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Stability of thick two-component galactic discs

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SUMMARY

The crucial role that the cold interstellar gas can play in the dynamics and structure of early normal spiral galaxies has been shown in our previous works, where finite-thickness effects have not been taken into account. In view of the importance that such effects might have in the self-regulation mechanisms which are expected to operate in galactic discs and to be at the basis of their secular heating, we have tried to evaluate them. This can be done only after their vertical structure at equilibrium has been carefully investigated. A detailed analysis has thus been carried out to study the thickness-scales and the local parameters relevant to both the equilibrium and stability of two-component galactic discs in regimes of astrophysical interest.

The results obtained, as regards the vertical structure at equilibrium of two-component galactic discs, are used to investigate their local linear stability properties. Under reasonable assumptions, finite-thickness corrections to the local dispersion relation can be expressed in terms of two reduction factors weakening the response of the two components or, equivalently, lowering their equilibrium surface densities. An ansatz for such reduction factors, justified by extending the analysis performed for one-component purely stellar discs, is made, and the corresponding two-fluid marginal stability curve is studied in standard star-dominated and peculiar gas-dominated regimes. It is found that the stabilizing role of finite-thickness effects can partially counterbalance the destabilizing role of the cold interstellar gas in linear regimes.

Key words: instabilities – galaxies: interstellar matter – galaxies: kinematics and dynamics – galaxies: spiral.

1 INTRODUCTION

The spiral structure theory relies on a number of working assumptions which make the linearized system of the coupled fluid (or Vlasov) and Poisson equations more tractable, but in some situations of astrophysical interest the validity of such assumptions may be questioned. For instance, we know that real galactic discs have finite, although small, thickness and the possibility of regarding them as infinitesimally thin depends on the relevant wavelengths of the perturbations excited. In some cases, when the underlying spiral structure has a high winding degree, finite-thickness effects should be taken into account in the stability analysis. In view of the importance that such effects may have in the *self-regulation* mechanisms which are expected to operate in galactic discs and to be at the basis of their secular heating (Romeo 1987, 1990b), we have tried to evaluate them. This can be done only after their vertical structure at equilibrium has been carefully investigated.

In performing this analysis, we have made use of simplified models of galactic discs in which the stars and the cold interstellar gas are treated as two different components. Although such models might be thought of as being inadequate to describe actual galactic discs, which are known to consist of different populations of stars and gas components, they nevertheless incorporate the most essential features as regards their stability properties. In this context it should be noted that such a single equivalent stellar component is taken to be representative of the whole active stellar disc, consisting of stars with low velocity dispersion (stars with high velocity dispersion do not participate appreciably in spiral structure), whereas the single gaseous component is taken to simulate H I regions of neutral atomic hydrogen and giant molecular clouds and complexes. Generally, even more drastically simplified galactic models are used, in which the cold interstellar gas is not taken into account. In some situations of

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astrophysical interest this further simplification is not justified, because the cold interstellar gas is expected to play an important or even *crucial* role in the stability of galactic discs due to its low turbulent velocities (*cf.* the infinitesimally thin case discussed by Romeo 1985, 1989, 1990b, and Bertin & Romeo 1988). For a more detailed account of the results presented in this paper, see Romeo (1990b). A brief account was given by Romeo (1991a, b, c, d); see also Romeo (1990a).

2 VERTICAL STRUCTURE AT EQUILIBRIUM

The vertical structure of galactic discs has been investigated in detail by a number of authors, who made use of multicomponent locally isothermal models (e.g., Woolley 1957; van der Kruit & Searle 1981; Bahcall 1984; Bahcall & Casertano 1984; Bahcall, Flynn & Gould 1991; see Boulares 1989 for a discussion of non-thermal effects). Alternative approaches have also been employed (e.g., Bienaymé, Robin & Crézé 1987; van der Kruit 1988; Crézé, Robin & Bienaymé 1989; Kuijken & Gilmore 1989a, b, c; see also Gilmore, Wyse & Kuijken 1989; Statler 1989; Wainscoat, Freeman & Hyland 1989; Gould 1990; Amendt & Cuddeford 1991; Cuddeford & Amendt 1991). These analyses take into account the fact that galactic discs are nearly self-gravitating perpendicular to their symmetry plane, so that standard asymptotic expansion techniques can be employed. While, in the case of one-component stellar discs, this is all that is needed to make the problem analytically tractable (see Vandervoort 1967, 1970a for the most rigorous analysis in this context), when multicomponent models are considered, further assumptions must be made. Generally, the component with the largest scaleheight is taken to have the largest mass density, so that a perturbative approach can be employed. In our two-component model this assumption is certainly satisfied, but a perturbative approach of this kind may not always be suitable because in the outermost regions of galactic discs the mass density of the cold interstellar gas becomes comparable to that of disc stars. For this reason we have relaxed this assumption and made a detailed analysis in regimes of astrophysical interest. The results obtained in this section form the basis of the further investigation which we shall carry out in Section 3.

While the investigation of the one-component case is more or less straightforward to the non-trivial lowest order of approximation, the same is not true for two-component galactic discs: several complications arise, some of which are already hidden in the one-component case.

The two components are denoted by different subscripts, H (for hot) and C (for cold), in order to recall that they are characterized by different vertical velocity dispersions. Having in mind cases of astrophysical interest, we refer to them as the stars of the active disc (H) and the cold interstellar gas (C). We could, however, also consider the case in which gas is absent but two stellar populations with different scaleheights can be identified. It is assumed that the system is in an axisymmetric and plane-symmetric equilibrium state, and that each component is locally isothermal perpendicular to the galactic plane. The only interaction between the two components is taken to occur via the gravitational field (Poisson equation). We have adopted the following parametrization:

$$\gamma \equiv \frac{\rho_{0\mathrm{C}}}{\rho_{0\mathrm{H}}}, \qquad \beta_z \equiv \frac{c_{z\mathrm{C}}^2}{c_{z\mathrm{H}}^2} \qquad (0<\gamma<+\infty, \quad 0<\beta_z<1). \tag{1}$$

The cases $\gamma=0$, $\gamma\to+\infty$ represent the limit of a one-component system, and the case $\beta_z=1$ represents the limit of a system in which the two components have the same scaleheight.

The total volume density in the plane determines the *Gaussian thickness-scales* of the two components:

$$\rho_i \underset{|z|\to 0}{\simeq} \rho_{0i} \exp\left(-\frac{z^2}{z_{\mathrm{G}i}^2}\right), \qquad \text{where} \quad z_{\mathrm{G}i} = \sqrt{\frac{c_{zi}^2}{2\pi G\rho_0}}, \qquad \rho_0 \equiv \rho_{0\mathrm{H}} + \rho_{0\mathrm{C}}; \tag{2}$$

while the total surface density determines their *exponential thickness-scales:*

$$\rho_i \underset{|z|\to +\infty}{\simeq} 4^{K_i}\rho_{0i} \exp\left(-\frac{|z|}{z_{\mathrm{E}i}}\right), \qquad \text{where} \qquad z_{\mathrm{E}i} = \frac{c_{zi}^2}{2\pi G\sigma}, \qquad \sigma \equiv \sigma_{\mathrm{H}} + \sigma_{\mathrm{C}}; \tag{3}$$

and hereafter

$$(i=\mathrm{H}, \mathrm{C}). \tag{4}$$

The z-independent exponents K_i depend on higher order corrections, and can only formally be expressed as series in terms of the local parameters γ and β_z. The *global effective thickness-scale* of the system, which can be identified with the *expansion thickness-scale* $\langle z\rangle$, can easily be expressed in terms of the asymptotic thickness-scales of the two components:

$$\langle z\rangle \equiv \frac{\sigma}{2\rho_0} = \frac{z_{\mathrm{G}i}^2}{2z_{\mathrm{E}i}} = \frac{z_{\mathrm{GH}} z_{\mathrm{GC}}}{\sqrt{2z_{\mathrm{EH}} \cdot 2z_{\mathrm{EC}}}}. \tag{5}$$

Equations (2), (3) and (5) admit a trivial generalization to the case of n-component systems.

As regards the *effective thickness-scales* of the two components:

$$z_{\mathrm{eff\,H}} \equiv \frac{\sigma_{\mathrm{H}}}{2\rho_{0\mathrm{H}}} = 2z_{\mathrm{EH}} \left[\frac{1}{2}\sqrt{1+\gamma\beta_z} \int_0^1 \frac{du}{\sqrt{(1-u)+\gamma\beta_z(1-u^{\beta_z^{-1}})}}\right], \tag{6}$$

$$z_{\mathrm{eff\,C}} \equiv \frac{\sigma_{\mathrm{C}}}{2\rho_{0\mathrm{C}}} = 2z_{\mathrm{EC}}\left[\frac{1}{2}\sqrt{1+\gamma\beta_z}\int_0^1 \frac{dv}{\sqrt{(1-v^{\beta_z})+\gamma\beta_z(1-v)}}\right], \tag{7}$$

the situation is not so straightforward: they cannot be expressed explicitly in terms of elementary or special functions for arbitrary values of the local parameter β_z (see, e.g., Gradshteyn & Ryzhik 1980; Prudnikov, Brychkov & Marichev 1986). The determination of the effective thickness-scales of the two components is indeed a delicate point. Some authors (e.g., Talbot & Arnett 1975; see also Jog & Solomon 1984a), in fact, have tried to generalize the equation found in the one-component case and obtained the wrong relation $\sigma_i = \rho_{0i}(2z_{\mathrm{G}i})$, in which the effective thickness-scales are identified with the Gaussian thickness-scales. On the other hand, the tricky relation

$$\sigma = \rho_{0\mathrm{H}}(4z_{\mathrm{EH}}) + \rho_{0\mathrm{C}}(4z_{\mathrm{EC}}) \tag{8}$$

does not imply the further relation $\sigma_i = \rho_{0i}(4z_{\mathrm{E}i})$, in which the effective thickness-scales are identified with twice the exponential thickness-scales (*cf.* Mihalas & Binney 1981). The reason for the existence of such a simple relation involving ρ_{0i} and $z_{\mathrm{E}i}$, as expressed by equation (8), lies in the fact that, while the volume densities are not separately integrable in terms of elementary or special functions, a proper linear combination of them is elementarily integrable. The Gaussian and the (self-consistent) exponential approximations to the effective thickness-scales of the two components are thoroughly justified only in the asymptotic regimes, in which the system can suitably be represented by a single equivalent component or by two components with the same vertical velocity dispersions. This difficulty is not apparent in the one-component case, where the Gaussian, the exponential and the effective thickness-scales are identical apart from numerical factors.

Let us now introduce the local parameters

$$\alpha \equiv \frac{\sigma_{\mathrm{C}}}{\sigma_{\mathrm{H}}} = \gamma\beta_{z\,\mathrm{eff}}, \qquad \beta_{z\,\mathrm{eff}} \equiv \frac{z_{\mathrm{eff\,C}}}{z_{\mathrm{eff\,H}}} \qquad (0<\alpha<+\infty, \quad 0<\beta_{z\,\mathrm{eff}}<1), \tag{9}$$

since in a flat galactic disc the surface densities are more relevant than the volume densities. More restrictive limitations can be given, where the lower and the upper bounds are nothing but the estimates provided by the (*self-consistent*) *exponential approximation*

$$z_{\mathrm{eff}\,i} \sim 2z_{\mathrm{E}i} \Rightarrow \alpha \sim \gamma\beta_z, \qquad \beta_{z\,\mathrm{eff}} \sim \beta_z, \tag{10}$$

and the *Gaussian approximation*

$$z_{\mathrm{eff}\,i} \sim z_{\mathrm{G}i} \Rightarrow \alpha \sim \gamma\sqrt{\beta_z}, \qquad \beta_{z\,\mathrm{eff}} \sim \sqrt{\beta_z}, \tag{11}$$

respectively. In Fig. 1 the 'exact' α is compared to its Gaussian and exponential approximations. It is apparent that, although only the exponential approximation is self-consistent, as expressed by equation (8), the Gaussian approximation works better in regimes of astrophysical interest.

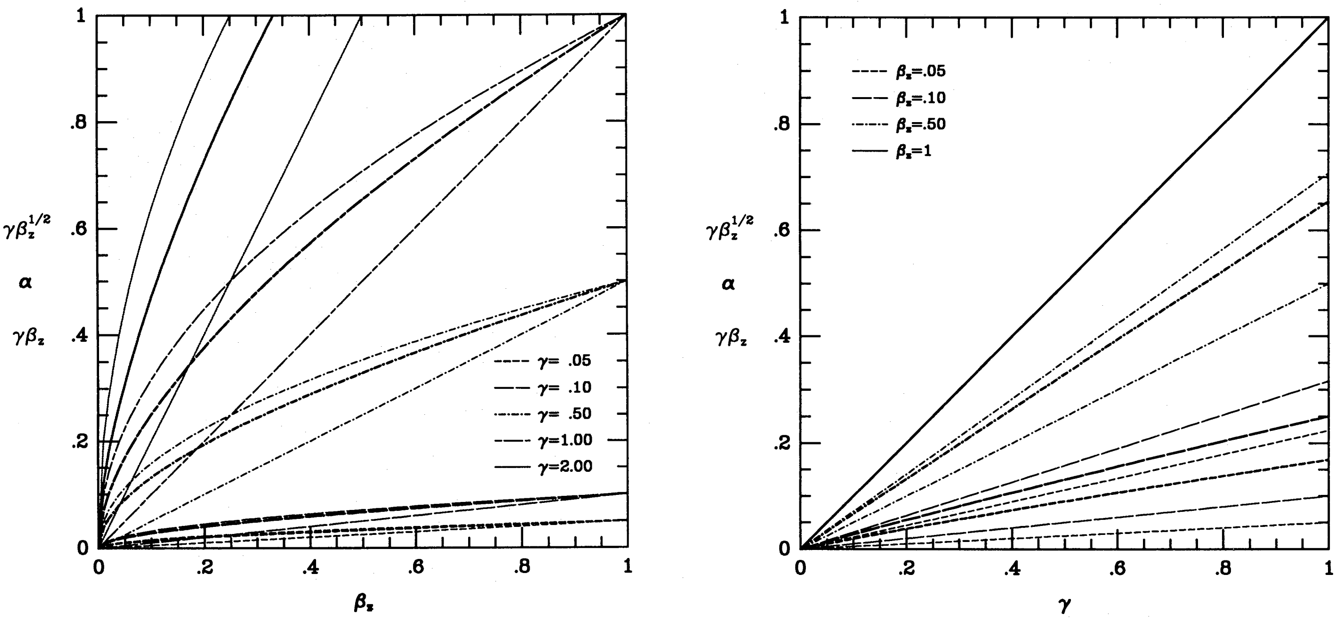

Figure 1. Ratio between the surface densities of the two components as a function of the local parameters γ and β_z in regimes of astrophysical interest, compared to its Gaussian and (self-consistent) exponential approximations. Note that none of these analytical approximations is particularly accurate throughout the ranges considered.

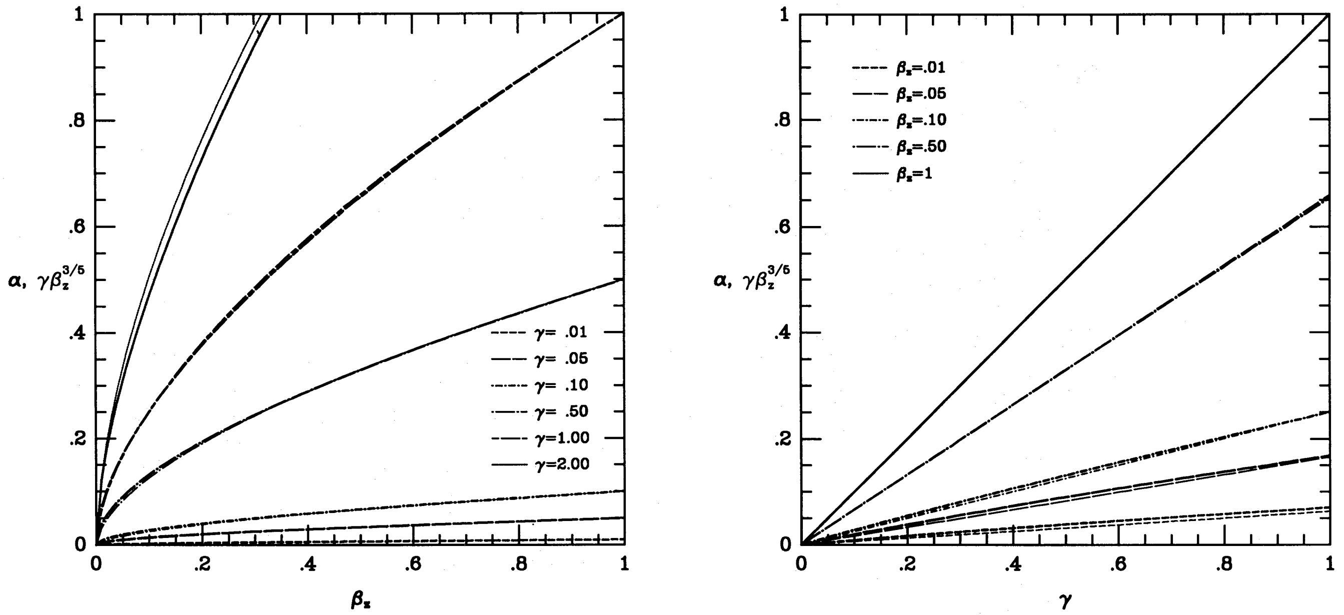

Figure 2. Comparison between the 'exact' α and its fit approximation. Note the remarkable accuracy of such a simple approximation throughout the ranges considered, except for $\gamma \gtrsim 1$ and $\beta_z \lesssim 5 \times 10^{-2}$.

A simple numerical approximation, much more accurate than those considered so far, has been derived employing a rough fitting procedure in the ranges $0 < \gamma \leq 1$ and $0 < \beta_z < 1$, which are those most relevant from an astrophysical point of view. In the following we shall refer to it as the *fit approximation:*

$$\alpha \simeq \gamma \beta_z^{3/5}, \qquad \beta_{z\,\mathrm{eff}} \simeq \beta_z^{3/5}. \tag{12}$$

No intuitive physical interpretation can be given for this specific power law. The remarkable accuracy of such a simple approximation is shown in Fig. 2. When $\gamma \gtrsim 1$ and $\beta_z \lesssim 5 \times 10^{-2}$ (these estimates are crude and mutually dependent), this approximation no longer adequately represents the real physical situation. More accurate approximations can be obtained by sacrificing the mathematical feature which it shares with the Gaussian and the exponential approximations and makes it so attractive: it avoids the γ-dependence of $\beta_{z\,\mathrm{eff}}$.

Using the fit approximation, we have expressed all the dimensionless quantities introduced so far (the ratios between the relevant thickness-scales and the remaining local parameters) in terms of α, β_z and α, $\beta_{z\,\mathrm{eff}}$. These parametrizations will be used in Section 3 for investigating the stability of galactic discs when finite-thickness effects are taken into account. Note that this 'inversion of coordinates' is made possible by the fact that the three parametrizations considered are diffeomorphic. Finally, we have performed a test to check *a posteriori* the accuracy of the fit approximation. We have first recalculated the relevant local parameters at the second step of iteration, and then we have compared these values to those obtained at the first step of iteration. In standard regimes the agreement is satisfactory, whereas significant discrepancies occur near the 'forbidden' asymptotic regimes mentioned in the previous paragraph, as expected.

3 LOCAL FINITE-THICKNESS EFFECTS

Several attempts have already been made to estimate finite-thickness corrections to the local dispersion relation in one-component galactic discs (e.g., Sweet & McGregor 1964; Toomre 1964, 1974; Goldreich & Lynden-Bell 1965a, b; Vandervoort 1970b; Genkin & Safronov 1975; Bertin & Casertano 1982; Yue 1982a, b, c; Balbus & Cowie 1985; Morozov & Khoperskov 1986; Peng 1988; Fridman 1989; see also Fridman & Polyachenko 1984 for a review). While it is generally agreed that the form of the dispersion relation remains the same provided the unperturbed surface density is multiplied by a suitable reduction factor, different estimates of this reduction factor have been given by the various authors. The most reliable and complete analysis is that performed by Vandervoort (1970b), which is local in the galactic plane and global perpendicular to it. The reduction factor, found by solving an eigenvalue problem, has been shown to be very well approximated by the simple expression

$$\mathfrak{T} = \frac{1}{1 + |k|\langle z \rangle}, \qquad |k|\langle z \rangle = O(1), \tag{13}$$

where k is the local radial wavenumber of the perturbation and $\langle z \rangle$ is the thickness-scale of the galactic disc. This estimate should be compared to that naively obtained by Toomre (1964):

$$\mathfrak{T} = \frac{1 - \mathrm{e}^{-|k|\langle z \rangle}}{|k|\langle z \rangle}, \qquad |k|\langle z \rangle \ll 1. \tag{14}$$

The aim of our calculations is just to extend the rigorous partially global analysis carried out by Vandervoort (1970b) in such a way as to include the cold interstellar gas as well. The investigation performed by Yue (1982a, b, c) is also of considerable interest, because it represents an attempt to take finite-thickness effects into account at a fully global level. As a compromise of this extension, the k-dependent reduction factor has been approximated by replacing the wavenumber with its characteristic local value corresponding to the maximum of the marginal stability curve.

As regards more realistic models of galactic discs in which more (than one) components are present, no such rigorous partially global analysis has been performed (e.g., Shu 1968; see also Lin 1970; Vandervoort 1970c; Lin & Shu 1971; Nakamura 1978; Jog & Solomon 1984a, b). Among these attempts, the contribution of Shu (1968) is surely the most important one, while Vandervoort (1970c) refers only to a particular continuous model of stellar populations without performing a proper stability analysis. The basis of our investigation, has already been mentioned, is the stability analysis performed by Vandervoort (1970b) in the case of one-component purely stellar discs. The method we have employed is in fact a straightforward extension of that developed by Vandervoort (1970b) to the case in which two components are present (the consideration of a fluid component does not give rise to difficulties), but the resulting analysis is much more complicated than in the one-component case.

3.1 Ansatz for the reduction factors

The working assumptions made by Vandervoort (1970b) are the same as those used in the local kinetic formulation of the spiral structure theory, except that concerning the infinitesimal thickness of the models. The method employed to solve the Vlasov equation when z-motions are taken into account is based on the existence of an adiabatic invariant J_z, whose approximate constancy characterizes the vertical motion of disc stars. This is a characteristic of highly flattened galactic discs, where the frequency of the oscillation in the z-direction is large compared to the frequency of the epicyclic motion in the symmetry plane, which in turn is generally of the same order as the pattern frequency of spiral waves.

When two stellar populations are considered instead of one, the method employed by Vandervoort (1970b), to solve the Vlasov equation when z-motions are taken into account, applies separately to each component. The same relation between the perturbations induced in the volume density and in the potential can be obtained in the framework of a fluid approach to the non-trivial lowest order of approximation, provided a proper reinterpretation of the relevant quantities is given (*cf.* Shu 1968). Therefore, assuming that the ordering

$$\frac{\langle z\rangle}{\langle r\rangle}\ll 1, \qquad \frac{m}{|k|r}\sim\frac{c_{r\mathrm{H}}}{r\kappa}\ll 1, \qquad |k|\langle z\rangle=O(1) \tag{15}$$

holds, we find that the wave equation (65) of Vandervoort (1970b) is replaced by

$$\frac{\partial^2\Phi_1}{\partial z^2}-k^2\Phi_1=-\left(\frac{\Lambda_\mathrm{H}}{z^2_{\mathrm{eff\,H}}}\frac{\rho_\mathrm{H}}{\rho_{0\mathrm{H}}}+\frac{\Lambda_\mathrm{C}}{z^2_{\mathrm{eff\,C}}}\frac{\rho_\mathrm{C}}{\rho_{0\mathrm{C}}}\right)\Phi_1,$$

$$\lim_{|z|\to+\infty}\Phi_1(r,\theta,z;t)=0, \qquad \text{where} \qquad \Lambda_i\equiv|k|z_{\mathrm{eff}\,i}\mathscr{D}_i. \tag{16}$$

In analogy with the one-component case we have defined:

$$\mathscr{D}_i\equiv\frac{4\pi G\rho_{0i}z_{\mathrm{eff}\,i}|k|}{\kappa^2-(\omega-m\Omega)^2}F_{i\nu}(x_i)=\frac{2\pi G\sigma_i|k|}{\kappa^2-(\omega-m\Omega)^2}F_{i\nu}(x_i), \tag{17}$$

$$F_{\mathrm{kin}\,i\nu}(x_{\mathrm{kin}\,i})\equiv\frac{1-\nu^2}{x_{\mathrm{kin}\,i}}\left[1-\frac{\nu e^{-x_{\mathrm{kin}\,i}}}{2\sin(\nu\pi)}\int_{-\pi}^{+\pi}e^{-x_{\mathrm{kin}\,i}\cos p}\cos(\nu p)\,dp\right], \tag{18}$$

$$F_{\mathrm{fl}\,i\nu}(x_{\mathrm{fl}\,i})\equiv\frac{1}{1+(x_{\mathrm{fl}\,i}/1-\nu^2)}, \tag{19}$$

$$\nu\equiv\frac{\omega-m\Omega}{\kappa}, \qquad x_{\mathrm{kin}\,i}\equiv k^2\frac{c^2_{ri}}{\kappa^2}, \qquad x_{\mathrm{fl}\,i}\equiv k^2\frac{c^2_i}{\kappa^2}. \tag{20}$$

In these formulae, $\mathscr{D}_\mathrm{H}+\mathscr{D}_\mathrm{C}=1$ is just the uncorrected local dispersion relation; $F_{i\nu}(x_i)=F_{\mathrm{kin}\,i\nu}(x_{\mathrm{kin}\,i})$, $F_{\mathrm{fl}\,i\nu}(x_{\mathrm{fl}\,i})$ are reduction factors which lower the response of components with high random motion in the plane, and whose dimensionless arguments involve the radial velocity dispersion c_{ri} and the equivalent planar acoustic speed c_i, respectively, depending on the kind of approach employed to describe each component; ν is the dimensionless Doppler-shifted frequency of the spiral wave; Ω is the angular velocity; κ is the epicyclic frequency; ω is the frequency of the spiral wave; m is the number of spiral arms. In what follows we shall adopt the usual fluid description, obtained by imposing a proper closure of the moments of the collisionless Boltzmann equation, which is more convenient for investigating the stability properties of galactic discs analytically and reproduces the main results almost completely except near the Lindblad resonances (see, e.g., Berman & Mark 1977, 1979; Sygnet, Pellat & Tagger 1987).

In contrast to the one-component case which is exactly soluble (see, e.g., Landau & Lifshitz 1977 for the corresponding quantum-mechanical problem), for arbitrary values of the local parameter $\beta_z \equiv c_{zC}^2/c_{zH}^2$ or, equivalently, $\beta_{z\text{eff}} \equiv z_{\text{eff C}}/z_{\text{eff H}}$, the Schrödinger-type wave equation (16) cannot be reduced to a Fuchsian differential equation, or to other well-known classes of differential equations whose solutions are expressible in terms of elementary or special functions, even in the Gaussian and exponential asymptotic limits (see, e.g., Morse & Feshbach 1953; Erdélyi 1956, 1981; Smirnov 1964; Abramowitz & Stegun 1970; Bender & Orszag 1978; Gradshteyn & Ryzhik 1980). In order to overcome such difficulties connected with the solution of this *double-eigenvalue problem* for Λ_H and Λ_C, we have developed a perturbative method (Romeo 1987) similar but not identical to that employed in quantum mechanics to solve the time-independent Schrödinger equation when the potential is of the form $U(z) = V(z) + W(z)$, $W(z)$ representing a small perturbation to $V(z)$ (see, e.g., Bender & Orszag 1978). Differences between our perturbative method and that used in quantum mechanics arise because, in quantum-mechanical language, the energy of the particle is kept fixed and *two* eigenvalues are associated with a *double*-term potential. This method has been further investigated and refined by the author, but the results are still in a preliminary form. Note that a WKBJ approach leading to a quantum condition of the Bohr–Sommerfeld type cannot be employed here because it fails for small values of the quantum number (recall that we are interested in $n=0$), as can be deduced by a comparison with the one-component case. For what follows it is sufficient to make some general remarks concerning the wave equation (16), without going into the details of the perturbative method. The quantum condition of this double-eigenvalue problem, which determines the corrected local dispersion relation, is expected to be of the form

$$\mathscr{C}(\Lambda_i, |k| z_{\text{eff}\,j}, \alpha; n) = 0. \tag{21}$$

While it seems reasonable that in this two-component case also, only the lowest eigenvalues ($n=0$) are physically relevant by virtue of their consistency with the infinitesimally thin limit, it cannot be expected *a priori* that such a relation can be reduced to the particular form

$$\mathscr{D}_H \mathfrak{T}_H(|k| z_{\text{eff}\,i}, \alpha) + \mathscr{D}_C \mathfrak{T}_C(|k| z_{\text{eff}\,i}, \alpha) = 1, \tag{22}$$

in analogy with the one-component case, where $\mathfrak{T}_i(|k| z_{\text{eff}\,j}, \alpha)$ are the reduction factors of the two components. If this is not the case, finite-thickness corrections to the local dispersion relation cannot simply be expressed in terms of two reduction factors, one for each component. In other words, the local dispersion relation cannot be reduced to the form obtained in the case of infinitesimally thin discs by a suitable scaling of the surface densities of the two components. It is interesting to note in this context that the condition for the occurrence of a restricted quantum condition, giving rise to a local dispersion relation of the 'reduced' form (22), can be identified with the convergence criterion of the perturbative method, which is expected to take place only in particular asymptotic regimes of the local parameters α, β_z or equivalently $\beta_{z\text{eff}}$ (Romeo 1987, and unpublished results; *cf.* Shu 1968).

From the considerations made above it follows that, in contrast to the one-component case, finite-thickness corrections to the local dispersion relation do not generally reduce to a simple scaling of the surface densities of the two components. The use of two corresponding reduction factors can reasonably be justified only when the two components are not strongly coupled. For the moment we are not able to express this statement in a precise mathematical form. In most situations of astrophysical interest, such as in the case in which the two components are identified with the stars of the active disc and the cold interstellar gas, this condition is rather general and is fulfilled throughout the galactic disc, except possibly in the outermost parts where the mass density of the cold interstellar gas becomes comparable to that of low velocity dispersion stars. It is to such cases that our analysis is devoted.

We shall now make a simple ansatz concerning the form of the two reduction factors, which for the moment can only be justified at an intuitive level. A comparison between the wave equations derived in the one-component and in the two-component cases [equation (65) of Vandervoort (1970b) and equation (16) of this paper, respectively] shows that Δ and $z_{\text{eff}\,i}$ play in a sense a similar role: they are the effective thickness-scales which allow us to express the local dispersion relations in terms of the unperturbed surface densities, introduced in place of the unperturbed volume densities in the plane. On the other hand, the different z-dependence of the equilibrium volume densities in the two cases is a source of dissimilarity in such a role. This argument suggests that when the contribution of the last effect can be neglected, or in other words when the two components are not strongly coupled (*cf.* the non-restrictive condition mentioned above), the corresponding reduction factors should be fairly well approximated by the *effective ansatz*:

$$\mathfrak{T}_i \equiv \frac{\sigma_{\text{red}\,i}}{\sigma_i} \simeq \mathfrak{T}_{\text{eff}\,i} \equiv \frac{1}{1+|k| z_{\text{eff}\,i}} \qquad [|k| z_{\text{eff}\,i} = O(1)], \tag{23}$$

where we have indicated the formal ordering which we expect to be required by the underlying approximation. Note that the orderings specified in equations (15) and (23) are to be understood in the maximal sense (*cf.* maximal orderings in asymptotic perturbation expansions), and hence are not mutually exclusive. Asymptotically, our ansatz matches the reduction factors derived by Shu (1968) when the system approaches the limiting cases in which only one component is present or two components with the same vertical velocity dispersions can be identified, apart from the obvious infinitesimally thin case. Other reasonable ansatz involving the asymptotic thickness-scales of the two components and characterized by a lower level of accuracy can be justified heuristically.

3.2 Local stability

The qualitative discussion given in the previous subsection will now be quantified by studying, in the framework of a fluid approach, the marginal stability curve corresponding to the ansatz introduced above. The results obtained in the infinitesimally thin case discussed by Romeo (1985, 1989, 1990b) and Bertin & Romeo (1988) are at the basis of this further investigation. We shall introduce two parametrizations, the use of which is suggested by their astrophysical relevance and/or mathematical convenience, as will be explained in detail later.

The reduction factors of the two components can be expressed in the following general dimensionless form:

$$\mathfrak{T}_i^{-1} = 1 + \frac{Q_{\rm H}^2}{2\bar{\lambda}} \mathfrak{A}_i, \tag{24}$$

where the functions $\mathfrak{A}_i$ depend on the ansatz employed:

$$\mathfrak{A}_{\rm eff\,H} \equiv \frac{\delta_{\rm H}}{1+\alpha} \frac{z_{\rm eff\,H}}{2z_{\rm EH}}, \qquad \mathfrak{A}_{\rm eff\,C} \equiv \frac{\beta\delta_{\rm C}}{1+\alpha} \frac{z_{\rm eff\,C}}{2z_{\rm EC}}, \tag{25}$$

in the same subscript notations used for the reduction factors. In these formulae we have adopted the following scaling and basic parametrization (*cf.* the infinitesimally thin case):

$$\bar{\lambda} \equiv \frac{k_{\rm H}}{|k|}, \qquad \text{where} \quad k_{\rm H} \equiv \frac{\kappa^2}{2\pi G\sigma_{\rm H}}; \tag{26}$$

$$\alpha \equiv \frac{\sigma_{\rm C}}{\sigma_{\rm H}}, \qquad \beta \equiv \frac{c_{\rm C}^2}{c_{\rm H}^2} \qquad (0<\alpha<+\infty, \quad 0<\beta<1); \tag{27}$$

$$Q_{\rm H} \equiv \frac{c_{\rm H}\kappa}{\pi G\sigma_{\rm H}} \qquad (\textit{local stability parameter}); \tag{28}$$

and the additional *velocity-dispersion parametrization:*

$$\delta_i \equiv \frac{c_{zi}^2}{c_i^2} \qquad (0<\beta\delta_{\rm C}<\delta_{\rm H}<1, \quad 0<\delta_{\rm C}\le 1), \tag{29}$$

based on the parametrization involving α, β_z introduced in Section 2 and so related to it:

$$\beta_z = \beta\frac{\delta_{\rm C}}{\delta_{\rm H}}. \tag{30}$$

The case $\delta_i = 0$ represents the limit of an infinitesimally thin system. When the two components are identified with the stars of the active disc and the cold interstellar gas ($\delta_{\rm C}=1$, due to its collisional nature), the cases $\delta_{\rm H}=\beta$ (i.e., $\beta_z=1$) and $\delta_{\rm H}=1$ correspond to a totally ineffective vertical heating and to an isotropic heating, respectively. The marginal stability curve in the ($\bar{\lambda}$, $Q_{\rm H}^2$) plane can be derived from its uncorrected expression, by applying the reduction factors to each dimensionless quantity involving the surface densities:

$$\beta\frac{Q_{\rm H}^4}{\mathfrak{T}_{\rm H}^4} + 4\frac{\bar{\lambda}}{\mathfrak{T}_{\rm H}}\frac{Q_{\rm H}^2}{\mathfrak{T}_{\rm H}^2}\left[\frac{\bar{\lambda}}{\mathfrak{T}_{\rm H}}(1+\beta) - \left(\alpha\frac{\mathfrak{T}_{\rm C}}{\mathfrak{T}_{\rm H}}+\beta\right)\right] + 16\frac{\bar{\lambda}^3}{\mathfrak{T}_{\rm H}^3}\left[\frac{\bar{\lambda}}{\mathfrak{T}_{\rm H}} - \left(1+\alpha\frac{\mathfrak{T}_{\rm C}}{\mathfrak{T}_{\rm H}}\right)\right] = 0,$$

therefore

$$\begin{aligned}&\beta\mathfrak{A}_{\rm H}\mathfrak{A}_{\rm C}Q_{\rm H}^8 + 2\bar{\lambda}Q_{\rm H}^6[2\bar{\lambda}(1+\beta)\mathfrak{A}_{\rm H}\mathfrak{A}_{\rm C}+\beta(\mathfrak{A}_{\rm H}+\mathfrak{A}_{\rm C})] + 4\bar{\lambda}^2Q_{\rm H}^4\{4\bar{\lambda}^2\mathfrak{A}_{\rm H}\mathfrak{A}_{\rm C}+2\bar{\lambda}(1+\beta)(\mathfrak{A}_{\rm H}+\mathfrak{A}_{\rm C})+[\beta-2(\alpha\mathfrak{A}_{\rm H}+\beta\mathfrak{A}_{\rm C})]\}\\ &\quad + 16\bar{\lambda}^3Q_{\rm H}^2\{2\bar{\lambda}^2(\mathfrak{A}_{\rm H}+\mathfrak{A}_{\rm C})+\bar{\lambda}[(1+\beta)-2(\alpha\mathfrak{A}_{\rm H}+\mathfrak{A}_{\rm C})]-(\alpha+\beta)\}+64\bar{\lambda}^5[\bar{\lambda}-(1+\alpha)]=0,\end{aligned} \tag{31}$$

which we consider in the range $0\le\bar{\lambda}\le 1+\alpha$. From this relation a local stability criterion can be stated in analogy with the case of infinitesimally thin one-component systems. A function $\bar{Q}^2=\bar{Q}^2(\alpha,\beta,\delta_i)$, which reduces to unity when $\alpha=0$ and $\delta_i=0$, can be defined in such a way that when $Q_{\rm H}^2>\bar{Q}^2$ the system is locally stable at all wavelengths. For $Q_{\rm H}^2<\bar{Q}^2$ we expect the system to be locally unstable in ranges of wavelengths defined by the marginal stability condition. Note in this context that the ratio

$$Q_{\rm eff} \equiv \frac{Q_{\rm H}}{\bar{Q}} \qquad (\textit{effective local stability parameter}) \tag{32}$$

plays the role of an effective Q-parameter (*cf.* the infinitesimally thin one-component case). Consistently with a mechanism of self-regulation, we expect

$$Q_{\rm eff}(r) = 1 \qquad (\textit{self-regulation constraint}) \tag{33}$$

throughout the active disc. The self-regulation constraint on the profile of the effective local stability parameter will be considered in future applications (see Section 4).

Alternatively, the reduction factors of the two components can be expressed in the following dimensionless form:

$$\mathfrak{T}_{\mathrm{eff\,H}}^{-1}=1+\frac{\zeta_{\mathrm{H}}}{\bar{\lambda}},\qquad \mathfrak{T}_{\mathrm{eff\,C}}^{-1}=1+\frac{\zeta_{\mathrm{H}}}{\bar{\lambda}}\,\beta_{z\,\mathrm{eff}}, \tag{34}$$

where we have adopted the same scaling and basic parametrization used above, and the additional *wavenumber parametrization:*

$$\zeta_{\mathrm{H}}\equiv k_{\mathrm{H}}z_{\mathrm{eff\,H}},\qquad \beta_{z\,\mathrm{eff}}\equiv\frac{z_{\mathrm{eff\,C}}}{z_{\mathrm{eff\,H}}}\qquad (0<\beta_{z\,\mathrm{eff}}<1), \tag{35}$$

based on the parametrization involving α, $\beta_{z\,\mathrm{eff}}$ introduced in Section 2 and similar to that adopted by Shu (1968). The case $\zeta_{\mathrm{H}}=0$ represents the limit of an infinitesimally thin system, and the case $\beta_{z\,\mathrm{eff}}=1$ represents the limit of a system in which the two components have the same scaleheight (and hence the same vertical velocity dispersions). Proceeding along the same line as before, we derive the following expression for the marginal stability curve:

$$\beta Q_{\mathrm{H}}^{4}[\bar{\lambda}^{2}+\bar{\lambda}\zeta_{\mathrm{H}}(1+\beta_{z\,\mathrm{eff}})+\zeta_{\mathrm{H}}^{2}\beta_{z\,\mathrm{eff}}]+4\bar{\lambda}^{2}Q_{\mathrm{H}}^{2}\{\bar{\lambda}^{2}(1+\beta)-\bar{\lambda}[(\alpha+\beta)-(1+\beta)\zeta_{\mathrm{H}}(1+\beta_{z\,\mathrm{eff}})]+\zeta_{\mathrm{H}}[(1+\beta)\zeta_{\mathrm{H}}\beta_{z\,\mathrm{eff}}-(\alpha+\beta\beta_{z\,\mathrm{eff}})]\}$$
$$+16\bar{\lambda}^{4}\{\bar{\lambda}^{2}-\bar{\lambda}[(1+\alpha)-\zeta_{\mathrm{H}}(1+\beta_{z\,\mathrm{eff}})]+\zeta_{\mathrm{H}}[\zeta_{\mathrm{H}}\beta_{z\,\mathrm{eff}}-(\alpha+\beta_{z\,\mathrm{eff}})]\}=0, \tag{36}$$

which we consider in the range $0\leq\bar{\lambda}\leq\bar{\lambda}_{*}$, where the upper zero is given by

$$\bar{\lambda}_{*}=\frac{1}{2}\left\{(1+\alpha)-\zeta_{\mathrm{H}}(1+\beta_{z\,\mathrm{eff}})+\sqrt{[(1+\alpha)-\zeta_{\mathrm{H}}(1+\beta_{z\,\mathrm{eff}})]^{2}-4\zeta_{\mathrm{H}}[\zeta_{\mathrm{H}}\beta_{z\,\mathrm{eff}}-(\alpha+\beta_{z\,\mathrm{eff}})]}\right\}<1+\alpha. \tag{37}$$

Note that in the context of this parametrization, the marginal stability curve can degenerate into the origin of the $(\bar{\lambda},\,Q_{\mathrm{H}}^{2})$ plane in the critical regimes of the local parameters ζ_{H} and $\beta_{z\,\mathrm{eff}}$ corresponding to the vanishing of its upper zero. The condition which should be satisfied for avoiding this singular behaviour of the marginal stability curve can thus be expressed by imposing the positivity of its upper zero:

$$\bar{\lambda}_{*}>0\Longrightarrow\zeta_{\mathrm{H}}<1+\frac{\alpha}{\beta_{z\,\mathrm{eff}}}. \tag{38}$$

This is not, however, a physical limitation for ζ_{H} because it corresponds to the obvious conditions $\delta_{i}<+\infty$, as can be deduced by comparing the dimensionless forms of the reduction factors of the two components, evaluated according to the effective ansatz in the context of these two parametrizations. Moreover, values of ζ_{H} sufficiently close to such a critical upper bound are not consistent with the working assumption $|k|\langle z\rangle=O(1)$ in the range of wavelengths relevant to the marginal stability curve. Taking these considerations into account, the same local stability criterion stated in the context of the velocity-dispersion parametrization and subsequent discussion apply, with $\bar{Q}^{2}=\bar{Q}^{2}(\alpha,\beta,\zeta_{\mathrm{H}},\beta_{z\,\mathrm{eff}})$.

We shall now discuss the advantages and drawbacks inherent in these two parametrizations. A considerable advantage of the wavenumber parametrization lies in the fact that the equation for the marginal stability curve can be solved analytically according to the standard technique employed in the case of quadratic algebraic equations. The equation for its stationary points can thus be given in a relatively simple explicit form, which turns out to be extremely useful for investigating the corrected two-phase region of the parameter space where the marginal stability curve exhibits two maxima (*cf.* the infinitesimally thin case). Instead, when the velocity-dispersion parametrization is adopted, the stationarity condition can only be given in the implicit form of a system of two non-linear (quartic and cubic) algebraic equations, the alternative explicit relation being of no practical utility. An advantage of the velocity-dispersion parametrization which should not be underestimated lies in the fact that, when the two components are identified with the stars of the active disc and the cold interstellar gas, the lower and the upper physical limitations (corresponding to a totally ineffective vertical heating and to an isotropic heating of the stellar component, respectively, as derived on observational grounds and suggested by stability considerations) can be included directly, and the collisional nature of the gaseous component can trivially be taken into account. Instead, when the wavenumber parametrization is adopted, the previous conditions do not admit any straightforward translation.

Even though the wavenumber parametrization is not as physically transparent as the velocity-dispersion parametrization, its use can reasonably be justified in view of its considerable mathematical convenience. Therefore, in order to make up for such a drawback, it is of extreme interest to study the way in which the results obtained in the context of one of them can be translated into corresponding results for the other. The first step consists in noting that these two parametrizations are so related:

$$\zeta_{\mathrm{H}}=\frac{1}{2}\,Q_{\mathrm{H}}^{2}\,\frac{\delta_{\mathrm{H}}}{1+\alpha}\,\frac{z_{\mathrm{eff\,H}}}{2z_{\mathrm{EH}}}, \tag{39}$$

as can be established by comparing equations (24), (25) and (34). Now the question arises as to which value of the local stability parameter should be used in this relation. If the two-component equilibrium model is specified (global stability analysis), there is no ambiguity: the relevant value of the local stability parameter is that corresponding to the point where all the other profiles are calculated. If the two-component equilibrium model is not specified (local stability analysis), and if we want to compare the marginal stability curves derived in the context of these two parametrizations, a natural and physically meaningful choice is represented by the *global prescription*:

$$\zeta_{\rm H}=\bar{\zeta}_{\rm H}\equiv\frac{1}{2}\,\bar{Q}^2\,\frac{\delta_{\rm H}}{1+\alpha}\,\frac{z_{\rm eff\,H}}{2z_{\rm EH}}, \tag{40}$$

involving the global maximum of the marginal stability curve, which refers to situations characterized by the same critical level of stability. Note, however, that this is not the only reasonable choice if peculiar gas-dominated regimes (*cf.* the infinitesimally thin case) are involved: depending on the various situations, it could be of more interest to consider one of the other stationary points of the marginal stability curve. If the global prescription is used to relate these two parametrizations, the corresponding marginal stability curves are characterized by a global maximum having the same location and height. On the other hand, any other point $(\bar{\lambda},\,Q_{\rm H}^2)$ belonging to the marginal stability curve derived in the context of the wavenumber parametrization can be viewed as belonging to the marginal stability curve derived in the context of the velocity-dispersion parametrization for correspondingly larger values of the local parameters δ_i:

$$\delta'_i=\bar{\delta}_i\equiv\delta_i\,\frac{\bar{Q}^2}{Q_{\rm H}^2}>\delta_i, \tag{41}$$

as can be deduced by comparing equations (39) and (40) once the monotonic behaviour of $\beta_{z\rm eff}$ with varying β_z at fixed α is taken into account. Unfortunately, these descriptive complications cannot be avoided.

As an immediate consequence of the considerations made above, it can be shown that the wavenumber parametrization exhibits some unphysical features already suspected in the discussion concerning the upper limitation (38). Here we do not want to consider this problem in detail, but just to give a correct idea of the reason for such a singular behaviour. For any given value of $\zeta_{\rm H}$, however small, the upper physical limitations $\delta_i\lesssim 1$ can only be fulfilled in a part of the range of wavelengths relevant to the marginal stability curve, because as $\bar{\lambda}$ approaches the lower and upper zeros the corresponding δ_i take arbitrarily large values [see in particular equation (41) and related discussion]. Note in this context that, as regards the consistency with the working assumption expressed by the maximal ordering $|k|\langle z\rangle=O(1)$, the upper bounds $\delta_i=1$ roughly correspond to the onset of a 'dangerous' situation at the global maximum of the marginal stability curve or at the gaseous peak (*cf.* the infinitesimally thin case), if present (the gaseous peak is always more 'dangerous' than the stellar peak, even in the case in which it is less high, because it occurs at much shorter wavelengths). On the other hand, the lower physical limitations $\delta_{\rm H}>\beta\delta_{\rm C}>0$ do not give rise to such difficulties when the global prescription is used. From these considerations it follows that the marginal stability curve derived in the context of the wavenumber parametrization does not provide a faithful representation of the local stability properties of galactic discs.

In presenting the results of the local stability analysis performed in this paper, we have considered the standard star-dominated and the peculiar gas-dominated regimes already investigated in the less general context of infinitesimally thin two-component models of galactic discs (Romeo 1985, 1989, 1990b; Bertin & Romeo 1988). Moreover, other regimes of the relevant local parameters which are typical in the solar neighbourhood have been considered. The marginal stability curves derived according to the effective ansatz in the context of the velocity-dispersion and of the wavenumber parametrizations are compared in Figs 3 and 4, respectively. We have used the global prescription to relate these two parametrizations, and the fit approximation to express the equilibrium-related dimensionless quantities in terms of α and β_z. The non-perfect agreement between the critical levels of stability characterizing corresponding situations (in the sense of the global prescription) can be ascribed to two facts: the low accuracy of the fit approximation in peculiar gas-dominated regimes, and the evaluation of the input values for the wavenumber parametrization only up to the second decimal digit. Apart from these numerical effects, the qualitatively different behaviour of corresponding marginal stability curves can be traced back to the fact that the wavenumber parametrization contains the local value of the stability parameter implicitly, as already explained. As a result, when finite-thickness effects are taken into account, the system tends to be more stable. More precisely, the stabilizing role of such effects can partially counterbalance the destabilizing role of the cold interstellar gas in linear regimes. As a consequence of these competing roles, in some cases the monotonicity properties of the marginal stability curve are qualitatively different from the infinitesimally thin case. Note in this context that the profile of the effective local stability parameter and the shape of the marginal stability curve are of *primary* importance in determining the propagation properties of the spiral waves and thus the *global* stability properties of galactic discs.

We shall now use some qualitative arguments, based on the first step of an iterative method for calculating $Q_{\rm H}^2$ from its uncorrected value, to provide a simple justification of these results. Such arguments are only indicative, and may be quite inaccurate if the convergence of the iterative method is slow. For what follows it is sufficient to understand the way in which the points of the marginal stability curve relevant to the stability analysis are modified when finite-thickness effects are taken into account. However, it should be borne in mind that such simple arguments are indeed able to account for all the basic properties of the marginal stability curve. Specifically, we find for the *upper zero*:

$$\bar{\lambda}_{*}\sim\mathfrak{T}_{\rm H}+\alpha\mathfrak{T}_{\rm C}\lesssim 1+\alpha; \tag{42}$$

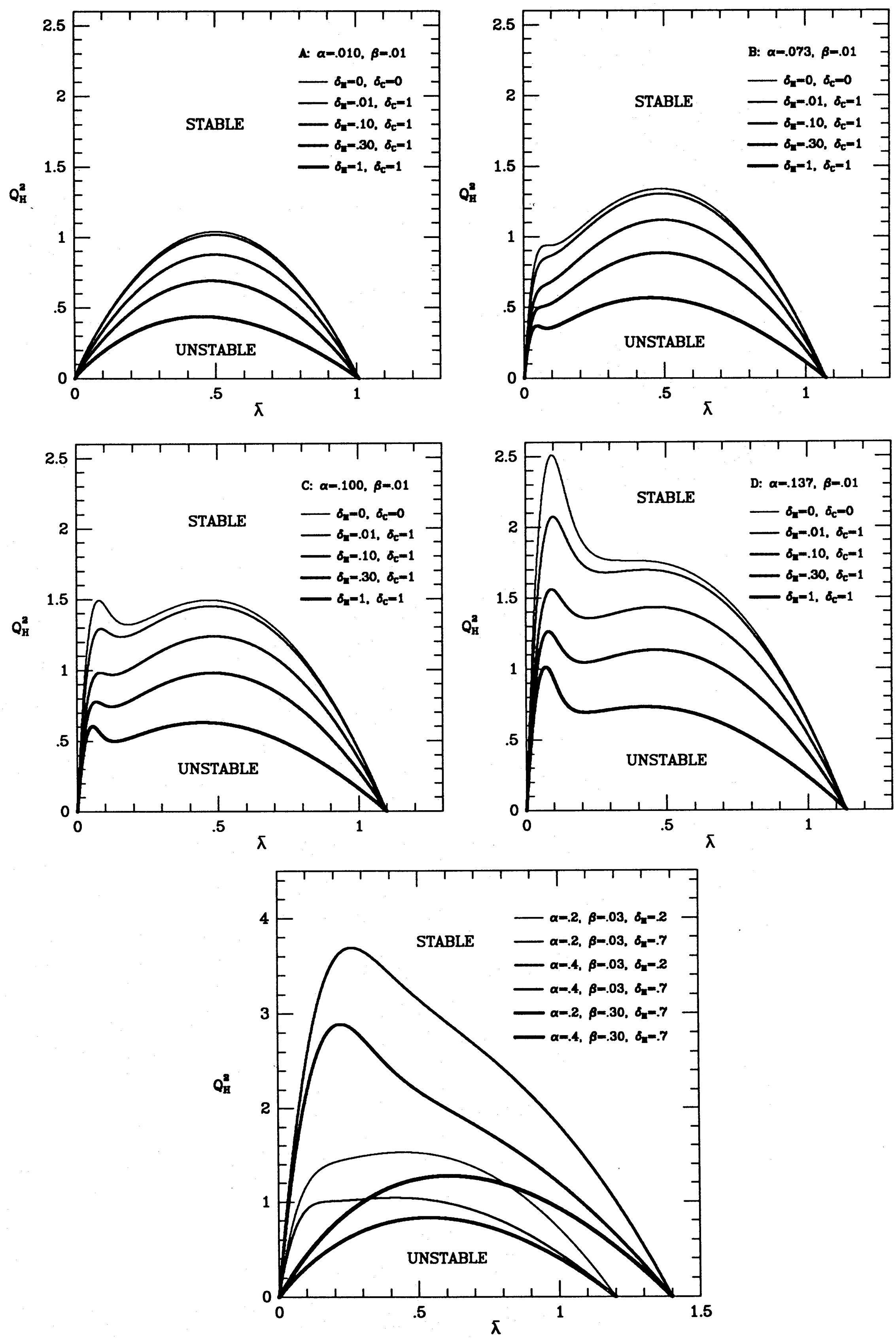

Figure 3. Two-fluid marginal stability curves in the $(\bar{\lambda}, Q_{\mathrm{H}}^2)$ plane derived by evaluating the reduction factors of the two components according to the effective ansatz for some values of the local parameters δ_i, α and fixed $\beta=0.01$ (top), and for values of the same local parameters corresponding to typical lower and upper bounds in the solar neighbourhood (bottom). The fit approximation has been used to express the equilibrium-related dimensionless quantities in terms of α and β_z. The case $\delta_i=0$ represents the limit of an infinitesimally thin system. The cases $\delta_{\mathrm{H}}=\beta$ and $\delta_{\mathrm{H}}=1$ correspond to a totally ineffective vertical heating and to an isotropic heating, respectively.

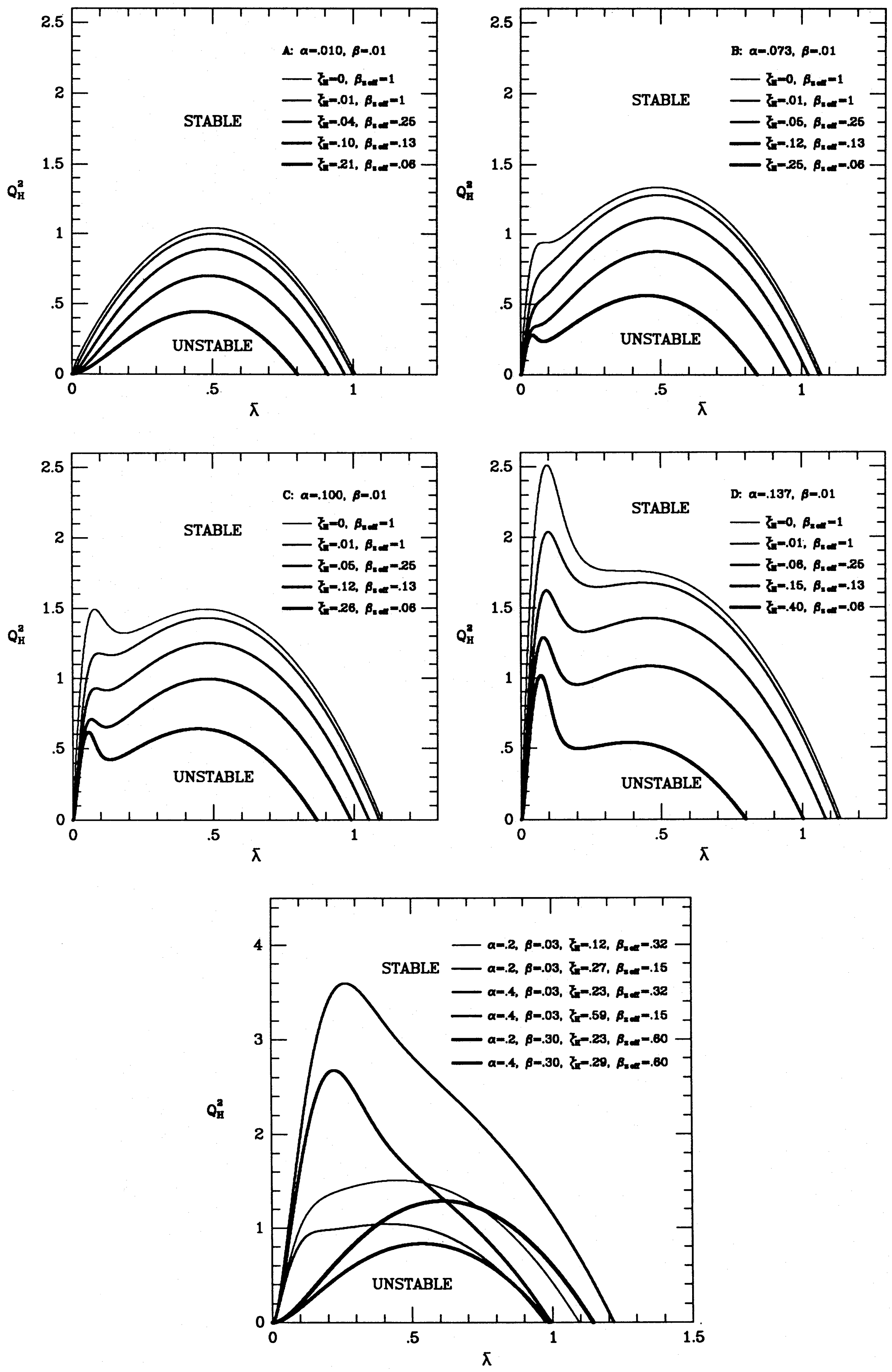

Figure 4. Same as Fig. 3 for the wavenumber parametrization. The input values of $\bar{\zeta}_H$ and $\beta_{z\,\mathrm{eff}}$ correspond to the input values of δ_i in Fig. 3. They are calculated according to the global prescription discussed in the text, which refers to situations characterized by the same critical level of stability, with an accuracy of two decimal digits.

for the *stellar peak*:

$$\bar{\lambda} \sim \frac{1}{2} \mathfrak{T}_H \lesssim \frac{1}{2}, \qquad Q_H^2 \sim \mathfrak{T}_H^2 + 4\alpha \mathfrak{T}_H \mathfrak{T}_C \lesssim 1 + 4\alpha; \tag{43}$$

and for the *gaseous peak*:

$$\bar{\lambda} \sim \frac{1}{2} \alpha \mathfrak{T}_C \lesssim \frac{1}{2} \alpha, \qquad Q_H^2 \sim \frac{\alpha^2}{\beta} \mathfrak{T}_C^2 + 4\alpha \mathfrak{T}_H \mathfrak{T}_C \lesssim \frac{\alpha^2}{\beta} + 4\alpha. \tag{44}$$

In these formulae, the reduction factors of the two components are intended to be calculated to the non-trivial lowest order of iteration. From these expressions follows, in agreement with physical intuition, the *decoupling* of the two components in *peculiar* gas-dominated regimes, as expressed by the leading terms in the stellar and in the gaseous peaks. Although such qualitative arguments account for the stabilization of these two peaks and their shift toward shorter wavelengths, a more sophisticated asymptotic expansion analysis is required to obtain more reliable estimates. Specifically, in the context of the wavenumber parametrization, we find for the *upper zero*:

$$\bar{\lambda}_* \approx 1 + (\alpha - \zeta_H), \qquad [\alpha \ll 1; \quad \beta = O(\alpha) \vee \beta = O(\alpha^2); \quad \zeta_H, \beta_{z\,\mathrm{eff}} = O(\alpha)]; \tag{45}$$

for the *stellar peak*:

$$\bar{\lambda} \approx \frac{1}{2}, \qquad Q_H^2 \approx 1 + 4(\alpha - \zeta_H), \qquad [\alpha \ll 1; \quad \beta = O(\alpha) \vee \beta = O(\alpha^2); \quad \zeta_H, \beta_{z\,\mathrm{eff}} = O(\alpha)]; \tag{46}$$

and for the *gaseous peak*:

$$\bar{\lambda} \approx \frac{1}{2} \alpha, \qquad Q_H^2 \approx \frac{\alpha^2}{\beta} + 4\alpha \left[1 - \left(\frac{2\zeta_H}{\alpha + 2\zeta_H} + \frac{\zeta_H \beta_{z\,\mathrm{eff}}}{\beta} \right) \right], \qquad [\alpha \ll 1; \quad \beta = O(\alpha^2); \quad \zeta_H, \beta_{z\,\mathrm{eff}} = O(\alpha)]. \tag{47}$$

The orderings indicated beside the corresponding asymptotic expressions are suggested by the analysis performed in the infinitesimally thin case, where they characterize the *two-phase region* of the (β, α) plane. Note in this context that the formal ordering involving the local parameters ζ_H and $\beta_{z\,\mathrm{eff}}$ is to be viewed as a maximal ordering, which at this stage we are not able to specify more precisely.

4 CONCLUDING REMARKS

We shall now make some simple points concerning the astrophysical relevance of the two parametrizations introduced in this paper. Even though the velocity-dispersion and wavenumber parametrizations cannot easily be compared in a local stability analysis, when the two-component equilibrium galactic model is specified and a global stability analysis is performed, these two parametrizations turn out to be equivalent (mathematical complications apart). As suggested by observations, for our Galaxy the proper input local parameters might be $\delta_i(r)$ while for external galaxies $\zeta_H(r)$ and $\beta_{z\,\mathrm{eff}}(r)$ might be more appropriate. Thus, apparently, the situation seems to be more complicated for our Galaxy: the local stability analysis which should be carried out to determine the global maximum $\bar{Q}^2(r)$ of the marginal stability curve in the context of the velocity-dispersion parametrization is considerably more difficult. If we seek to bypass this difficulty by adopting the wavenumber parametrization, we should recall that relating the former to the latter is also not an easy task and, indeed, is equivalently difficult because it requires the knowledge of $\bar{Q}^2(r)$. The situation would certainly be more tractable, but still complicated, if the input profiles were $c_{zi}(r)$, in which case it would be convenient to derive $z_{\mathrm{eff}\,i}(r)$ and to adopt the wavenumber parametrization. In our opinion, since the vertical and planar heating mechanisms in galactic discs are almost decoupled as well as their relevant stability properties (Romeo 1987, 1990b), the constancy of $\delta_H(r)$ throughout the Galactic disc, commonly invoked to fit or deduce the relevant stellar velocity-dispersion profiles, should be viewed as a convenient observational working assumption rather than a well-established observational constraint. Therefore $\delta_i(r)$ might not be so meaningful as input parameters even for our Galaxy. Also in the case of external galaxies there might be some interpretative complications in finding out whether the observed $z_{\mathrm{eff}\,i}(r)$ have the same physical meaning as those inferred on theoretical grounds: generally Gaussian, exponential or sech-squared profiles are used to fit the observed brightness distribution of the stellar component, and a constant mass-to-light ratio is assumed to derive the corresponding volume mass density profiles. Consequently, these fitting analyses are not able to discriminate between the true $z_{\mathrm{eff}\,H}(r)$ and $z_{GH}(r)$ or $2z_{EH}(r)$, so that we should carefully reinterpret all the related local parameters and derive their correct values. Alternatively, it could be wise to abandon the idea of performing a 'strictly' correct analysis, and to be content with performing an 'approximately' correct analysis in the spirit of the Gaussian and (self-consistent) exponential approximations discussed in Section 2 and of the homonymous ansatz discussed by Romeo (1990b): the essential physics is not missed. Other discussions of related topics were given by Bahcall (1984) and Bahcall & Casertano (1984).

In view of a future comparison between theoretical predictions and observational results, we now want to draw attention to some delicate points which are often not given sufficient consideration, and which make such a comparison not straightforward. From a theoretical point of view, a single equivalent stellar component is taken to be representative of the whole active stellar disc consisting of stars with low velocity dispersion. The consideration of more stellar populations would give rise to several complications due to their gravitational coupling via the Poisson equation, as required by the self-consistency condition. On the other hand, it should be noted that observations tend to overestimate the effect of stars with high velocity dispersion, which are not so dynamically relevant as regards their participation in spiral structure (see Lin & Bertin 1985; Romeo 1985). From a theoretical point of view, a single gaseous component is taken to simulate H I regions of neutral atomic hydrogen and giant molecular clouds and complexes. On the other hand, observational surveys often provide significantly different estimates as regards the molecular hydrogen.

The analysis performed in this paper represents the first step of a long-term project devoted to the study of self-regulation mechanisms, which are expected to drive the dynamical evolution of galactic discs and, in particular, to be responsible for their secular heating (Romeo 1987, 1990b). A deep understanding of the problem which we have tackled can only be attained by a scrupulous investigation into the stability properties of galactic discs, to which the above-mentioned self-regulation mechanisms are intimately related, taking into account effects which are generally neglected for mathematical convenience (namely those related to the presence of the cold interstellar gas and to the finite thickness of galactic discs) but which we have shown to be important or even *crucial*. As a practical application of the analysis carried out in this paper, the profiles of the local stability parameter Q_{H} for two-component models of the Galaxy will be derived consistently with a mechanism of self-regulation in the active disc, and compared to those inferred on observational grounds. In particular, this extension might require a deep knowledge of the monotonicity properties of the marginal stability curve and of the behaviour of its global maximum in the peculiar gas-dominated regimes characterizing the two-phase region, which at present is not available.

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