FULL DISCRETISATION OF SEMI-LINEAR STOCHASTIC WAVE EQUATIONS DRIVEN BY MULTIPLICATIVE NOISE

RIKARD ANTON*, DAVID COHEN†, STIG LARSSON‡, AND XIAOJIE WANG§

Abstract. A fully discrete approximation of the semi-linear stochastic wave equation driven by multiplicative noise is presented. A standard linear finite element approximation is used in space and a stochastic trigonometric method for the temporal approximation. This explicit time integrator allows for mean-square error bounds independent of the space discretisation and thus do not suffer from a step size restriction as in the often used Störmer-Verlet-leap-frog scheme. Furthermore, it satisfies an almost trace formula (i. e., a linear drift of the expected value of the energy of the problem). Numerical experiments are presented and confirm the theoretical results.

Key words. Semi-linear stochastic wave equation, Multiplicative noise, Strong convergence, Trace formula, Stochastic trigonometric methods, Geometric numerical integration

AMS subject classifications. 65C20, 60H10, 60H15, 60H35, 65C30

1. Introduction. We consider the numerical discretisation of semi-linear stochastic wave equations of the form

$$\begin{aligned} &\mathrm{d}\dot{u} - \Delta u \,\mathrm{d}t = f(u)\,\mathrm{d}t + g(u)\,\mathrm{d}W && \text{in } \mathscr{D}\times(0,\infty), \\ &u = 0 && \text{in } \partial\mathscr{D}\times(0,\infty), \\ &u(\cdot,0) = u_0,\ \dot{u}(\cdot,0) = v_0 && \text{in } \mathscr{D}, \end{aligned} \tag{1.1}$$

where $u = u(x,t)$ and $\mathscr{D}\subset\mathbb{R}^d$, $d = 1,2,3$, is a bounded convex domain with polygonal boundary $\partial\mathscr{D}$. The "$\cdot$" denotes the time derivative $\frac{\partial}{\partial t}$. Assumptions on the smoothness of the nonlinearities f and g will be given below. The stochastic process $\{W(t)\}_{t\geq 0}$ is an $L_2(\mathscr{D})$-valued (possibly cylindrical) Q-Wiener process with respect to a normal filtration $\{\mathscr{F}_t\}_{t\geq 0}$ on a filtered probability space $(\Omega,\mathscr{F},\mathbb{P},\{\mathscr{F}_t\}_{t\geq 0})$. The initial data u_0 and v_0 are $\mathscr{F}_0$-measurable random variables. We will numerically solve this problem with a linear finite element method in space and a stochastic trigonometric method in time.

We refer to the introductions of [15] and [5] for the relevant literature on the spatial, respectively temporal, discretisation of stochastic (linear) wave equations. Further, the recent publication [21] presents a full discretisation of the wave equation with additive noise: a spectral Galerkin approximation is used in space and an adapted stochastic trigonometric method, using linear functionals of the noise as in [11], is employed in time. Furthermore, the time discretisation of nonlinear stochastic wave equations by stochastic trigonometric methods is analysed in [20]. Finally, let us mention the recent preprint [6] which analyses convergence in $L^p(\Omega)$ of the stochastic trigonometric method applied to the one-dimensional nonlinear stochastic wave equation.

In the present publication, we prove mean-square convergence for the full discretisation to the exact solution to the nonlinear problem (1.1). Furthermore, using this result, we derive a geometric property of our numerical integrator, namely a trace formula. The trace formula

*Department of Mathematics and Mathematical Statistics, Umeå University, SE–901 87 Umeå, Sweden (rikard.anton@math.umu.se).

†Department of Mathematics and Mathematical Statistics, Umeå University, SE–901 87 Umeå, Sweden (david.cohen@math.umu.se).

‡Department of Mathematical Sciences, Chalmers University of Technology and University of Gothenburg, SE–412 96 Gothenburg, Sweden (stig@chalmers.se).

§School of Mathematics and Statistics, and School of Geosciences and Info-Physics, Central South University, CN–410083 Changsha, Hunan, China (x.j.wang7@csu.edu.cn)

(the linear drift of the expected value of the energy) for the exact solution of (1.1) as well as for the finite element solution and the completely discrete solution are presented.

The paper is organised as follows. We introduce some notations and mention some useful results in the next section. Section 3 presents a mean-square convergence analysis for our numerical discretisation. A trace formula for the exact and numerical solutions is given in Section 4. Finally, numerical experiments illustrating the rates of convergence and the trace formula of the numerical solution are given in the final section.

2. Notations and useful results. Let U and H be separable Hilbert spaces with norms $\|\cdot\|_U$ and $\|\cdot\|_H$ respectively. We denote the space of bounded linear operators from U to H by $\mathscr{L}(U,H)$, and we let $\mathscr{L}_2(U,H)$ be the set of Hilbert-Schmidt operators with norm

$$\|T\|_{\mathscr{L}_2(U,H)} := \left(\sum_{k=1}^{\infty} \|Te_k\|_H^2\right)^{1/2},$$

where $\{e_k\}_{k=1}^{\infty}$ is an arbitrary orthonormal basis of U. If $H = U$, then we write $\mathscr{L}(U) = \mathscr{L}(U,U)$ and $\mathrm{HS} = \mathscr{L}_2(U,U)$. Let $Q \in \mathscr{L}(U)$ be a self-adjoint, positive semidefinite operator. We denote the space of Hilbert-Schmidt operators from $Q^{1/2}(U)$ to $H = U$ by $\mathscr{L}_2^0$ with norm

$$\|T\|_{\mathscr{L}_2^0} = \|TQ^{1/2}\|_{\mathrm{HS}}.$$

For the stochastic wave equation (1.1), we define $U := L_2(\mathscr{D})$ and denote the $L_2(\mathscr{D})$-norm by $\|\cdot\| := \|\cdot\|_{L_2(\mathscr{D})}$. Further, we set $\Lambda = -\Delta$ with $D(\Lambda) = H^2(\mathscr{D}) \cap H_0^1(\mathscr{D})$.

Let $(\Omega, \mathscr{F}, \mathbb{P}, \{\mathscr{F}_t\}_{t\geq 0})$ be a filtered probability space and $L_2(\Omega, H)$ the space of H-valued square integrable random variables with norm

$$\|v\|_{L_2(\Omega,H)} := \mathbb{E}[\|v\|_H^2]^{1/2}.$$

Next, we define the space $\dot{H}^\alpha = D(\Lambda^{\alpha/2})$, for $\alpha \in \mathbb{R}$, with norm

$$\|v\|_\alpha := \|\Lambda^{\alpha/2} v\|_{L_2(\mathscr{D})} = \left(\sum_{j=1}^{\infty} \lambda_j^\alpha (v, \varphi_j)_{L_2(\mathscr{D})}^2\right)^{1/2},$$

where $\{(\lambda_j, \varphi_j)\}_{j=1}^{\infty}$ are the eigenpairs of Λ with orthonormal eigenvectors. We also introduce the space

$$H^\alpha := \dot{H}^\alpha \times \dot{H}^{\alpha-1},$$

with norm $|||v|||_\alpha^2 := \|v_1\|_\alpha^2 + \|v_2\|_{\alpha-1}^2$, for $\alpha \in \mathbb{R}$ and $v = [v_1, v_2]^T$. Note that $\dot{H}^0 = U := L_2(\mathscr{D})$ and $H := H^0 = \dot{H}^0 \times \dot{H}^{-1}$. In the following we denote $(\cdot,\cdot) = (\cdot,\cdot)_{L_2(\mathscr{D})}$ and $\|\cdot\| = \|\cdot\|_{L_2(\mathscr{D})}$.

Denoting the velocity of the solution to our stochastic partial differential equation by $u_2 := \dot{u}_1 := \dot{u}$, one can rewrite (1.1) as

$$\begin{aligned} &\mathrm{d}X(t) = AX(t)\,\mathrm{d}t + F(X(t))\,\mathrm{d}t + G(X(t))\,\mathrm{d}W(t), \quad t > 0, \\ &X(0) = X_0, \end{aligned} \tag{2.1}$$

where $X := \begin{bmatrix} u_1 \\ u_2 \end{bmatrix}$, $A := \begin{bmatrix} 0 & I \\ -\Lambda & 0 \end{bmatrix}$, $F(X) := \begin{bmatrix} 0 \\ f(u_1) \end{bmatrix}$, $G(X) := \begin{bmatrix} 0 \\ g(u_1) \end{bmatrix}$ and $X_0 := \begin{bmatrix} u_0 \\ v_0 \end{bmatrix}$. The operator A with $D(A) = H^1 = \dot{H}^1 \times \dot{H}^0$ is the generator of a strongly continuous semigroup of bounded linear operators $E(t) = \mathrm{e}^{tA}$ on $H = H^0 = \dot{H}^0 \times \dot{H}^{-1}$, in fact, a unitary group.

Let $\{\mathscr{T}_h\}$ be a quasi-uniform family of triangulations of the convex polygonal domain $\mathscr{D}$ with $h_{\mathrm{K}} = \mathrm{diam}(K)$ and $h = \max_{\mathrm{K}\in\mathscr{T}_h} h_{\mathrm{K}}$. Let $V_h \subset H_0^1(\mathscr{D}) = \dot{H}^1$ be the space of piecewise linear continuous functions with respect to $\mathscr{T}_h$ which are zero on the boundary of $\mathscr{D}$, and let $\mathscr{P}_h : \dot{H}^0 \to V_h$ denote the orthogonal projector onto the finite element space. The discrete Laplace operator $\Lambda_h\colon V_h \to V_h$ is then defined by

$$(\Lambda_h v_h, w_h) = (\nabla v_h, \nabla w_h) \quad \forall w_h \in V_h.$$

We also define discrete variants of $\|\cdot\|_\alpha$ and $\dot{H}^\alpha$ by

$$\|v_h\|_{h,\alpha} = \|\Lambda_h^{\alpha/2} v_h\|, \quad v_h \in V_h$$

and $\dot{H}_h^\alpha = V_h$ equipped with the norm $\|\cdot\|_{h,\alpha}$. Finally, the finite element approximation of (1.1) can then be written as

$$\begin{aligned} &\mathrm{d}\dot{u}_{h,1}(t) + \Lambda_h u_{h,1}(t)\,\mathrm{d}t = \mathscr{P}_h f(u_{h,1}(t))\,\mathrm{d}t + \mathscr{P}_h g(u_{h,1}(t))\,\mathrm{d}W(t), \;\; t>0, \\ &u_{h,1}(0) = u_{h,0}, \; u_{h,2}(0) = v_{h,0}, \end{aligned} \tag{2.2}$$

or in the abstract form

$$\begin{aligned} &\mathrm{d}X_h(t) = A_h X_h(t)\,\mathrm{d}t + \mathscr{P}_h F(X_h(t))\,\mathrm{d}t + \mathscr{P}_h G(X_h(t))\,\mathrm{d}W(t), \;\; t>0, \\ &X_h(0) = X_{h,0}, \end{aligned} \tag{2.3}$$

where $A_h := \begin{bmatrix} 0 & I \\ -\Lambda_h & 0 \end{bmatrix}$, $X_h := \begin{bmatrix} u_{h,1} \\ u_{h,2} \end{bmatrix}$, F and G are as before, and $X_{h,0} := \begin{bmatrix} u_{h,0} \\ v_{h,0} \end{bmatrix}$ with $u_{h,0} = \mathscr{P}_h u_0, v_{h,0} = \mathscr{P}_h v_0 \in V_h$. Note the abuse of notation for the projection $\mathscr{P}_h F(X_h) = (0, \mathscr{P}_h f(u_{h,1}))^T$ and similarly for $\mathscr{P}_h G(X_h)$. This will be used throughout the paper. Again, A_h is the generator of a C_0-semigroup $E_h(t) = \mathrm{e}^{tA_h}$ on $H_h := \dot{H}_h^0 \times \dot{H}_h^{-1}$.

We study the equations (2.1) and (2.3) in their mild form

$$X(t) = E(t)X_0 + \int_0^t E(t-s)F(X(s))\,\mathrm{d}s + \int_0^t E(t-s)G(X(s))\,\mathrm{d}W(s), \tag{2.4}$$

$$X_h(t) = E_h(t)X_{h,0} + \int_0^t E_h(t-s)\mathscr{P}_h F(X_h(s))\,\mathrm{d}s + \int_0^t E_h(t-s)\mathscr{P}_h G(X_h(s))\,\mathrm{d}W(s), \tag{2.5}$$

where the semigroups can be expressed as

$$E(t) = \begin{bmatrix} C(t) & \Lambda^{-1/2}S(t) \\ -\Lambda^{1/2}S(t) & C(t) \end{bmatrix}, \tag{2.6}$$

$$E_h(t) = \begin{bmatrix} C_h(t) & \Lambda_h^{-1/2}S_h(t) \\ -\Lambda_h^{1/2}S_h(t) & C_h(t) \end{bmatrix}, \tag{2.7}$$

with $C(t) = \cos(t\Lambda^{1/2})$, $S(t) = \sin(t\Lambda^{1/2})$, $C_h(t) = \cos(t\Lambda_h^{1/2})$ and $S_h(t) = \sin(t\Lambda_h^{1/2})$.

In order to ensure existence and uniqueness of problem (1.1) we shall assume that $u_0 \in L_2(\Omega, \dot{H}^\beta)$ and $v_0 \in L_2(\Omega, \dot{H}^{\beta-1})$ for some regularity parameter $\beta \geq 0$, and that the functions $f\colon L_2(\mathscr{D}) \to L_2(\mathscr{D})$ and $g\colon L_2(\mathscr{D}) \to \mathscr{L}_2^0$ satisfy

$$\begin{aligned} \|f(u)-f(v)\| + \|g(u)-g(v)\|_{\mathscr{L}_2^0} &\leq C\|u-v\|, && \text{if } \beta \geq 0, \\ \|f(u)\| + \|g(u)\|_{\mathscr{L}_2^0} &\leq C(1+\|u\|), && \text{if } 0 \leq \beta \leq 1, \\ \|\Lambda^{(\beta-1)/2}f(u)\| + \|\Lambda^{(\beta-1)/2}g(u)\|_{\mathscr{L}_2^0} &\leq C(1+\|\Lambda^{(\beta-1)/2}u\|), && \text{if } \beta > 1, \end{aligned} \tag{2.8}$$

for all $u, v \in L_2(\mathscr{D})$ in the first two inequalities and for all $u \in \dot{H}^{\beta-1}$ in the last one. Through the text, C (or C_1, C_2, K_1, K_2 etc.) denotes a generic positive constant that may vary from line to line.

LEMMA 2.1. *Assume that $u_0 \in L_2(\Omega, \dot{H}^\beta)$, $v_0 \in L_2(\Omega, \dot{H}^{\beta-1})$, and the functions f and g satisfy* (2.8) *for some $\beta \geq 0$. Then there exists a unique solution to the stochastic wave equation* (2.1) *and the finite element equation* (2.3) *given by the solution of their respective mild equation, i. e., equations* (2.4) *and* (2.5).

The proof of this lemma follows from [8, Theorem 7.4], see also the proof of Theorem 2.1 in [20].

We now collect some results that we will use later on.

• The error estimates for the cosine and sine operators (Corollary 4.2 in [15]): Denote $X_0 = [u_0, v_0]^T$ and let

$$F_h(t)X_0 = \big(C_h(t)\mathscr{P}_h - C(t)\big)u_0 + \big(\Lambda_h^{-1/2} S_h(t)\mathscr{P}_h - \Lambda^{-1/2} S(t)\big)v_0,$$
$$\dot{F}_h(t)X_0 = -\big(\Lambda_h^{1/2} S_h(t)\mathscr{P}_h - \Lambda^{1/2} S(t)\big)u_0 + \big(C_h(t)\mathscr{P}_h - C(t)\big)v_0.$$

Then we have

$$\begin{aligned} \|F_h(t)X_0\| &\leq C\cdot(1+t)\cdot h^{\frac{2}{3}\beta}|||X_0|||_\beta, \qquad && t\geq 0, \quad \beta\in[0,3],\\ \|\dot{F}_h(t)X_0\| &\leq C\cdot(1+t)\cdot h^{\frac{2}{3}(\beta-1)}|||X_0|||_\beta, && t\geq 0, \quad \beta\in[1,4]. \end{aligned} \tag{2.9}$$

Note that we have $\dot{F}_h$ and not $\dot{G}_h$ as in Corollary 4.2 in [15]. The only difference lies in the projector present in the above equation.

• The temporal Hölder continuity of the sine and cosine operators, see (4.1) in [5]:

$$\begin{aligned} \|(S_h(t)-S_h(s))\Lambda_h^{-\beta/2}\|_{\mathscr{L}(V_h)} &\leq C\cdot|t-s|^\beta, \qquad && \beta\in[0,1],\\ \|(C_h(t)-C_h(s))\Lambda_h^{-(\beta-1)/2}\|_{\mathscr{L}(V_h)} &\leq C\cdot|t-s|^{\beta-1}, && \beta\in[1,2], \end{aligned} \tag{2.10}$$

together with its continuous version:

$$\begin{aligned} \|(S(t)-S(s))\Lambda^{-\beta/2}\|_{\mathscr{L}(U)} &\leq C\cdot|t-s|^\beta, \qquad && \beta\in[0,1],\\ \|(C(t)-C(s))\Lambda^{-(\beta-1)/2}\|_{\mathscr{L}(U)} &\leq C\cdot|t-s|^{\beta-1}, && \beta\in[1,2]. \end{aligned} \tag{2.11}$$

• The equivalence of Λ_h and Λ, see the proof of Theorem 4.4 in [14]: This uses an inverse inequality, hence our assumption about the quasi-uniformity of the mesh family.

$$\|\Lambda_h^\alpha \mathscr{P}_h \Lambda^{-\alpha} v\|^2 \leq \|v\|^2, \;\; \alpha\in[-\tfrac{1}{2},1], \;\; v\in\dot{H}^0 = L_2(\mathscr{D}). \tag{2.12}$$

• The equivalence of the discrete and continuous norm, see (2.13) in [1]:

$$c\|\Lambda_h^\gamma v_h\| \leq \|\Lambda^\gamma v_h\| \leq C\|\Lambda_h^\gamma v_h\| \quad \text{for} \quad v_h\in V_h \quad \text{and} \quad \gamma\in[-\tfrac{1}{2},\tfrac{1}{2}]. \tag{2.13}$$

Using the above estimates, one can deduce the following regularity results for the exact solution to our stochastic wave equation (1.1) and for the exact solution of the finite element approximation (2.2).

PROPOSITION 2.2. *Let $[u_1, u_2]^T$ be the solution to* (1.1)*, where the initial values satisfy $u_0 \in L_2(\Omega, \dot{H}^\beta)$, $v_0 \in L_2(\Omega, \dot{H}^{\beta-1})$, and the functions f and g satisfy* (2.8) *for some $\beta \geq 0$. Then it holds that*

$$\sup_{0\leq t\leq T} \mathbb{E}[\|u_1(t)\|_\beta^2 + \|u_2(t)\|_{\beta-1}^2] \leq C$$

and, for $0 \le s \le t \le T$,

$$\mathbb{E}[\|u_1(t)-u_1(s)\|^2] \le C\,|t-s|^{2\min(\beta,1)}\Big(\mathbb{E}[\|u_0\|_\beta^2+\|v_0\|_{\beta-1}^2] + \sup_{r\in[0,T]}\mathbb{E}[1+\|u_1(r)\|_\beta^2]\Big).$$

The proof of this proposition is very similar to the proof of Proposition 2.3 given below and is therefore omitted (see also the proofs of Proposition 3.1 and Lemma 3.3 in [20]).

The next result will be useful in Section 4 when we will deal with the trace formula of the numerical solution.

PROPOSITION 2.3. *Let* $[u_{h,1},u_{h,2}]^T$ *be the solution to the finite element problem* (2.2), *where the initial values satisfy* $u_0 \in L_2(\Omega,\in \dot{H}^\beta)$, $v_0 \in L_2(\Omega,\dot{H}^{\beta-1})$, *and the functions* f *and* g *satisfy* (2.8) *for some* $\beta\in[0,2]$. *Then it holds that*

$$\sup_{0\le t\le T}\mathbb{E}[\|u_{h,1}(t)\|_{h,\beta}^2+\|u_{h,2}(t)\|_{h,\beta-1}^2] \le C$$

and for $0\le s\le t\le T$

$$\mathbb{E}[\|u_{h,1}(t)-u_{h,1}(s)\|^2] \le C\,|t-s|^{2\min(\beta,1)}\Big(\mathbb{E}[\|u_{h,0}\|_{h,\beta}^2+\|v_{h,0}\|_{h,\beta-1}^2] + \sup_{r\in[0,T]}\mathbb{E}[1+\|u_{h,1}(r)\|_{h,\beta}^2]\Big),$$

where we recall that $u_{h,0}$ *and* $v_{h,0}$ *are the initial position and velocity to the finite element problem.*

Proof. Let us start with the first estimate of the norm of $\Lambda_h^{\beta/2}u_{h,1}(t)$ and consider the expression

$$\begin{aligned}\Lambda_h^{\beta/2}u_{h,1}(t) &= \Lambda_h^{\beta/2}C_h(t)u_{h,0}+\Lambda_h^{(\beta-1)/2}S_h(t)v_{h,0}\\ &\quad+\int_0^t \Lambda_h^{(\beta-1)/2}S_h(t-r)\mathscr{P}_h f(u_{h,1}(r))\,\mathrm{d}r\\ &\quad+\int_0^t \Lambda_h^{(\beta-1)/2}S_h(t-r)\mathscr{P}_h g(u_{h,1}(r))\,\mathrm{d}W(r).\end{aligned}$$

Using the fact that Λ_h and $C_h(t)$ commute, the boundedness of the cosine operator, together with our assumptions on the initial values for the finite element problem, we get

$$\mathbb{E}[\|\Lambda_h^{\beta/2}C_h(t)u_{h,0}\|^2] \le C \quad \text{for} \quad \beta\in[0,2].$$

Similarly, one obtains

$$\mathbb{E}[\|\Lambda_h^{(\beta-1)/2}S_h(t)v_{h,0}\|^2] \le C.$$

To estimate the third term, we use (2.12), the assumptions on f given in (2.8), and the equivalence of the norms stated in (2.13). First for $\beta\in[0,1]$, we get

$$\begin{aligned}&\mathbb{E}\Big[\Big\|\int_0^t S_h(t-r)\Lambda_h^{(\beta-1)/2}\mathscr{P}_h\Lambda^{-(\beta-1)/2}\Lambda^{(\beta-1)/2}f(u_{h,1}(r))\,\mathrm{d}r\Big\|^2\Big]\\ &\le C_1+C_2\int_0^t\mathbb{E}[\|u_{h,1}(r)\|^2]\,\mathrm{d}r\\ &\le C_3+C_4\int_0^t\mathbb{E}[\|u_{h,1}(r)\|_{h,\beta}^2]\,\mathrm{d}r.\end{aligned}$$

For $\beta \in [1,2]$, we have

$$\begin{aligned}
&\mathbb{E}\Big[\Big\| \int_0^t S_h(t-r)\Lambda_h^{(\beta-1)/2}\mathscr{P}_h\Lambda^{-(\beta-1)/2}\Lambda^{(\beta-1)/2} f(u_{h,1}(r))\,\mathrm{d}r\Big\|^2\Big] \\
&\quad\le C\int_0^t \mathbb{E}[1+\|\Lambda^{(\beta-1)/2}u_{h,1}(r)\|^2]\,\mathrm{d}r \le C_1 + C_2\int_0^t \mathbb{E}[\|u_{h,1}(r)\|_{h,\beta-1}^2]\,\mathrm{d}r \\
&\quad\le C_3 + C_4\int_0^t \mathbb{E}[\|u_{h,1}(r)\|_{h,\beta}^2]\,\mathrm{d}r.
\end{aligned}$$

Finally, Ito isometry, equations (2.13) and (2.12), and the assumptions (2.8) on g give us

$$\mathbb{E}\Big[\Big\| \int_0^t \Lambda_h^{(\beta-1)/2} S_h(t-r)\mathscr{P}_h g(u_{h,1}(r))\,\mathrm{d}W(r)\Big\|^2\Big] \le C_3 + C_4\int_0^t \mathbb{E}[\|u_{h,1}(r)\|_{h,\beta}^2]\,\mathrm{d}r.$$

All together, for $\beta \in [0,2]$, one thus obtains

$$\mathbb{E}[\|u_{h,1}(t)\|_{h,\beta}^2] \le K_1 + K_2\int_0^t \mathbb{E}[\|u_{h,1}(r)\|_{h,\beta}^2]\,\mathrm{d}r$$

and an application of Gronwall's lemma give the desired bound for $\mathbb{E}[\|u_{h,1}(t)\|_{h,\beta}^2]$.

The proofs for the other bound is done in a similar fashion.

We now prove a Hölder regularity property of the finite element solution. We write, for $0 \le s \le t \le T$,

$$\begin{aligned}
u_{h,1}(t) - u_{h,1}(s) &= (C_h(t)-C_h(s))u_{h,0} + \Lambda_h^{-1/2}(S_h(t)-S_h(s))v_{h,0} \\
&\quad + \int_0^s \Lambda_h^{-1/2}(S_h(t-r)-S_h(s-r))\mathscr{P}_h f(u_{h,1}(r))\,\mathrm{d}r \\
&\quad + \int_s^t \Lambda_h^{-1/2} S_h(t-r)\mathscr{P}_h f(u_{h,1}(r))\,\mathrm{d}r \\
&\quad + \int_0^s \Lambda_h^{-1/2}(S_h(t-r)-S_h(s-r))\mathscr{P}_h g(u_{h,1}(r))\,\mathrm{d}W(r) \\
&\quad + \int_s^t \Lambda_h^{-1/2} S_h(t-r)\mathscr{P}_h g(u_{h,1}(r))\,\mathrm{d}W(r).
\end{aligned}$$

To estimate the first term we use (2.10) to get

$$\begin{aligned}
\mathbb{E}[\|(C_h(t)-C_h(s))u_{h,0}\|^2] &= \mathbb{E}[\|(C_h(t)-C_h(s))\Lambda_h^{-\beta/2}\Lambda_h^{\beta/2}u_{h,0}\|^2] \\
&\le C|t-s|^{2\beta}\mathbb{E}[\|\Lambda_h^{\beta/2}u_{h,0}\|^2],
\end{aligned}$$

for $\beta \in [0,1]$. For $\beta \in (1,2]$ we note that $\Lambda_h^{-\beta/2} = \Lambda_h^{-1/2}\Lambda_h^{-(\beta-1)/2}$ and that $\Lambda_h^{-(\beta-1)/2}$ is bounded in the operator norm. Using a similar argument for the second term, we get the following estimate for the first two terms

$$\begin{aligned}
&\mathbb{E}[\|(C_h(t)-C_h(s))u_{h,0} + \Lambda_h^{-1/2}(S_h(t)-S_h(s))v_{h,0}\|^2] \\
&\quad\le C|t-s|^{2\min(\beta,1)}\mathbb{E}[\|u_{h,0}\|_{h,\beta}^2 + \|v_{h,0}\|_{h,\beta-1}^2],
\end{aligned}$$

for $\beta \in [0,2]$. In order to estimate the third term, we use (2.10), the assumptions on f, and the equivalence of the norms given in (2.13). First for $\beta \in [0,1]$, we obtain

$$\begin{aligned}
&\mathbb{E}\Big[\Big\| \int_0^s \Lambda_h^{-1/2}(S_h(t-r)-S_h(s-r))\mathscr{P}_h f(u_{h,1}(r))\,\mathrm{d}r\Big\|^2\Big] \\
&\quad\le C|t-s|^2 \sup_{t\in[0,T]} \mathbb{E}[1+\|u_{h,1}(t)\|^2] \\
&\quad\le C|t-s|^2 \sup_{t\in[0,T]} \mathbb{E}[1+\|u_{h,1}(t)\|_{h,\beta}^2].
\end{aligned}$$

For $\beta \in [1,2]$ we have, using (2.10), (2.12), (2.13) and the fact that $\Lambda_h^{-(\beta-1)/2}$ is bounded in the operator norm

$$\begin{aligned}
\mathbb{E}\Big[\Big\|\int_0^s \Lambda_h^{-1/2}(S_h(t-r)-S_h(s-r))\mathscr{P}_h f(u_{h,1}(r))\,\mathrm{d}r\Big\|^2\Big] \\
&\le \int_0^s \mathbb{E}[\|(S_h(t-r)-S_h(s-r))\Lambda_h^{-1/2}\Lambda_h^{-(\beta-1)/2}\Lambda_h^{(\beta-1)/2}\mathscr{P}_h\Lambda^{-(\beta-1)/2} \\
&\qquad \times \Lambda^{(\beta-1)/2} f(u_{h,1}(r))\|^2]\,\mathrm{d}r \\
&\le C|t-s|^2 \sup_{t\in[0,T]} \mathbb{E}[1+\|u_{h,1}(t)\|^2_{h,\beta-1}] \\
&\le C|t-s|^2 \sup_{t\in[0,T]} \mathbb{E}[1+\|u_{h,1}(t)\|^2_{h,\beta}].
\end{aligned}$$

Similarly we get for the fourth term

$$\begin{aligned}
\mathbb{E}\Big[\Big\|\int_s^t \Lambda_h^{-1/2}S_h(t-r)\mathscr{P}_h f(u_{h,1}(r))\,\mathrm{d}r\Big\|^2\Big] \\
&\le C|t-s|^{2\min(\beta,1)} \sup_{t\in[0,T]} \mathbb{E}[1+\|u_{h,1}(t)\|^2_{h,\beta}].
\end{aligned}$$

To estimate terms five and six we use Ito isometry, (2.10), (2.12), (2.13) and the assumptions on g to get, for $\beta \in [0,1]$,

$$\begin{aligned}
\mathbb{E}\Big[\Big\|\int_0^s \Lambda_h^{-1/2}(S_h(t-r)-S_h(s-r))\mathscr{P}_h g(u_{h,1}(r))\,\mathrm{d}W(r)\Big\|^2\Big] \\
&\le \int_0^s \mathbb{E}[\|(S_h(t-r)-S_h(s-r))\Lambda_h^{-\beta/2}\Lambda_h^{(\beta-1)/2}\mathscr{P}_h \\
&\qquad \times g(u_{h,1}(r))\|^2_{\mathscr{L}_2^0}]\,\mathrm{d}r \\
&\le C|t-s|^{2\beta} \sup_{t\in[0,T]} \mathbb{E}[\|g(u_{h,1}(t))\|^2_{\mathscr{L}_2^0}] \\
&\le C|t-s|^{2\beta} \sup_{t\in[0,T]} \mathbb{E}[1+\|u_{h,1}(t)\|^2_{h,\beta}]
\end{aligned}$$

and

$$\begin{aligned}
\mathbb{E}\Big[\Big\|\int_s^t \Lambda_h^{-1/2}S_h(t-r)\mathscr{P}_h g(u_{h,1}(r))\,\mathrm{d}W(r)\Big\|^2\Big] \\
&\le \int_s^t \mathbb{E}[\|S_h(t-r)\Lambda_h^{-\beta/2}\Lambda_h^{(\beta-1)/2}\mathscr{P}_h g(u_{h,1}(r))\|^2_{\mathscr{L}_2^0}]\,\mathrm{d}r \\
&\le C|t-s|^{2\beta} \sup_{t\in[0,T]} \mathbb{E}[\|g(u_{h,1}(t))\|^2_{\mathscr{L}_2^0}] \\
&\le C|t-s|^{2\beta} \sup_{t\in[0,T]} \mathbb{E}[1+\|u_{h,1}(t)\|^2_{h,\beta}].
\end{aligned}$$

For $\beta \in [1,2]$ we again use that $\Lambda_h^{-(\beta-1)/2}$ is bounded in the operator norm.

Collecting the above estimates give us the statement about the regularity of the finite element solution. □

3. Mean-square convergence analysis. Recall that the exact solutions to (2.1) and (2.3) solve the following equations

$$
\begin{aligned}
X(t) &= E(t)X_0 + \int_0^t E(t-s)F(X(s))\,\mathrm{d}s + \int_0^t E(t-s)G(X(s))\,\mathrm{d}W(s),\\
X_h(t) &= E_h(t)X_{h,0} + \int_0^t E_h(t-s)\mathscr{P}_hF(X_h(s))\,\mathrm{d}s + \int_0^t E_h(t-s)\mathscr{P}_hG(X_h(s))\,\mathrm{d}W(s),
\end{aligned}
$$

where

$$
E(t) = \begin{bmatrix} C(t) & \Lambda^{-1/2}S(t) \\ -\Lambda^{1/2}S(t) & C(t) \end{bmatrix}, \quad E_h(t) = \begin{bmatrix} C_h(t) & \Lambda_h^{-1/2}S_h(t) \\ -\Lambda_h^{1/2}S_h(t) & C_h(t) \end{bmatrix},
$$

with $C(t) = \cos(t\Lambda^{1/2})$, $S(t) = \sin(t\Lambda^{1/2})$, $C_h(t) = \cos(t\Lambda_h^{1/2})$ and $S_h(t) = \sin(t\Lambda_h^{1/2})$.

The explicit time discretisation of the finite element solution (2.3) of the stochastic wave equation using a stochastic trigonometric method with stepsize k reads

$$
U^{n+1} = E_h(k)U^n + E_h(k)\mathscr{P}_hF(U^n)k + E_h(k)\mathscr{P}_hG(U^n)\Delta W^n,
$$

that is,

$$
\begin{aligned}
\begin{bmatrix} U_1^{n+1} \\ U_2^{n+1} \end{bmatrix} &= \begin{bmatrix} C_h(k) & \Lambda_h^{-1/2}S_h(k) \\ -\Lambda_h^{1/2}S_h(k) & C_h(k) \end{bmatrix} \begin{bmatrix} U_1^n \\ U_2^n \end{bmatrix} + \begin{bmatrix} \Lambda_h^{-1/2}S_h(k) \\ C_h(k) \end{bmatrix} \mathscr{P}_hf(U_1^n)k \\
&\quad + \begin{bmatrix} \Lambda_h^{-1/2}S_h(k) \\ C_h(k) \end{bmatrix} \mathscr{P}_hg(U_1^n)\Delta W^n, \qquad (3.1)
\end{aligned}
$$

where $\Delta W^n = W(t_{n+1}) - W(t_n)$ denotes the Wiener increments. Here we thus get an approximation $U_j^n \approx u_{h,j}(t_n)$ of the exact solution of our finite element problem at the discrete times $t_n = nk$. Further, a recursion gives

$$
U^n = E_h(t_n)U^0 + \sum_{j=0}^{n-1} E_h(t_n - t_j)\mathscr{P}_hF(U^j)\,k + \sum_{j=0}^{n-1} E_h(t_n - t_j)\mathscr{P}_hG(U^j)\,\Delta W^j.
$$

We now look at the error between the numerical and the exact solutions $U^n - X(t_n)$. We follow the same approach as in [22] for parabolic problems, see also [16], and obtain

$$
\mathbb{E}[\|U^n - X(t_n)\|^2] \le 3\big(\mathbb{E}[\|\mathrm{Err}_0\|^2] + \mathbb{E}[\|\mathrm{Err}_\mathrm{d}\|^2] + \mathbb{E}[\|\mathrm{Err}_\mathrm{s}\|^2]\big),
$$

where we define

$$
\mathrm{Err}_0 := (E_h(t_n)\mathscr{P}_h - E(t_n))X_0,
$$

$$
\mathrm{Err}_\mathrm{d} := \sum_{j=0}^{n-1} \int_{t_j}^{t_{j+1}} \Big(E_h(t_n - t_j)\mathscr{P}_hF(U^j) - E(t_n - s)F(X(s)) \Big)\,\mathrm{d}s
$$

and

$$
\mathrm{Err}_\mathrm{s} := \sum_{j=0}^{n-1} \int_{t_j}^{t_{j+1}} \Big(E_h(t_n - t_j)\mathscr{P}_hG(U^j) - E(t_n - s)G(X(s)) \Big)\,\mathrm{d}W(s).
$$

We next estimate the above three terms.

Estimate for the initial error $\mathbf{Err}_0$. By (2.9), the first component reads

$$
\begin{aligned}
&\mathbb{E}[\|(C_h(t_n)\mathscr{P}_h - C(t_n))u_0 + (\Lambda_h^{-1/2}S_h(t_n)\mathscr{P}_h - \Lambda^{-1/2}S(t_n))v_0\|^2] \\
&\quad \le C(1+t_n)^2 h^{\frac{4}{3}\beta}\left(\mathbb{E}[\|u_0\|_\beta + \|v_0\|_{\beta-1}]\right)^2,
\end{aligned}
$$

for $\beta \in [0,3]$. Similarly for the second component

$$
\begin{aligned}
&\mathbb{E}[\|-(\Lambda_h^{1/2}S_h(t_n)\mathscr{P}_h - \Lambda^{1/2}S(t_n))u_0 + (C_h(t_n)\mathscr{P}_h - C(t_n))v_0\|^2] \\
&\quad \le C(1+t_n)^2 h^{\frac{4}{3}(\beta-1)}\left(\mathbb{E}[\|u_0\|_\beta + \|v_0\|_{\beta-1}]\right)^2,
\end{aligned}
$$

for $\beta \in [1,4]$.

Estimate for the deterministic part, $\mathbf{Err_d}$. We write the deterministic error as

$$
\begin{aligned}
\mathrm{Err_d} &= \sum_{j=0}^{n-1}\int_{t_j}^{t_{j+1}} \Big(E_h(t_n-t_j)\mathscr{P}_h F(U^j) - E(t_n-s)F(X(s))\Big)\,\mathrm{d}s \\
&= \sum_{j=0}^{n-1}\int_{t_j}^{t_{j+1}} E_h(t_n-t_j)\mathscr{P}_h(F(U^j) - F(X(t_j)))\,\mathrm{d}s \\
&\quad + \sum_{j=0}^{n-1}\int_{t_j}^{t_{j+1}} E_h(t_n-t_j)\mathscr{P}_h(F(X(t_j)) - F(X(s)))\,\mathrm{d}s \\
&\quad + \sum_{j=0}^{n-1}\int_{t_j}^{t_{j+1}} \big(E_h(t_n-t_j)\mathscr{P}_h - E(t_n-t_j)\big)F(X(s))\,\mathrm{d}s \\
&\quad + \sum_{j=0}^{n-1}\int_{t_j}^{t_{j+1}} \big(E(t_n-t_j) - E(t_n-s)\big)F(X(s))\,\mathrm{d}s \\
&=: I_1 + I_2 + I_3 + I_4,
\end{aligned}
$$

and estimate the second moment of each term in the above equation. For the first component of the first term we get the following estimate by using (2.11) and (2.8)

$$
\begin{aligned}
\left(\mathbb{E}[\|I_{[1,1]}\|^2]\right)^{1/2} &\le \sum_{j=0}^{n-1}\int_{t_j}^{t_{j+1}} \left(\mathbb{E}[\|\Lambda_h^{-1/2}S_h(t_n-t_j)\mathscr{P}_h(f(U_1^j) - f(u_1(t_j)))\|^2]\right)^{1/2}\,\mathrm{d}s \\
&\le C\sum_{j=0}^{n-1} k\left(\mathbb{E}[\|U_1^j - u(t_j)\|^2]\right)^{1/2},
\end{aligned}
$$

so that

$$
\mathbb{E}[\|I_{[1,1]}\|^2] \le \left(Ck\sum_{j=0}^{n-1}\left(\mathbb{E}[\|U_1^j - u(t_j)\|^2]\right)^{1/2}\right)^2 \le Ck\sum_{j=0}^{n-1}\mathbb{E}[\|U_1^j - u(t_j)\|^2].
$$

The second component is estimated in the same way

$$
\begin{aligned}
\left(\mathbb{E}[\|I_{[1,2]}\|^2]\right)^{1/2} &\le \sum_{j=0}^{n-1}\int_{t_j}^{t_{j+1}} \left(\mathbb{E}[\|C_h(t_n-t_j)\mathscr{P}_h(f(U_1^j) - f(u_1(t_j)))\|^2]\right)^{1/2}\,\mathrm{d}s \\
&\le C\sum_{j=0}^{n-1} k\left(\mathbb{E}[\|U_1^j - u(t_j)\|^2]\right)^{1/2}.
\end{aligned}
$$

For the second term, using Proposition 2.2, we get

$$
\begin{aligned}
&\left(\mathbb{E}[\|I_{[2,1]}\|^2]\right)^{1/2}\\
&\leq \sum_{j=0}^{n-1}\int_{t_j}^{t_{j+1}} \left(\mathbb{E}[\|\Lambda_h^{-1/2}S_h(t_n-t_j)\mathscr{P}_h(f(u_1(t_j))-f(u_1(s)))\|^2]\right)^{1/2}\,\mathrm{d}s\\
&\leq C\sum_{j=0}^{n-1}\int_{t_j}^{t_{j+1}} \left(\mathbb{E}[\|u_1(t_j)-u_1(s)\|^2]\right)^{1/2}\,\mathrm{d}s\\
&\leq C\sum_{j=0}^{n-1}\int_{t_j}^{t_{j+1}} |t_j-s|^{\min(\beta,1)}\,\mathrm{d}s\Big(\mathbb{E}[\|u_0\|_\beta^2+\|v_0\|_{\beta-1}^2]+\sup_{t\in[0,T]}\mathbb{E}[1+\|u_1(t)\|_\beta^2]\Big)^{1/2}\\
&\leq Ck^{\min(\beta,1)},
\end{aligned}
$$

for $\beta\in[0,3]$. Thus

$$
\mathbb{E}[\|I_{[2,1]}\|^2]\leq Ck^{2\min(\beta,1)}.
$$

The second component $I_{[2,2]}$ is estimated in the same way.

The third term reads, using $F_h(t)$ in (2.9) with $u_0=0$ and $\beta\in[1,3]$,

$$
\begin{aligned}
\left(\mathbb{E}[\|I_{[3,1]}\|^2]\right)^{1/2} &\leq \sum_{j=0}^{n-1}\int_{t_j}^{t_{j+1}} \left(\mathbb{E}[\|(\Lambda_h^{-1/2}S_h(t_n-t_j)\mathscr{P}_h-\Lambda^{-1/2}S(t_n-t_j))f(u_1(s))\|^2]\right)^{1/2}\,\mathrm{d}s\\
&= \sum_{j=0}^{n-1}\int_{t_j}^{t_{j+1}} \left(\mathbb{E}[\|F_h(t_n-t_j)f(u_1(s))\|^2]\right)^{1/2}\,\mathrm{d}s\\
&\leq Ch^{\frac{2}{3}\beta}\sum_{j=0}^{n-1}\int_{t_j}^{t_{j+1}} \left(\mathbb{E}[\|\Lambda^{(\beta-1)/2}f(u_1(s))\|^2\right)^{1/2}\,\mathrm{d}s\\
&\leq Ch^{\frac{2}{3}\beta}\left(\sup_{t\in[0,T]}\mathbb{E}[1+\|u_1(t)\|_{\beta-1}^2]\right)^{1/2}\\
&\leq Ch^{\frac{2}{3}\beta}.
\end{aligned}
$$

For $\beta\in[0,1]$ we simply note that

$$
\mathbb{E}[\|\Lambda^{(\beta-1)/2}f(u_1(s))\|^2]\leq C\mathbb{E}[\|f(u_1(s))\|^2]\leq C.
$$

The estimate for the second component is done in a similar way using now $\dot{F}_h(t)$ in (2.9) with $u_0=0$ to get, for $\beta\in[1,4]$,

$$
\mathbb{E}[\|I_{[3,2]}\|^2]\leq Ch^{\frac{4}{3}(\beta-1)},
$$

For the fourth term with $\beta\in[0,3]$, using (2.11) and the assumption on the function f in

(2.8), we get

$$\begin{aligned}\left(\mathbb{E}[\|I_{[4,1]}\|^2]\right)^{1/2} &\leq \sum_{j=0}^{n-1}\int_{t_j}^{t_{j+1}} \Big(\mathbb{E}[\|(S(t_n-t_j)-S(t_n-s))\Lambda^{-1/2}f(u_1(s))\|^2]\Big)^{1/2}\,\mathrm{d}s\\ &\leq C\sum_{j=0}^{n-1}\int_{t_j}^{t_{j+1}} \Big(\|(S(t_n-t_j)-S(t_n-s))\Lambda^{-1/2}\|^2_{\mathscr{L}(U)}\mathbb{E}[\|f(u_1(s))\|^2]\Big)^{1/2}\,\mathrm{d}s\\ &\leq C\sum_{j=0}^{n-1}\int_{t_j}^{t_{j+1}} \left(|s-t_j|^2\mathbb{E}[1+\|u_1(s)\|^2]\right)^{1/2}\,\mathrm{d}s\\ &\leq Ck.\end{aligned}$$

Thus we obtain

$$\mathbb{E}[\|I_{[4,1]}\|^2] \leq Ck^2.$$

For the second component we get

$$\begin{aligned}\left(\mathbb{E}[\|I_{[4,2]}\|^2]\right)^{1/2} &\leq \sum_{j=0}^{n-1}\int_{t_j}^{t_{j+1}} \left(\mathbb{E}[\|(C(t_n-t_j)-C(t_n-s))f(u_1(s))\|^2]\right)^{1/2}\,\mathrm{d}s\\ &\leq C\sum_{j=0}^{n-1}\int_{t_j}^{t_{j+1}} \Big(\|(C(t_n-t_j)-C(t_n-s))\Lambda^{-(\beta-1)/2}\|^2_{\mathscr{L}(U)}\\ &\qquad \times \mathbb{E}[\|\Lambda^{(\beta-1)/2}f(u_1(s))\|^2]\Big)^{1/2}\,\mathrm{d}s\\ &\leq C\sum_{j=0}^{n-1}\int_{t_j}^{t_{j+1}} \Big(|s-t_j|^{2(\beta-1)}\mathbb{E}[1+\|u_1(s)\|^2_{\beta-1}]\Big)^{1/2}\,\mathrm{d}s\\ &\leq Ck^{\min(\beta-1,1)},\end{aligned}$$

for $\beta \geq 1$.

Altogether we thus obtain

$$\begin{aligned}\mathbb{E}[\|\mathrm{Err}_{\mathrm{d},1}\|^2] &\leq C\cdot\Big(h^{\frac{4\beta}{3}}+k^{2\min(\beta,1)}+k\sum_{j=0}^{n-1}\mathbb{E}[\|U_1^j-u_1(t_j)\|^2]\Big) \qquad \text{for } \beta\in[0,3],\\ \mathbb{E}[\|\mathrm{Err}_{\mathrm{d},2}\|^2] &\leq C\cdot\Big(h^{\frac{4(\beta-1)}{3}}+k^{2\min(\beta-1,1)}+k\sum_{j=0}^{n-1}\mathbb{E}[\|U_1^j-u_1(t_j)\|^2]\Big) \qquad \text{for } \beta\in[1,4].\end{aligned}$$

Estimate for the stochastic part, $\mathbf{Err_s}$. We rewrite the stochastic part as we did for the deterministic part of the error:

$$\begin{aligned}
\mathrm{Err_s} &= \sum_{j=0}^{n-1} \int_{t_j}^{t_{j+1}} \Big(E_h(t_n - t_j)\mathscr{P}_h G(U^j) - E(t_n - s) G(X(s)) \Big) \,\mathrm{d}W(s) \\
&= \sum_{j=0}^{n-1} \int_{t_j}^{t_{j+1}} E_h(t_n - t_j)\mathscr{P}_h (G(U^j) - G(X(t_j))) \,\mathrm{d}W(s) \\
&\quad + \sum_{j=0}^{n-1} \int_{t_j}^{t_{j+1}} E_h(t_n - t_j)\mathscr{P}_h (G(X(t_j)) - G(X(s))) \,\mathrm{d}W(s) \\
&\quad + \sum_{j=0}^{n-1} \int_{t_j}^{t_{j+1}} \big(E_h(t_n - t_j)\mathscr{P}_h - E(t_n - t_j) \big) G(X(s)) \,\mathrm{d}W(s) \\
&\quad + \sum_{j=0}^{n-1} \int_{t_j}^{t_{j+1}} \big(E(t_n - t_j) - E(t_n - s) \big) G(X(s)) \,\mathrm{d}W(s) \\
&=: J_1 + J_2 + J_3 + J_4.
\end{aligned}$$

The estimate for the first term follows by using the Ito isometry, the boundedness of $\mathscr{P}_h$, S_h and $\Lambda_h^{-1/2}$, and the Lipschitz condition on the function g in (2.8)

$$\begin{aligned}
\mathbb{E}[\|J_{[1,1]}\|^2] &= \sum_{j=0}^{n-1} \int_{t_j}^{t_{j+1}} \mathbb{E}[\|\Lambda_h^{-1/2} S_h(t_n - t_j)\mathscr{P}_h (g(U_1^j) - g(u_1(t_j)))\|_{\mathscr{L}_2^0}^2] \,\mathrm{d}s \\
&\le Ck \sum_{j=0}^{n-1} \mathbb{E}[\|U_1^j - u_1(t_j)\|^2]
\end{aligned}$$

for $\beta \in [0,3]$. The same estimate holds for the second component $J_{[1,2]}$ with $\beta \in [1,4]$. For the first component of the second term, using Proposition 2.2, we obtain

$$\begin{aligned}
\mathbb{E}[\|J_{[2,1]}\|^2] &= \sum_{j=0}^{n-1} \int_{t_j}^{t_{j+1}} \mathbb{E}[\|\Lambda_h^{-1/2} S_h(t_n - t_j)\mathscr{P}_h (g(u_1(t_j)) - g(u_1(s))\|_{\mathscr{L}_2^0}^2] \,\mathrm{d}s \\
&\le C \sum_{j=0}^{n-1} \int_{t_j}^{t_{j+1}} \mathbb{E}[\|u_1(t_j) - u_1(s)\|^2] \,\mathrm{d}s \\
&\le C \sum_{j=0}^{n-1} \int_{t_j}^{t_{j+1}} \big|t_j - s\big|^{2\min(\beta,1)} \,\mathrm{d}s \Big(\mathbb{E}[\|u_0\|_\beta^2 + \|v_0\|_{\beta-1}^2] \\
&\qquad + \sup_{t \in [0,T]} \mathbb{E}[1 + \|u_1(t)\|_\beta^2] \Big) \\
&\le Ck^{2\min(\beta,1)},
\end{aligned}$$

for $\beta \in [0,3]$. Similarly, the estimate for the second component of J_2 reads

$$\begin{aligned}
\mathbb{E}[\|J_{[2,2]}\|^2] &= \sum_{j=0}^{n-1} \int_{t_j}^{t_{j+1}} \mathbb{E}[\|C_h(t_n - t_j)\mathscr{P}_h (g(u_1(t_j)) - g(u_1(s))\|_{\mathscr{L}_2^0}^2] \,\mathrm{d}s \\
&\le Ck^2 \Big(\mathbb{E}[\|u_0\|_\beta^2 + \|v_0\|_{\beta-1}^2] + \sup_{t \in [0,T]} \mathbb{E}[1 + \|u_1(t)\|_\beta^2] \Big).
\end{aligned}$$

For the second component we have $\beta \in [1,4]$, so that $\min(\beta,1) = 1$ and $\max(\beta-1,0) = \beta - 1$. For the first component of the third term we use (2.9) with $u_0 = 0$ and $\beta \in (1,3]$ to get

$$\begin{aligned}\mathbb{E}[\|J_{[3,1]}\|^2] &= \sum_{j=0}^{n-1}\int_{t_j}^{t_{j+1}} \mathbb{E}[\|(\Lambda_h^{-1/2}S_h(t_n-t_j)\mathscr{P}_h - \Lambda^{-1/2}S(t_n-t_j))g(u_1(s))\|^2_{\mathscr{L}_2^0}]\,\mathrm{d}s \\ &\le Ch^{\frac{4}{3}\beta}\sum_{j=0}^{n-1}\int_{t_j}^{t_{j+1}} \mathbb{E}[\|\Lambda^{(\beta-1)/2}g(u_1(s))\|^2]\,\mathrm{d}s \\ &\le Ch^{\frac{4\beta}{3}}\sup_{t\in[0,T]}\mathbb{E}[1+\|u_1(t)\|_\beta^2] \le Ch^{\frac{4\beta}{3}}\end{aligned}$$

by Proposition 2.2. The estimate for $\beta \in [0,1]$ is obtained in the same way. For the second component, we also obtain

$$\mathbb{E}[\|J_{[3,2]}\|^2] \le Ch^{\frac{4(\beta-1)}{3}}\sup_{t\in[0,T]}\mathbb{E}[1+\|u_1(t)\|_\beta^2] \le Ch^{\frac{4(\beta-1)}{3}}$$

for $\beta \in [1,4]$. Finally, for the first component of the fourth term, we get

$$\begin{aligned}\mathbb{E}[\|J_{[4,1]}\|^2] &= \sum_{j=0}^{n-1}\int_{t_j}^{t_{j+1}} \mathbb{E}[\|(S(t_n-t_j)-S(t_n-s))\Lambda^{-\beta/2}\Lambda^{(\beta-1)/2}g(u_1(s))\|^2_{\mathscr{L}_2^0}]\,\mathrm{d}s \\ &\le C\sum_{j=0}^{n-1}\int_{t_j}^{t_{j+1}} |s-t_j|^{2\beta}\,\mathrm{d}s\sup_{t\in[0,T]}\mathbb{E}[1+\|u_1(t)\|^2] \\ &\le Ck^{2\beta}\sup_{t\in[0,T]}\mathbb{E}[1+\|u_1(t)\|^2],\end{aligned}$$

for $\beta \in [0,1]$. For $\beta > 1$, we note that $\Lambda^{-\beta/2} = \Lambda^{-1/2}\Lambda^{-(\beta-1)/2}$ and that $\Lambda^{-(\beta-1)/2}$ is bounded so that we get

$$\mathbb{E}[\|J_{[4,1]}\|^2] \le Ck^{2\min(\beta,1)}.$$

Similarly, for the second component, we obtain

$$\mathbb{E}[\|J_{[4,2]}\|^2] \le Ck^{2\min(\beta-1,1)},$$

for $\beta \ge 1$. Altogether the estimate for the stochastic error reads

$$\mathbb{E}[\|\mathrm{Err}_{s,1}\|^2] \le C\cdot\Big(h^{\frac{4\beta}{3}} + k^{2\min(\beta,1)} + k\sum_{j=0}^{n-1}\mathbb{E}[\|U_1^j - u_1(t_j)\|^2]\Big) \qquad \text{for } \beta\in[0,3],$$

$$\mathbb{E}[\|\mathrm{Err}_{s,2}\|^2] \le C\cdot\Big(h^{\frac{4(\beta-1)}{3}} + k^{2\min(\beta-1,1)} + k\sum_{j=0}^{n-1}\mathbb{E}[\|U_1^j - u_1(t_j)\|^2]\Big) \qquad \text{for } \beta\in[1,4].$$

Collecting the estimates of the three parts of the error, we thus obtain the following estimate for the error in the position and velocity of the stochastic wave equation

$$\mathbb{E}[\|U_1^n - u_1(t_n)\|^2] \le C\cdot\Big(h^{\frac{4\beta}{3}} + k^{2\min(\beta,1)} + k\sum_{j=0}^{n-1}\mathbb{E}[\|U_1^j - u_1(t_j)\|^2]\Big), \quad \beta\in[0,3],$$

$$\mathbb{E}[\|U_2^n - u_2(t_n)\|^2] \le C\cdot\Big(h^{\frac{4}{3}(\beta-1)} + k^{2\min(\beta-1,1)} + k\sum_{j=0}^{n-1}\mathbb{E}[\|U_1^j - u_1(t_j)\|^2]\Big), \quad \beta\in[1,4].$$

Using the above error bounds and an application of the discrete Gronwall lemma proves the following result for the mean-square errors of the full discretisation of the semi-linear stochastic wave equation with a multiplicative noise.

THEOREM 3.1. *Consider the numerical discretisation of the semi-linear stochastic wave equation with a multiplicative noise* (1.1) *by a linear finite element method in space and the stochastic trigonometric method* (3.1) *in time. Assume that* $u_0 \in L_2(\Omega, \dot{H}^\beta)$, $v_0 \in L_2(\Omega, \dot{H}^{\beta-1})$ *and that the functions f and g satisfy* (2.8) *for some $\beta \geq 0$ for the error in the position (and for some $\beta \geq 1$ for the error in the velocity). Then, the mean-square errors read*

$$\begin{aligned}
\|U_1^n - u_{h,1}(t_n)\|_{L_2(\Omega,\dot{H}^0)} &\leq C \cdot k^{\min(\beta,1)} \quad \text{for } \beta \in [0,2], \\
\|U_2^n - u_{h,2}(t_n)\|_{L_2(\Omega,\dot{H}^0)} &\leq C \cdot k^{\min(\beta-1,1)} \quad \text{for } \beta \in [1,2], \\
\|U_1^n - u_1(t_n)\|_{L_2(\Omega,\dot{H}^0)} &\leq C \cdot \left(h^{\frac{2\beta}{3}} + k^{\min(\beta,1)}\right) \quad \text{for } \beta \in [0,3], \\
\|U_2^n - u_2(t_n)\|_{L_2(\Omega,\dot{H}^0)} &\leq C \cdot \left(h^{\frac{2(\beta-1)}{3}} + k^{\min(\beta-1,1)}\right) \quad \text{for } \beta \in [1,4].
\end{aligned}$$

Observe that the error estimates between the finite element solutions and the solutions given by the stochastic trigonometric method are proven in a similar way as above, using in addition Proposition 2.3.

4. A trace formula. In this section, we will only consider the problem (1.1) with additive noise ($g \equiv 1$ in (1.1)) and the nonlinearity $f(u) = -V'(u)$ for a smooth potential V. We will further consider a trace-class Q-Wiener process W, i. e., $\mathrm{Tr}(Q) = \|Q^{1/2}\|_{\mathrm{HS}}^2 < \infty$. In this case, the exact solution of our nonlinear stochastic wave equation satisfies a trace formula (see for example [2, 5] for linear stochastic wave equations), where, in analogy to deterministic problems, the "Hamiltonian" function is defined on $H^1 = \dot{H}^1 \times \dot{H}^0$ as

$$\begin{aligned}
H(X) &= \frac{1}{2}\int_{\mathscr{D}} (|u_2|^2 + |\nabla u_1|^2)\,\mathrm{d}x + \int_{\mathscr{D}} V(u_1)\,\mathrm{d}x \\
&= \frac{1}{2}\|u_2\|^2 + \frac{1}{2}\|\Lambda^{1/2}u_1\|^2 + \int_{\mathscr{D}} V(u_1)\,\mathrm{d}x.
\end{aligned}$$

PROPOSITION 4.1. *Consider the nonlinear stochastic wave equation* (1.1) *with additive noise, that is with $g \equiv 1$. Further, let $f(u) = -V'(u)$ for a smooth potential V, let W be a trace-class Q-Wiener process, and let the Hamiltonian H be defined as above. Then the exact solution, $X(t)$ in equation* (2.4)*, of the nonlinear stochastic wave equation* (1.1)*, satisfies the trace formula*

$$\mathbb{E}[H(X(t))] = \mathbb{E}[H(X(0))] + t\frac{1}{2}\mathrm{Tr}(Q), \quad t \geq 0. \tag{4.1}$$

Proof. Indeed, using Ito's formula (Theorem 4.17 in [8]) for the above Hamiltonian, we obtain

$$\begin{aligned}
H(X(t)) = H(X(0)) &+ \int_0^t (H'(X(s)), G\,\mathrm{d}W(s)) + \int_0^t (H'(X(s)), AX + F(X))\,\mathrm{d}s \\
&+ \frac{1}{2}\int_0^t \mathrm{Tr}[H''(X(s))(GQ^{1/2})(GQ^{1/2})^*]\,\mathrm{d}s
\end{aligned}$$

for all time t. Here we have $G = \begin{bmatrix} 0 \\ I \end{bmatrix}$, since we are concerned with additive noise. The expected value of the second term in the above formula is seen to be zero. Using the definition of A and of the nonlinearity F, the integrand present in the third term reads

$$(\Lambda u_1, u_2) + (V'(u_1), u_2) + (u_2, -\Lambda u_2 - V'(u_1)) = 0.$$

Finally, using the above definition of G and the fact that the operator Q is self-adjoint, the last term in the above formula is seen to be equal to

$$\frac{1}{2}\int_0^t \mathrm{Tr}(Q^{1/2}(Q^{1/2})^*)\,\mathrm{d}s = t\frac{1}{2}\mathrm{Tr}(Q).$$

This shows the trace formula (4.1) for the exact solution of our problem. □

REMARK 4.2. *The trace formula is also related to the energy equation, a tool that can be used to analyse the existence, or nonexistence, of solutions to stochastic nonlinear wave equations, see [3] for further details on this topic.*

We next observe that, for the finite element solution X_h, one has

$$H(X_h) = \frac{1}{2}\|u_{h,2}\|^2 + \frac{1}{2}\|\Lambda_h^{1/2}u_{h,1}\|^2 + \int_{\mathscr{D}} V(u_{h,1})\,\mathrm{d}x,$$

because $\|\nabla v_h\| = \|\Lambda^{1/2}v_h\| = \|\Lambda_h^{1/2}v_h\|$ for finite element functions v_h. This results from the definitions of $\Lambda^{1/2}$ and $\Lambda_h^{1/2}$, see Section 2. Using similar arguments as in the proof of the above result, one can now show that the finite element solution $X_h(t)$, defined in (2.5), also possesses a trace formula.

PROPOSITION 4.3. *Let f, g and W be as in Proposition 4.1. The solution of the finite element approximation of problem* (1.1)*, $X_h(t)$ in equation* (2.5)*, satisfies the trace formula*

$$\mathbb{E}[H(X_h(t))] = \mathbb{E}[H(X_h(0))] + t\frac{1}{2}\mathrm{Tr}(\mathscr{P}_h Q \mathscr{P}_h), \quad t \geq 0. \tag{4.2}$$

We will now prove that the full discretisation of the stochastic wave equation, that is the numerical solution given by (3.1), satisfies an almost trace formula. Indeed, as seen in the theorem below, we get a small defect of size $\mathscr{O}(k^{\min(2(\beta-1),1)})$. However, due to the use of Gronwall's inequality, the defect term is not uniform in time.

THEOREM 4.4. *Let f, g and W be as in Proposition 4.1 and 4.3. Let further the assumptions in Theorem 3.1 be fulfilled. Then the stochastic trigonometric method* (3.1) *satisfies an almost trace formula*

$$\mathbb{E}[H(U^n)] = \mathbb{E}[H(U^0)] + t_n\frac{1}{2}\mathrm{Tr}(\mathscr{P}_h Q \mathscr{P}_h) + \mathscr{O}(k^{\min(2(\beta-1),1)}) \tag{4.3}$$

for $0 \leq t_n \leq T$ *and* $\beta \in [1,2]$.

Proof. The proof uses similar techniques as the ones used to prove the mean-square error estimates for the numerical solution in Section 3.

To prove the almost trace formula (4.3), we first add and subtract the expectation of the Hamiltonian for the finite element solution $X_h(t)$

$$\begin{aligned}\mathbb{E}[H(U^n)] &= \mathbb{E}[H(U^n) - H(X_h(t_n))] + \mathbb{E}[H(X_h(t_n))]\\ &= \mathbb{E}[H(U^n) - H(X_h(t_n))] + \mathbb{E}[H(X_h(0))] + t_n\frac{1}{2}\mathrm{Tr}(\mathscr{P}_h Q \mathscr{P}_h)\end{aligned}$$

using Proposition 4.3. We will next show that

$$\mathbb{E}[H(U^n) - H(X_h(t_n))] = \mathscr{O}(k^{\min(2(\beta-1),1)}) \tag{4.4}$$

for $\beta \in [1,2]$. Indeed, we have that

$$\mathbb{E}[H(U^n) - H(X_h(t_n))] = \mathbb{E}\left[\frac{1}{2}\int_{\mathscr{D}} (|U_2^n|^2 - |u_{h,2}(t_n)|^2)\,\mathrm{d}x \right.$$
$$+\frac{1}{2}\int_{\mathscr{D}} (|\Lambda_h^{1/2} U_1^n|^2 - |\Lambda_h^{1/2} u_{h,1}(t_n)|^2)\,\mathrm{d}x$$
$$\left. + \int_{\mathscr{D}} (V(U_1^n) - V(u_{h,1}(t_n)))\,\mathrm{d}x\right]. \tag{4.5}$$

Thus we get three terms to estimate. Using Cauchy-Schwartz inequality, the first term in the above equation can be estimated by (neglecting the factor $\frac{1}{2}$ for ease of presentation)

$$\begin{aligned}
\left|\mathbb{E}[\|U_2^n\|^2 - \|u_{h,2}(t_n)\|^2]\right| &= \left|\mathbb{E}[(U_2^n + u_{h,2}(t_n), U_2^n - u_{h,2}(t_n))]\right| \\
&\leq \left(\mathbb{E}[\|U_2^n + u_{h,2}(t_n)\|^2_{h,\beta-1}]\right)^{1/2} \left(\mathbb{E}[\|U_2^n - u_{h,2}(t_n)\|^2_{h,1-\beta}]\right)^{1/2} \\
&\leq C\left(\mathbb{E}[\|\Lambda_h^{(1-\beta)/2}(U_2^n - u_{h,2}(t_n))\|^2]\right)^{1/2},
\end{aligned}$$

where we have used the discrete norm, the fact that the finite element solution $u_{h,2}(t)$ is bounded in the mean-square sense (see Proposition 2.3), and the fact that the numerical solution given by the stochastic trigonometric method is also bounded, i. e.

$$\mathbb{E}[\|U_1^n\|^2_{h,\beta} + \|U_2^n\|^2_{h,\beta-1}] \leq C < \infty \quad \text{for} \quad n = 0,1,\ldots,N-1.$$

The proof of these estimates is similar to the one for the finite element solution given in Proposition 2.3.

Using the definition of the time integrator and similar techniques as in the proof of the mean-square convergence, one next estimates

$$\begin{aligned}
\Lambda_h^{(1-\beta)/2}(U_2^n - u_{h,2}(t_n)) &= \sum_{j=0}^{n-1}\int_{t_j}^{t_{j+1}} \Lambda_h^{(1-\beta)/2}(C_h(t_n - t_j) - C_h(t_n - s))\mathscr{P}_h\,\mathrm{d}W(s) \\
&\quad + \sum_{j=0}^{n-1}\int_{t_j}^{t_{j+1}} \Lambda_h^{(1-\beta)/2} C_h(t_n - t_j)\mathscr{P}_h(f(U_1^j) - f(u_{h,1}(t_j)))\,\mathrm{d}s \\
&\quad + \sum_{j=0}^{n-1}\int_{t_j}^{t_{j+1}} \Lambda_h^{(1-\beta)/2} C_h(t_n - t_j)\mathscr{P}_h(f(u_{h,1}(t_j)) - f(u_{h,1}(s)))\,\mathrm{d}s \\
&\quad + \sum_{j=0}^{n-1}\int_{t_j}^{t_{j+1}} \Lambda_h^{(1-\beta)/2}(C_h(t_n - t_j) - C_h(t_n - s))\mathscr{P}_h f(u_{h,1}(s))\,\mathrm{d}s \\
&=: J_1 + J_2 + J_3 + J_4.
\end{aligned}$$

Using the temporal regularity of the cosine operator, see (2.10), equation (2.12), and assumptions on g given in (2.8), one gets

$$\begin{aligned}
\mathbb{E}[\|J_1\|^2] &= \sum_{j=0}^{n-1}\int_{t_j}^{t_{j+1}} \|\Lambda_h^{(1-\beta)}(C_h(t_n - t_j) - C_h(t_n - s))\Lambda_h^{(\beta-1)/2}\mathscr{P}_h\|^2_{\mathscr{L}_2^0}\,\mathrm{d}s \\
&\leq C\sum_{j=0}^{n-1}\int_{t_j}^{t_{j+1}} \|\Lambda_h^{(1-\beta)}(C_h(t_n - t_j) - C_h(t_n - s))\Lambda_h^{(\beta-1)/2}\mathscr{P}_h\Lambda^{-(\beta-1)/2} \\
&\qquad \times \Lambda^{(\beta-1)/2}Q^{1/2}\|^2_{\mathrm{HS}}\,\mathrm{d}s \\
&\leq Ck^{2\min(2(\beta-1),1)} \quad \text{for} \quad \beta \in [1,2].
\end{aligned}$$

Next, using the convergence results from Theorem 3.1 and the Lipschitz assumption on f, we observe that

$$\begin{aligned}\big(\mathbb{E}[\|J_2\|^2]\big)^{1/2} &= \sum_{j=0}^{n-1}\int_{t_j}^{t_{j+1}}\big(\mathbb{E}[\|\Lambda_h^{(1-\beta)/2}C_h(t_n-t_j)\mathscr{P}_h(f(U_1^j)-f(u_{h,1}(t_j)))\|^2]\big)^{1/2}\,\mathrm{d}s\\ &\le Ck\sum_{j=0}^{n-1}\big(\mathbb{E}[\|U_1^j-u_{h,1}(t_j)\|^2]\big)^{1/2}\le Ck^{\min(\beta,1)}\quad\text{for}\quad\beta\in[1,2].\end{aligned}$$

Similarly, using the assumptions on f given in (2.8), and the regularity property of the finite element solution stated in Proposition 2.3, one gets

$$\begin{aligned}\big(\mathbb{E}[\|J_3\|^2]\big)^{1/2} &= \sum_{j=0}^{n-1}\int_{t_j}^{t_{j+1}}\big(\mathbb{E}[\|\Lambda_h^{(1-\beta)/2}C_h(t_n-t_j)\mathscr{P}_h(f(u_{h,1}(t_j))-f(u_{h,1}(s)))\|^2]\big)^{1/2}\,\mathrm{d}s\\ &\le Ck^{\min(\beta,1)}\quad\text{for}\quad\beta\in[1,2].\end{aligned}$$

For the last term, J_4, we obtain the estimate for $\beta\in[1,2]$ as follows

$$\begin{aligned}\big(\mathbb{E}[\|J_4\|^2]\big)^{1/2} &= \sum_{j=0}^{n-1}\int_{t_j}^{t_{j+1}}\big(\mathbb{E}[\|\Lambda_h^{(1-\beta)}(C_h(t_n-t_j)-C_h(t_n-s))\\ &\qquad\times\Lambda_h^{(\beta-1)/2}\mathscr{P}_hf(u_{h,1}(s))\|^2]\big)^{1/2}\,\mathrm{d}s\\ &\le Ck^{\min(2(\beta-1),1)},\end{aligned}$$

where we have used equation (2.10), the equivalence between the norms (2.13), the assumptions on the nonlinearity f, and the fact that the finite element solution $u_{h,1}$ is bounded in the norm $\|\cdot\|_{h,\beta-1}$.

Collecting all the above estimates and observing that $2(\beta-1)\le\beta$ for $\beta\in[1,2]$, we finally get

$$\big|\mathbb{E}[\|U_2^n\|^2-\|u_{h,2}(t_n)\|^2]\big|\le Ck^{\min(2(\beta-1),1)}\quad\text{for}\quad\beta\in[1,2].$$

The second term in (4.5) can be estimated in a similar way as above and we obtain

$$\Big|\mathbb{E}[\|\Lambda_h^{1/2}U_1^n\|^2-\|\Lambda_h^{1/2}u_{h,1}(t_n)\|^2]\Big|\le Ck^{\min(2(\beta-1),1)}\quad\text{for}\quad\beta\in[1,2].$$

For the third and final term in (4.5), using the mean value theorem we get first obtain

$$\begin{aligned}\mathbb{E}[\|V(U_1^n)-V(u_{h,1}(t_n))\|_{L_1(\mathscr{D})}] &\le C\mathbb{E}[\|V(U_1^n)-V(u_{h,1}(t_n))\|_{L_2(\mathscr{D})}]\\ &\le C\|V'(\xi)(U_1^n-u_{h,1}(t_n))\|_{L_2(\Omega,\dot H^0)}.\end{aligned}$$

Recalling that $f(u)=-V'(u)$, using Hölder's inequality, using the fact the numerical solutions are bounded in the mean-square sense, and the error bounds stated in Theorem 3.1, we estimate the following expression

$$\begin{aligned}\mathbb{E}[\|V(U_1^n)-V(u_{h,1}(t_n))\|_{L_1(\mathscr{D})}] &\le C\|V'(\xi)(U_1^n-u_{h,1}(t_n))\|_{L_2(\Omega,\dot H^0)}\\ &\le C\Big(\mathbb{E}[\|U_1^n-u_{h,1}(t_n)\|^2_{L_2(\mathscr{D})}]\Big)^{1/2}\le Ck^{\min(\beta,1)}.\end{aligned}$$

Putting all these estimates together we obtain equation (4.4) and the theorem is proven. □

5. Numerical experiments. This section illustrates numerically the main results of the paper. We first present the time integrators we will consider, then test their mean-square orders of convergence on various problems and finally illustrate their behaviours with respect to the trace formula from the previous section.

5.1. Setting. The solution of our stochastic wave equation (1.1) will now be numerically approximated using the method of lines, i. e., with a linear finite element method in space and then with various time integrators (see below). Further, we will consider two kinds of noise: a space-time white noise with covariance operator $Q = I$ and a correlated one with $Q = \Lambda^{-s}$ for some $s > 0$. We refer for example to [5] for a discussion on the approximation of the noise.

We shall compare the stochastic trigonometric method (3.1) with the following classical numerical schemes for stochastic differential equations. When applied to the wave equation in the form (2.1), these numerical integrators are:

1. The forward Euler-Maruyama scheme, see for example [13] or [17],
$$X^{n+1} = X^n + kAX^n + kF(X^n) + G(X^n)\Delta W^n.$$
2. The semi-implicit Euler-Maruyama scheme, see for example [10] or [19],
$$X^{n+1} = X^n + kAX^{n+1} + kF(X^n) + G(X^n)\Delta W^n.$$
3. The backward Euler-Maruyama scheme, see for example [13] or [17],
$$X^{n+1} = X^n + kAX^{n+1} + kF(X^{n+1}) + G(X^n)\Delta W^n.$$
4. The semi-implicit Crank-Nicolson-Maruyama scheme, [10] or [19],
$$X^{n+1} = X^n + \frac{k}{2}A(X^{n+1} + X^n) + kF(X^n) + G(X^n)\Delta W^n.$$

Note that the backward Euler-Maruyama scheme, the semi-implicit Euler-Maruyama scheme, and the semi-implicit Crank-Nicolson-Maruyama scheme are implicit numerical integrators.

All the numerical experiments were performed in Matlab using specially designed software and the random numbers were generated with the command `randn('state',100)`.

5.2. Multiplicative noise. Let us first consider the one-dimensional hyperbolic Anderson model [7, 9]:

$$\begin{aligned}
&\mathrm{d}\dot{u}(x,t) - u_{xx}(x,t)\,\mathrm{d}t = u(x,t)\,\mathrm{d}W(x,t) && \text{for} \quad (x,t) \in (0,1)\times(0,1),\\
&u(0,t) = u(1,t) = 0, && t \in (0,1),\\
&u(x,0) = \sin(2\pi x),\ \dot{u}(x,0) = \sin(3\pi x), && x \in (0,1).
\end{aligned}$$

This stochastic partial differential equation with multiplicative noise is now discretised in space by a linear finite element method with mesh size h. This leads to a system of stiff stochastic differential equations. The latter problem is then discretised in time by various integrators with time step k.

Figure 5.1 illustrates the results on the spatial discretisation of the finite element method as stated in Theorem 3.1. The spatial mean-square errors at time $T_{\text{end}} = 1$,

$$\sqrt{\mathbb{E}\left[\|u_h(x,T_{\text{end}}) - u(x,T_{\text{end}})\|^2\right]},$$

are displayed for various values of the parameter $h = 2^{-\ell}$, $\ell = 2,\dots,9$. The covariance operator is chosen as $Q = \Lambda^{-s}$ for $s = 0, 1/2, 1/3, 1/4$. In the present situation, $f(u) = 0$ and

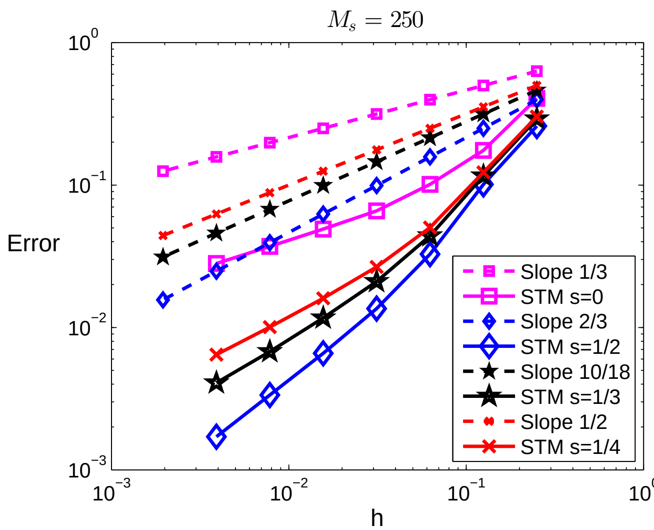

FIGURE 5.1. *The Anderson model: Spatial rates of convergence for the covariance operators* $Q = \Lambda^{-s}$ *with* $s = 0, 1/2, 1/3, 1/4$. *The dotted lines are reference lines of slopes* $1/3, 2/3, 10/18, 1/2$.

$g(u) = u$ satisfy the assumptions (2.8) with $\beta < s + \frac{1}{2}$. This can be seen using the computations done in Subsection 4.1 from [12] (with $\rho = 2s$ and $\alpha = \frac{\beta-1}{2}$). A clear dependence of the spatial convergence rates with respect to the covariance operator can be observed in this figure, in agreement with Theorem 3.1. Here, we simulate the exact solution $u(x,t)$ with the numerical one using the stochastic trigonometric method (STM) (3.1) with a small time step $k_{\text{exact}} = 2^{-9}$ (in order to neglect the error from the discretisation in time) and $h_{\text{exact}} = 2^{-9}$ for the mesh of the FEM. The expected values are approximated by computing averages over $M_{\text{s}} = 250$ samples.

We are now interested in the time discretisation of the above stochastic partial differential equation with space-time white noise ($Q = I$ and thus $\beta < 1/2$). In Figure 5.2, one can observe the rates of mean-square convergence of various time integrators. The expected rate of convergence $\mathcal{O}(k^{1/2})$ of the stochastic trigonometric method as stated in Theorem 3.1 can be confirmed. Again, the exact solution is approximated by the stochastic trigonometric method with a very small time step $k_{\text{exact}} = 2^{-12}$ and uses $h_{\text{exact}} = 2^{-10}$ for the spatial discretisation. $M_{\text{s}} = 250$ samples are used for the approximation of the expected values. The numerical results for the forward and backward Euler-Maruyama schemes are not displayed since these numerical schemes would have to use very small time steps for such an h_{exact} (see also Subsection 5.4 below).

5.3. Semi-linear problem with additive space-time white noise. We next consider the sine-Gordon equation driven by additive space-time white noise ($Q = I$ and thus $\beta < 1/2$)

$$\begin{aligned} &\mathrm{d}\dot{u} - \Delta u \,\mathrm{d}t = -\sin(u)\,\mathrm{d}t + \mathrm{d}W, && (x,t) \in (0,1)\times(0,1),\\ &u(0,t) = u(1,t) = 0, && t \in (0,1),\\ &u(x,0) = 0,\ \dot{u}(x,0) = 1_{[\frac{1}{4},\frac{3}{4}]}(x), && x \in (0,1), \end{aligned}$$

where $1_I(x)$ denotes the indicator function for the interval I.

Figure 5.3 displays the rates of mean-square convergence of various time integrators. The expected temporal rate of convergence $\mathcal{O}(k^{1/2})$ of the stochastic trigonometric method

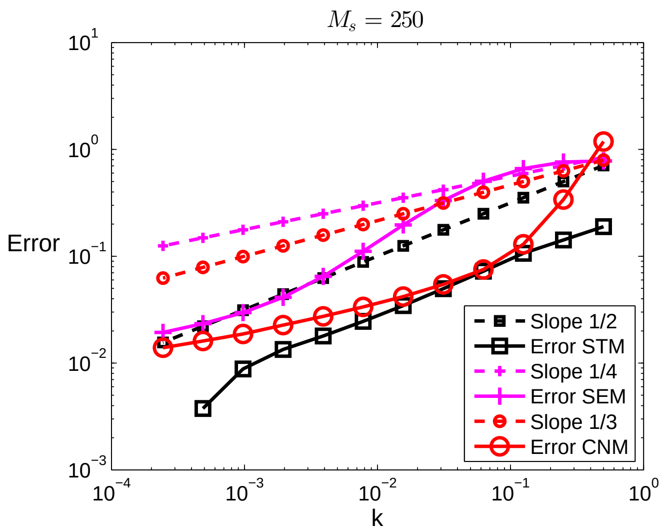

FIGURE 5.2. *The Anderson model (space-time white noise): Temporal rates of convergence of the stochastic trigonometric method (STM), the semi-implicit Euler-Maruyama scheme (SEM) and the Crank-Nicolson-Maruyama scheme (CNM). The reference lines have slopes* $1/4, 1/3$ *and* $1/2$.

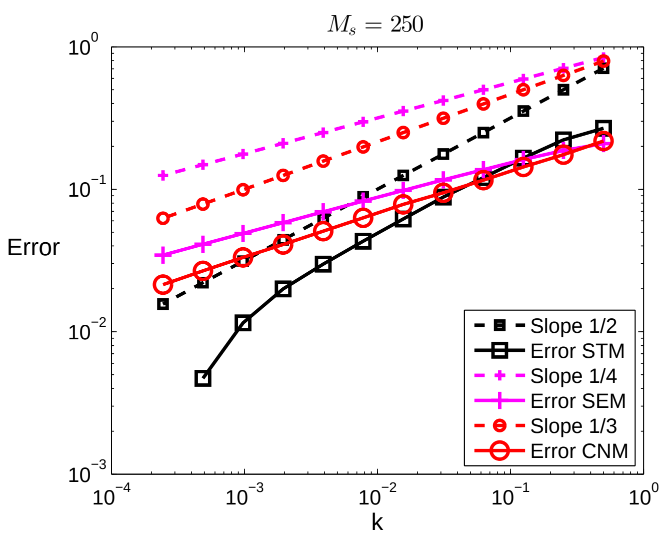

FIGURE 5.3. *The sine-Gordon equation (space-time white noise): Temporal rates of convergence of the stochastic trigonometric method (STM), the semi-implicit Euler-Maruyama scheme (SEM) and the Crank-Nicolson-Maruyama scheme (CNM). The dotted lines have slopes* $1/4, 1/3$ *and* $1/2$.

as stated in Theorem 3.1 can be confirmed. Again, the exact solution is approximated by the stochastic trigonometric method with a very small step size $k_{\text{exact}} = 2^{-12}$ and uses $h_{\text{exact}} = 2^{-10}$ for the spatial discretisation. $M_s = 250$ samples are used for the approximation of the expected values.

5.4. Trace formula. We will now illustrate the trace formula from Section 4. To do this, we again consider the above sine-Gordon equation and solve this problem with a linear finite element method in space and in time we use the stochastic trigonometric method (3.1) with $f(u) = -\sin(u)$, $g(u) = 1$. Figure 5.4 (top) displays the expected value of the Hamiltonian along the numerical solutions of the above stochastic sine-Gordon equation where the covariance operator is given by $Q = \Lambda^{-2}$. In the present situation, the Lipschitz function $f(u) = -\sin(u)$ and the function $g(u) = 1$ satisfy the assumptions (2.8) with $\beta = 2$. This is seen using the fact that the eigenvalues of the Laplace operator with Dirichlet boundary condition satisfy $\lambda_j \sim j^2$ and the eigenvectors are given by $\{\sqrt{2}\sin(j\pi x)\}_j$. The meshes are $h = 0.1$ and $k = 0.01$, the time interval is $[0,5]$, and $M_s = 500$ samples are used for the approximation of the expected values. In this figure, one can observe the unsatisfactory behaviour of classical Euler-Maruyama-type methods. This is not a big surprise, since, already for stochastic ordinary differential equations, the growth rate of the expected energy along solutions given by these numerical solutions is incorrect [18, 4]. The Crank-Nicolson-Maruyama scheme however seems to reproduce very well the linear drift in the expected value of the Hamiltonian. Let us see what happens when one uses bigger time step and longer time interval. Figure 5.4 (bottom) displays the same quantities on the longer time interval $[0,250]$ for the Crank-Nicolson-Maruyama and the stochastic trigonometric methods with a larger time step $k = 0.1$. Excellent behaviour of the stochastic trigonometric method (3.1) is still observed although this does not follow from the result presented in Theorem 4.4.

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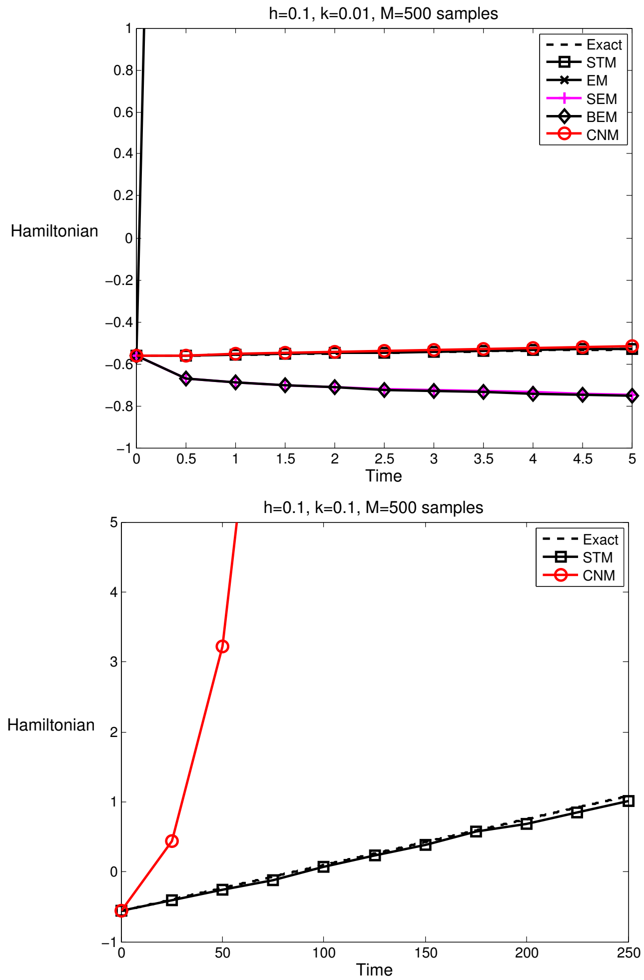

FIGURE 5.4. *Trace formula for the sine-Gordon equation: Expected values of the Hamiltonian along the numerical solutions given by the stochastic trigonometric method (STM), the forward Euler-Maruyama scheme (EM), the semi-implicit Euler-Maruyama scheme (SEM), the backward Euler-Maruyama scheme (BEM) and the Crank-Nicolson-Maruyama scheme (CNM). Time intervals and time steps:* $[0,5]$, $k = 0.01$ *(top) and* $[0,250]$, $k = 0.1$ *(bottom).*

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