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Scattering by an anisotropic circle

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HIGHLIGHTS

- A series solution is obtained for the wave modes inside an anisotropic circle.
- An analytical solution is given for the scattering by an anisotropic circle.
- The anisotropy has strong effects on the scattering except at low frequencies.

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ABSTRACT

The scattering by a circle is considered when the outside medium is isotropic and the inside medium is anisotropic (orthotropic). The problem is a scalar one and is phrased as a scattering problem for elastic waves with polarization out of the plane of the circle (SH wave), but the solution is with minor modifications valid also for scattering of electromagnetic waves. The equation inside the circle is first transformed to polar coordinates and it then explicitly contains the azimuthal angle through trigonometric functions. Making an expansion in a trigonometric series in the azimuthal coordinate then gives a coupled system of ordinary differential equations in the radial coordinate that is solved by power series expansions. With the solution inside the circle complete the scattering problem is solved essentially as in the classical case. Some numerical examples are given showing the influence of anisotropy, and it is noted that the effects of anisotropy are generally strong except at low frequencies where the dominating scattering only depends on the mean stiffness and not on the degree of anisotropy.

1. Introduction

The 2D scattering of a wave from a circle is an old problem in mathematical physics, see e.g. the classical book by Morse and Feshbach [1]. It requires that both the medium outside and inside the circle is homogeneous and isotropic (or cylindrically anisotropic), although a layered circle may also be considered (or a void or rigid inclusion). The scattering by anisotropic (with fixed directions of anisotropy) objects is, however, also of great interest. In mechanics this may be the scattering by fibres in a composite, the grains in a metal, or, on a larger scale, an anisotropic formation on the ground.

Waves in anisotropic media have mostly been treated in Cartesian coordinates. It is then straightforward to investigate the propagation of waves in layered media. For bounded anisotropic media (with fixed directions of anisotropy) little has been done, and most interest for such problems seems to arise for electromagnetic problems, not mechanical ones. Thus Ren [2] has derived the cylindrical and spherical wave functions in anisotropic electromagnetic media and these were used by Wu and Ren [3] to investigate the scattering by an anisotropic circle in an isotropic medium. They have been further used to treat the scattering by a sphere, see e.g. Wan and Li [4], and to derive the null field approach (T matrix method), see Doicu [5] and Wang et al. [6]. The basic idea in the derivation of these wave functions is a plane wave expansion that is

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transformed into polar or spherical coordinates, which leads to quite complicated expressions involving integrals that have to be computed numerically. It seems that the method has not been used for mechanical scattering problems. Very recently Zharnikov and Syresin [7] developed a very different approach (leading to a Riccati equation for the impedance operator) which they applied to the determination of the modes in an anisotropic elastic waveguide. For cylindrical orthotropy, on the other hand, more investigations have been performed; an interesting example is given by Martin and Berger [8], who investigate the eigenfrequencies in a wooden pole using somewhat similar ideas as in the present paper.

In this paper the scattering by an anisotropic circle is treated in the scalar case. This is phrased in terms of mechanical waves so it is antiplane shear waves that are assumed, i.e. the displacement is perpendicular to the plane of the circle and the problem is 2D. In contrast to the methods mentioned above the starting point is to state the anisotropic wave equation in polar coordinates. The equation then becomes more complicated than in rectangular coordinates in that the azimuthal angle φ appears in some places, but only as factors $\cos 2\varphi$ or $\sin 2\varphi$. Expanding the field in a trigonometric series in φ leads to a set of coupled ordinary differential equations. These can be solved by a power series ansatz, which leads to a very efficient way of calculating the field inside the anisotropic circle. In the isotropic medium outside the circle the classical expansions of the incident and scattered fields in terms of Bessel and Hankel functions, respectively, are made and invoking the boundary conditions this solves the problem, although it is noted that the stress boundary condition leads to a coupling between different azimuthal orders.

2. Problem formulation

Consider the scattering by an anisotropic circle of radius a residing in an isotropic infinite medium in the simplest possible setting, i.e. let the incoming field be an antisymmetric plane shear wave. Introduce a rectangular coordinate system xy with the origin at the centre of the circle and the x and y axes along the principal directions of the anisotropic medium. Also polar coordinates $r\varphi$ in the xy plane are used. The infinite medium has density ρ_0 and shear modulus μ_0. The anisotropic medium has density ρ and the shear moduli c_1 and c_2 with respect to the x and y directions, respectively. Time harmonic conditions are assumed with the time factor $\exp(-\mathrm{i}\omega t)$, where t is time and ω the angular frequency. The wave number in the infinite medium is then $k_0 = \omega\sqrt{\rho_0/\mu_0}$.

The displacement field has only an out-of-plane component u that in the medium outside the circle satisfies the 2D Helmholtz equation

$$\nabla^2 u + k_0^2 u = 0. \tag{1}$$

The medium inside the circle is assumed to be orthotropic with the plane of the circle as a symmetry plane so that the constitutive equations are

$$\sigma_{xz} = 2c_1\epsilon_{xz}, \tag{2}$$

$$\sigma_{yz} = 2c_2\epsilon_{yz}. \tag{3}$$

The shear stresses are σ_{xz} and σ_{yz} and ϵ_{xz} and ϵ_{yz} are corresponding shear strains. The equation of motion inside the circle is then

$$c_1\frac{\partial^2 u}{\partial x^2} + c_2\frac{\partial^2 u}{\partial y^2} + \rho\,\omega^2 u = 0. \tag{4}$$

The boundary conditions at $r = a$ between the isotropic medium outside the circle and the anisotropic one inside the circle are that the displacement u and the shear stress σ_{rz} are continuous. The incoming field u^{in} is taken as a plane wave with unit amplitude propagating in a direction making the angle φ_0 with the x axis

$$u^{\mathrm{in}} = \exp(\mathrm{i}k_0 r\cos(\varphi - \varphi_0)). \tag{5}$$

To fully specify the scattering problem the scattered field $u^{\mathrm{sc}} = u - u^{\mathrm{in}}$ must satisfy radiation conditions.

3. Solution inside the circle

Because the anisotropic medium resides within a circle it is convenient to formulate the equation of motion in polar coordinates. Expressed in terms of stresses this equation is

$$\frac{\partial\sigma_{rz}}{\partial r} + \frac{1}{r}\frac{\partial\sigma_{\varphi z}}{\partial\varphi} + \frac{\sigma_{rz}}{r} + \rho\,\omega^2 u = 0. \tag{6}$$

Using the transformations for the stresses and strains between the two coordinate systems and the definition of the strains, the stresses can be given in polar coordinates as

$$\sigma_{rz} = \frac{\partial u}{\partial r}\left(c_1\cos^2\varphi + c_2\sin^2\varphi\right) + \frac{1}{r}\frac{\partial u}{\partial\varphi}(c_2 - c_1)\sin\varphi\cos\varphi, \tag{7}$$

$$\sigma_{\varphi z} = \frac{\partial u}{\partial r}(c_2 - c_1)\sin\varphi\cos\varphi + \frac{1}{r}\frac{\partial u}{\partial\varphi}\left(c_1\sin^2\varphi + c_2\cos^2\varphi\right). \tag{8}$$

Inserting into the equation of motion and rearranging a little gives

$$(1+\beta\cos 2\varphi)\frac{\partial^2 u}{\partial r^2}+(1-\beta\cos 2\varphi)\left(\frac{1}{r^2}\frac{\partial^2 u}{\partial\varphi^2}+\frac{1}{r}\frac{\partial u}{\partial r}\right)+2\beta\sin 2\varphi\left(\frac{1}{r^2}\frac{\partial u}{\partial\varphi}-\frac{1}{r}\frac{\partial^2 u}{\partial r\partial\varphi}\right)+k^2u=0. \quad (9)$$

Here $k=\omega\sqrt{\rho/c}$ is the wave number in the anisotropic medium determined with the mean stiffness $c=(c_1+c_2)/2$, and

$$\beta=\frac{c_1-c_2}{c_1+c_2}, \quad (10)$$

is a measure of the degree of anisotropy, with $\beta=0$ in the special case of isotropy. It is noted that the equation of motion in polar coordinates has terms that depend explicitly on the angular coordinate φ.

Due to symmetry the solution inside the circle can be divided into four parts that are even or odd with respect to the x and y axes. For the part that is even–even the solution can be assumed in the form

$$u(r,\varphi)=\sum_{m=0,\,2,\ldots} f_m(r)\cos m\varphi. \quad (11)$$

Inserting this into the equation of motion, using trigonometric relations for products, and using the orthogonality of the trigonometric system gives

$$f_m''+\frac{1}{r}f_m'+\left(k^2-\frac{m^2}{r^2}\right)f_m+\frac{\beta}{2}\left(f_{m+2}''+\frac{2m+3}{r}f_{m+2}'+\frac{m^2+2m}{r^2}f_{m+2}\right)$$
$$+\frac{\beta}{2}\left(f_{m-2}''-\frac{2m-3}{r}f_{m-2}'+\frac{m^2-2m}{r^2}f_{m-2}\right)=0, \quad (12)$$

where a prime denotes differentiation with respect to r. This equation is valid also for $m=0$ with the condition $f_{-2}=0$. The result is thus a system of coupled ordinary differential equations for the functions $f_m(r),\ m=0,2,4,\ldots$. To solve this system the functions are expanded into power series in r. To obtain solutions that are analytical at the origin the function $f_m(r)$ must behave as r^m as r approaches $r=0$ and furthermore only even m can be included. Thus the following power series expansion can be made

$$f_m(r)=\sum_{j=m,\,m+2,\ldots}\alpha_j^m(r/a)^j. \quad (13)$$

Inserting this into Eq. (12) and identifying equal powers of r gives

$$\left(j^2-m^2\right)\alpha_j^m+(ka)^2\alpha_{j-2}^m+\frac{1}{2}\beta(j+m)(j+m+2)\alpha_j^{m+2}+\frac{1}{2}\beta(j-m)(j-m+2)\alpha_j^{m-2}=0. \quad (14)$$

Here $j=m+2, m+4,\ldots$ and $m=0,2,\ldots$ and $\alpha_j^{m-2}=0$ for $m=0$. This system of equations can be used to determine all α_j^m for $j=m+2, m+4,\ldots$ in terms of the α_m^m. Thus the equation for $m=0$ and $j=2$ determines α_2^0 in terms of α_0^0 and α_2^2. Then the two equations for $m=0, j=4$ and $m=2, j=4$ determine α_4^0 and α_4^2, and so on. The even–even solution inside the circle is then complete and contains the unknown coefficients α_m^m, $m=0,2,\ldots$.

At this stage wave functions can be introduced in the following way. Set all $\alpha_{m'}^{m'}=0$ except for α_m^m, which could be set to any convenient normalization. Then it is easily seen that $\alpha_j^{m'}=0$ when $m'<m$ and $j<m+2$. The wave function then has the following form

$$\psi_m(\beta,r)=\sum_{m'=0,\,2,\ldots}^{m-2}\ \sum_{j=m+2,\,m+4,\ldots}\alpha_j^{m'}(r/a)^j+\sum_{j=m,\,m+2,\ldots}\alpha_j^m(r/a)^j\cos m\varphi$$
$$+\sum_{m'=m+2,\,m+4,\ldots}\ \sum_{j=m'+2,\,m'+4,\ldots}\alpha_j^{m'}(r/a)^j. \quad (15)$$

It is seen that these wave functions behave as r^m when r is small. For $\beta=0$ they reduce to Bessel functions (if the normalization is chosen correctly; note that for $\beta=0$ Eq. (13) reduces to the usual recursion relation for the coefficients in the power series expansion of Bessel functions). It seems that these wave functions are not the same as those of Wu and Ren [3]. At least for the present purpose, however, there seems to be no point in using these functions, but instead work directly with the expansions Eq. (13) and the system of equation (14).

For the part of the solution that is even in x and odd in y the same expansion Eq. (11) is made except that now m is summed over the odd integers $m=1,3,\ldots$. The differential equation (12) remains the same, but now the interpretation $f_{-1}=f_1$ must be made. The expansion into power series Eq. (13) also remains valid, as does the resulting equation (14), now with the requirement that $\alpha_j^{-1}=\alpha_j^1$. The parts that are odd in x can be treated in exactly the same way. Instead of an expansion in $\cos m\varphi$, the expansion is in $\sin m\varphi$ and $m=0$ is of course not included in the even part. Otherwise the equations remain the same with the same requirement for $m=1$ and with $\alpha_j^{m-2}=0$ for $m=2$.

4. The scattering problem

With the solution inside the circle determined, the scattering problem can now be solved. The solution in the isotropic medium outside the circle is standard and concentrating on the part that is even in both x and y it is

$$u(r,\varphi)=\sum_{m=0,\,2,\dots}\left[a_m J_m(k_0 r)+b_m H_m(k_0 r)\right]\cos m\varphi. \tag{16}$$

Here J_m and H_m are the Bessel and Hankel (of the first kind) functions, respectively. The coefficients b_m of the scattered field are to be determined. The expansion coefficients a_m of the incoming field are known and for the incoming plane wave they are

$$a_m=\epsilon_m \mathrm{i}^m\cos m\varphi_0, \tag{17}$$

where φ_0 is the angle of incidence of the incoming wave and the Neumann factor is $\epsilon_0=1$ and $\epsilon_m=2$ otherwise.

The boundary conditions on the circle $r=a$ are the continuity of displacement u and radial shear stress σ_{rz}. Continuity of displacement gives

$$a_m J_m(k_0 a)+b_m H_m(k_0 a)=f_m(a). \tag{18}$$

Using Eq. (7) for the stress and some trigonometric identities the continuity of stress gives

$$\gamma k_0\left[a_m J'_m(k_0 a)+b_m H'_m(k_0 a)\right]=f'_m(a)+\frac{1}{2}\beta\left(f'_{m-2}(a)+\frac{m-2}{a}f_{m-2}(a)\right)+\frac{1}{2}\beta\left(f'_{m+2}(a)+\frac{m+2}{a}f_{m+2}(a)\right), \tag{19}$$

where $\gamma=2\mu_0/(c_1+c_2)$ is the quotient between the stiffness outside the circle and the mean stiffness inside the circle. Inserting the expansion for f_m Eq. (13) the two continuity equations become

$$a_m J_m(k_0 a)+b_m H_m(k_0 a)=\sum_{j=m,\,m+2,\dots}\alpha_j^m, \tag{20}$$

$$\gamma k_0 a\left[a_m J'_m(k_0 a)+b_m H'_m(k_0 a)\right]=\sum_{j=m,\,m+2,\dots}\left(\alpha_j^m+\frac{1}{2}\beta(j-m+2)\alpha_j^{m-2}+\frac{1}{2}\beta(j+m+2)\alpha_j^{m+2}\right), \tag{21}$$

for $m=0,2,\dots$, and again with $\alpha_j^{-2}=0$. Together with Eq. (14) this is enough to solve for all unknowns. The natural way of truncating the equations seems to first fix the number of azimuthal terms needed, i.e. to include m values up to some $m_{\max}$. Then the j index is also restricted to some $j_{\max}$, which must be larger than $m_{\max}$, and this means that fewer j values are used for higher m.

For the other symmetries the solution is essentially the same, with the special cases for $m=1$ and $m=2$ noted above.

5. Numerical examples

The numerical implementation is straightforward from the equations given, although for general incidence four (very similar) subproblems are to be solved. It is noted, however, that numerical problems are to be expected for higher frequencies. For an isotropic circle the present approach reduces to the standard isotropic solution with the Bessel functions inside the circle expanded in their series form. This series is well behaved for small and moderate arguments, but for larger arguments so large cancellations occur as to make the series more or less useless. A corresponding problem occurs with the present approach when increasing the frequency so that problems start to appear when the radius of the circle is a couple of wavelengths (as measured with the larger stiffness inside the circle).

To illustrate the effects that appear due to the anisotropy inside the circle the far field amplitude is computed for a few cases. The far field amplitude is defined from the asymptotic behaviour of the scattered field

$$u^{\mathrm{sc}}=F(\varphi)\frac{\mathrm{e}^{\mathrm{i}k_0 r}}{\sqrt{k_0 r}}, \tag{22}$$

where the far field amplitude is

$$F(\varphi)=\sqrt{\frac{2}{\pi}}\,\mathrm{e}^{-\mathrm{i}\pi/4}\sum_{m=0}^{\infty}\epsilon_m b_m\cos m\varphi. \tag{23}$$

In general a corresponding sum with $\sin m\varphi$ instead of $\cos m\varphi$ must be added. It is the absolute value of F that is plotted in the following.

First it is appropriate to point out that at low frequencies (approximately $\max(k_0 a, ka)<0.5$) there is no dependence on the anisotropy at all, it is only the mean stiffness inside the circle that determines the scattering, so in this limit the circle behaves as though it is isotropic. This can easily be seen by keeping only the first one or two terms in the expansions inside

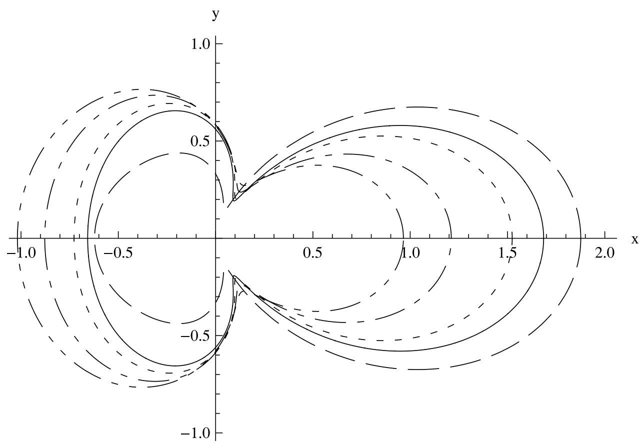

Fig. 1. The far field amplitude for the frequency $k_0a = 2$ for the stiffness ratios $c_1/c_2 = 1$ (full-drawn), 10 (dashed), 0.5 (dotted), 0.2 (dash–dotted), and 0.1 (dash-double-dotted). Density ratio $\rho/\rho_0 = 2$ and stiffness ratio $c/\mu_0 = 4$.

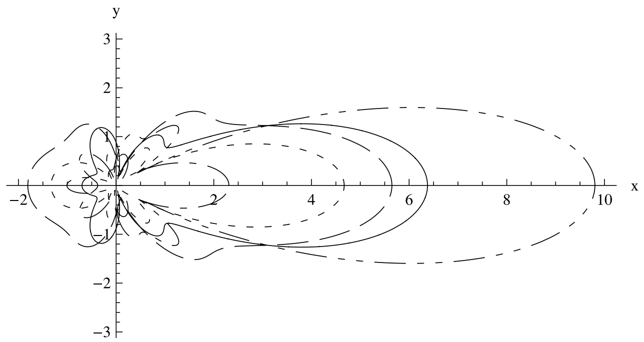

Fig. 2. The far field amplitude for the frequency $k_0a = 5$ for the stiffness ratios $c_1/c_2 = 1$ (full-drawn), 10 (dashed), 0.5 (dotted), 0.2 (dash–dotted), and 0.1 (dash-double-dotted). Density ratio $\rho/\rho_0 = 2$ and stiffness ratio $c/\mu_0 = 4$.

the circle. This is dependent on the fact that $m = 0$ and $m = 1$ do not couple, and is therefore not expected to be true for general anisotropy and for elastic vector problems.

First a case with the material of the circle denser and stiffer than the surrounding is considered; thus $\rho/\rho_0 = 2$ and $c/\mu_0 = 4$ is chosen. Fig. 1 shows the far field amplitude for $k_0a = 2$ and an incident wave in the positive x direction and five different values of the stiffness quotient inside the circle: $c_1/c_2 = 1$ (full-drawn), i.e. isotropy, 10, (dashed) 0.5, (dotted) 0.2, (dash–dotted) and 0.1 (dash-double-dotted). It is apparent that the anisotropy has a strong effect already at this relatively low frequency, this being particularly true when $c_1/c_2 < 1$, i.e. when the wavelength in the x direction is smaller than in the y direction. Fig. 2 shows the same situation for the somewhat higher frequency $k_0a = 5$. The effects due to anisotropy are similar, but generally speaking they are now even stronger.

To stress the importance of anisotropy Figs. 3 and 4 show the same situation as Figs. 1 and 2 with the only change that the material inside the circle has the same density and mean stiffness as the surrounding material. Thus the scattering is only due to the anisotropy and not to any change in density or mean stiffness. It is seen that the scattering is about equally strong as in Figs. 1 and 2, and that the anisotropy thus is a very important variable in the scattering process.

To further illustrate the anisotropy Fig. 5 shows the scattering for $k_0a = 5$ for four different angles of incidence: $\varphi = 0°$, (dash–dotted) 30°, (full-drawn) 60°, (dotted) and 90° (dashed). The material inside the circle still has the same density and mean stiffness as the surrounding material and $c_1/c_2 = 0.2$. Due to the anisotropy the scattering is strongly dependent on the angle of incidence, and it is also noted that for $\varphi = 30°$ and 60° there is no mirror symmetry around the direction of incidence.

6. Concluding remarks

The scattering by an anisotropic circle in an isotropic surrounding is solved by formulating the equation of motion in polar coordinates and expanding in a trigonometric series in the angular coordinate and a power series in the radial coordinate inside the circle. The numerical results show that the influence of anisotropy is strong except at very low frequencies.

The method should be straightforward to extend to more complex cases such as the scattering by a circle for P-SV waves and a sphere, and such work is in progress. A more challenging task is to find the outgoing solutions to the anisotropic wave

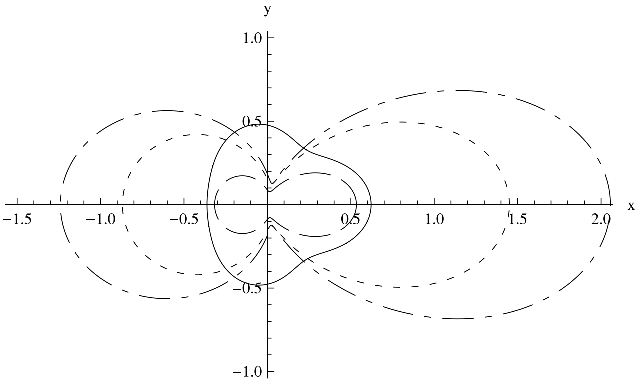

Fig. 3. The far field amplitude for the frequency $k_0a = 2$ for the stiffness ratios $c_1/c_2 = 10$ (full-drawn), 0.5 (dashed), 0.2 (dotted), and 0.1 (dash–dotted). Density ratio $\rho/\rho_0 = 1$ and stiffness ratio $c/\mu_0 = 1$.

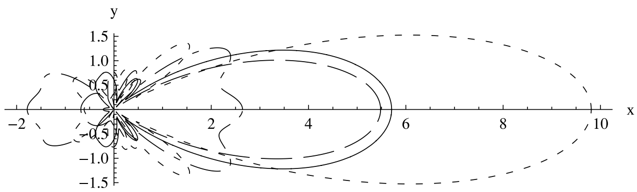

Fig. 4. The far field amplitude for the frequency $k_0a = 5$ for the stiffness ratios $c_1/c_2 = 10$ (full-drawn), 0.5 (dashed), 0.2 (dotted), and 0.1 (dash–dotted). Density ratio $\rho/\rho_0 = 1$ and stiffness ratio $c/\mu_0 = 1$.

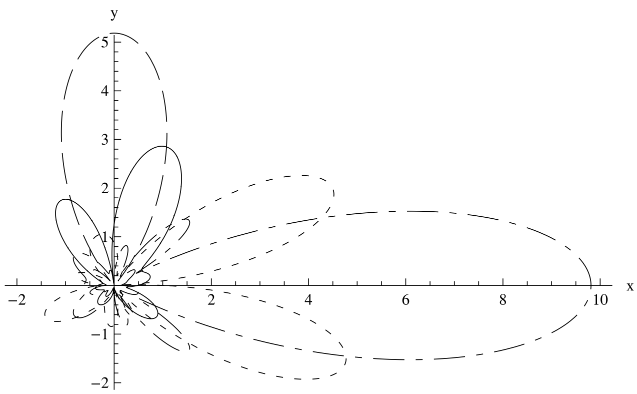

Fig. 5. The far field amplitude for the frequency $k_0a = 5$ for the angles of incidence 0° (dash–dotted), 30° (full-drawn), 60° (dotted), and 90° (dashed). Density ratio $\rho/\rho_0 = 1$ and stiffness ratios $c/\mu_0 = 1$ and $c_1/c_2 = 0.2$.

equation and thus to solve the scattering with an anisotropic material in the surrounding of a scatterer. It should also be possible to formulate the null field method using the present approach, cf. Doicu [5] and Wang et al. [6].

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