E_8 geometry

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ABSTRACT: We investigate exceptional generalised diffeomorphisms based on $E_{8(8)}$ in a geometric setting. The transformations include gauge transformations for the dual gravity field. The surprising key result, which allows for a development of a tensor formalism, is that it is possible to define field-dependent transformations containing connection, which are covariant. We solve for the spin connection and construct a curvature tensor. A geometry for the Ehlers symmetry $\mathrm{SL}(n+1)$ is sketched. Some related issues are discussed.

Contents

1 Introduction

Doubled geometry and exceptional geometry provide a means to include all massless gauge fields in string theory or M-theory into a unified setting, providing a geometric origin of T-duality [1–26] or U-duality [27–44].

The purpose of the present paper is to extend the concept of extended geometry to E_8, the U-duality group obtained when M-theory is dimensionally reduced to 3 dimensions. We will however focus on the geometric picture for the "internal" dimensions. Some work on this case has been done previously. In refs. [34, 36], it was noted that a naïve attempt to extend the definition of generalised diffeomorphisms fail to close — the commutator of two such transformations produce a local E_8 transformation of a restricted kind. Hohm and Samtleben [45] nevertheless managed to base a description of 11-dimensional supergravity in a 3+8 split on such transformations, however with the drawback that a geometric understanding was lacking. It was observed by one of the present authors [46] that the form of the "extra" E_8 transformations suggests an interpretation in terms of a connection. We will build on the latter observation, and develop an E_8 geometry. It essentially vindicates the conclusions in ref. [45].

The difficulty with E_8 is sometimes attributed to the occurrence of a dual gravity field. Constructing a geometry for E_8 may be a first step towards incorporating dual gravity. If contact is to be made with the infinite-dimensional cases of E_9, E_{10} (and maybe E_{11}), this is essential, especially since there are no-go theorems to circumvent [47]. We will comment

more on this in the discussion section. A solution to the problem is also relevant for lower n, where non-covariance occurs not at the level of the algebra of generalised diffeomorphisms, but higher in their reducibility, where mixed symmetry fields arise.

The paper is organised as follows: in section 2, we discuss exceptional geometry (the arguments are valid also for ordinary and doubled geometry) from the perspective of covariance and closure. This helps us, in section 3, to get a better geometric understanding of what happens for E_8, and leads us to a candidate field-dependent transformation. In section 4, it is shown that this transformation, quite surprisingly, has the required covariance property, and the algebra is examined. Section 5 is devoted to the development of the geometric framework of models based on this symmetry. We define torsion, solve for the spin connection and find a Ricci scalar. Section 6 deals with reducibility and covariance, for the E_8 case, but also for lower n. Section 7 sketches the situation for the simpler case of Ehlers symmetry, where the dual gravity field also is present. We end with a summary and discussion.

2 Covariance and closure for generalised diffeomorphisms

In this preparatory section, we will revisit the concepts of covariance and closure, especially how they are linked together, for the cases known to work: ordinary diffeomorphisms, double diffeomorphisms, and exceptional diffeomorphisms for $n \leq 7$. The exceptional cases, of which $n = 8$ is continuing the series, will however be our model examples. This will give us tools to use when analysing the case $n = 8$.

Consider some generalised diffeomorphism, which is generated by $\mathring{\mathcal{L}}_\xi$ constructed with naked derivatives. Let its action on a vector in the module R_1 of $E_{n(n)}$ be defined by

$$
\begin{aligned}
\mathring{\mathcal{L}}_\xi V^M &= L_\xi V^M + Y^{MN}{}_{PQ} \partial_N \xi^P V^Q \\
&= \xi^N \partial_N V^M + Z^{MN}{}_{PQ} \partial_N \xi^P V^Q \ ,
\end{aligned}
\tag{2.1}
$$

where the Z in the second term ensures that the indices $^M{}_Q$ are projected on $\mathfrak{e}_{n(n)} \oplus \mathbb{R} \subset \mathfrak{gl}(|R_1|)$ [34, 36]. Of course the expression applies also for ordinary diffeomorphisms and for double diffeomorphisms. The invariant tensors Y or Z for the exceptional series have been given in diverse papers, e.g. ref. [36], and will not be repeated here. They satisfy the important identity

$$
\left(Y^{MN}{}_{TQ} Y^{TP}{}_{RS} - Y^{MN}{}_{RS} \delta^P_Q \right) \partial_{(N} \otimes \partial_{P)} = 0 \ ,
\tag{2.2}
$$

which can equivalently be written

$$
\left(Z^{MN}{}_{TQ} Z^{TP}{}_{RS} + Z^{MP}{}_{RQ} \delta^N_S \right) \partial_{(N} \otimes \partial_{P)} = 0 \ .
\tag{2.3}
$$

The tensor Y governs the section condition $(\partial \otimes \partial)|_{\overline{R}_2} = 0$, which reads

$$
Y^{MN}{}_{PQ} \partial_M \otimes \partial_N = 0 \ .
\tag{2.4}
$$

While eq. (2.2) manifests the R_2 and $\overline{R}_2$ projections of the index pairs ^{MN} and $_{RS}$, the form (2.3) manifests the $\mathfrak{e}_8 \oplus \mathbb{R}$ projections in the pairs $^M{}_Q$ and $^P{}_R$.

There is a close connection between covariance and closure of the algebra, and the former may be used to prove the latter. Let us first formalise what covariance and closure means. The latter is simple, it means that the generators commute to a transformation with some parameter:

$$[\mathring{\mathcal{L}}_\xi, \mathring{\mathcal{L}}_\eta]V = \mathring{\mathcal{L}}_{[\![\xi,\eta]\!]}V \ , \tag{2.5}$$

where $[\![\cdot,\cdot]\!]$ for the moment is an unspecified bracket encoding the structure constants. Covariance means, on the other hand, that the transformed vector $\mathring{\mathcal{L}}_\xi V$ is a vector, when the vectorial transformation of both V and ξ are taken into account. This may be written

$$\hat{\delta}_\eta(\mathring{\mathcal{L}}_\xi V) \equiv \mathring{\mathcal{L}}_\xi \mathring{\mathcal{L}}_\eta V + \mathring{\mathcal{L}}_{\mathring{\mathcal{L}}_\eta \xi}V = \mathring{\mathcal{L}}_\eta \mathring{\mathcal{L}}_\xi V \tag{2.6}$$

(the convention is that "$\hat{\delta}$" is used when also parameters transform, unlike "δ", which only transforms fields). Assuming covariance immediately means that

$$[\mathring{\mathcal{L}}_\xi, \mathring{\mathcal{L}}_\eta]V = -\mathring{\mathcal{L}}_{\mathring{\mathcal{L}}_\eta \xi}V \ . \tag{2.7}$$

This implies that the algebra closes, with $[\![\cdot,\cdot]\!] = \frac{1}{2}(\mathring{\mathcal{L}}_\xi \eta - \mathring{\mathcal{L}}_\eta \xi)$. In addition, the left hand side is antisymmetric, so one also gets

$$\mathring{\mathcal{L}}_{((\xi,\eta))}V = 0 \ , \tag{2.8}$$

where $((\cdot,\cdot)) = \frac{1}{2}(\mathring{\mathcal{L}}_\xi \eta + \mathring{\mathcal{L}}_\eta \xi)$. Ordinary diffeomorphisms of course already have $((\xi,\eta)) = 0$, but for generalised diffeomorphisms for $O(d,d)$ and $E_{n(n)} \times \mathbb{R}^+$, $n \leq 7$, eq. (2.8) is non-trivially satisfied. The covariance equation (2.6) can be used to show that the Jacobiator $[\![\xi,\eta,\zeta]\!] \equiv [\![\xi, [\![\eta,\zeta]\!]]\!] + \text{cycl}$ then is non-zero, but equal to such a null parameter:

$$[\![\xi,\eta,\zeta]\!] = -\frac{1}{3}((\xi, [\![\eta,\zeta]\!])) + \text{cycl} \ . \tag{2.9}$$

Thus, checking covariance is enough to ensure closure. If the naked derivatives in $\mathring{\mathcal{L}}$ are replaced by covariant derivatives, covariance becomes manifest. To show covariance therefore amounts to demonstrating the absence of (the non-torsion part of) the connection in

$$\mathcal{L}_\xi V^M = \xi^N D_N V^M + Z^{MN}{}_{PQ} D_N \xi^P V^Q \ , \tag{2.10}$$

where $D = \partial + \Gamma$. One only has to consider the inhomogeneous transformation of a connection Γ. We use the convention $D_M V_N = \partial_M V_N + \Gamma_{MN}{}^P V_P$. Denote any inhomogeneous transformation (deviation from tensorial) by $\Delta_\xi \phi \equiv \delta_\xi \phi - \mathring{\mathcal{L}}_\xi \phi$. Then

$$\Delta_\xi \Gamma_{MN}{}^P = Z^{PQ}{}_{RN} \partial_M \partial_Q \xi^R \ . \tag{2.11}$$

Inserting this in the transformation (2.10) leads to

$$\Delta_\eta(\mathcal{L}_\xi V^M) = -\left(Z^{MN}{}_{TQ} Z^{TP}{}_{RS} + Z^{MP}{}_{RQ} \delta_S^N\right) \partial_N \partial_P \eta^R \xi^S V^Q \ . \tag{2.12}$$

If this vanishes, with the help of the section condition, the transformation is covariant, and $\mathcal{L} = \mathring{\mathcal{L}}$ for a torsion-free connection. This can be shown explicitly for all the cases up to $n = 7$ (eq. (2.3) above). We should stress that the reasoning only holds for transformations constructed with naked derivatives.

3 Beginning of a geometric construction for E_8

Let us now reconsider the E_8 case. The coordinate representation R_1 is the adjoint, and R_2 is $\mathbf{1} \oplus \mathbf{3875}$, leaving only $\mathbf{27000}$ in the symmetrised product of two derivatives. Any solution implies that also the antisymmetrised $\mathbf{248}$ vanishes. (For more details about E_8 representations and tensor products, see e.g. ref. [48].) It is known that the natural candidate for a transformation,

$$\mathring{\mathcal{L}}_\xi V^M = \xi^N \partial_N V^M + Z^{MN}{}_{PQ} \partial_N \xi^P V^Q \ , \tag{3.1}$$

with $Z_{MN}{}^{PQ} = -f_{AM}{}^Q f^A{}_N{}^P + \delta_M^Q \delta_N^P$, does not lead to a closed algebra. On the other hand, there is no "better" form with naked derivatives. Eq. (3.1) has precisely the property that it can be written in terms of a Y tensor, projecting on modules vanishing due to the section condition:

$$\mathring{\mathcal{L}}_\xi V^M = L_\xi V^M + (14 P_{(\mathbf{3875})} - 30 P_{(\mathbf{248})} + 62 P_{(\mathbf{1})})^{MN}{}_{PQ} \partial_N \xi^P V^Q \ . \tag{3.2}$$

A direct calculation [36, 45] shows that

$$[\mathring{\mathcal{L}}_\xi, \mathring{\mathcal{L}}_\eta] V^M = \mathring{\mathcal{L}}_{\frac{1}{2}(\mathring{\mathcal{L}}_\xi \eta - \mathring{\mathcal{L}}_\eta \xi)} V^M + \frac{1}{2} f^{MN}{}_P f^Q{}_{RS} (\partial_N \partial_Q \xi^R \eta^S - \partial_N \partial_Q \eta^R \xi^S) V^P \ . \tag{3.3}$$

The anomalous term takes the form of a local $\mathfrak{e}_8$ transformation with a parameter carrying an index obeying the section condition. This was used by Hohm and Samtleben in ref. [45].

In view of the connection between covariance and closure discussed in section 2, let us examine the failure in geometric terms. Here it is important to keep in mind that the analysis is performed with respect to the naïve transformation (3.1), which is known to have problems. The considerations concerning covariance etc. are not the final ones, only helpful steps on the way.

The occurrence of a two-derivative term points strongly to the transformation of a connection [46]. Let us perform a geometric check of the covariance (and, thereby, the closure), which we know will fail, but which will give interesting information. Define torsion as the part of the connection Γ that transforms covariantly under the transformation (3.1). Since the connection is a one-form taking values in $\mathfrak{e}_8 \oplus \mathbb{R}$, the possible E_8 modules in $\Gamma_{MN}{}^P$ are

$$\mathbf{248} \otimes (\mathbf{1} \oplus \mathbf{248}) = \mathbf{248} \oplus (\mathbf{1} \oplus \mathbf{3875} \oplus \mathbf{27000})_s \oplus (\mathbf{248} \oplus \mathbf{30380})_a \ , \tag{3.4}$$

where the subscripts denote the symmetric and antisymmetric tensor products of the two $\mathbf{248}$'s. In order for the connection to produce a covariant derivative, its transformation must contain an inhomogeneous term

$$\Delta_\xi \Gamma_{MN}{}^P \equiv (\delta_\xi - \mathring{\mathcal{L}}_\xi) \Gamma_{MN}{}^P = Z^{PQ}{}_{RN} \partial_M \partial_Q \xi^R \ . \tag{3.5}$$

This expression of course transforms in $\mathbf{248} \otimes (\mathbf{1} \oplus \mathbf{248})$, but thanks to the section condition it must also lie in $\mathbf{27000} \otimes \mathbf{248}$. The irreducible modules in the overlap are $\mathbf{248} \oplus \mathbf{27000} \oplus \mathbf{30380}$. This means that the remaining $\mathbf{1} \oplus \mathbf{248} \oplus \mathbf{3875}$ will transform covariantly, and are

torsion. The torsion combination of the two **248**'s turns out to be the linear combination $\Gamma_{NM}{}^N + \frac{1}{248}\Gamma_{MN}{}^N$ of the two **248**'s appearing in eq. (3.4).

What now goes wrong with the proposed naïve transformation (3.1) is that even a torsion-free connection does not drop out of the covariantised expression

$$\mathcal{L}_\xi^{(\Gamma)} V^M = \xi^N D_N V^M + Z^{MN}{}_{PQ} D_N \xi^P V^Q \ , \tag{3.6}$$

as it did for $n \leq 7$. Using the projection operators of the appendix, a connection is torsion-free if

$$f^{MN}{}_P \Gamma_{MN}{}^P = 0 \ ,$$

$$\left(f_{A(M}{}^P f^A{}_{N)}{}^Q - 2\delta^P_{(M}\delta^Q_{N)} \right) f_P{}^R{}_S \Gamma_{QR}{}^S = 0 \ , \tag{3.7}$$

$$\Gamma_{NM}{}^N + \frac{1}{248}\Gamma_{MN}{}^N = 0$$

(the last relation is a linear combination of the two **248**'s). The transformation fails to be covariant, and in light of the previous section, the algebra will not close. We can investigate precisely how the different irreducible modules in Γ enter in $\mathcal{L}_\xi^{(\Gamma)} V^M$. For this purpose we use the projection operators listed in the appendix. It now turns out that a torsion-free connection, i.e., one satisfying eq. (3.7), with vanishing $\mathbf{1} \oplus \mathbf{248} \oplus \mathbf{3875}$, but remaining components in $\mathbf{248} \oplus \mathbf{27000} \oplus \mathbf{30380}$, will satisfy

$$\mathcal{L}_\xi^{(\Gamma)} V^M = \mathring{\mathcal{L}}_\xi V^M - \frac{1}{60} f^{MQ}{}_P f_N{}^R{}_S \Gamma_{QR}{}^S \xi^N V^P \tag{3.8}$$

(the number 60 is twice the Coxeter number). The inhomogeneous transformation of a connection gives precisely the failure of closure. The extra term in eq. (3.8) is an $\mathfrak{e}_8$ transformation of V with parameter

$$\Sigma_{\xi M} = -\frac{1}{60} f^N{}_{PQ} \Gamma_{MN}{}^P \xi^Q \ . \tag{3.9}$$

Instead of the equality of $\mathcal{L}_\xi^{(\Gamma)}$ with $\mathring{\mathcal{L}}_\xi$, that holds for $n \leq 7$, we now have

$$\mathcal{L}_\xi^{(\Gamma)} = \mathring{\mathcal{L}}_\xi + \mathrm{ad}\,\Sigma_\xi \ . \tag{3.10}$$

Note that eq. (3.10) holds for a torsion-free connection. But since torsion, per definition, is covariant, not only $\mathcal{L}_\xi^{(\Gamma)}$, but also the right hand side of eq. (3.10), is covariant, with Σ constructed from *any* connection as in eq. (3.9).

So far, the "geometric" considerations have been performed with respect to the transformations $\mathring{\mathcal{L}}_\xi$. We know that $\mathcal{L}_\xi^{(\Gamma)}$, per definition, is covariant with respect to $\mathring{\mathcal{L}}_\xi$, but this is not the goal (and it is not really a statement that makes geometric sense). We need an expression for the transformations that is covariant with respect to *itself* (like for $n \leq 7$). Can $\mathcal{L}_\xi^{(\Gamma)}$ have this property?

We drop the superscript "(Γ)", and let

$$\mathcal{L}_\xi = \mathring{\mathcal{L}}_\xi + \mathrm{ad}\,\Sigma_\xi \ . \tag{3.11}$$

This is our candidate transformation for $n = 8$. It is highly unconventional in that it depends on a connection.

4 Covariance and algebra

Before checking for the covariance of the transformation (3.11) with respect to itself, we would like to consider connections in this setting. All connections above are connections transforming with the appropriate inhomogeneous terms under $\mathring{\mathcal{L}}$, not $\mathcal{L}$. We want to check how a covariant derivative $D = \partial + \Gamma$ must transform in order to take tensors to tensors. By a tensor we mean an object that transforms under scaling as is induced by the transformation of a vector (whose scaling weight we normalise to 1). Tensor densities may transform with other weights. The connection Γ is not necessarily the same one as is used in Σ. It is straight-forward to check that the presence of the Σ term in the transformation leads to one more inhomogeneous term in the transformation of a connection:

$$\Delta_\xi \Gamma_{MN}{}^P \equiv (\delta_\xi - \mathcal{L}_\xi)\Gamma_{MN}{}^P = Z^{PQ}{}_{RN}\partial_M\partial_Q\xi^R + f_N{}^{PQ}\partial_M\Sigma_{\xi Q} . \tag{4.1}$$

As mentioned, this transformation rule holds for any connection, in particular for the one used to define Σ. This can be used quite trivially to obtain the inhomogeneous transformation of Σ on the form

$$\Delta_\xi \Sigma_\eta = X^{\xi,\eta} + Y^{\xi,\eta} , \tag{4.2}$$

where the inhomogeneous terms X and Y,

$$\begin{aligned} X_M^{\xi,\eta} &= f^N{}_{PQ}\partial_M\partial_N\xi^P\eta^Q , \\ Y_M^{\xi,\eta} &= \partial_M\Sigma_{\xi N}\eta^N , \end{aligned} \tag{4.3}$$

have been introduced for convenience in the following calculation.

When now the (candidate) transformation (3.11) is no longer linear, but contains explicit fields (connection) through Σ, closure and covariance are not equivalent. Covariance is essential for the geometric framework, so we will first focus on that, and then check what the implications for the algebra are. The condition for covariance of this expression with respect to the transformations it generates reads

$$\hat{\delta}_\eta(\mathcal{L}_\xi V) = \delta_\eta(\mathcal{L}_\xi V) + \mathcal{L}_{\mathcal{L}_\eta\xi}V = \mathcal{L}_\eta\mathcal{L}_\xi V . \tag{4.4}$$

We will go through the full check of covariance, even if part of it (the covariance of $\mathcal{L}$ with respect to $\mathring{\mathcal{L}}$) follows from the considerations above. Remember that δ only acts on fields. The second term on the left hand side is the additional covariant transformation of the parameter. The $\mathcal{L}$'s, on the other hand, are just operators, acting on everything on the right. This can be rewritten as

$$([\mathcal{L}_\xi, \mathcal{L}_\eta] + \mathrm{ad}(\delta_\eta\Sigma_\xi) + \mathcal{L}_{\mathcal{L}_\eta\xi})V = 0 . \tag{4.5}$$

We use only the Leibniz rule (i.e., "the product rule") for $\mathring{\mathcal{L}}$ and the Jacobi identity for the adjoint action, which together provide the Leibniz rule for $\mathcal{L}$. After inserting the split (3.11) into eq. (4.5) and throwing away some cancelling terms, we get the condition

$$\begin{aligned} 0 = {}& [\mathring{\mathcal{L}}_\xi, \mathring{\mathcal{L}}_\eta] + \mathring{\mathcal{L}}_{\mathring{\mathcal{L}}_\eta\xi} \\ &+ \mathrm{ad}(\delta_\eta\Sigma_\xi + \Sigma_{\mathring{\mathcal{L}}_\eta\xi+[\Sigma_\eta,\xi]} - \mathring{\mathcal{L}}_\eta\Sigma_\xi - [\Sigma_\eta, \Sigma_\xi]) \\ &+ \mathring{\mathcal{L}}_{[\Sigma_\eta,\xi]} + \mathrm{ad}(\mathring{\mathcal{L}}_\xi\Sigma_\eta) . \end{aligned} \tag{4.6}$$

The first line, with the parameters in this order, and no (anti-)symmetrisation understood, states the failure of covariance for $\mathring{\mathcal{L}}$ (with respect to itself). Let us check it first (although it follows from the calculation above). It is convenient to introduce

$$[\![\xi,\eta]\!]^\circ = \frac{1}{2}(\mathring{\mathcal{L}}_\xi\eta - \mathring{\mathcal{L}}_\eta\xi) , \tag{4.7}$$

$$(\!(\xi,\eta)\!)^\circ = \frac{1}{2}(\mathring{\mathcal{L}}_\xi\eta + \mathring{\mathcal{L}}_\eta\xi) .$$

The action of $\mathring{\mathcal{L}}_{(\!(\xi,\eta)\!)^\circ}$ on a vector does not vanish. $(\!(\xi,\eta)\!)^{\circ M}$ contains, in addition to reducibility in $\mathbf{1} \oplus \mathbf{3875}$, a part

$$\frac{1}{4}f_A{}^{MN}f^A{}_{PQ}(\partial_N\xi^P\eta^Q + \partial_N\eta^P\xi^Q) , \tag{4.8}$$

which leads to

$$\mathring{\mathcal{L}}_{(\!(\xi,\eta)\!)^\circ} = -\frac{1}{2}\mathrm{ad}(X^{\xi,\eta} + X^{\eta,\xi}) . \tag{4.9}$$

We know from earlier that

$$[\mathring{\mathcal{L}}_\xi, \mathring{\mathcal{L}}_\eta] = \mathring{\mathcal{L}}_{[\![\xi,\eta]\!]^\circ} + \frac{1}{2}\mathrm{ad}(X^{\xi,\eta} - X^{\eta,\xi}) , \tag{4.10}$$

so the full first row or eq. (4.7) becomes

$$[\mathring{\mathcal{L}}_\xi, \mathring{\mathcal{L}}_\eta] + \mathring{\mathcal{L}}_{\mathring{\mathcal{L}}_\eta\xi} = [\mathring{\mathcal{L}}_\xi, \mathring{\mathcal{L}}_\eta] - \mathring{\mathcal{L}}_{[\![\xi,\eta]\!]^\circ} + \mathring{\mathcal{L}}_{(\!(\xi,\eta)\!)^\circ} = -\mathrm{ad}\,X^{\eta,\xi} . \tag{4.11}$$

It is essential that the $\partial^2\xi\eta$ terms from $X^{\xi,\eta}$ cancel, as they can not appear in the additional inhomogeneous terms coming from the transformation of a connection.

The second line of eq. (4.7) states the deviation of Σ_ξ alone from being covariant under the full transformation $\mathcal{L}_\eta$ (note that the weight of Σ_ξ is 0, since it is constructed from a vector ξ with weight 1 and a connection with weight -1). We have already calculated this in eq. (4.2); the second line becomes $\mathrm{ad}(X^{\eta,\xi} + Y^{\eta,\xi})$. We note that the X terms cancel between the first and second lines. This is not surprising, and only a consequence of the covariance of $\mathcal{L}$ under $\mathring{\mathcal{L}}$ discussed above.

Finally, the cross terms of the third line. Its first term contains a translation term, which acting on a vector V_M gives $f^{NP}{}_Q\Sigma_{\eta P}\xi^Q\partial_N V_M$. It must disappear if cancellation with the remaining terms, which are $\mathfrak{e}_8$ transformations, is to be possible. If Σ_M fulfills the section condition, the translation term goes away. This means that the connection used to define Σ has to respect the section condition regarding its first index. Unless such an extra condition is introduced by hand (which may be possible)[1], there is only one possibility, namely the Weitzenböck connection

$$W_{MN}{}^P = -(\partial_M E E^{-1})_N{}^P , \tag{4.12}$$

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[1]Such a condition is used e.g. in ref. [49]. It seems to be allowed, since the concept of torsion can be extended to allow for such a choice to be made, in view of the transformation (4.1). We do not find it practical, however, since its solution demands splitting the connection into $GL(n)$ modules.

defined for a generalised vielbein E. It is a flat but torsionful connection. From now on, we will assume that Σ is constructed with the Weitzenböck connection,

$$\Sigma_{\xi M} = -\frac{1}{60} f^N{}_{PQ} W_{MN}{}^P \xi^Q = \frac{1}{60} f^N{}_{PQ} (\partial_M E E^{-1})_N{}^P \xi^Q \ . \tag{4.13}$$

With this assumption, one easily derives the identity

$$\mathring{\mathcal{L}}_{[\Sigma_\eta, \xi]} + \mathrm{ad}(\mathring{\mathcal{L}}_\xi \Sigma_\eta) = -\mathrm{ad} Y^{\eta, \xi} \ . \tag{4.14}$$

All anomalous terms thus cancel, and $\mathcal{L}$ is covariant with respect to itself, which is quite remarkable. What was expected, and more or less trivial, was that $\mathcal{L}$ should be covariant with respect to $\mathring{\mathcal{L}}$. This happens for any connection, not just Weitzenböck. That the other anomalous terms (with Y) also cancel is more surprising.

It should be noted that even if we needed the Weitzenböck connection in the definition of the transformation, any connection, for example a torsion-free connection compatible with the covariant constancy of a vielbein, may be used for the construction of covariant derivatives. Which part of the connection is now torsion, in the sense that it transforms covariantly under the new transformations $\mathcal{L}_\xi$? With the previous definition, under $\mathring{\mathcal{L}}_\xi$, we had torsion in $\mathbf{1} \oplus \mathbf{248} \oplus \mathbf{3875}$. A direct inspection of the second inhomogeneous term in eq. (4.1) shows that precisely these modules drop out due to the section condition, since Σ is formed from the Weitzenböck connection. The notion of torsion remains unchanged. This is of course essential when it comes to defining torsion-free connections, curvatures etc.

We can now consider the commutator of two transformations. We get

$$\delta_\eta(\delta_\xi V) = \delta_\eta(\mathcal{L}_\xi V) = \mathcal{L}_\xi \mathcal{L}_\eta V + \mathrm{ad}(\delta_\eta \Sigma_\xi) V \ . \tag{4.15}$$

A simple comparison with eq. (4.5) gives at hand that

$$\delta_\eta(\delta_\xi V) - \delta_\xi(\delta_\eta V) = (\mathcal{L}_{[\![\xi, \eta]\!]} - \mathrm{ad}(\delta_{[\xi} \Sigma_{\eta]})) V \ . \tag{4.16}$$

The expression

$$\delta_{[\xi} \Sigma_{\eta]} = \mathcal{L}_{[\xi} \Sigma_{\eta]} - \Sigma_{[\![\xi, \eta]\!]} + (X + Y)^{[\xi, \eta]} \tag{4.17}$$

should now be a tensor. Note that $\mathcal{L}_\xi \Sigma_\eta = \mathring{\mathcal{L}}_\xi \Sigma_\eta$, but that the second term in $\Sigma_{[\![\xi, \eta]\!]} = \Sigma_{[\![\xi, \eta]\!]^\circ} + \Sigma_{\frac{1}{2}([\Sigma_\xi, \eta] - [\Sigma_\eta, \xi])}$ can not be dropped, and will be quadratic in connections.

The tensorial property seems intuitively natural, considering that it is a variation of a connection. It follows directly from the definition of the transformation of a connection,

$$\delta_\xi \Gamma_{MN}{}^P = \mathcal{L}_\xi (D_M V_N) - D_M \mathcal{L}_\xi V_N \ , \tag{4.18}$$

but it may of course be spelt out more concretely. The result of ref. [45] that the commutator of two generalised diffeomorphisms contains a section-restricted $\mathfrak{e}_8$ transformation remains, but has been given a covariant formulation. In the present formalism, there is however no need to introduce a separate connection for the $\mathfrak{e}_8$ transformations, since the geometric connection already transforms in the appropriate way.

5 Torsion, spin connection and curvature

Let us now consider curvature, and try to construct it for a general connection. This attempt, which will not be entirely successful, for reasons we will come back to, goes along the same lines as constructions for lower n in several papers, e.g. refs. [34, 35, 38, 41]. The starting point would be the inhomogeneous transformation of the connection,

$$\Delta_\xi \Gamma_{MN}{}^P = Z^{PQ}{}_{RN}\partial_M\partial_Q\xi^R + f_N{}^{PQ}\partial_M\Sigma_{\xi Q} \ . \tag{5.1}$$

It leads to the transformation of the derivative of a connection:

$$\Delta_\xi \partial_M\Gamma_{NP}{}^Q = Z^{QR}{}_{SP}\partial_M\partial_N\partial_R\xi^S + f_P{}^{QR}\partial_M\partial_N\Sigma_{\xi R} \tag{5.2}$$
$$+ \Delta_\xi\Gamma_{MR}{}^Q\Gamma_{NP}{}^R - \Delta_\xi\Gamma_{MN}{}^R\Gamma_{RP}{}^Q - \Delta_\xi\Gamma_{MP}{}^R\Gamma_{NR}{}^Q \ .$$

The first two terms can be removed by antisymmetrisation $[MN]$, leading to

$$\Delta_\xi \left(2\partial_{[M}\Gamma_{N]} + 2\Gamma_{[M}\Gamma_{N]}\right)_P{}^Q = -2\Delta_\xi\Gamma_{[MN]}{}^R\Gamma_{RP}{}^Q \ . \tag{5.3}$$

In ordinary geometry, the right hand side vanishes, since $\Gamma_{[MN]}{}^P$ is torsion, and the co-variance of the ordinary Riemann tensor is obtained. Here we need to use the identity for torsion, eq. (3.8), which states that

$$T_{MN}{}^P \equiv \Gamma_{MN}{}^P + Z^{PQ}{}_{RN}\Gamma_{QM}{}^R - \frac{1}{60}f^{PQ}{}_N f_M{}^R{}_S\Gamma_{QR}{}^S \tag{5.4}$$

is torsion. It can equivalently be written

$$T_{MN}{}^P = 2\Gamma_{[MN]}{}^P + Y^{PQ}{}_{RN}\Gamma_{QM}{}^R - \frac{1}{60}f^{PQ}{}_N f_M{}^R{}_S\Gamma_{QR}{}^S \ . \tag{5.5}$$

Using $\Delta_\xi T = 0$ in the right hand side of eq. (5.3) gives

$$\Delta_\xi \left(2\partial_{[M}\Gamma_{N]} + 2\Gamma_{[M}\Gamma_{N]}\right)_P{}^Q \tag{5.6}$$

$$= Y_{NR}{}^{ST}\Delta_\xi\Gamma_{TM}{}^R\Gamma_{SP}{}^Q - \frac{1}{60}f_N{}^{RS}f_M{}^T{}_U\Delta_\xi\Gamma_{ST}{}^U\Gamma_{RP}{}^Q \ . \tag{5.7}$$

The first term on the right hand side can be written as a total transformation if contracted with δ_Q^N and symmetrised (MP). This reflects the usual phenomenon that a full "Riemann" 4-index tensor can not be formed, only a 2-index "Ricci" tensor. We then have

$$\Delta_\xi \left(2\partial_{[M}\Gamma_{N]} + 2\Gamma_{[M}\Gamma_{N]}\right)_P{}^N\Big|_{(MP)} \tag{5.8}$$

$$= Y_{NR}{}^{ST}\Delta_\xi\Gamma_{T(M}{}^R\Gamma_{|S|P)}{}^N - \frac{1}{60}f_N{}^{RS}f^T{}_{U(M}\Gamma_{|R|P)}{}^N\Delta_\xi\Gamma_{ST}{}^U \ .$$

The first term on the right hand side can be written as $\frac{1}{2}Y_{NR}{}^{ST}\Delta_\xi(\Gamma_{T(M}{}^R\Gamma_{|S|P)}{}^N)$. The second term does not display this symmetry, but we note that it can be expressed as

$-\frac{1}{60}f_N{}^{RS}f^T{}_{U(M}\Delta_\xi(\Gamma_{|R|P)}{}^N W_{ST}{}^U)$, since $\Delta_\xi W = \Delta_\xi \Gamma$, and the inhomogeneous transformation of the first Γ drops out due to W (and not only its inhomogeneous variation) satisfying the section condition. We have thus showed that the "Ricci" tensor

$$R_{MN} = \partial_{(M}\Gamma_{|P|N)}{}^P - \partial_P\Gamma_{(MN)}{}^P + \Gamma_{(MN)}{}^Q\Gamma_{PQ}{}^P - \Gamma_{P(M}{}^Q\Gamma_{N)Q}{}^P \qquad (5.9)$$
$$- \frac{1}{2}Y^{PQ}{}_{RS}\Gamma_{PM}{}^R\Gamma_{QN}{}^S + \frac{1}{60}f_P{}^{QR}f^S{}_{T(M}\Gamma_{|Q|N)}{}^P W_{RS}{}^T$$

transforms covariantly. It vanishes for the Weitzenböck connection, as expected. The appearance of W in the expression for the curvature is peculiar, but maybe not more so than its appearance in the transformations. The value of the expression (5.9) is however doubtful — it is quite likely that it will lead to curvature in the $\mathrm{Spin}(16)/\mathbb{Z}_2$ representation **128**, the candidate for the equation of motion for the generalised metric or vielbein, which contains undefined connection. We have however not been able to strictly show that this is the case, so it is still an open question whether the tensor of eq. (5.9) can give rise to e.g. a well-defined scalar curvature. We also note that this kind of "fake curvature" with explicit W's, can be constructed also for a full Riemann (4-index) tensor, by using the section condition on the right hand side of eq. (5.7).

It is our impression that the last term in the torsion, which is traced back to the "mismatch" between the transformations and the torsion, causing the failure of closure of the transformation with naked derivatives ($\mathring{\mathcal{L}}$), cannot be compensated for by a pure "$\Gamma\Gamma$" term. The absence of some (at least 2-index) curvature for an arbitrary connection may not be a disaster, however. The important thing in the end is to have some curvature in **128** of $\mathrm{Spin}(16)/\mathbb{Z}_2$, which contains only connection which is well defined by compatibility, and this is far from excluded. Compatible and solvable (affine or spin) connections will contain explicit derivatives, which means that the section condition can be at work, to a higher degree than above, when constructing curvature.

We therefore change our strategy, and focus on solving, as far as possible, for a spin connection $\Omega_{MA}{}^B$. The covariant constancy of the generalised vielbein reads

$$D_M E_N{}^A = \partial_M E_N{}^A + \Gamma_{MN}{}^P E_P{}^A - E_N{}^B \Omega_{MB}{}^A = 0 \ . \qquad (5.10)$$

Using the vanishing torsion in Γ the equation for the spin connection becomes

$$T(E\Omega E^{-1} + W) = 0 \ , \qquad (5.11)$$

where $T(\Gamma)$ is the torsion combination of eq. (5.4). This equation contains the E_8 modules with $\mathrm{Spin}(16)$ decomposition:

$$\begin{aligned} \mathbf{1} &\to \mathbf{1} \\ \mathbf{248} &\to \mathbf{120} \oplus \mathbf{128} \\ \mathbf{3875} &\to \mathbf{135} \oplus \mathbf{1820} \oplus \mathbf{1920} \end{aligned} \qquad (5.12)$$

The modules appearing in the decomposition of **3875** are a symmetric traceless tensor, a 4-form, and a Γ-traceless vector-cospinor. The modules that can appear in Ω are (with

flattened form index):

$$\Omega_{ab}: \qquad \mathbf{120} \otimes \mathbf{120} = \mathbf{1} \oplus \mathbf{120} \oplus \mathbf{135} \oplus \mathbf{1820} \oplus \mathbf{5304} \oplus \mathbf{7020} \qquad (5.13)$$

$$\Omega_{\alpha}: \qquad \mathbf{128} \otimes \mathbf{120} = \mathbf{128} \oplus \mathbf{1920} \oplus \mathbf{13312} \qquad (5.14)$$

The modules in **black** may be solved for, and the ones in grey, which in order of appearance are traceless tensors of types ⊞ and ⬓ and a Γ-traceless 2-form-spinor, remain undetermined. In any equation used for fields or transformations, one should in the end only use covariant derivatives where the undefined connection components drop out. Examples of such well-defined covariant derivatives are $\mathbf{16} \to \mathbf{16}$, $\mathbf{128}' \to \mathbf{128}'$ and $\mathbf{16} \leftrightarrow \mathbf{128}'$. This will be important when considering local supersymmetry (which is beyond the scope of this paper).

In order to solve for the solvable part of the spin connection, we need to decompose our tensors into Spin(16) modules. A choice for the $E_{8(8)}$ structure constants and metric corresponding to the normalisation used is

$$f^{ab,cd}{}_{ef} = -2\sqrt{2}\delta^{[a}_{[e}\delta^{b][c}\delta^{d]}_{f]} \; ,$$

$$f^{ab,\alpha}{}_{\beta} = \frac{1}{2\sqrt{2}}(\Gamma^{ab})^{\alpha}{}_{\beta} \; , \qquad (5.15)$$

$$\eta_{ab,cd} = -\delta^{ab}_{cd} \; , \qquad \eta_{\alpha\beta} = \delta_{\alpha\beta}$$

(the overall normalisation is $\sqrt{2}$ times the one most commonly used in the physics literature). It will be useful to know the form of the decomposition of the projections of two-index tensors on $\mathbf{248}$ (antisymmetric) and $\mathbf{1} \oplus \mathbf{3875}$ used in the section condition, and also in the torsion. An antisymmetric tensor A_{AB} will have vanishing component in $\mathbf{248} \to \mathbf{120} \oplus \mathbf{128}$ if

$$A_{ac,bc} - \frac{1}{8}(\Gamma_{ab})^{\alpha\beta}A_{\alpha\beta} = 0 \; , \qquad (5.16)$$

$$(\Gamma^{ab})_{\alpha}{}^{\beta}A_{ab,\beta} = 0 \; , \qquad (5.17)$$

and the part of a symmetric tensor S_{AB} in $\mathbf{1} \oplus \mathbf{3875} \to \mathbf{1} \oplus \mathbf{135} \oplus \mathbf{1820} \oplus \mathbf{1920}$ vanishes if

$$S_{ai,bi} - \frac{1}{16}\delta_{ab}S_{\alpha\alpha} = 0 \; ,$$

$$S_{[ab,cd]} + \frac{1}{48}(\Gamma_{abcd})^{\alpha\beta}S_{\alpha\beta} = 0 \; , \qquad (5.18)$$

$$(\Gamma^{i})_{\dot{\alpha}}{}^{\alpha}S_{ai,\alpha} - \frac{1}{16}(\Gamma_a\Gamma^{ij})_{\dot{\alpha}}{}^{\alpha}S_{ij,\alpha} = 0 \; .$$

As a consistency check of our structure constants (5.16), the middle equation turns up both in the ab, cd and the $\alpha\beta$ part of the projection on $\mathbf{3875}$, with the same combination, and the Γ-tracelessness of the last equation is reproduced correctly.

We can now solve for the well-defined part of the spin connection. If we use a one-index notation both for the spin connection and the E_8 part of the Weitzenböck connection, so

that (with flattened indices)

$$W_{AB}{}^C = f_B{}^{CD} W_{AD} + \delta_B^C w_A \ , \tag{5.19}$$
$$\Omega_{AB}{}^C = f_B{}^{C,ab} \Omega_{A,ab} \ ,$$

the solution is obtained using eqs. (3.7), (5.17) and (5.18), and reads:

$$\mathbf{1} \oplus \mathbf{135} \oplus \mathbf{120} : \Omega_{ai,bi} \qquad = - W_{ai,bi} + \frac{1}{8} (\Gamma_{ab})^{\alpha\beta} W_{\alpha\beta} + \frac{1}{16} \delta_{ab} W_{\alpha\alpha} - \frac{1}{\sqrt{2}} w_{ab} \ ,$$

$$\mathbf{1820} : \Omega_{[ab,cd]} \qquad = - W_{[ab,cd]} - \frac{1}{48} (\Gamma_{abcd})^{\alpha\beta} W_{\alpha\beta} \ ,$$

$$\mathbf{128} \oplus \mathbf{1920} : (\Gamma^i)_{\dot\alpha}{}^\alpha \Omega_{\alpha,ai} \quad = - (\Gamma^i)_{\dot\alpha}{}^\alpha (W_{\alpha,ai} + W_{ai,\alpha}) + \frac{1}{8} (\Gamma_a \Gamma^{ij})_{\dot\alpha}{}^\alpha W_{ij,\alpha}$$
$$- \frac{1}{2\sqrt{2}} (\Gamma_a w)_{\dot\alpha} \ . \tag{5.20}$$

The right hand sides represent the torsion components $-T_{a,b}$, $-T_{abcd}$ and $-T_{a\dot\alpha}$ of the Weitzenböck connection, sometimes referred to as fluxes (not to be confused with the torsion of the affine connection which we have chosen to vanish).

Since Ω is constructed from derivatives of the connection, there will be implications from the section condition that may help in the construction of curvature. We have checked, through a rather long calculation, that there is a spinorial constraint

$$\frac{1}{24} (\Gamma^{abcd})_\alpha{}^\beta \Omega_{ab,cd} \partial_\beta + \frac{1}{2} (\Gamma^{ab})_\alpha{}^\beta \Omega_{ai,bi} \partial_\beta - \frac{5}{12} \Omega_{ij,ij} \partial_\alpha - \frac{1}{12} (\Gamma^c \Gamma^{ab} \Gamma^d)_\alpha{}^\beta \Omega_{\beta,cd} \partial_{ab} = 0 \ . \tag{5.21}$$

This is shown by direct insertion of the solution (5.20) for Ω into an Ansatz, and using the section condition between the derivative and the first index on W (or the index on w) on the forms (5.17), (5.18). The relation relies on the Fierz identity

$$F^{\alpha\beta\gamma\delta} = F^{\alpha\delta\gamma\beta} \ , \tag{5.22}$$

where

$$F^{\alpha\beta\gamma\delta} = \frac{1}{24} (\Gamma^{abcd})^{\alpha\beta} (\Gamma_{abcd})^{\gamma\delta} - 3 (\Gamma^{ab})^{\alpha\beta} (\Gamma_{ab})^{\gamma\delta} + 20 \delta^{\alpha\beta} \delta^{\gamma\delta} \tag{5.23}$$

(note the absence of $\Gamma^{(6)}$ and $\Gamma^{(8)}$, which would lead to terms where the section condition can not be used).

Instead of attempting to construct curvature from Ω, which is possible (see below), we will take another approach, namely to construct a scalar K which is quadratic in the torsion of the Weitzenböck connection. This scalar will play the rôle analogous to that of a curvature scalar in an action. This procedure is possible in Einstein gravity, and has been used earlier in extended geometry [50, 51]. In addition to the restriction to torsion, dictated by covariance, it is also important that the only parts of W that appear are $W_{M\alpha}$ and w_M. Then the expression will be invariant under local Spin(16) transformations; when a variation of the vielbein is performed, δK will not contain $(E^{-1} \delta E)_{ab}$, only $(E^{-1} \delta E)_\alpha$ and $\mathrm{tr}(E^{-1} \delta E)$.

Naïvely, the first terms on the right hand sides of eq. (5.20) seem to present obstructions to such a construction, but the section condition may (and will) help. The contribution of these first terms must vanish altogether, both the quadratic and linear ones. By considering the quadratic part, we first find that only 2 out of the 4 possible scalars from $(\mathbf{1} \oplus \mathbf{135} \oplus \mathbf{120} \oplus \mathbf{1820})^2$ are possible, if the first indices are to arrange in ways that can give a cancellation with $(\mathbf{128} \oplus \mathbf{1920})^2$ using the section condition. These combinations are

$$
\begin{aligned}
W_{[ab,cd]}W_{ab,cd} &- \frac{2}{3}W_{ai,bi}W_{bj,aj} + \frac{1}{6}W_{ij,ij}W_{kl,kl} \\
&= W_{[ab,|ab|}W_{cd],cd} + \frac{1}{6}W_{ab,cd}W_{ab,cd} - \frac{2}{3}W_{ac,bd}W_{ab,cd} \qquad (5.24)
\end{aligned}
$$

and $\quad W_{ai,bi}W_{aj,bj}$.

When they are matched to the two possible contributions from $(\mathbf{128} \oplus \mathbf{1920})^2$, we obtain a unique combination, modulo an overall constant, where the terms quadratic in $W_{M,ab}$ cancel, namely

$$
\begin{aligned}
K = T_{abcd}T_{abcd} + \frac{4}{3}T_{a,b}T_{a,b} &- \frac{2}{3}T_{a,b}T_{b,a} + \frac{1}{6}T_{a,a}T_{b,b} \qquad (5.25) \\
&- \frac{3}{16}T_{a\dot{\alpha}}T_{a\dot{\alpha}} - \frac{1}{48}(\Gamma^{ab})^{\dot{\alpha}\dot{\beta}}T_{a\dot{\alpha}}T_{b\dot{\beta}} \ .
\end{aligned}
$$

Then one has to check for the terms linear in $W_{M,ab}$. It turns out that these cancel, through what looks like a long series of numerical coincidences, using the section condition to switch the first indices on a pair of W's.

This shows that the covariant scalar (5.25) can be written in a (non-covariant) form, where each T is replaced by a (non-tensorial) $\tilde{T}$ obtained by omitting the first terms from eq. (5.20), i.e.,

$$
\begin{aligned}
\tilde{T}_{a,b} &= -\frac{1}{8}(\Gamma_{ab})^{\alpha\beta}W_{\alpha\beta} - \frac{1}{16}\delta_{ab}W_{\alpha\alpha} + \frac{1}{\sqrt{2}}w_{ab} \ , \\
\tilde{T}_{abcd} &= \frac{1}{48}(\Gamma_{abcd})^{\alpha\beta}W_{\alpha\beta} \ , \qquad (5.26) \\
\tilde{T}_{a\dot{\alpha}} &= (\Gamma^i)_{\dot{\alpha}}{}^{\alpha}W_{ai,\alpha} - \frac{1}{8}(\Gamma_a\Gamma^{ij})_{\dot{\alpha}}{}^{\alpha}W_{ij,\alpha} + \frac{1}{2\sqrt{2}}(\Gamma_a w)_{\dot{\alpha}} \ .
\end{aligned}
$$

If the $\tilde{T}$'s are used, covariance is not manifest. If the T's are used, covariance is manifest, but local Spin(16) only arises thanks to the section condition. This behaviour seems to support our speculation that a curvature tensor (in terms of only a general torsion-free connection Γ, or in terms of a spin connection Ω) does not exist before the section condition is used on the solution of the compatibility equation. Since the scalar we have found is unique, it must coincide with the one given in ref. [45].

A "Ricci tensor" in $\mathbf{1} \oplus \mathbf{128}$, governing the equation of motion for the vielbein, is obtained by the formal variation of an "action"

$$
S \sim \int |E|^{-\frac{1}{248}}K \ . \qquad (5.27)
$$

The power of the determinant of the vielbein is dictated by the correct weight of the integrand, allowing for partial integration [38]. In view of the Spin(16)-invariance, the variation will only contain $(E^{-1}\delta E)_\alpha$ and $\mathrm{tr}(E^{-1}\delta E)$, and an "Einstein tensor" is obtained after partial integration. It can in turn be contracted to a scalar curvature R, such that $|E|^{-\frac{1}{248}}K$ and $|E|^{-\frac{1}{248}}R$ differ by a total derivative.

6 Covariant reducibility

As mentioned in the introduction, the situation has been unclear not only concerning the generalised diffeomorphisms for E_8, but also for a complete understanding of the symmetries for lower n. Although the generalised diffeomorphisms and the tensor formalism work fine, there have been questions concerning the (infinite) reducibility, and its associated (infinite) tower of ghosts for ghosts. This may seem like a technical detail, but is important if an understanding of the global properties is to be taken to the same level as the one for $O(d,d)$. It concerns e.g. the formulation of transition functions in terms of gerbes [19]. As shown in ref. [38], there is a sequence of modules

$$R_1 \leftarrow R_2 \leftarrow \ldots \leftarrow R_{8-n} \;, \tag{6.1}$$

where the arrows denote action of the derivative, for which torsion-free connection drops out of the covariant derivatives. In this precise sense, the modules are analogous to forms in ordinary geometry. These are modules appearing in tensor hierarchies (see e.g. refs. [42, 52]), coinciding with the reducibility of the generalised diffeomorphisms [36], with parameter in R_1, and also of tensor fields, with parameters in higher R_k. The sequence of R_k's coincides with the generators of Borcherds superalgebras [53–57]. The sequence of ghosts, corresponding to reducibility, does not stop where the connection-free window closes, however. It has been somewhat disturbing that the complete ghost structure, formulated with naked derivatives, has been non-covariant, beginning with $R_{8-n} \leftarrow R_{9-n}$, R_{9-n} being the adjoint. Of course, this is also an indication to why the problem comes all the way down to the algebra of generalised diffeomorphisms for $n = 8$. In the light of the solution for E_8, the solution for lower n becomes clear: the higher reducibilities should be formulated with covariant derivatives containing the Weitzenböck connection. This is consistent, since such a covariant derivative automatically obeys the section condition. We are not in a position to say what this implies for the gerbe structure of exceptional geometry.

For the generalised diffeomorphisms of the present paper, the same construction holds. Reducibility can be obtained covariantly, with covariant derivatives containing the Weitzenböck connection. At the first step, $R_2 = \mathbf{1} \oplus \mathbf{3875}$, and a transformation $\mathcal{L}_\xi$ with $\xi^M = D_N\Lambda^{MN}$, $\Lambda \in R_2$, generates a null transformation, $\mathcal{L}_\xi V = 0$.

7 Ehlers symmetry

What has been done for E_8 above applies in spirit to enhanced symmetries arising on dimensional reduction of gravity from $3 + n$ to 3 dimensions due to the appearance of a dual gravity field. This is the Ehlers symmetry $\mathrm{SL}(n+1)$. $E_{8(8)}$ of course contains an $\mathrm{SL}(9)$

subgroup, but it is possible to construct an extended geometry for all n. A series of models built on SL algebras was considered previously in ref. [37]. It is different from the present one in the choice of coordinate representation etc., and does not contain dual gravity.

The transformations are obtained exactly as the ones for E_8, with the invariant $SL(n+1)$ tensor Z given by the same expression, $Z_{MN}{}^{PQ} = -f_{AM}{}^Q f^A{}_N{}^P + \delta_M^Q \delta_N^P$. Tensors are most conveniently written in fundamental indices, where the structure constants are

$$f_m{}^n{}_{,p}{}^q{}_{,r}{}^s = \delta_p^n \delta_r^q \delta_m^s - \delta_r^n \delta_m^q \delta_p^s , \qquad (7.1)$$

and the invariant metric is

$$\eta_m{}^n{}_{,p}{}^q = \delta_p^n \delta_m^q - \frac{1}{n+1} \delta_m^n \delta_p^q . \qquad (7.2)$$

The steps of sections 3 and 4 can be followed, where the numerical factor $\frac{1}{60}$ is replaced by $\frac{1}{2(n+1)}$, in all cases equalling $\frac{1}{2h}$, h being the Coxeter number. The projection operators can of course not be copied, but the important relations used in the calculation can; it is straightforward to show that the section condition also in this case implies

$$\eta^{MN} \partial_M \otimes \partial_N = 0 ,$$
$$f_M{}^{NP} \partial_N \otimes \partial_P = 0 , \qquad (7.3)$$
$$(f_{AM}{}^P f^A{}_N{}^Q - 2\delta_{(M}^P \delta_{N)}^Q) \partial_P \otimes \partial_Q = 0 .$$

The coordinate representation is also here the adjoint, with highest weight Dynkin label $(10\ldots01)$. This choice can be inferred by considering the SL(9) subgroup of $E_{8(8)}$, but is ultimately motivated by the solution of the section condition and of the field content in the coset described below. The representations for coordinates and for the section condition also appears in the classification of ref. [58].

Although the symmetric product of two elements in the adjoint generically contains four irreducible representations (instead of three, for E_8),

$$\vee^2 (10\ldots01) = (0\ldots0) \oplus (10\ldots01) \oplus (010\ldots010) \oplus (20\ldots02) , \qquad (7.4)$$

the section condition removes all but the largest one, the one with highest weight twice the one of the adjoint. It effectively sets to zero any contraction of fundamental indices, and a solution can be taken as $\partial_{m'}{}^0 = \partial_{m'}$ and all other derivatives vanishing, so that fields locally depend on a set of coordinates $x^{m'}$, $m' = 1, \ldots, n$. The section condition reads

$$\frac{1}{2}(\partial_m{}^n \otimes \partial_p{}^q + \partial_p{}^q \otimes \partial_m{}^n) = \partial_{(m}{}^{(n} \otimes \partial_{p)}{}^{q)} \qquad (7.5)$$

$$\partial_m{}^p \otimes \partial_p{}^n = 0 , \qquad (7.6)$$

eq. (7.3) is verified by the short calculation

$$\frac{1}{2}\big(\mathrm{tr}(A\partial_1)\mathrm{tr}(B\partial_2) + \mathrm{tr}(B\partial_1)\mathrm{tr}(A\partial_2)\big)$$
$$= \frac{1}{4}\big(\mathrm{tr}(A\partial_1)\mathrm{tr}(B\partial_2) + \mathrm{tr}(B\partial_1)\mathrm{tr}(A\partial_2) + \mathrm{tr}(A\partial_1 B\partial_2) + \mathrm{tr}(B\partial_1 A\partial_2)\big) ;$$

$$\mathrm{tr}([A, \partial_1][B, \partial_2]) = \mathrm{tr}(A\partial_1 B\partial_2 + \partial_1 A\partial_2 B) = \mathrm{tr}(A\partial_1)\mathrm{tr}(B\partial_2) + \mathrm{tr}(B\partial_1)\mathrm{tr}(A\partial_2) , \qquad (7.7)$$

using eq. (7.5) to derive the last equation in (7.3).

Here, it is technically less complicated to isolate the dual gravity field, which is the only field apart from the ordinary vielbein. An $SL(n+1)/SO(n+1)$ vielbein can, fixing the $SO(n+1)$ gauge, be parametrised as

$$E_m{}^a = \begin{bmatrix} |e|^{-1} & 0 \\ |e|^{-1}\phi_{m'} & e_{m'}{}^{a'} \end{bmatrix} , \tag{7.8}$$

where $\phi_{m'}$ represents the dual gravity field. The action on the generalised vielbein by a restricted $SL(n+1)$ transformation

$$T_m{}^n = \begin{bmatrix} 0 & 0 \\ t_{m'} & 0 \end{bmatrix} \tag{7.9}$$

amounts to a shift in ϕ. The dual gravity field does not carry any local degrees of freedom, and it becomes clear that the restricted $SL(n+1)$ transformations should not be counted as removing any local degrees of freedom beyond the ones removed by the generalised diffeomorphisms.

This is also verified by a counting of the effective number of degrees of freedom removed by a generalised diffeomorphism. By the method of ref. [57] (see also ref. [36], where the corresponding counting is performed for E_8 and for lower n), the relevant Borcherds algebra is related to a bosonic object λ in $(10\ldots01)$, constrained so that the only module appearing at λ^2 is $(20\ldots02)$. The partition function of λ then becomes

$$Z_n(t) = \sum_{k=0}^{\infty} \dim(k0\ldots0k)t^k = \sum_{k=0}^{\infty} \left(\frac{(n+k-1)!}{k!}\right)^2 \frac{n+2k}{n!(n-1)!} t^k \tag{7.10}$$

$$= (1-t)\,{}_2F_1(n+1, n+1; 1; t) = (1-t)^{-2n}\,{}_2F_1(-n, -n; 1; t) \tag{7.11}$$

$$= (1-t)^{-n} P_n\left(\frac{1+t}{1-t}\right) = (1-t)^{-2n} \sum_{i=0}^{n} \binom{n}{i}^2 t^i . \tag{7.12}$$

The partition functions for all n are governed by a simple generating function

$$\mathcal{Z}(s,t) = \sum_{n=0}^{\infty} Z_n(t)s^n = \frac{1}{\sqrt{1 - \frac{2(1+t)s}{(1-t)^2} + \frac{s^2}{(1-t)^2}}} . \tag{7.13}$$

$Z_n(t)$ is also the inverse partition function for the subalgebra of the Borcherds superalgebra at positive levels [57], with generators in R_k. The effective number of gauge parameters, modulo reducibility, equals the number of degrees of freedom in λ, and may be read off as the power of the pole of the partition function at $t = 1$. The partition functions are rational functions with denominators $(1-t)^{2n}$. Of the $2n$ gauge degrees of freedom, ordinary diffeomorphisms and dual diffeomorphisms make up n each, which completely removes the local degrees of freedom for the dual gravity field.

8 Discussion

We have shown how it is possible to define covariant field-dependent generalised diffeomorphisms for E_8, and used them to understand the dynamics in a geometric way. We

would like to stress that the conclusions of Hohm and Samtleben [45] remain true, but are given a geometric framework. The solution also provides a covariant formulation of the reducibility for lower n. A very similar construction, which we have only sketched, is valid for the Ehlers symmetry $SL(n+1)$.

The transformations needed in order to achieve covariance and build a tensor formalism are field-dependent, and depend on a generalised vielbein through the Weitzenböck connection $W_{MN}{}^P = -(\partial_M E E^{-1})_N{}^P$. This is very unconventional, but in this case necessary, and unlike any previously encountered generalised diffeomorphisms. Our analysis this far is entirely local, and it is not yet clear to us what the consequences for global structures will be when such transformations are used to relate overlapping patches. In double geometry, a double manifold has a manifold structure before any fields (e.g. generalised vielbein or metric) are introduced, which on the introduction of generalised metric data acquires a gerbe structure (visible already at the level of the algebra) [19]. In the present situation, no such distinction is possible, since there is no way of constructing covariant transformations without the presence of a vielbein. We would however like to remind again that this is true also for the lower exceptional cases. Although the field dependence there enters at higher ghost levels, it will be necessary in order to understand the full reducibility and the full "gerbe" structure.

Hopefully, the present treatment can open the road towards higher n and infinite-dimensional algebras, starting with E_9. There may be reason to wonder if the "dual gravity barrier" really has been broken, in a way that will persist for higher n, or if new difficulties (apart from infinite dimensionality) will arise. A reason for hope may be the unexpected covariance of the field-dependent transformations, including the gauge symmetry for the dual gravity field. A reason for doubt, on the other hand, may be the observation that we have not yet reached a situation where the dual gravity field becomes dynamical.

A   Projection operators for E_8 tensors

The tensor products of two adjoint $\mathbf{248}$'s of E_8 contains the irreducible modules $\mathbf{1} \oplus \mathbf{3875} \oplus \mathbf{27000}$ in the symmetric part and $\mathbf{248} \oplus \mathbf{30380}$ in the antisymmetric part. The projection operators on the irreducible modules are

$$
\begin{aligned}
P_{(\mathbf{1})}^{MN}{}_{PQ} &= \frac{1}{248}\eta^{MN}\eta_{PQ} \ , \\
P_{(\mathbf{3875})}^{MN}{}_{PQ} &= \frac{1}{7}\delta_P^{(M}\delta_Q^{N)} - \frac{1}{14}f^{A(M}{}_P f_A{}^{N)}{}_Q - \frac{1}{56}\eta^{MN}\eta_{PQ} \ , \\
P_{(\mathbf{27000})}^{MN}{}_{PQ} &= \frac{6}{7}\delta_P^{(M}\delta_Q^{N)} + \frac{1}{14}f^{A(M}{}_P f_A{}^{N)}{}_Q + \frac{3}{217}\eta^{MN}\eta_{PQ} \ , \\
P_{(\mathbf{248})}^{MN}{}_{PQ} &= -\frac{1}{60}f_A{}^{MN}f^A{}_{PQ} \ , \\
P_{(\mathbf{30380})}^{MN}{}_{PQ} &= \delta_{PQ}^{MN} + \frac{1}{60}f_A{}^{MN}f^A{}_{PQ} \ ,
\end{aligned}
\tag{A.1}
$$

where the structure constants are normalised so that $f^{MAB}f_{NAB} = -60\delta_N^M$.

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