})^{-1/2}$	$e^+,\ e^-,$ $\ldots$

Table 7.1: *Table describing the decomposition of the spinor representation under the four-dimensional geometric Langlands-twist. Bold face numbers denote representations under the $SU(2)$'s and superscripts the charges under the $U(1)$.*

The bosonic fields of the theory naturally also decompose under this choice of subgroup. The six scalar fields split into one one-form and one complex scalar, whereas the gauge field becomes a one-form. (The fermion fields naturally decompose in the same manner as the supersymmetry charges.) Using the knowledge of how all fields transform under the twist, the action of the twisted theory may be obtained by rewriting the ordinary $\mathcal{N} = 4$ Lagrangian in terms of these new twisted fields. Any possible curvature terms are then fixed by the requirement that it be supersymmetric. This twisted theory was presented by Kapustin and Witten in 2007, where they showed it to have applications to the geometric Langlands program [48]. Since its introduction, this theory has been extensively studied (an incomplete list of references includes [52, 53, 54, 55]), and shown to be related to a vast number of different areas in mathematics and physics. It is this four-dimensional theory that is the inspiration of the five-dimensional theory that has been the focus of papers I and II.

1.2 Twisting in five dimensions

In five dimensions, the situation is slightly different. If we consider maximally supersymmetric Yang-Mills theory on a general (Euclidean) five-manifold, we have both Lorentz- and R-symmetry groups given by $Spin(5)$, and there is thus one unique way to twist this theory. However, if we instead consider a five-manifold with a product structure, $M_5 = M_4 \times I$, where I is an interval and M_4 is an Euclidean four-manifold, the theory may admit several inequivalent twists, one of which is the five-dimensional analog of the GL-twist described above [49]. Furthermore, by restricting ourselves to manifolds with such a product structure, the overall requirement of five-dimensional Euclidean signature

[ii] This basically means that the boundary conditions preserve half of the supersymmetries.

is now superfluous for our purposes, since we only twist along M_4. Thus, as long as any time-like direction is taken to lie along the interval I, the twist may be carried out without problems.

Again, we use the fact that the Lorentz group along M_4 is isomorphic to $SU(2)_l \times SU(2)_r$, and that the R-symmetry group now has a subgroup of $U(1) \times SU(2)_R$. The twisting then amounts to identifying the $SU(2)_r$ with $SU(2)_R$, and, as in the four-dimensional case, this is summarised in table (7.2), illustrating how the representations of the different fields decompose under this identification.

	$SU(2)_l \times SU(2)_r \times U(1) \times SU(2)_R$	$\xrightarrow{twist}$	$SU(2)_l \times SU(2)' \times U(1)$	
A_μ	$(\mathbf{2{,}2{,}1})^{\mathbf{0}}$		$(\mathbf{2{,}2})^{\mathbf{0}}$	A_μ
A_y	$(\mathbf{1{,}1{,}1})^{\mathbf{0}}$		$(\mathbf{1{,}1})^{\mathbf{0}}$	A_y
Φ_I	$(\mathbf{1{,}1{,}1})^{\mathbf{+1}} \oplus (\mathbf{1{,}1{,}1})^{\mathbf{-1}} \oplus (\mathbf{1{,}1{,}3})^{\mathbf{0}}$		$(\mathbf{1{,}1})^{\mathbf{+1}} \oplus (\mathbf{1{,}1})^{\mathbf{-1}} \oplus (\mathbf{1{,}3})^{\mathbf{0}}$	$\sigma, \bar{\sigma}, B_{\mu\nu}$
λ_α	$(\mathbf{1{,}2{,}2})^{\mathbf{+1/2}} \oplus (\mathbf{1{,}2{,}2})^{\mathbf{-1/2}}$		$(\mathbf{1{,}1})^{\mathbf{\pm1/2}} \oplus (\mathbf{1{,}3})^{\mathbf{\pm1/2}}$	$\eta, \tilde{\eta}, \chi_{\mu\nu}, \tilde{\chi}_{\mu\nu}$
	$\oplus(\mathbf{2{,}1{,}2})^{\mathbf{+1/2}} \oplus (\mathbf{2{,}1{,}2})^{\mathbf{-1/2}}$		$\oplus(\mathbf{2{,}2})^{\mathbf{\pm1/2}}$	$\psi_\mu, \tilde{\psi}_\mu$

Table 7.2: *Table describing the decomposition of the different representations during the twisting procedure of the five-dimensional GL-twist. Bold face numbers denote representations under the $SU(2)$'s and superscripts the charges under the $U(1)$.*

Again, just as in the four-dimensional case, this twist gives rise to two scalar supersymmetries under the twisted symmetry group, and we have once more obtained a family of topological field theories. These five-dimensional theories are much less studied than their four-dimensional counterparts, and different aspects of this five-dimensional twist were studied in the first two papers appended to this thesis. For example, to obtain the full action of these theories was the objective of paper II.

2 *Twisting the* (2,0) *theory and lower-dimensional correspondences*

Another example of topological twisting was studied in paper III: the twist of the free tensor multiplet of (2,0) theory on a six-manifold with a product structure. Such a twisted theory is of physical interest since it may be the origin of the AGT-correspondence [32], which relates observables in a certain class of supersymmetric four-dimensional theories to quantities in two-dimensional Liouville theory (see section 1 of chapter 5). The hypothesis is as follows: If we take (2,0) theory on some (Euclidean) manifold $M_6 = \mathcal{C} \times M_4$, we can choose to compactify on either M_4 or the Riemann surface $\mathcal{C}$, giving rise to a two- or four-dimensional theory respectively. The idea is to then look for protected quantities in these lower-dimensional theories. However, on a general six-manifold, there will not be any supersymmetry, and so the lower-dimensional theories would not have any protected quantities.

One way of solving this problem would be to twist the (2,0) theory to create a topological field theory, and consider the compactification of such a theory on M_4 or $\mathcal{C}$. Then, there would be some protected quantities after compactification, which may just give us a correspondence between four- and two-dimensional supersymmetric theories as desired [32, 56, 57]. This idea is schematically described in figure 7.2. In the appended paper (III), focus lies on the left hand side of this diagram: the compactification on $\mathcal{C}$.

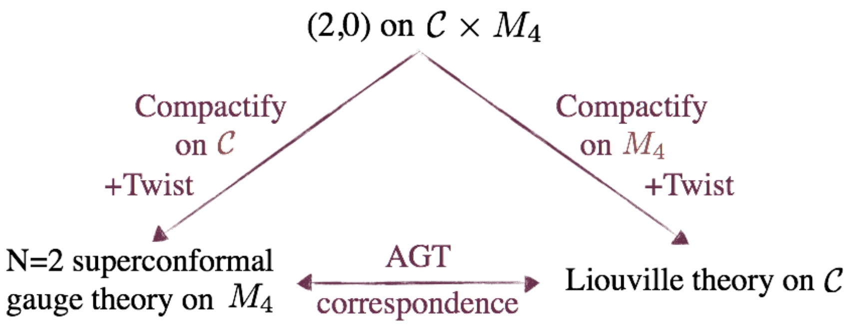

Figure 7.2: *Schematic picture of the approach to the AGT-correspondence taken in paper* III*, where we focus on the left arrow of the diagram.*

However, there are two immediate problems with this approach: there is no satisfactory formulation of the full (2,0) theory in any signature, and not even of the free tensor multiplet in Euclidean signature. So, like in the case of twisting five-dimensional Yang-Mills theory on $M_5 = M_4 \times I$ described in the previous section, we use the product structure of the base manifold and confine the time-like direction to lie along the Riemann surface $\mathcal{C}$. This allows us to carry out a twist along the four-manifold following the usual recipe, giving rise to two supercharges scalar on M_4 with parameters ε^+ and ε^-. This twisting is unique, and has been conjectured to be equivalent to the Donaldson-Witten twist of four-dimensional $\mathcal{N} = 2$ super Yang-Mills [41]. The two supercharges $\varepsilon^{\pm}$, though scalar on M_4, still transform under the holonomy of $\mathcal{C}$. By studying the transformation properties these supercharges would have *if* $\mathcal{C}$ had Euclidean signature (and therefore had holonomy group $U(1)$), we can hand-pick the supercharge that *would* become scalar if we could also twist along $\mathcal{C}$ (which turns out to be the one with $U(1)$-charge of $-1/2$ on M_4, that is, ε^-). We then consider our observables to lie in the cohomology of this supercharge. However, in the Donaldson-Witten twist, the supercharge used is rather a linear combination of ε^+ and ε^- (with non-vanishing coefficient in front of ε^+). The representations of the fields and supersymmetry parameters under the holonomy group of M_4 (which is $SU(2)_l \times SU(2)_r$) and R-symmetry group ($SU(2)_R \times U(1)_R$) before twisting, and under the twisted symmetry group after twisting is represented in tables 7.3 and 7.4 respectively. Bold faced numbers once more indicate representations under the $SU(2)$'s and superscripts the $U(1)$-charge.

Another difficulty in carrying out this twist is the absence of a Lagrangian for the

	$SU(2)_l \times SU(2)_r \times SU(2)_R \times U(1)_R$
Φ	$(\mathbf{1},\mathbf{1},\mathbf{3})^0 \oplus (\mathbf{1},\mathbf{1},\mathbf{1})^{\pm 1}$
Ψ	$(\mathbf{2},\mathbf{1},\mathbf{2})^{\pm 1/2} \oplus (\mathbf{1},\mathbf{2},\mathbf{2})^{\pm 1/2}$
H	$(\mathbf{3},\mathbf{1},\mathbf{1})^0 \oplus (\mathbf{1},\mathbf{3},\mathbf{1})^0 \oplus (\mathbf{2},\mathbf{2},\mathbf{1})^0$
ε	$(\mathbf{2},\mathbf{1},\mathbf{2})^{\pm 1/2} \oplus (\mathbf{1},\mathbf{2},\mathbf{2})^{\pm 1/2}$

Table 7.3: *Representations of the fields and supersymmetry parameters under both the holonomy group of M_4 and R-symmetry group before twisting. Bold faced numbers indicate representations under the $SU(2)$'s and the superscript the $U(1)$-charge.*

	$SU(2)_l \times SU(2)' \times U(1)_R$	Twisted fields
Φ	$(\mathbf{1},\mathbf{3})^0 \oplus (\mathbf{1},\mathbf{1})^{\pm 1}$	$E_{\mu\nu}, \bar{\sigma}, \sigma$
Ψ	$(\mathbf{2},\mathbf{2})^{\pm 1/2} \oplus (\mathbf{1},\mathbf{3},)^{\pm 1/2} \oplus (\mathbf{1},\mathbf{1})^{\pm 1/2}$	$\psi_\mu, \tilde{\psi}_\mu, \chi_{\mu\nu}, \tilde{\chi}_{\mu\nu}, \eta, \tilde{\eta}$
H	$(\mathbf{3},\mathbf{1})^0 \oplus (\mathbf{1},\mathbf{3})^0 \oplus (\mathbf{2},\mathbf{2})^0$	$F^-_{\mu\nu}, F^+_{\mu\nu}, A_\mu$
ε	$(\mathbf{2},\mathbf{2})^{\pm 1/2} \oplus (\mathbf{1},\mathbf{3})^{\pm 1/2} \oplus (\mathbf{1},\mathbf{1})^{\pm 1/2}$	$\ldots, (\varepsilon^+), \varepsilon^-$

Table 7.4: *Representations of fields and supersymmetry parameters under the twisted symmetry group.*

original (2,0) theory. Even for the free tensor multiplet, we are restricted to carrying out our calculations on the level of the equations of motion. Instead of attempting to construct a Lagrangian for the twisted theory, the goal is therefore constrained to finding a stress tensor satisfying all properties required of a topological field theory, i.e., that it is covariantly conserved and vanishes in cohomology. This work was started in paper III and later on finished and extended in [58]. It is our hope that considering this partial twist of the free tensor multiplet may in extension shed some light on the full AGT correspondence.

Chapter 8:

Localisation: handling infinite-dimensional integrals

We have now reviewed the concept of symmetries, of quantum field theory in general and some specific theories in particular, and seen that the way observable quantities are described in modern physics is by infinite-dimensional path integrals (as for example the partition function of equation (6.18)). These effectively describe the theory, but, are as mentioned hard, and often bordering on impossible, to compute. So even though we have a very nice formalism for describing the world on a fundamental level, actually computing observables, such as for example the free energy of a system, is still far from easy.

The concept of localisation in the context of supersymmetric theories was first suggested in the early 1980's [59], and applied shortly thereafter [60]. Ever since then, it has been a valuable tool for explicit calculations of observables. It boils down to the following basic idea: in theories with enough (super)symmetry, it is possible to reduce the infinite-dimensional integrals describing observable quantities to finite-dimensional ones. All the appended papers in this thesis are in some way related to this technique, though some more than others. We may make a rough division between two different situations in which localisation is applied:

- ⋆ Localisation in topological field theories
- ⋆ Localisation in supersymmetric field theories on symmetric manifolds

In both of these cases, the technique works roughly in the same way: By using the symmetry of the theory, we may localise the path integral onto a finitely-dimensional space of field configurations. We shall below describe the theory in a general fashion, and we will in the following sections go into further details on how it is used in the individual papers and the theories considered therein. But first, let us return to the general recipe of localisation.

Let δ be a Grassmann-odd symmetry of the theory (here, this will always be some supersymmetry), squaring to a bosonic symmetry (for example a linear combination of a Lorentz- and gauge transformation). Furthermore, let V be a quantity (often taken to be fermionic) which is invariant under this bosonic symmetry, δ^2. Then we may add a term $-t\delta V$ to the action, such that we consider the perturbed partition function $Z(t)$, defined by

$$Z(t) = \int \mathcal{D}\varphi \; e^{-S-t\delta V}. \tag{8.1}$$

We straight away see that $Z(t)$ is independent of t since

$$\partial_t Z(t) = \int \mathcal{D}\varphi \; \delta V e^{-S-t\delta V} = -\int \mathcal{D}\varphi \; \delta\Big(V e^{-S-t\delta V}\Big) = 0 \; \Big(+ \text{boundary terms}\Big). \tag{8.2}$$

Here, we have used that the measure is assumed to be invariant under the symmetry δ, and that $\delta S = \delta^2 V = 0$ (since both δ, δ^2 are symmetries of the theory). In this way, one may interpret the integrand as a total derivative, and by neglecting the boundary terms[i], we find that the perturbed path integral indeed is independent of t. This is the key point of localisation. We may thus compute the path integral in the the limit $t \to \infty$ instead of computing it for the original case of $t = 0$. Furthermore, if the bosonic part of δV is positive definite, $e^{-t\delta V}$ is dominated by field configurations satisfying $(\delta V)_{\text{Bosonic}} = 0$, and in the limit $t \to \infty$, the path integral thus localises onto these field configurations. If we are lucky, this is a finitely-dimensional space. In general, this recipe may be applied to compute the correlation functions of any δ-invariant operators, not just the partition function, by inserting the operator in question in (8.1).

However, there are several non-trivial requirements which must be satisfied for us to be able to use this technique − we must have a theory in which we may find a suitable δ and V. If we wish to take the fermionic symmetry δ to be a supersymmetry transformation, this poses strict requirements on the manifold on which the theory is placed to admit enough supersymmetry (unless the theory is topological). This requirement may be formulated such that the manifold must admit a constant Killing spinor. In some cases, there may be several choices of perturbation δV which fulfil all the requirements. It should then be pointed out that these will not all give rise to the same expression for the partition integral, but rather capture different aspects of the theory.

We will begin by considering the localisation technique in the context of topological field theories of cohomological type before moving on to an explicit localisation of the path integral: Chern-Simons matter theories on the three-sphere.

1 Localisation in topological field theories

If the theory we wish to localise is a topological field theory of cohomological type, there exists a natural choice for the fermionic symmetry δ: the scalar nilpotent supersymmetry Q.

The perturbation of the action, δV, is however slightly more complicated. A clever way of choosing this deformation is to take δV to be given by the sum of squares of the

[i] In rare cases, this cannot be done, but assuming that the fields vanish quickly enough at infinity, the perturbed path integral may be considered independent of t.

supersymmetry transformations of all fermionic fields of the theory. This is then clearly a positive definite bosonic quantity, and by the standard localisation argument, the theory localises onto field configurations such that the supersymmetry variations of all fermionic fields vanish.

However, the problem of finding these field configurations − of solving the localisation equations of the theory to obtain the localisation locus − is in general a non-trivial problem, and is the subject of paper I: finding the localisation locus of five-dimensional GL-twisted Yang-Mills theory. The localisation equations in this theory are a set of coupled, non-linear partial differential equations, and they are in the paper solved in the special case when the four-manifold has the form of a three sphere times an interval, $M_4 = S^3 \times I$. The solution to the five-dimensional equations (on $M_5 = \mathbb{R}^+ \times S^3 \times I$) may be seen as a tunnelling solution to a set of solutions to the localisation equations of the corresponding four-dimensional theory. When the interval I is small, there are no solutions at all in four dimensions (nor any in five), but as it increases, a pair of four-dimensional solutions show up, which are connected by a tunnelling solution in five dimensions.

Finding these field configurations is the first step towards an explicit expression for the partition function. This has however not yet been found in the case of five-dimensional GL-twisted Yang-Mills theory, but we will in the next section consider another theory in which this may be done explicitly.

2 An explicit example of localisation: Chern-Simons-matter theories

Consider a Chern-Simons type theory with matter (as described in chapter 6) placed on $\mathbb{S}^3$. Then, we may take the fermionic symmetry δ to be given by the supersymmetry generated by the two constant Killing spinors $\varepsilon, \bar{\varepsilon}$, in the special case where $\bar{\varepsilon} = \varepsilon$. The transformations of the fields under this symmetry are given by equation (6.9) and (6.16) of chapter 6, and it satisfies $\delta^2 = 0$, which trivially is a bosonic symmetry of the theory.

The localisation of the path integral, which is now given by

$$Z(t)_{CS} = \int \mathcal{D}\varphi \, e^{-\frac{ik}{4\pi}\mathcal{S}_{CS}}, \tag{8.3}$$

will here be done in two steps: first we shall compute the contribution from the vector multiplet, also known as localising the gauge sector of the theory, after which we will turn our attention to the effects of the matter fields. This was first done in [61], for a review see [8].

Localising the gauge sector

We choose the perturbation $-t\delta V$ of the path integral such that $\delta V = \mathcal{S}_{YM} = \int d^3x\sqrt{g}\,\mathcal{L}_{YM}$, where we recall that

$$\bar{\varepsilon}\varepsilon\mathcal{L}_{YM} = \delta_{\bar{\varepsilon}}\delta_{\varepsilon}\,\mathrm{Tr}\Big(\frac{1}{2}\bar{\lambda}\lambda - 2D\sigma\Big), \tag{8.4}$$

by equation (6.13) of chapter 6 (note that D here is an auxiliary field and should not be confused with the covariant derivative D_μ). By the localisation argument above, the partition function will not depend on t, so taking the limit $t \to \infty$ localises the path integral on field configurations that make the bosonic part of δV vanish. Since we know δV to be given by the Yang-Mills action, the bosonic part of this is simply given by a sum of squares as:

$$\delta V = \mathcal{S}_{YM} = \int d^3x\sqrt{g}\,\mathrm{Tr}\Big(\frac{1}{4}F_{\mu\nu}F^{\mu\nu} + \frac{1}{2}D_\mu\sigma D^\mu\sigma + \frac{1}{2}\left(D + \frac{\sigma}{r}\right)^2 + \dots\Big), \tag{8.5}$$

(where the dots represent fermionic quantities). The localisation equations are therefore in this case simply given by:

$$F_{\mu\nu}F^{\mu\nu} = D_\mu\sigma D^\mu\sigma = \left(D + \frac{\sigma}{r}\right)^2 = 0, \tag{8.6}$$

which are straight-forward to solve with solutions given by:

$$F_{\mu\nu} = 0 \qquad D_\mu\sigma = 0 \qquad D + \frac{\sigma}{r} = 0. \tag{8.7}$$

The requirement on the field strength forces the gauge connection to be flat, but on $\mathbb{S}^3$, the only possible flat connection is zero[ii]. This reduces the covariant derivative to an ordinary one, and we see that σ must be a constant, σ_0. In turn, this allows us to write D as $D = -\frac{\sigma_0}{r}$. Thus the localisation locus of the theory is indeed a finitely-dimensional space, where D and σ are Hermitian (dependent) matrices and the gauge connection vanishes.

If we perturb around these field configurations, such that

$$\begin{aligned} \sigma =&\sigma_0 + \frac{1}{\sqrt{t}}\sigma' \\ D =& -\frac{\sigma_0}{r} + \frac{1}{\sqrt{t}}D' \\ A \to& \frac{1}{\sqrt{t}}A, \end{aligned} \tag{8.8}$$

[ii]If the field strength vanishes, the gauge field can always be taken to vanish locally. However, on the three-sphere, we can define a basis for tangential vector fields globally (i.e., we can comb the hairs of $\mathbb{S}^3$), which means that we can extend our local definition of the gauge field to hold globally. This is due to the fact that $\mathbb{S}^3$ inherits a differentiable structure from $\mathbb{H}$, since we can naturally view $\mathbb{S}^3$ as the set of quaternions with unit norm, just as the unit circle can be taken to be all complex numbers of unit norm.

and do the same rescaling for the fermionic fields as for the gauge connection, we find that the perturbation, δV, in terms of these new rescaled and perturbed fields becomes:

$$t\delta V = t\mathcal{S}_{YM} = \frac{1}{2}\int d^3x\sqrt{g}\ \mathrm{Tr}\left(-A^\mu\Delta A_\mu - [A_\mu,\sigma_0]^2 + \partial_\mu\sigma'\partial^\mu\sigma' + \frac{1}{2}\left(D' + \frac{\sigma'}{r}\right)^2 + \ \ldots\right) \\ + \mathcal{O}(t^{-1/2}), \tag{8.9}$$

and thus, as $t \to \infty$, the only terms in equation (8.9) that will contribute to $e^{-t\delta V}$ are those quadratic in the fields. This means that the integrals we need to compute are Gaussian integrals, and they may thus be carried out exactly. This contribution from the quadratic fluctuations around the saddle points of δV is what is commonly known as the *one loop determinant*, Z_{1-loop}.

Here is however a small subtlety which we need to point out. What one really wishes to localise is the gauge fixed theory, but this was shown in [61] to not give any additional contributions, and we may continue our calculations without restricting to a particular gauge.

We now wish to compute the contribution of the Chern-Simons action[iii] of equation (6.11) to the path integral. On the perturbed field configurations (8.8), this gives us:

$$\exp\left[-\frac{ik}{4\pi}S_{CS}[\sigma_0]\right] = \exp\left[-\frac{ik}{4\pi}\int d^3x\sqrt{g}\ 2\mathrm{Tr}\left(\sigma_0^2\right) + \mathcal{O}(t^{-1/2})\right] = \exp\left[-i\pi k\mathrm{Tr}\left(\sigma_0^2\right)\right], \tag{8.10}$$

where the factors of 2π are cancelled by the volume of the unit sphere, and higher-order terms in $1/t$ vanish identically.

Thus we may now write the partition function, in the limit of large t, as

$$Z(t) = \int d\sigma_0\ e^{-\frac{ik}{4\pi}S_{CS}[\sigma_0]}\ Z_{1-\mathrm{loop}}[\sigma_0], \tag{8.11}$$

where the dependance on all other fields has been incorporated into the one-loop determinant. One can view the integrations over the fields which decouple from σ_0 as some overall normalisation, which may be ignored for now and instead be computed at weak coupling [61]. The only bosonic field which couples to σ_0 in the one-loop determinant is A_μ, and we thus need to carry out this integration. There will also be some contribution from the fermions, but the calculations are more or less identical and shall be omitted here. The contribution to $Z_{1-\mathrm{loop}}[\sigma_0]$ from the bosons will therefore be given by:

$$Z^{bos}_{1-\mathrm{loop}}[\sigma_0] = \int dA_\mu \exp[\ \frac{1}{2}\ \int d^3x\sqrt{g}\ \ \mathrm{Tr}\left(A^\mu\Delta A_\mu + [A_\mu,\sigma_0]^2\right). \tag{8.12}$$

[iii]Recall that this was given by: $\mathcal{S}_{CS} = \int d^3x\sqrt{g}\ \mathrm{Tr}\Big(\epsilon^{\mu\nu\rho}\left(A_\mu\partial_\nu A_\rho + \frac{2i}{3}A_\mu A_\nu A_\rho\right) - \bar{\lambda}\lambda + 2D\sigma\Big)$.

To carry out this integration, it is convenient to note that since $\mathcal{S}_{YM}$ is invariant under gauge transformations, we may diagonalise σ_0 to take values in some chosen Cartan subalgebra of the gauge group, and we denote the eigenvalues of this matrix by $\sigma_{i,j,\dots}$. Doing this introduces a Vandermonde determinant,

$$\Delta(\sigma) = \prod_{\alpha>0} \alpha(\sigma_0)^2, \tag{8.13}$$

coming from the measure of the integral. Here, $\alpha(\sigma_0)$ are the roots of the gauge group in question, (more specifically $\alpha(\sigma_0) = \sigma^i H_i$, where the matrices H_i form a basis of the Cartan subalgebra). There is however some gauge symmetry remaining, namely the Weyl group $\mathcal{W}$ of G. In order to prevent ourselves from double counting equivalent states, we should divide the whole path integral by $|\mathcal{W}|$, the order of this group.

Furthermore, using the fact that we have diagonalised σ_0 to lie in the Cartan algebra, we may decompose the gauge field A_μ into components along the Cartan, and components along the root space (spanned by vectors X_α satisfying $\mathrm{Tr}(X_\alpha X_\beta) = \delta_{\alpha+\beta}$), which allows us to write the commutator of σ_0 with A_μ as

$$[\sigma_0, A_\mu] = \alpha(\sigma_0) A_\mu^\alpha X_\alpha. \tag{8.14}$$

This may be seen as follows: The basis elements of the root space may be taken to be the non-zero eigenvectors of the basis of the Cartan subalgebra (H_i)[iv], with corresponding eigenvalues α_i (i.e. $[H_i, X_\alpha] = \alpha_i X_\alpha$). And so for any element in the Cartan (such as σ_0), we have

$$[\sigma_0, X_\alpha] = \sum_i \sigma^i \alpha_i X_\alpha = \alpha(\sigma_0) X_\alpha, \tag{8.15}$$

which gives us just relation (8.14).

Thus we may now write down the bosonic contribution to the one-loop determinant as:

$$Z^{bos}_{1-\mathrm{loop}}[\sigma_0] = \int dA_\mu \exp[-\frac{1}{2} \int d^3x\sqrt{g}\ \mathrm{Tr}\left(\sum_\alpha (g^{\mu\nu} A_\mu^{-\alpha}(-\Delta + \alpha(\sigma_0)^2) A_\nu^\alpha\right), \tag{8.16}$$

which is a Gaussian integral, and it may as such be evaluated by computing the determinant of the operator $(\Delta - \alpha(\sigma_0)^2)$. The eigenfunctions of the Laplacian on $\mathbb{S}^3$ are given by the vector spherical harmonics[v], with eigenvalues $(l+1)^2$ and degeneracies $2l(l+1)$. Thus the eigenvalues of the operator $(-\Delta+\alpha(\sigma_0)^2)$ are simply given by $((l+1)^2+\alpha(\sigma_0)^2)$, again with degeneracies $2l(l+2)$. Hence, the determinant for the bosons is given by:

$$\det(\text{bosons}) = \prod_\alpha \prod_{l>0} \left(((l+1)^2 + \alpha(\sigma_0)^2)^{2l(l+2)}\right). \tag{8.17}$$

[iv]The root space is exactly the dual space to the Cartan, and thus the orthogonal (linearly independent) vectors to the matrices H_i span this space.

[v] A_μ may here be treated as a divergenceless vector field, for a more detailed description, see [61].

By standard arguments for Gaussian integrals [8], the bosonic one-loop contribution to the path integral of equation (8.16) may thus be written as:

$$Z^{\text{bosons}}_{1-\text{loop}}[\sigma_0] = \Big(\prod_{\alpha}\prod_{l>0}\Big(((l+1)^2+\alpha(\sigma_0)^2)^{2l(l+2)}\Big)\Big)^{-\frac{1}{2}}. \tag{8.18}$$

By following the same approach for the fermions of the Chern-Simons-vector multiplet, one finds that their contribution to the one-loop determinant is given by:

$$Z^{\text{fermions}}_{1-\text{loop}}[\sigma_0] = \prod_{\alpha}\prod_{l>0}\Big((l+i\alpha(\sigma_0))(-l-1+i\alpha(\sigma_0))\Big)^{l(l+1)}. \tag{8.19}$$

So, the one-loop contribution from the vector multiplet in its entirety may be written as:

$$Z_{1-\text{loop}}[\sigma_0] = \Delta(\sigma_0)\prod_{\alpha}\prod_{l>0}\frac{\Big((l+i\alpha(\sigma_0))(-l-1+i\alpha(\sigma_0))\Big)^{l(l+1)}}{\Big((l+1)^2+\alpha(\sigma_0)^2\Big)^{l(l+2)}}, \tag{8.20}$$

which may be simplified to yield:

$$Z_{1-\text{loop}}[\sigma_0] = \Delta(\sigma_0)\prod_{\alpha}\prod_{l>0}\frac{(l+i\alpha(\sigma_0))^{l+1}}{(l-i\alpha(\sigma_0))^{l-1}}, \tag{8.21}$$

where we recall that $\Delta(\sigma_0)$ is the Vandermonde determinant of (8.13) introduced by the diagonalisation of σ_0. This may be even further simplified by the fact that for each root $\alpha(\sigma_0)$, $-\alpha(\sigma_0)$ will also be a root. Using this, we may rewrite the product over only the positive roots, giving:

$$Z_{1-\text{loop}}[\sigma_0] = \Delta(\sigma_0)\prod_{l>0}l^4\prod_{\alpha>0}\Big(1+\frac{\alpha(\sigma_0)^2}{l^2}\Big)^2 = \Delta(\sigma_0)\prod_{l>0}l^4\prod_{\alpha>0}\Big(\frac{2\ \sinh(\pi\alpha(\sigma_0))}{\pi\alpha(\sigma_0)}\Big)^2. \tag{8.22}$$

The infinite product over l may be handled by zeta regularisation to give a finite number, and the denominator is cancelled by the Vandermonde determinant.

It is now convenient to introduce the variables μ and g_s, defined by:

$$g_s = \frac{2\pi i}{k} \qquad\qquad \sigma_0 = \frac{\mu}{2\pi}. \tag{8.23}$$

In terms of these, the contribution to the path integral coming from the gauge fields may be written as:

$$Z^{\text{gauge}}(t) \propto \int \mathrm{d}\mu\, e^{-\frac{1}{2g_s}\,\mathrm{Tr}\left(\mu^2\right)}\prod_{\alpha>0}\Big(2\ \sinh(\alpha(\frac{\mu}{2}))\Big)^2. \tag{8.24}$$

For Chern-Simons theory, where the gauge group is $U(N)$, this simplifies even further as the Cartan may be taken to be the set of diagonal matrices and the roots α can be labelled by two integers, $i \neq j$, and are then given by:

$$\alpha_{i,j}(\sigma_0) = \sigma_i - \sigma_j. \tag{8.25}$$

Hence

$$Z_{CS}^{\text{gauge}}(t) \propto \int \prod_i \mathrm{d}\mu_i \, e^{-\frac{1}{2g_s}\sum_{i=1}^N \mu_i^2} \prod_{i<j} \left(2 \, \sinh \frac{\mu_i - \mu_j}{2}\right)^2, \tag{8.26}$$

with $\mu_i = 2\pi\sigma_i$, and the normalisation constant coming from the integrations over all fields which decouple from σ_0.

Localising the matter fields

If we add matter multiplets to our theory, with supersymmetry transformations given by equation (6.16), the action for these fields may on its own be written as a superderivative (6.17). Hence, we may simply introduce the parameter t in front of this term in the partition function, and, by the now well-known localisation argument, the value of the partition function at $t = 1$ (i.e., the original partition function for Chern-Simons theory with matter), and for $t \to \infty$, are the same.

On the localisation locus of the gauge fields, the (real) bosonic part of the matter Lagrangian is positive definite, and minimised by

$$\phi = 0, \tag{8.27}$$

in which case it simply vanishes. Hence, there will be no contribution of the matter multiplets to the classical action. Rather, the only contribution to the path integral they will give is through the one-loop determinant, which, following the same technique used for the gauge sector, may be computed to give

$$Z_{1-\text{loop}}^{\text{matter}}[\sigma_0] = \prod_{m>0} \left(\frac{m+1-\Delta+ir\sigma_0}{m-1+\Delta-ir\sigma_0}\right)^m, \tag{8.28}$$

where we recall that Δ was the dimension of the scalar ϕ.

For matter in a general representation R of the gauge group, and general values of Δ, this is a very complicated expression, but for the case when there are no quantum corrections to Δ (that is, $\Delta = 1/2$), and the representation is self-conjugate, it simplifies drastically, and may, after regularisation, be written as:

$$Z_{1-\text{loop}}^{\text{matter}}[\mu] = \prod_\lambda \left(2\cosh\frac{\Lambda(\mu)}{2}\right)^{-1/2}, \tag{8.29}$$

where Λ are the weights of the representation of the matter multiplet, and $\mu = 2\pi\sigma$ as before [8].

Chapter 9:

The ABJM matrix model and large N techniques

We have now arrived at the point where we may write down the ABJ matrix model, which has been studied in the limit of infinite mass in the last two papers appended to this thesis. So far, all calculations have been for the massless case, but the generalisation to the situation with non-vanishing mass is quite straight-forward.

In this case, the gauge groups and levels are $U_k(N_1) \times U_{-k}(N_2)$, and so there are two vector multiplets in the adjoint representation of the two gauge groups, each contributing with one sinh-factor to the one-loop determinant, and appearing with opposite signs in the exponential due to the opposite Chern-Simons levels of the two gauge groups. Thus the total contribution from the vector multiplets to the partition function may be written as:

$$\begin{aligned} Z_{ABJ}(\mathbb{S}^3) \propto \int \prod_{i=1}^{N_1} \mathrm{d}\mu_i \prod_{a=1}^{N_2} \mathrm{d}\nu_a \, e^{-\frac{1}{2g_s}\left(\sum_{i=1}^{N_1}\mu_i^2 - \sum_{a=1}^{N_2}\nu_a^2\right)} \\ \prod_{i<j}\left(2\, \sinh\frac{\mu_i-\mu_j}{2}\right)^2 \prod_{a<b}\left(2\, \sinh\frac{\nu_a-\nu_b}{2}\right)^2 Z_{1-\text{loop}}^{\text{matter}}[\mu_i,\nu_a]. \end{aligned} \tag{9.1}$$

Furthermore, there are four hypermultiplets in the bifundamental representation, which in total give a contribution to the one-loop determinant of the form:

$$Z_{1-\text{loop}}^{\text{matter}}[\mu_i,\nu_a] = \prod_{i=1}^{N_1}\prod_{a=1}^{N_2}\left(2\cosh\frac{\mu_i-\nu_a}{2}\right)^{-2}. \tag{9.2}$$

The generalisation to the case where the hypermultiplets all are given the same mass m simply amounts to a shift in the arguments of the hyperbolic cosines in the matter one-loop determinant:

$$Z_{1-\text{loop}}^{\text{matter}}[\mu_i,\nu_a] = \prod_{i=1}^{N_1}\prod_{a=1}^{N_2}\left(4\, \cosh\frac{\mu_i-\nu_a+m}{2}\, \cosh\frac{\mu_i-\nu_a-m}{2}\right)^{-1}. \tag{9.3}$$

Thus, by using localisation, we have reduced the infinite-dimensional integral expression for the partition function of ABJ theory, given by equation (6.18), to a finite-dimensional one, which, though complicated, may be solved exactly, at least in the limit where the number of colours, N, approaches infinity[i]. Note here that we have only determined the partition function up to a normalisation constant, but this will only depend on N and may be obtained by computing the partition function for any value of the localisation parameter t. In all its glory, (following the normalisation of [62]), the ABJ matrix model is given by:

$$\begin{aligned} Z_{ABJ}(\mathbb{S}^3) =& \frac{1}{N_1!N_2!}\int \prod_{i=1}^{N_1}\frac{\mathrm{d}\mu_i}{2\pi}\prod_{a=1}^{N_2}\frac{\mathrm{d}\nu_a}{2\pi} \\ &\times \frac{\prod\limits_{i\neq j}\sinh^2\frac{\mu_i-\mu_j}{2}\prod\limits_{a\neq b}\sinh^2\frac{\nu_a-\nu_b}{2}}{\prod\limits_{i,a}\cosh\frac{\mu_i-\nu_a+m}{2}\cosh\frac{\mu_i-\nu_a-m}{2}} e^{i\zeta\left(\sum\limits_i\mu_i+\sum\limits_a\nu_a\right)-\frac{1}{2g_s}\left(\sum_{i=1}^{N_1}\mu_i^2-\sum_{a=1}^{N_2}\nu_a^2\right)}, \end{aligned} \tag{9.4}$$

where we recall that g_s is related to the Chern-Simons level by $g_s = \frac{2\pi i}{k}$. ζ is the so-called Fayet-Illiopoulus parameter, which is present in the most general form of deformation of the theory (it is possible to add a term of the form $+\zeta D$ to the action, where D is the auxiliary scalar of the vector multiplet).

This matrix model was first obtained in [61], where the expectation values of Wilson loops were derived. In [62], the free energy was computed exactly in the case $N_1 = N_2 \to \infty$, where the $N^{3/2}$ scaling behaviour was found to agree perfectly in the strong coupling limit with the results of geometrical calculations in the dual theory, IIA string theory on $AdS_4\times\mathbb{CP}^3$. Since then, this matrix model has been extensively studied, see for example [63], (where it was shown that the ABJM partition function with $k = 1$ precisely corresponds to the partition function of $\mathcal{N} = 4$ SYM on $\mathbb{S}^3$) and [64, 65, 66] (where a new approach of studying these matrix models was developed and applied, namely by considering them as a Fermi gas).

1 *The limit of infinitely many colours*

As we saw in the previous chapter, even though the matrix models may be finite-dimensional integrals, these may still be very hard and it is only in specific cases that they may be solved exactly. One such example is in the limit where the number of colours tends to infinity. In this limit, the partition function of ABJ theory on the three sphere, which we derived in the previous chapter (9.4) may be drastically simplified by using a *saddle point approximation*, which in this limit becomes exact.

We start by noting that we may rewrite the partition function of (9.4) in terms of an effective action, $\mathcal{S}_{eff}$, such that:

$$Z_{ABJ}(\mathbb{S}^3) = \frac{1}{N_1!N_2!}\int \prod_{i=1}^{N_1}\frac{\mathrm{d}\mu_i}{2\pi}\prod_{a=1}^{N_2}\frac{\mathrm{d}\nu_a}{2\pi}\, e^{-\frac{1}{g_s^2}\mathcal{S}_{eff}}, \tag{9.5}$$

[i]Thankfully, we live in a world where the apparent number of colours is 3, which is quite close to infinity.

where g_s may be seen as $\hbar$, so when $g_s \to 0$, the partition function is simply given by its classical value, that is, the *saddle-point configuration* that extremises the effective action. However, for this argument to hold, we need the effective action to be independent of g_s. This may be achieved by introducing the new parameters $\lambda_{1,2} = g_s N_{1,2}$, known as the *'t Hooft couplings* of the theory. We then consider these as independent of g_s (a common limit to take is the so-called 't Hooft limit, where we consider $\lambda_{1,2}$ as fixed as we take the large N and small g_s−limit). In terms of these new parameters, the effective action may be written as:

$$\begin{aligned}\mathcal{S}_{eff} =& -2\frac{\lambda_1^2}{N_1^2}\sum_{i\neq j}\log\left(\sinh\frac{\mu_i-\mu_j}{2}\right)-2\frac{\lambda_2^2}{N_2^2}\sum_{a\neq b}\log\left(\sinh\frac{\nu_a-\nu_b}{2}\right)\\ &+\frac{\lambda_1\lambda_2}{N_1N_2}\sum_{i,a}\left(\log\left(\cosh\frac{\mu_i-\nu_a+m}{2}\right)+\log\left(\cosh\frac{\mu_i-\nu_a-m}{2}\right)\right)\\ &-i\zeta\frac{\lambda_1\lambda_2}{N_1N_2}\left(\sum_i\mu_i+\sum_a\nu_a\right)+\frac{1}{2}\left(\frac{\lambda_1}{N_1}\sum_{i=1}^{N_1}\mu_i^2-\frac{\lambda_2}{N_2}\sum_{a=1}^{N_2}\nu_a^2\right).\end{aligned} \tag{9.6}$$

So, in the classical limit of $g_s \to 0$, for large N, the saddle-point approximation will be exact, since, for fixed λ, the effective action will be roughly of order one.

Finding the field configurations that extremise the effective action amounts to solving the equations of motion for the eigenvalues μ_i, ν_a, however, they are commonly referred to as the *saddle-point equations* of the theory (rather than the equations of motion). For the ABJ matrix model, these are given by:

$$\begin{aligned}\mu_i &= \frac{\lambda_1}{N_1}\sum_{i\neq j}\coth\frac{\mu_i-\mu_j}{2}-\frac{\lambda_2}{2N_2}\sum_a\left(\tanh\frac{\mu_i-\nu_a+m}{2}+\tanh\frac{\mu_i-\nu_a-m}{2}\right)+\frac{i\zeta\lambda_2}{N_2}\\ \nu_a &= -\frac{\lambda_2}{N_2}\sum_{a\neq b}\coth\frac{\nu_a-\nu_b}{2}+\frac{\lambda_1}{2N_1}\sum_i\left(\tanh\frac{\nu_a-\mu_i-m}{2}+\tanh\frac{\nu_a-\mu_i+m}{2}\right)-\frac{i\zeta\lambda_1}{N_1}.\end{aligned} \tag{9.7}$$

In general, these saddle-point equations cannot be solved analytically, even in the large N limit, but there are some technical manipulations we may do to simplify the situation. First of all, we may absorb the FI-parameter into the arguments of the hyperbolic tangent-terms by a change of integration variables in the partition function such that $\mu \to \mu - \zeta,\ \ \nu \to \nu + \zeta$, which simplifies the situation slightly. Furthermore, it has been shown to be useful to perform an analytic continuation in the 't Hooft couplings, such that $\lambda_2 \to -\lambda_2$, and thereafter consider both $\lambda_{1,2}$ to be real. This is naturally not a physical situation, since it forces either the Chern-Simons levels, or the gauge group ranks, to be purely imaginary, but it is non the less helpful from a computational perspective. This is because the eigenvalues μ_i,ν_a also become real in this analytically continued model, whereas they in general for ABJ theories are distributed along some cuts in the complex plane, which results in the saddle-point equations becoming unstable. Thus we may use the analytically continued model to carry out our calculations and obtain results, which, under favourable conditions, then may be analytically continued back to the physical theory.

After these manipulations, we land in the following saddle-point equations:

$$\begin{aligned}\mu_i &= \frac{\lambda_1}{N_1}\sum_{i\neq j}\coth\frac{\mu_i-\mu_j}{2}+\frac{\lambda_2}{2N_2}\sum_a\left(\tanh\frac{\mu_i-\nu_a+m_1}{2}+\tanh\frac{\mu_i-\nu_a-m_2}{2}\right) \qquad (9.8)\\ \nu_a &= \frac{\lambda_2}{N_2}\sum_{a\neq b}\coth\frac{\nu_a-\nu_b}{2}+\frac{\lambda_1}{2N_1}\sum_i\left(\tanh\frac{\nu_a-\mu_i-m_1}{2}+\tanh\frac{\nu_a-\mu_i+m_2}{2}\right),\end{aligned}$$

where $m_1 = m + 2\zeta, m_2 = m - 2\zeta$. Again using the fact that N is large, we pass to the continuum limit by introducing density functions for the eigenvalues (instead of considering discrete sets of eigenvalues).

$$\rho_\mu(\mu) = \frac{1}{N_1}\sum_i\delta(\mu-\mu_i) \qquad \rho_\nu(\nu) = \frac{1}{N_2}\sum_a\delta(\nu-\nu_a). \qquad (9.9)$$

These are supported on some intervals $\mathcal{C}_\mu = [-A,B]$, $\mathcal{C}_\nu = [-C,D]$ along the real axis, and normalised such that:

$$\int\rho_\mu(\mu)\,d\mu = 1 = \int\rho_\nu(\nu)\,d\nu. \qquad (9.10)$$

We can now rewrite the discrete saddle-point equations in equation (9.8) as integral equations for the densities, ρ_μ, ρ_ν:

$$\begin{aligned}\mu &= \lambda_1\int_{\mathcal{C}_\mu}\mathrm{d}\mu'\,\rho_\mu(\mu')\,\coth\frac{\mu-\mu'}{2}+\frac{\lambda_2}{2}\int_{\mathcal{C}_\nu}\mathrm{d}\nu\,\rho_\nu(\nu)\left(\tanh\frac{\mu-\nu+m_1}{2}+\tanh\frac{\mu-\nu-m_2}{2}\right)\\ &\qquad (9.11)\\ \nu &= \lambda_2\int_{\mathcal{C}_\nu}\mathrm{d}\nu'\,\rho_\nu(\nu')\,\coth\frac{\nu-\nu'}{2}+\frac{\lambda_1}{2}\int_{\mathcal{C}_\mu}\mathrm{d}\mu\,\rho_\mu(\mu)\left(\tanh\frac{\nu-\mu-m_1}{2}+\tanh\frac{\nu-\mu+m_2}{2}\right).\end{aligned}$$

In the special case of massless ABJM theory with vanishing FI-parameter, these saddle-point equations have been extensively studied, and several observables have been computed [64]. However, the situation which has been studied in the papers appended to this thesis is the more general case with non-vanishing mass (paper IV) and FI-parameter (paper V).

2 Phase transitions in massive theories

In general, the equations (9.11) cannot be solved analytically for $m_1,m_2 \neq 0$, but, in massive theories, interesting things have been found to occur in a particular limit known as the *decompactification limit*, where the saddle-point equations simplify. To be able to take this limit in a self-consistent manner, it requires us to rescale the 't Hooft coupling with the inverse radius of the sphere, and we therefore introduce $t_{1,2} \equiv \lambda_{1,2}/R$.

Apart from simplifying the saddle-point equations, the decompactification limit brings with it new and interesting quantitative features of the theory. This limit consists of taking the radius of the sphere used to localise the theory to infinity, and in this infinite-volume limit, phase transitions occur at certain values of the couplings t_1 and t_2. The physical origin of these phase transitions is the appearance of new massless particles that contribute to the saddle-points at these critical points, $t_{1,2}^{(c)}$. These correspond to the situations when the arguments of the hyperbolic tangent functions in the saddle-point equations (9.11) lie within the support of ρ_μ, ρ_ν respectively, in some part of the integration regime. That is, when some of the points $A \pm m_{2,1}$ or $m_2 - B$ move inside $[-C,D]$ (or similarly when shifted points from $\mathcal{C}_\nu$ moves inside $\mathcal{C}_\mu$). This is illustrated for the case where $t_1 = t_2$ (which also gives $\mathcal{C}_\mu = \mathcal{C}_\nu$ and $\rho_\nu = \rho_\mu$) in figure 9.1. There, the situation of small but still positive m_2 is illustrated (in this case, $A \approx 0$), and the region of support of the eigenvalue density is shaded in pink. A phase transition occurs as m_2 becomes positive, since the shifted point $-A + m_2$ moves inside the interval. The shaded blue region illustrates the region in which one of the shifted tanh-terms in equation (9.11) contributes. This picture complicates significantly as m_2 grows, see for example figure 2 of paper V.

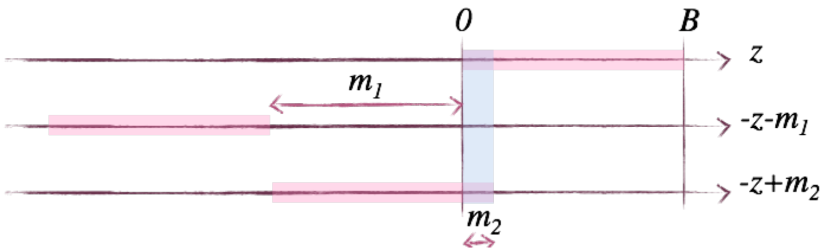

Figure 9.1: *Schematic picture of when a phase transition occurs in the decompactification limit of (analytically continued) ABJM. This illustrates the simplest case of $t_1 = t_2$ and m_2 small and positive. A phase transition occurs as m_2 becomes positive, since the shifted point $-z + m_2$ then moves inside the region of support of the eigenvalue density.*

This appearance of new massless particles is a sort of resonance phenomena, and occurs every time intervals increase to admit shifts of the latest resonance points. This is further explained in section 3.1 of [67], and it is not a phenomena unique to (analytical continuations of) ABJM theory.

In recent years, this kind of phase transitions have been found in a variety of theories: In four-dimensional $\mathcal{N} = 2^*$ super Yang-Mills[ii], an infinite series of phase transitions was found in [67], and these transitions have since then been further studied in [68, 69, 70, 71,

[ii] $\mathcal{N} = 2$ super Yang-Mills with matter in the adjoint representation.

72, 73]. Three-dimensional Chern-Simons theory exhibits a finite number of phases with running coupling, [74, 75], as does five-dimensional $\mathcal{N} = 1$ super Yang-Mills [76, 77].

For the case of ABJ theories, the situation is similar to that of $\mathcal{N} = 2^*$: for vanishing FI-parameter, the theory exhibits an infinite number of phase transitions as the coupling runs from zero to infinity, and furthermore, they appear to accumulate at strong coupling (paper IV). They should as such be visible even in the gravity dual theory, which is something that has been investigated for its four-dimensional cousin, but not yet for the analytically continued ABJ models. When the FI-parameter is introduced (as in paper V), this serves to regularise the theory, and the number of phase transitions is no longer infinite. Actually, for an FI-parameter larger than a certain multiple of the mass, the phase transitions vanish completely.

All of these results are however only valid for the analytically continued model, whereas for the physical ABJM theory, the phase transitions appear to be absent. This is shown in paper V, where the partition function of the physical theory is shown to be symmetric under the exchange of mass- and FI-parameter. Since a large FI-parameter rids the theory of any phase transitions, we expect the free energy of the theory to be a nice and smooth function for large ζ, and for there to be no problems with analytically continuing this back to the original, physical theory. Then using the symmetry property for the partition function implies that there can be no phase transitions, even for smaller values of the FI-parameter (in relation to the mass). Explicit calculations in the case where the gauge group is taken to be $U(2) \times U(2)$ also supports this conclusion.

Even though much progress on understanding these phase transitions has been made in recent years, there is still a lot which remains unknown. Since they appear in such a broad class of theories, it would be highly interesting to attempt to take a more general approach: What are the requirements on the theory for it to contain phase transitions in the decompactification limit? What determines if they are infinite or finite in number? Hopefully, time will tell.

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