JOURNAL OF MATHEMATICAL PHYSICS **56**, 043303 (2015)

Decay of transverse correlations in quantum Heisenberg models

Jakob E. Björnberg[1,a)] and Daniel Ueltschi[2,a)]
[1]*Department of Mathematical Sciences, Chalmers and University of Gothenburg, 41296 Göteborg, Sweden*
[2]*Department of Mathematics, University of Warwick, Coventry CV4 7AL, United Kingdom*

(Received 9 January 2015; accepted 8 April 2015; published online 20 April 2015)

We study a class of quantum spin systems that include the $S = \frac{1}{2}$ Heisenberg and XY-models and prove that two-point correlations exhibit exponential decay in the presence of a transverse magnetic field. The field is not necessarily constant, it may be random, and it points in the same direction. Our proof is entirely probabilistic and it relies on a random loop representations of the correlation functions, on stochastic domination and on first-passage percolation. © *2015 AIP Publishing LLC.* [http://dx.doi.org/10.1063/1.4918675]

I. SETTING AND RESULTS

It has been known since the work of Tóth[18] and Aizenman and Nachtergaele[1] that certain quantum spin systems may be represented in terms of a collection of random loops. The two representations were recently combined so as to be included in a larger family of models.[19] We study this family with the addition of positive transverse fields and use the loop representation to prove that two-point correlations decay exponentially. Some results can alternatively be obtained as a consequence of the Lee–Yang theorem (as we remark below). However, our method of proof, which uses techniques from modern probability theory, is new and interesting in itself.

This work is one of the growing number of contributions to the understanding of quantum spin systems using probabilistic graphical representations. This includes the recent work by Crawford, Ng, and Starr[7] on emptiness formation in the XXZ model as well as work on the transverse field Ising model.[3–6] We note, in particular, that Crawford and Ioffe[6] establish exponential decay of truncated correlations in the presence of an external field, using an argument which has some similarities with our method.

We consider the following class of quantum spin systems. Let L be an even integer and $\Lambda = \{-\frac{1}{2}L, \dots, \frac{1}{2}L\}^d \subset \mathbb{Z}^d$. Write $\mathcal{E}_\Lambda$ for the set of nearest neighbors in Λ. The Hilbert space is $\mathcal{H}_\Lambda = \otimes_{x\in\Lambda}\mathbb{C}^2$ and the Hamiltonian is

$$H_{\Lambda,\boldsymbol{h}} = -2\sum_{xy\in\mathcal{E}_\Lambda}\left(S_x^1S_y^1 + (2u-1)S_x^2S_y^2 + S_x^3S_y^3 - \tfrac{1}{4}\right) - \sum_{x\in\Lambda} h_x S_x^3. \tag{1.1}$$

Here, S_x^i are the usual spin operators that satisfy the commutation relations $[S_x^1, S_y^2] = \mathrm{i}\delta_{x,y}S_x^3$ and further relations obtained by cyclic permutation of the indices 1, 2, and 3. The parameters $\boldsymbol{h} = (h_x)_{x\in\Lambda}$ represent external magnetic fields; we assume that they take values in $[0,\infty)$. The parameter u belongs to $[0,1]$ and well-known models are obtained for certain values. The main examples are the $S = \frac{1}{2}$ Heisenberg and XY models in transverse fields, obtained by taking $u = 1$ for the Heisenberg ferromagnet, $u = 0$ for the Heisenberg anti-ferromagnet (up to unitary equivalence), and $u = \frac{1}{2}$ for the XY model.

We actually discuss a more general setting allowing $S \in \frac{1}{2}\mathbb{N}$ that is compatible with the loop representation (the case $S = \frac{1}{2}$ is physically the most relevant). Let $S \in \frac{1}{2}\mathbb{N}$, and let us consider

[a)]Electronic addresses: jakob.bjornberg@gmail.com and daniel@ueltschi.org

the Hilbert space $\mathcal{H}_\Lambda = \otimes_{x\in\Lambda}\mathbb{C}^{2S+1}$ and the Hamiltonian

$$H_{\Lambda,\boldsymbol{h}} = -\sum_{xy\in\mathcal{E}_\Lambda}\big(uT_{xy} + (1-u)Q_{xy} - 1\big) - \sum_{x\in\Lambda} h_x S_x^3. \tag{1.2}$$

In order to define the operators T and Q that appear above, let $|a\rangle$, $a = -S, -S+1, \dots, S$ denote a basis of $\mathbb{C}^{2S+1}$ of eigenvectors for S_x^3. Then $S_x^3|a\rangle_x = a|a\rangle_x$. The transposition operator T_{xy} acts on $\mathbb{C}^{2S+1}\otimes\mathbb{C}^{2S+1}$ as follows:

$$T_{xy}|a\rangle_x \otimes |b\rangle_y = |b\rangle_x \otimes |a\rangle_y, \tag{1.3}$$

and the operator Q_{xy} has matrix elements

$$\langle a|_x \otimes \langle b|_y Q_{xy}|c\rangle_x \otimes |d\rangle_y = \delta_{a,b}\delta_{c,d}. \tag{1.4}$$

In the case $S = \frac{1}{2}$, one can check that the Hamiltonian of (1.2) is equal to that of (1.1).

For suitable observables M, the finite-volume states are defined by

$$\langle M\rangle_{\Lambda,\boldsymbol{h}} = \frac{\mathrm{Tr}\, M e^{-\beta H_{\Lambda,\boldsymbol{h}}}}{Z(\beta,\Lambda,\boldsymbol{h})}, \qquad \text{where } Z(\beta,\Lambda,\boldsymbol{h}) = \mathrm{Tr}\, e^{-\beta H_{\Lambda,\boldsymbol{h}}},$$

and where $\beta > 0$ denotes the inverse temperature.

Our results consist of two theorems. In Theorem 1.1, we assume a uniform lower bound on all h_xs, and we obtain a bound for the transverse correlations that is uniform in the size of the system and in β.

Theorem 1.1. *Assume that $h_x \geq \alpha$, for all $x \in \Lambda$ and some $\alpha > 0$. Then there exist constants $C, c > 0$ (they depend on S, d, α, but not on L, β) such that*

$$0 < \langle S_0^1 S_x^1\rangle_{\Lambda,\boldsymbol{h}} < C\mathrm{e}^{-c\|x\|}$$

for all $x \in \Lambda$.

Let us remark that similar results follow from the Lee–Yang theorem, as observed by Lebowitz and Penrose.[14] Let $s\boldsymbol{h} = (sh_x)_{x\in\Lambda}$ with $s \in \mathbb{C}$. It can be shown that the two point function $\langle S_0^1 S_x^1\rangle_{\Lambda,s\boldsymbol{h}}$ is analytic in s, when $\mathrm{Re}\; s \neq 0$. Assume that $\boldsymbol{h}$ is such that the thermodynamic limit $\langle S_0^1 S_x^1\rangle_{s\boldsymbol{h}}$ exists. The inverse correlation length

$$\xi^{-1}(s) = -\limsup_{\|x\|\to\infty} \frac{1}{\|x\|}\log\big|\langle S_0^1 S_x^1\rangle_{s\boldsymbol{h}}\big| \tag{1.5}$$

is therefore subharmonic. A cluster expansion shows that $\xi^{-1}(s) > 0$, when $\mathrm{Re}\; s$ is large; then it never vanishes in the domain of analyticity. We refer to Refs. 8, 14, and 16 for more information.

Our proof of Theorem 1.1 is new and very different. We use the random loop representation of Refs. 1, 18, and 19 in order to obtain a suitable expression for the two-point function. It can be bounded by the two-point function of a model of dependent percolation. Stochastic domination allows to remove dependence, and our theorem follows from results about first-passage percolation. This method of proof seems more robust.

The second result deals with a quenched disordered system, where the h_xs are independent and identically distributed (i.i.d.) random variables taking values in $[0,\infty)$. Let $\mathbf{E}$ denote expectation with respect to the magnetic fields $\boldsymbol{h}$. The transverse two-point function is defined as

$$\langle\!\langle S_0^1 S_x^1\rangle\!\rangle_\Lambda = \mathbf{E}\Big(\frac{1}{Z(\beta,\Lambda,\boldsymbol{h})}\mathrm{Tr}\, S_0^1 S_x^1 \mathrm{e}^{-\beta H_{\Lambda,\boldsymbol{h}}}\Big). \tag{1.6}$$

We allow a small fraction of magnetic fields to be zero. The Lee–Yang method does not seem to apply any more. Our result is not uniform in β; the situation of the ground state remains to be clarified.

Theorem 1.2. *For every S, d, β, there exists $\varepsilon > 0$ such that, if $\mathbf{P}(h_x < \alpha) < \varepsilon$ for some $\alpha > 0$, there exist $C, c > 0$ (they depend on S, d, β, α, but not on L) such that*

$$0 < \langle\!\langle S_0^1 S_x^1\rangle\!\rangle_\Lambda \leq C\mathrm{e}^{-c\|x\|},$$

for all $x \in \Lambda$.

Straightforward modifications of our argument also give the same result for the Schwinger functions $\langle S_0^1 e^{-(\beta-t)H_{\Lambda,\boldsymbol{h}}} S_x^1 e^{-tH_{\Lambda,\boldsymbol{h}}}\rangle_{\Lambda,\boldsymbol{h}}$ and $\langle\!\langle S_0^1 e^{-(\beta-t)H_{\Lambda,\boldsymbol{h}}} S_x^1 e^{-tH_{\Lambda,\boldsymbol{h}}}\rangle\!\rangle_\Lambda$. Under the assumptions of Theorem 1.1, we can show that for all $t \in [0,\beta]$,

$$\langle S_0^1 e^{-(\beta-t)H_\Lambda} S_x^1 e^{-tH_{\Lambda,\boldsymbol{h}}}\rangle_{\Lambda,\boldsymbol{h}} < C\mathrm{e}^{-c(\|x\|+t)}. \tag{1.7}$$

The constants C, c are positive and they do not depend on L,β,t,x. Under the assumptions of Theorem 1.2, we obtain a similar upper bound on $\langle\!\langle S_0^1 e^{-(\beta-t)H_\Lambda} S_x^1 e^{-tH_{\Lambda,\boldsymbol{h}}}\rangle\!\rangle_\Lambda$ (in this case, the constants are not uniform in β).

We explain the random loop representation in Sec. II and use it to prove Theorems 1.1 and 1.2 in Sec. III.

II. RANDOM LOOPS

We now describe an ensemble of random loops. Its relevance for the spin system is explained in Theorem 2.1.

The loops live in $\Lambda \times [0,\beta)$, and we regard the interval $[0,\beta)$ as a circle of length β. Points in $\Lambda \times \{0\}$ are identified with the corresponding elements of Λ and denoted $0, x$, etc. We consider two independent Poisson processes in the set $\mathcal{E}_\Lambda \times [0,\beta)$. The first process has intensity u and is called the process of *crosses*; the second process has intensity $1-u$ and is called the process of *double bars* (or *bars* for short). The joint realization of bars and crosses is denoted by ω and its distribution is denoted by ρ. Note that ω, taken as a whole, is a realization of a Poisson process of intensity 1.

The realization ω decomposes $\Lambda \times [0,\beta)$ into a collection of disjoint loops. Informally, these loops are obtained as follows. One starts at a point $(x,t) \in \Lambda \times [0,\beta)$ and proceeds "upwards" (or "downwards") until hitting the endpoint of a bar or a cross. One then moves to the other endpoint and proceeds in the same direction if it was a cross, alternatively changes direction if it was a bar. The loop is completed when one returns to the starting point (x,t). We write $\mathcal{L}(\omega)$ for the collection of loops defined by ω. For more details, and illustrations, see Refs. 10 and 19.

Let us define the relevant loop activities. Given $\gamma \in \mathcal{L}(\omega)$, let $\ell_y(\gamma)$ denote the vertical length of γ at the site y (that is, the length of $\gamma \cap (\{y\} \times [0,\beta))$). Notice the following identity that holds for all realizations ω:

$$\sum_{\gamma \in \mathcal{L}(\omega)} \sum_{y \in \Lambda} \ell_y(\gamma) = \beta|\Lambda|. \tag{2.1}$$

If there is a loop $\gamma_{0,x} \in \mathcal{L}(\omega)$ that contains both 0 and x, we let $\ell_y^+(\gamma_{0,x})$ denote the vertical length at y of the component of the loop that links $(0,0+)$ with $(x,0\pm)$; that is, the component obtained by starting in the "upwards" direction at $(0,0)$ and continuing until the first visit to $(x,0)$. We also let $\ell_y^-(\gamma_{0,x})$ denote the length at y of the other component that links $(0,0-)$ with $(x,0\pm)$; note that $\ell_y(\gamma_{0,x}) = \ell_y^+(\gamma_{0,x}) + \ell_y^-(\gamma_{0,x})$. Define

$$\begin{aligned} z_{\boldsymbol{h}}(\gamma) &= \sum_{a=-S}^{S} \exp\Big(a \sum_y h_y \ell_y(\gamma)\Big), \\ \tilde{z}_{\boldsymbol{h}}(\gamma_{0,x}) &= \tfrac{1}{4} \sum_{a=-S}^{S-1} \big(S(S+1) - a(a+1)\big)\Big[\exp\Big((a+1)\sum_y h_y \ell_y^+(\gamma_{0,x}) + a \sum_y h_y \ell_y^-(\gamma_{0,x})\Big) \\ &\qquad + \exp\Big(a \sum_y h_y \ell_y^+(\gamma_{0,x}) + (a+1)\sum_y h_y \ell_y^-(\gamma_{0,x})\Big)\Big]. \end{aligned} \tag{2.2}$$

(Here, and in all similar sums, the index a increases in steps of size 1.) We write $1_{0\leftrightarrow x}(\omega)$ for the indicator that 0 and x belong to the same loop $\gamma_{0,x} \in \mathcal{L}(\omega)$.

Theorem 2.1. *The partition function and the two-point function have the following representations:*

(a) $Z(\beta,\Lambda,\boldsymbol{h}) = \int \rho(\mathrm{d}\omega) \prod_{\gamma\in\mathcal{L}(\omega)} z_{\boldsymbol{h}}(\gamma)$,

(b) $\langle S_0^1 S_x^1\rangle_{\Lambda,\boldsymbol{h}} = \dfrac{1}{Z(\beta,\Lambda,\boldsymbol{h})} \int \rho(\mathrm{d}\omega)\; 1_{0\leftrightarrow x}(\omega)\; \tilde z_{\boldsymbol{h}}(\gamma_{0x}) \prod_{\gamma\in\mathcal{L}(\omega)\setminus\{\gamma_{0x}\}} z_{\boldsymbol{h}}(\gamma)$.

This theorem builds on Ref. 19, Theorems 3.2 and 3.3, and is proved at the end of this section. Notice that (b) shows that the two-point function is positive.

The following corollary "essentially" shows exponential decay—but it takes a surprising effort in order to turn it into a rigorous proof. We write $\mathbb{E}_{\boldsymbol{h}}$ for expectation with respect to the probability measure with density proportional to $\prod_{\gamma\in\mathcal{L}(\omega)} z_{\boldsymbol{h}}(\gamma)$ with respect to ρ.

Corollary 2.2. We have the estimate

$$\langle S_0^1 S_x^1\rangle_{\Lambda,\boldsymbol{h}} \le \tfrac13 S(S+1)(2S+1)\; \mathbb{E}_{\boldsymbol{h}}\Big(\; 1_{0\leftrightarrow x}(\omega)\; \mathrm{e}^{-\sum_y h_y \ell_y^+(\gamma_{0,x})}\Big).$$

Proof. By Theorem 2.1, we have $\langle S_0^1 S_x^1\rangle_{\Lambda,\boldsymbol{h}} = \mathbb{E}_{\boldsymbol{h}}\big(1_{0\leftrightarrow x}(\omega)\tilde z_{\boldsymbol{h}}(\gamma_{0,x})/z_{\boldsymbol{h}}(\gamma_{0,x})\big)$. The loop activity $z_{\boldsymbol{h}}$ satisfies the lower bound

$$z_{\boldsymbol{h}}(\gamma) \ge \mathrm{e}^{S\sum_y h_y \ell_y(\gamma)}. \tag{2.3}$$

As for $\tilde z_{\boldsymbol{h}}$, we have the upper bound

$$\tilde z_{\boldsymbol{h}}(\gamma_{0,x}) \le \tfrac14 \Big(\mathrm{e}^{\sum_y h_y (S\ell_y^+(\gamma_{0,x}) + (S-1)\ell_y^-(\gamma_{0,x}))} + \mathrm{e}^{\sum_y h_y ((S-1)\ell_y^+(\gamma_{0,x}) + S\ell_y^-(\gamma_{0,x}))}\Big) \cdot \sum_{a=-S}^{S-1} ((S(S+1) - a(a+1)). \tag{2.4}$$

One can check that the latter sum is equal to $\frac23 S(S+1)(2S+1)$. Thus,

$$\tilde z_{\boldsymbol{h}}(\gamma_{0,x}) \le \tfrac16 S(S+1)(2S+1) \mathrm{e}^{S\sum_y h_y \ell_y(\gamma_{0,x})} \Big(\mathrm{e}^{-\sum_y h_y \ell_y^+(\gamma_{0,x})} + \mathrm{e}^{-\sum_y h_y \ell_y^-(\gamma_{0,x})}\Big). \tag{2.5}$$

The corollary follows. □

Proof of Theorem 2.1. By the Trotter product formula, with ρ the Poisson point process described above, we have

$$\begin{aligned} \mathrm{e}^{-\beta H_{\Lambda,\boldsymbol{h}}} &= \lim_{N\to\infty} \Big[\Big(1 - \tfrac1N |\mathcal{E}| + \frac1N \sum_{xy\in\mathcal{E}} \big(uT_{xy} + (1-u)Q_{xy}\big)\Big) \mathrm{e}^{\frac1N \sum_x h_x S_x^3}\Big]^{\beta N} \\ &= \int \rho(\mathrm{d}\omega) \mathrm{e}^{(\beta - t_n)\sum_x h_x S_x^3} R_{x_n y_n} \mathrm{e}^{(t_n - t_{n-1})\sum_x h_x S_x^3} \dots \mathrm{e}^{(t_2 - t_1)\sum_x h_x S_x^3} R_{x_1 y_1} \mathrm{e}^{t_1 \sum_x h_x S_x^3}. \end{aligned} \tag{2.6}$$

Here, $(t_i; x_i, y_i)$ are the times and locations of the outcomes of the realization ω, ordered so that $0 < t_1 < \dots < t_n < \beta$. The operator $R_{x_i y_i}$ is equal to $T_{x_i y_i}$ if the outcome $(t_i; x_i, y_i)$ is a cross; it is equal to $Q_{x_i y_i}$ if the outcome is a double bar.

Inserting the expansion of unity $1\!\!1 = \sum_{(\sigma_x)} \otimes_x |\sigma_x\rangle\langle\sigma_x|$, with $\sigma_x \in \{-S, \dots, S\}$, one obtains

$$Z(\beta,\Lambda,\boldsymbol{h}) = \mathrm{Tr}\, \mathrm{e}^{-\beta H_{\Lambda,\boldsymbol{h}}} = \int \rho(\mathrm{d}\omega) \sum_{\sigma\in\Sigma(\omega)} \exp\Big\{\int_0^\beta \mathrm{d}t \sum_x h_x \sigma_x(t)\Big\}. \tag{2.7}$$

The last sum is over "space-time spin configurations" $\sigma = (\sigma_x : x \in \Lambda)$ that are compatible with ω. That is, σ is a (periodic) function $[0,\beta) \to \{-S, -S+1, \dots, S\}^\Lambda$ and satisfies

- $\sigma(t)$ is constant except possibly at times $t_1, \dots, t_n$ and
- at those times, we have $\langle\sigma(t_i+)|R_{x_i y_i}|\sigma(t_i-)\rangle = 1$.

It is not hard to check that $\sigma \in \Sigma(\omega)$ if and only if the spin values are constant on each loop of $\mathcal{L}(\omega)$. The sum over space-time spin configurations factorizes according to the loops, and we get the claim (a) of Theorem 2.1.

For the correlation function, let $\Sigma_{0,x}(\omega)$ be the set of space-time configurations $[0,\beta) \to \{-S,\dots,S\}^\Lambda$ that satisfy

- $\sigma(t)$ is constant except possibly at times $0, t_1, \dots, t_n$;
- at times $t_1,\dots,t_n$, we have $\langle \sigma(t_i+)|R_{x_i y_i}|\sigma(t_i-)\rangle = 1$; and
- at time 0, we have

$$\sigma_y(0+) = \begin{cases} \sigma_y(0-), & \text{if } y \neq 0, x \\ \sigma_y(0-) \pm 1, & \text{if } y = 0 \text{ or } x \end{cases}. \tag{2.8}$$

Since $\langle a|S^1|b\rangle = \frac{1}{2}\sqrt{S(S+1)-ab}$ if $a = b \pm 1$, and 0 otherwise, we have

$$\operatorname{Tr} S_0^1 S_x^1 \mathrm{e}^{-\beta H_{\Lambda,\boldsymbol{h}}} = \tfrac{1}{4}\int \rho(\mathrm{d}\omega) 1_{0\leftrightarrow x}(\omega) \sum_{\sigma\in\Sigma_{0,x}(\omega)} \big(S(S+1) - \sigma_0(0-)\sigma_0(0+)\big)\mathrm{e}^{\int_0^\beta \mathrm{d}t \sum_y h_y \sigma_y(t)}. \tag{2.9}$$

Extracting the contribution of the loop $\gamma_{0,x}$ and using the definition of $\tilde z_{\boldsymbol{h}}(\gamma_{0,x})$, one obtains Theorem 2.1 (b). □

The step from Corollary 2.2 to Theorems 1.1 and 1.2 is intuitively clear: on the event $0 \leftrightarrow x$, we expect $\gamma_{0,x}$ to have length proportional to $\|x\|$, so the right-hand-side in Corollary 2.2 should decay exponentially in $\|x\|$. The difficulty is that $\ell_y^+(\gamma_{0,x})$ denotes *vertical* length. For any $\varepsilon > 0$, it is possible for a loop to reach from 0 to x, yet still has total vertical length at most ε. This seems unlikely when ε is small, but obtaining a quantitative statement requires dealing with the dependencies under $\mathbb{E}_{\boldsymbol{h}}$.

III. PROOFS

We begin by noting that both $\langle S_0^1 S_x^1\rangle_{\Lambda,\boldsymbol{h}}$ and $\langle\!\langle S_0^1 S_x^1\rangle\!\rangle_\Lambda$ can be written in the general form $\mathbf{E}(\langle S_0^1 S_x^1\rangle_{\Lambda,\boldsymbol{h}})$, where now $\mathbf{E}$ is a measure governing the vector $\boldsymbol{h}$ under which the h_x are independent (but not necessarily identically distributed). Indeed, $\langle\!\langle S_0^1 S_x^1\rangle\!\rangle_\Lambda$ is obtained by letting the h_x be identically distributed, whereas $\langle S_0^1 S_x^1\rangle_{\Lambda,\boldsymbol{h}}$ is obtained when $\mathbf{E}$ is the degenerate measure under which the h_x are almost surely constant. In either case, by Corollary 2.2, the two-point function is bounded by the constant times

$$\mathbf{E}\Big(\mathbb{E}_{\boldsymbol{h}}\Big(1_{0\leftrightarrow x}(\omega)\, \mathrm{e}^{-\sum_y h_y \ell_y^+(\gamma_{0,x})}\Big)\Big). \tag{3.1}$$

We focus on bounding quantity (3.1), and at the end, deduce Theorem 1.1 and 1.2 by specializing to the specific choices for $\mathbf{E}$.

Let $\delta > 0$ be such that $N = \beta/\delta$ is an integer. In what follows, we no longer need to distinguish between bars and crosses and we use the term *bridges* to refer collectively to the two. We write $\Gamma = \Gamma(\Lambda,\beta,\delta)$ for the collection of intervals of the form,

$$I = \{x\} \times [k\delta, (k+1)\delta), \text{ for } x \in \Lambda \text{ and } 0 \le k \le N-1.$$

We view Γ as a graph, where intervals $\{x\}\times[k\delta,(k+1)\delta)$ and $\{y\}\times[\ell\delta,(\ell+1)\delta)$ are said to be *adjacent* if either

1. $xy \in \mathcal{E}_\Lambda$ and $k = \ell$ or
2. $x = y$ and $k = \ell \pm 1$ (viewed modulo N).

A *path* π in Γ is as usual a sequence of elements of Γ which are consecutively adjacent in this sense, and any such path thus corresponds to a sequence of "neighbouring" intervals.

Fix $\alpha > 0$, to be chosen later, and let $I = \{x\}\times[k\delta,(k+1)\delta) \in \Gamma$. Based on the random outcomes $\boldsymbol{h}$ and ω (i.e., the collection of bridges), we will declare the interval I to be

- $\boldsymbol{h}$-good if $h_x \ge \alpha$,
- ω-good if there is no bridge with an endpoint in I, and
- *good* if it is both $\boldsymbol{h}$-good and ω-good.

An interval which is not declared good is declared *bad*. We encode the collection of good and bad intervals as an element $\eta = (\eta(i) : i \in \Gamma)$ of $\{0,1\}^\Gamma$, where 0 denotes bad and 1 denotes good. This classification may be seen as a (dependent) percolation process in Γ.

It is convenient to use the fact that $\mathbb{Z}^d$ is bipartite: we may write $\mathbb{Z}^d = \mathcal{A} \cup \mathcal{B}$, where

$$\mathcal{A} = \{(x_1, \dots, x_d) \in \mathbb{Z}^d : x_1 + \cdots + x_d \equiv 0 \ (\mathrm{mod}\ 2)\}$$

and $\mathcal{B} = \mathbb{Z}^d \setminus \mathcal{A}$. We refer to $\mathcal{A}$ and $\mathcal{B}$ as the even and odd sublattices, respectively. Bipartiteness refers to the fact that a vertex in $\mathcal{A}$ is only adjacent to vertices in $\mathcal{B}$, and vice versa. If $I = \{x\} \times [k\delta, (k+1)\delta)$ is an interval belonging to Γ, we commit a small abuse of notation and write $I \in \mathcal{A}$ if $x \in \mathcal{A}$. We also simply write x for the unique interval $\{x\} \times [0,\delta)$ of Γ containing $(x, 0)$.

We define the *passage time* $T_\Lambda(x)$ from 0 to x in Λ as

$$T_\Lambda(x) = \min_{\pi : 0 \to x} \sum_{i \in \pi \cap \mathcal{A}} \eta(i), \tag{3.2}$$

where the minimum is over all paths π in Γ from $\{0\} \times [0,\delta)$ to $\{x\} \times [0,\delta)$. Thus, $T_\Lambda(x)$ is the minimal number of good intervals, indexed by the even sublattice, on a path from 0 to x. This is a slight variation of the standard definition of a (point-to-point) passage time, where the sum would usually go over *all* points on π. Summing over the even sublattice $\mathcal{A}$ only is a convenient way to avoid dependencies, as will be explained below.

Let $\varphi > 0$ be arbitrary, and assume that the event $0 \leftrightarrow x$ occurs (so $\gamma_{0,x}$ is well-defined). If, in addition, $T_\Lambda(x) \geq \varphi \|x\|$, then any path in Γ from 0 to x contains at least $\varphi\|x\|$ good intervals. In particular, it follows that $\sum_y \ell_y^+(\gamma_{0,x}) \geq \alpha\delta(\varphi\|x\| - 1)$. (We subtract 1 for the last interval, which may contribute less than $\alpha\delta$ even if it is good.) Thus, we have that

$$\mathbf{E}\Big(\mathbb{E}_{\boldsymbol{h}}\Big(\ 1_{0\leftrightarrow x}(\omega)\, \mathrm{e}^{-\sum_y h_y \ell_y^+(\gamma_{0,x})}\Big)\Big) \leq e^{-\alpha\delta(\varphi\|x\|-1)} + \mathbf{E}[\mathbb{P}_{\boldsymbol{h}}(T_\Lambda(x) < \varphi\|x\|)]. \tag{3.3}$$

The theorems follow if we show that the last term is exponentially small in $\|x\|$.

To establish this, we first simplify the probability measure by using the theory of stochastic domination. We begin by defining a partial ordering. We say that $\omega \leq \tilde\omega$ if one may obtain ω from $\tilde\omega$ by removing some bridges (in other words, the support of ω is a subset of the support of $\tilde\omega$). For any fixed $\boldsymbol{h}$, the event $A = \{T_\Lambda(x) < \varphi\|x\|\}$ is *increasing* in the ordering on ω, i.e., if $\omega \leq \tilde\omega$ and $\omega \in A$ then necessarily $\tilde\omega \in A$. This allows us to use results on stochastic domination for point processes[9,17] to bound $\mathbb{P}_{\boldsymbol{h}}(A)$.

The following result lets us get rid of the complicated density $\prod_{\gamma \in \mathcal{L}(\omega)} z_{\boldsymbol{h}}(\gamma)$ at the cost of increasing the intensity of bridges. Write $\theta = 2S + 1$.

Lemma 3.1. Let $\mathbb{P}'$ denote the probability measure under which the bridges form a Poisson process of intensity θ. Then for any realization of $\boldsymbol{h}$, we have that

$$\mathbb{P}_{\boldsymbol{h}}(T_\Lambda(x) < \varphi\|x\|) \leq \mathbb{P}'(T_\Lambda(x) < \varphi\|x\|).$$

Before turning to the proof, we note that $\mathbb{P}'$ may alternatively be described as follows. For each pair $xy \in \mathcal{E}_\Lambda$, the process of bridges on $\{xy\} \times [0,\beta)$ is (under $\mathbb{P}'$) a Poisson process with intensity θ. For all other pairs $x'y' \in \mathcal{E}_\Lambda$, the processes of bridges on $\{xy\} \times [0,\beta)$ and on $\{x'y'\} \times [0,\beta)$ are independent.

Proof. If γ_1, γ_2 are disjoint, measurable subsets of $\Lambda \times [0,\beta)$, a calculation shows that

$$\frac{1}{\theta} \leq \frac{z_{\boldsymbol{h}}(\gamma_1 \cup \gamma_2)}{z_{\boldsymbol{h}}(\gamma_1) z_{\boldsymbol{h}}(\gamma_2)} \leq 1 \leq \theta. \tag{3.4}$$

If $\tilde\omega$ is obtained from ω by adding a bridge, then either some loop $\gamma \in \mathcal{L}(\omega)$ decomposes into two loops γ_1, γ_2, or two loops $\gamma_1, \gamma_2 \in \mathcal{L}(\omega)$ are joined to form a larger loop γ, or the loops stay the same. In either case, (3.4) shows that

$$\frac{\prod_{\tilde\gamma \in \mathcal{L}(\tilde\omega)} z_{\boldsymbol{h}}(\tilde\gamma)}{\prod_{\gamma \in \mathcal{L}(\omega)} z_{\boldsymbol{h}}(\gamma)} \leq \theta = \frac{\theta^{|\tilde\omega|}}{\theta^{|\omega|}}.$$

It follows from Theorem 1.1 in Ref. 9 that, for any event A which is increasing in the ordering on ω, the probability $\mathbb{P}_{\boldsymbol{h}}(A)$ is dominated by the probability of A under the measure with density proportional to $\theta^{|\omega|}$ with respect to ρ. The latter measure is precisely $\mathbb{P}'$, so the result follows. □

In (3.3), we need to bound

$$\mathbf{E}[\mathbb{P}_{\boldsymbol{h}}(T_\Lambda(x) < \varphi\|x\|)] \le \mathbf{E}[\mathbb{P}'(T_\Lambda(x) < \varphi\|x\|)] = \mathbb{E}'[\mathbf{P}(T_\Lambda(x) < \varphi\|x\|)]. \tag{3.5}$$

Let $\mathbf{P}'$ (or $\mathbf{E}'$) denote the measure under which the classification of the intervals $I = \{x\} \times [k\delta, (k+1)\delta) \in \Gamma$ into $\boldsymbol{h}$-good and $\boldsymbol{h}$-bad is done independently over all such intervals, with probability $\mathbf{P}(h_x < \alpha)^{1/N}$ for $\boldsymbol{h}$-bad. Then a straightforward coupling shows that, for any fixed ω,

$$\mathbf{P}(T_\Lambda(x) < \varphi\|x\|) \le \mathbf{P}'(T_\Lambda(x) < \varphi\|x\|). \tag{3.6}$$

(For any x the $\mathbf{P}'$-probability that *all* intervals $I = \{x\} \times [k\delta, (k+1)\delta)$ are $\boldsymbol{h}$-bad is exactly the same as under $\mathbf{P}$, and for all other outcomes, the $\mathbf{P}'$-realization has more bad intervals than the $\mathbf{P}$-realization.) It follows that

$$\mathbf{E}[\mathbb{P}_{\boldsymbol{h}}(T_\Lambda(x) < \varphi\|x\|)] \le \mathbf{E}'[\mathbb{P}'(T_\Lambda(x) < \varphi\|x\|)]. \tag{3.7}$$

Under $\mathbb{P}'$, the labels ω-good and ω-bad assigned to the intervals in Γ are almost independent. In fact, they are independent if the intervals are at vertical distance at least 1, or at horizontal distance at least 2. This is because the labels assigned to such intervals are functions of the realization of the Poisson process ω in disjoint intervals and are therefore independent. Thus, since we look only at the even sublattice $\mathcal{A}$, under $\mathbf{E}' \times \mathbb{P}'$, the labels $\eta(i)$ assigned to the vertices in the sum in (3.2) are independent.

At this point, we comment on differences between the two theorems and on the choice of α. Each $i \in \Gamma$ is "good" with probability at least $p := (1 - \mathbf{P}(h_x < \alpha)^{1/N})e^{-2d\theta\delta}$. In the case of Theorem 1.1, we take α as in the statement of the result, so that all the h_x are uniformly bounded from below by α. Then $p = e^{-2d\theta\delta}$, which first does not depend on x and second approaches 1 uniformly in β as $\delta \to 0$. In the case of Theorem 1.2, the h_x are identically distributed, so again p does not depend on x (for any α). In this case, we need to pick $\delta > 0$ small enough and then $\alpha > 0$ small enough to make p close to 1.

The next step will be to use a general result from the theory of first-passage percolation. Recall that $\mathcal{A}$ is the even sublattice of $\mathbb{Z}^d$, and let

$$\Xi = \mathcal{A} \times \{0, \ldots, N-1\}.$$

We view Ξ as a graph as follows. If $x, y \in \mathcal{A}$ and $k, \ell \in \{0, \ldots, N-1\}$ then (x,k) and (y,ℓ) are adjacent in Ξ if either $x = y$ and $k = \ell \pm 1 \pmod N$ or if $k = \ell$ and x and y are *next*-nearest neighbours in $\mathbb{Z}^d$. Thus Ξ inherits the natural adjacency relation in $\mathcal{A}$ for the first coordinate and is also "periodic" in the last coordinate.

For convenience, we introduce the probability measure $\tilde{\mathbb{P}}$ which assigns values 0 or 1 to the elements of Ξ independently, with probability p (defined above) for 1. We also define the infinite-volume passage time $T(x)$ in the same way as in (3.2), except that we replace the minimum by an infimum taken over *all* paths in Ξ. Note that $T(x) \le T_\Lambda(x)$, so by the reasoning above, we have in (3.3) that

$$\mathbf{E}[\mathbb{P}(T_\Lambda(x) < \varphi\|x\|)] \le \tilde{\mathbb{P}}(T(x) < \varphi\|x\|). \tag{3.8}$$

The theorems will follow by applying the following lemma.

Lemma 3.2. There is $\kappa > 0$, depending only on d, such that if $p > 1 - \kappa$ then there are constants $\varphi > 0$ and $c_1, c_2 > 0$, depending only on p and d, such that

$$\tilde{\mathbb{P}}(T(x) < \varphi\|x\|) \le c_1 e^{-c_2\|x\|}. \tag{3.9}$$

In fact, here, it suffices if $\kappa \le (2d)^{-2}$. (The relevance of the number $(2d)^{-2}$ is that it is a lower bound on the critical probability for site percolation on Ξ, as can easily be proved using standard path-counting arguments, see, e.g., Ref. 12, Theorem 1.10.) Theorem 1.1 follows on taking $\delta > 0$

sufficiently small: as noted above, p is then close to 1 uniformly in β so all the constants in (3.9) are uniform in both Λ and β. For Theorem 1.2, we first take $\delta > 0$ small and find that there is $\varepsilon > 0$ such that if $\mathbf{P}(h_x < \alpha) < \varepsilon$ then $p > 1 - \kappa$, but ε will now depend on N and hence, β. (For an explicit bound, $\varepsilon \le (1 - (1 - \kappa)e^{2d\theta\delta})^N$ suffices.)

Sketch proof of Lemma 3.2. This can be proved by adapting Ref. 13, Proposition 5.8 (see also Ref. 11). That result deals with bond-first-passage-percolation on $\mathbb{Z}^d$, and our situation is slightly different since we are dealing with *site*-percolation on the sublattice $\mathcal{A}$ with the next-nearest-neighbour adjacency relation, and also the underlying graph is periodic in one direction. We give a rough outline of the main ideas.

Write $n = \|x\|$. On the event that $T(x) < \alpha n$, there must be a self-avoiding walk w in Ξ which starts at the origin and contains at least n steps, such that the passage time along w satisfies

$$\sum_{i \in w} \eta(i) < \alpha n.$$

One may decompose w into a finite sequence $w_1, w_2, \ldots$ of sub-walks, each of which traverses distance m for some fixed m, and the sum of whose passage times is "small." Since these paths are disjoint, we obtain an upper bound if we assume that the corresponding passage times are independent, by the BK-inequality.[2] Since $1 - p$ is subcritical for site-percolation in Ξ, the set of vertices with passage time 0 from the origin does not percolate, meaning that for suitable m, the passage times for distance m are very unlikely to be small, and exponential decay follows from a large deviations type estimate. □

For extension (1.7) to Schwinger functions, we note that small modifications of Theorem 2.1 and Corollary 2.2 give that

$$\langle S_0^1 e^{-(\beta - t)H_{\Lambda,\boldsymbol{h}}} S_x^1 e^{-tH_{\Lambda,\boldsymbol{h}}} \rangle_{\Lambda,\boldsymbol{h}} \le \tfrac{1}{3} S(S+1)(2S+1) \mathbb{E}_{\boldsymbol{h}}\big(1_{0 \leftrightarrow (x,t)} \mathrm{e}^{-\sum_y h_y \ell_y^+(\gamma_{0,x})}\big).$$

Here, $0 \leftrightarrow (x,t)$ denotes the event that $(0,0)$ and (x,t) lie in the same loop, $\gamma_{0,x}$ is the loop, and $\ell_y^+(\gamma_{0,x})$ is defined as before except that it concerns the part of the loop up to (x,t) rather than to $(x,0)$. The result then follows from straightforward adjustments of definition (3.2) of the passage time T_Λ as well as of the remaining arguments in this section.

Remark. Key inequality (3.8) can also be obtained by an appeal to the main result of Ref. 15. Indeed, as remarked after Lemma 3.1, the good/bad labelling η forms a 1-dependent random field under $\mathbf{E}' \times \mathbb{P}'$, with marginal density at least p (using the terminology of Ref. 15). Hence, there is $q > 0$, satisfying $q \to 1$ as $p \to 1$, such that η stochastically dominates an i.i.d. field with marginal density q. (A result of this form can alternatively be obtained by "hands-on" methods.)

With this approach, it is no longer necessary to restrict the sum in (3.2) to the even sublattice $\mathcal{A}$. Together with straightforward adaptations of the remaining arguments, this allows us to extend the results of this paper to arbitrary translation-invariant lattices with uniformly bounded degrees (even if they are not bipartite).

ACKNOWLEDGMENTS

The research of JEB is supported by the Knut and Alice Wallenberg Foundation. We are grateful for mobility support by The Leverhulme Trust through the International Network Laplacians, Random Walks, Quantum Spin Systems.

[1] M. Aizenman and B. Nachtergaele, "Geometric aspects of quantum spin states," Commun. Math. Phys. **164**, 17–63 (1994).
[2] J. van den Berg and H. Kesten, "Inequalities with applications to percolation and reliability," J. Appl. Probab. **22**, 556–569 (1985).
[3] J. Björnberg, "Infrared bound and mean-field behaviour in the quantum Ising model," Commun. Math. Phys. **323**(2), 329–366 (2013).
[4] J. Björnberg, "Vanishing critical magnetization in the quantum Ising model," Commun. Math. Phys. (published online).
[5] J. Björnberg and G. Grimmett, "The phase transition in the quantum Ising model is sharp," J. Stat. Phys. **136**(2), 231–273 (2009).

[6] N. Crawford and D. Ioffe, "Random current representation for transverse field Ising model," Commun. Math. Phys. **296**(2), 447–474 (2010).
[7] N. Crawford, S. Ng, and S. Starr, "Emptiness formation probability," e-print arXiv:1410.3928.
[8] J. Fröhlich and P.-F. Rodriguez, "Some applications of the Lee-Yang theorem," J. Math. Phys. **53**, 095218 (2012).
[9] H.-O. Georgii and T. Küneth, "Stochastic comparison of point random fields," J. Appl. Probab. **34**, 868–881 (1997).
[10] C. Goldschmidt, D. Ueltschi, and P. Windridge, "Quantum Heisenberg models and their probabilistic representations," *Entropy and the Quantum II*, Contemporary Mathematics (AMS, 2011), pp. 177–224.
[11] G. R. Grimmett and H. Kesten, "First-passage percolation, network flows, and electrical resistances," Z. Wahrsch. Verw. Geb. **66**, 335–366 (1984).
[12] G. R. Grimmett, *Percolation* (Springer, Berlin, 2010).
[13] H. Kesten, "Aspects of first passage percolation," Lect. Notes Math. **1180**, 125–264 (1986).
[14] J. L. Lebowitz and O. Penrose, "On the exponential decay of correlations," Commun. Math. Phys. **39**, 165–184 (1974).
[15] T. M. Liggett, R. H. Schonmann, and A. M. Stacey, "Domination by product measures," Ann. Probab. **25**, 71–95 (1997).
[16] C.-É. Pfister, "Analyticity properties of the correlation functions for the anisotropic Heisenberg model," Commun. Math. Phys. **41**, 109–117 (1975).
[17] C. Preston, "Spatial birth-and-death processes," Bull. Int. Stat. Inst. **46**, 371–390 (1975).
[18] B. Tóth, "Improved lower bound on the thermodynamic pressure of the spin $1/2$ Heisenberg ferromagnet," Lett. Math. Phys. **28**, 75–84 (1993).
[19] D. Ueltschi, "Random loop representations for quantum spin systems," J. Math. Phys. **54**, 083301 (2013).